Perfect Bayesian Equilibria in Repeated Sales *

Nikhil R. Devanur[†] Yuval Peres[‡] Balasubramanian Sivan[§]

Abstract

A special case of Myerson's classic result describes the revenue-optimal equilibrium when a seller offers a single item to a buyer. We study a *repeated sales* extension of this model: a seller offers to sell a single *fresh copy* of an item to the *same buyer every day* via a posted price. The buyer's private value for the item is drawn initially from a publicly known distribution F and remains the same throughout. A key aspect of this game is that the seller might try to learn the buyers private value to extract more revenue, while the buyer is motivated to hide it. We study the Perfect Bayesian Equilibria (PBE) in this setting with varying levels of commitment power to the seller. We find that the seller having the commitment power to not raise prices subsequent to a purchase significantly improves revenue in a PBE.

1 Introduction

Most interesting economic games are inherently dynamic and/or repetitive, with the same sellers repeatedly interacting with the same buyers. Such scenarios arise commonly in e-commerce platforms, such as eBay and Amazon, and online advertising markets, such as Google, Yahoo! and Microsoft, among others. Unfortunately, the game-theoretic aspects of such repeated interactions are poorly understood compared to their static, one-shot counterparts. In this paper, we develop the theory for one such fundamental setting.

Questions we study. There is a single seller of a certain good (say fish) and a single buyer who enjoys consuming a fresh fish every day. The buyer has a private value v for each day's fish, drawn from a publicly known distribution. However, this value is drawn $only\ once$, i.e., the buyer has the same unknown value on all days. Each day, the seller sets a price for that day's fish, which of course can depend on what happened on previous days. The buyer can then decide whether to buy a fish at that price or to reject. The goal of the buyer is to maximize his total utility (his value minus price on each day he buys and 0 on other days), and the goal of the seller is to maximize profit. How much money can the seller make in n days in a $Perfect\ Bayesian\ Equilibrium\ (PBE)$? The key point here is that the seller is unable to credibly commit to prices for the future days (we refer the reader to Appendix A for a gentle introduction to a Perfect\ Bayesian\ Equulibrium\ and the role of commitment in it). We study three different versions of this problem for arbitrary distributions: the n rounds version without any commitment, the n rounds version with partial commitment, and the time discounted infinite horizon version with partial commitment. Here are the formal definitions.

Definition 1 A 1 seller, 1 buyer Finite Horizon Repeated Sales game is a sequential (extensive form) game between a seller and a buyer, with n rounds. In each round, the buyer has a private valuation of v for a perishable item, with a quasilinear utility. The value v is initially drawn from a distribution F supported on $[\ell, \mathfrak{h}]$ ($0 \le \ell \le \mathfrak{h}$) and stays the same throughout; the seller only knows F. The seller can produce a fresh copy of the item in each round, at a publicly known cost (normalized to) 0. Each round has two stages: the seller first offers a price for the item, and then the buyer responds with an accept or a reject.

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[†]Microsoft Research. nikdev@microsoft.com.

[‡]Microsoft Research. peres@microsoft.com.

[§]Google Research. balusivan@google.com.

Definition 2 A 1 seller, 1 buyer Finite Horizon Partial Commitment Repeated Sales game is the same as a finite horizon repeated sales game with the additional condition that the seller cannot raise prices once a purchase has been made at a certain price. He still holds the freedom to lower prices.

Definition 3 A 1 seller, 1 buyer Time Discounted Infinite Horizon Partial Commitment Repeated Sales game with a discount factor $1 - \delta$ is a partial commitment repeated sales game that is played forever, with the buyer and the seller discounting their round i's utility by $(1 - \delta)^{i-1}$. Equivalently, it can be thought of as a partial commitment repeated sales game (without any discounting) whose stopping time is a geometrically distributed random variable: the probability that the game stops after any given round is δ .

Interpretation as a game with geometric stopping time. While the infinite horizon might appear as a mathematical curiosity with little practical relevance, it is actually the most realistic of the three models. The time discounted infinite horizon game is exactly equivalent to a game with geometric stopping time. With a discount factor of $1-\delta$ per round, the *i*-th round utility is discounted by a factor $(1-\delta)^{i-1}$. Equivalently, if the game stops after any given round with probability δ , the probability that the *i*-th round is reached is $(1-\delta)^{i-1}$, and therefore, any utility obtained in that round has to be discounted by a factor $(1-\delta)^{i-1}$. Often, a geometric stopping time is more realistic than a fixed n day horizon because the buyer or the seller may not be sure of the precise number of interactions that will take place.

A revenue upper bound. We already know that if the seller were able to fully commit to all future prices, he can commit to setting Myerson optimal price for every single day, thereby getting n times Myerson optimal revenue. By not committing to future prices, can the seller get more revenue, or at least as much revenue? Surprisingly, the seller cannot get any more revenue by not committing — this was shown in Baron and Besanko [1984]. The gist of the argument is that if it were possible to extract more than n times Myerson revenue, then it would be possible to extract more than Myerson optimal revenue in a single round game. We present this small and crisp argument in Proposition 1 for a very general mechanism design setting, with arbitrary objectives, arbitrary time discounting, with the number of repetitions of the game possibly being a random variable. We refer the reader to Appendix $\mathbb C$ for a formal description of the model and the proof of the proposition.

Proposition 1 (A simple generalization of the result in Baron and Besanko [1984]) In the general model of repeated mechanism design, the optimal objective value obtained without any commitment is never larger than the optimal objective value obtained when commitment is possible.

The power of commitment and a revenue benchmark. The goal of this paper is to explore the "power of commitment" from the seller's point of view: to what extent does varying levels of commitment impact seller revenue? Proposition 1 says that the revenue obtained by a seller with full commitment power cannot be exceeded by a seller without commitment power, suggesting it as a benchmark for studying the power of commitment. In particular, we seek to study the ratio of seller revenue obtained from a certain level of commitment to that of revenue from full commitment. This ratio quantifies the extent to commitment impacts seller revenue.

The finite horizon repeated sales game (the first version) has been previously considered by Hart and Tirole [1988] and Schmidt [1993]. (See also the survey by Fudenberg and Villas-Boas [2006].) Hart and Tirole [1988] consider the special case where F is a 2-point distribution and Schmidt [1993] generalizes it to any discrete distribution. These papers show that for the finite horizon version of the game, a PBE always exists, and, every PBE charges the minimum possible price ℓ on all but the final few constant number of rounds. For instance, for such a PBE in the n-rounds repeated sales problem where the buyer's value is U[0,1], the seller extracts only a constant amount of revenue, as opposed to our full-commitment-benchmark of n/4, namely, n times the Myerson 1-round revenue of 1/4. While the benchmark grows linearly with n, the no commitment revenue does not even grow with n. The question we ask is whether partial commitment from the seller not to raise prices can significantly improve the situation? A commitment not to raise prices is observed in several settings: for example annual and two-year contracts often offered by Internet and telephone service providers can be seen as a guarantee not to raise prices in the future — they could always offer a lower price if that would entice the buyer to purchase. On the other hand, a guarantee not to lower prices is rarely seen in practice, and is also much harder to enforce. After all, it may be difficult for the seller to resist the temptation

to lower prices if it will entice the buyer to purchase. Moreover while increasing the price may be beneficial to the seller, it is never in the interest of the buyer, but decreasing the price (when it is higher than the buyer's value) benefits both the buyer and the seller.

Our results. The main message of this paper is that the seller possessing the power to credibly commit not to raise prices can guarantee significantly higher revenue, than when he is unable to commit to anything, in very simple PBEs that we call threshold PBEs. We prove this via our analysis of settings 2 and 3 below. For the sake of completeness, we also analyze the well-studied setting 1 for the existence of threshold PBEs. Our main technical results are (a) the derivation of seller's PBE revenue, and the PBE prices posted for the linear demand case (i.e., for the U[0,1] distribution where the demand diminishes linearly with price) in the finite horizon partial commitment game (setting 2 below), and (b) the derivation of equivalence (and its interesting consequences) between infinite horizon partial commitment game (setting 3 below) and the durable goods monopoly / bargaining with one-sided offer setting.

- 1. Finite horizon with no commitment. PBEs being complex objects, we look for the existence of simple PBEs. One notion of simplicity in PBEs is that buyers follow threshold strategies: on each day, the buyer purchases only if his value is above a certain threshold (e.g. see Fudenberg and Tirole [1991], Fudenberg and Villas-Boas [2006]). Such a pure strategy threshold PBE has, among other things, a very simple representation: the strategy and the Bayesian updated seller beliefs are simple to represent. We show that such pure strategy threshold PBEs do not exist in the finite horizon setting with no commitment for any atomless bounded support distribution (supported in $[0, \mathfrak{h}]$). See Theorem 1.
- **2. Finite horizon with partial commitment.** What if the seller has the power to credibly commit not to raise prices upon purchase? Please see "The power of commitment" discussion in the introduction for the motivation for such one-sided commitment. In our second result, we show that with partial commitment from the seller not to raise prices upon purchase, threshold PBEs are guaranteed to exist for all atomless bounded support distributions (Theorem 2). For the case where the buyer value distribution F is U[0,1], the seller's revenue is $\sqrt{\frac{n}{2} + \frac{\log n}{8} + O(1)}$, with a horizon of n rounds. (See Theorem 3.) This is much better than the O(1) revenue that we get when there is no commitment.
- **3. Time discounted infinite horizon with partial commitment.** In our final result we consider the *infinite horizon game with time discounting*, combined with the power of partial commitment: the game is repeated forever but time discounting ensures that players' utilities are still finite, and the seller promises never to raise prices upon purchase.
 - a. We establish a close connection between the partial commitment repeated sales game and the literature on bargaining with one-sided offer and durable goods monopoly in Theorem 4 (see related work Section 1.1.1 for a definition of bargaining and durable goods monopoly settings). We show that for every atomless bounded distribution: for every PBE in a bargaining game (or equivalently the durable goods monopoly setting), there is a corresponding PBE in the repeated sales game with identical utility structure s.t. buyer utility and seller revenue in the repeated sales game are a factor $\frac{1}{\hbar}$ larger than in the bargaining game.
 - b. We use this connection to confirm the famous Coase conjecture (Coase [1972]) in our setting for the linear demand case (i.e., for the U[0,1] distribution) in Theorem 5. In more detail: prior work on bargaining and durable goods monopoly (Sobel and Takahashi [1983], Stokey [1981]) has established that when $\delta \to 0$, the seller's first round price approaches 0, and almost the entire mass of buyers accept in the first round itself in all PBEs where the buyer follows a stationary threshold strategy and seller follows a scale-invariant strategy (see Section 5.2 for definitions of stationarity and scale-invariance). We arrive at the same conclusions for our partial commitment repeated sales setting based on our reduction in Theorem 4. We note that although the revenue from Theorem 5 confirms the Coase conjecture, the revenue of $O(\frac{1}{\sqrt{\delta}})$ obtained from this PBE is quite high, and grows with δ , unlike the revenue from the PBEs in the finite horizon with no commitment case.
 - c. When no equilibrium selection is performed (i.e., not focusing on stationary strategies for buyers etc.), Ausubel and Deneckere [1989] prove a folk theorem that *any* revenue between 0 and Myerson optimal revenue benchmark can be ob-

- tained in a PBE in the infinite horizon bargaining/durable goods monopoly setting. Again, our Theorem 4 immediately implies that the folk theorem applies to our infinite horizon repeated sales setting (Theorem 6).
- d. For the linear demand case (i.e., U[0,1] distribution), Sobel and Takahashi [1983], Stokey [1981] focus on the PBEs where the seller follows scale-invariant strategies. What if we exogenously enforce that the seller follows scale-invariant strategies? I.e., seller's strategy space is restricted to scale-invariant strategies that post a price of pk when his belief is U[0,k] for all k, with p being independent of k. With this exogenous enforcement, it turns out there is a unique PBE in which the seller can extract at least $\frac{4}{3+2\sqrt{2}}\sim 69\%$ of Myerson optimal revenue benchmark, unlike the $O(\frac{1}{\sqrt{\delta}})$ predicted by the Coase-conjecture-confirming Theorem 5 (see Theorem 7 and the following discussion for this result). Further, the price in the first round as $\delta \to 0$ is about 0.586, unlike the $\sqrt{\delta}$ as the first round price predicted by Theorem 5.

1.1 Related Work

1.1.1 Closely related work

We discuss closely related work here. See Appendix B for a discussion of broader related work.

Bargaining with one-sided offer. In bargaining with one-sided offer, a single seller repeatedly makes price offers to a single buyer for the sale of a single unit of good, till a sale is made. The buyer's private value v for the good is drawn from distribution F and is publicly known. In each round, the seller posts a price, and the buyer can either accept or reject the offer the seller made. Once the buyer accepts at a price, the game ends, and the seller's revenue is that round's price. We are primarily interested in the infinite horizon bargaining game: the number of rounds in the game is unbounded, but the buyer and seller have a common discount rate of $1-\delta$ on their utilities, i.e., utilities in round i are scaled by $(1-\delta)^{i-1}$. Equivalently, one could think of the there being a $1-\delta$ probability of the game ending after each round.

Durable goods monopoly. The only difference of a durable goods monopoly from the bargaining with one-sided offer setting is that instead of a single buyer with a continuous type space, durable goods monopoly has a continuum of infinitesimal buyers. But apart from that, just like in bargaining, there is a single seller, and each buyer is interested in consuming exactly one good: once a buyer makes a purchase the same buyer never purchases again because the good is durable and lasts forever. The math for bargaining and durable goods monopoly are identical.

Bargaining, durable goods monopoly and Coase conjecture. In his analysis of the durable goods monopoly setting, Coase [1972] discussed several properties of the equilibria. These include the threshold behavior from buyers (resulting in higher value types buying earlier), the equilibrium path exhibiting a decreasing sequence of prices till the buyer accepts to purchase etc. Perhaps the most dramatic of these is that in a PBE the monopolist's profit (which is same as his revenue since we assumed fixed marginal cost of 0) tends to zero when the discount factor $1 - \delta$ tends to 1. This is surprising because a monopolist, who is by definition without competition, should expect to extract a non-trivial amount of surplus as profit. The reasoning is that the monopolist experiences competition from his own future offers at a lower price — in particular, the discount factor $1 - \delta$ tending to 1 implies that the seller makes his future offers at very quick succession. I.e., he is unable to commit that there won't be any reselling in the future at lower prices.

Bulow [1982] analyzed Coase's conjecture in a finite horizon model and showed that the monopolist's price in every round (except of course the last round) is indeed strictly smaller than the one-shot monopolist's price. Stokey [1981] further verified Coase's conjecture by studying the infinite horizon durable goods monopolist game, and constructed an equilibrium that is the limit of unique equilibria in the finite horizon models. Gul et al. [1986] proved that there is a continuum of PBEs for this setting, but that all of them being qualitatively equivalent, and confirming Coase's conjecture in that the initial price of the seller converges to 0. Sobel and Takahashi [1983] analyzed the infinite horizon model for bargaining with one-sided offer and obtained results much along the lines of Bulow [1982] and Stokey [1981], confirming Coase's conjecture. The PBE obtained is one where the seller follows a scale-invariant strategy,

and the buyer follows a threshold strategy with a stationary threshold (explained in the proof). Both Fudenberg et al. [1985] and Gul et al. [1986] confirm the Coase conjecture in the infinite horizon bargaining game with one-sided offer in the "gap" case, where the smallest point in the buyer's value distribution is strictly larger than the seller's cost.

Behavior based price discrimination. The particular model of repeated sales we study has been investigated in the economics literature, under the name *behavior based price discrimination* (BBPD) [Fudenberg and Villas-Boas, 2006]. The motivation there is that firms can offer personalized prices to consumers based on their past consumption pattern. Such consumption patterns could be collected in various ways, such as when the consumers use loyalty cards, or in an online world where the consumer identifies himself by logging in, or by the use of technology such as cookies. Fudenberg and Villas-Boas [2006] give several other markets where BBPD is observed, such as magazine subscriptions and labor markets. BBPD is also prevalent in government and corporate procurement, from raw materials to IT infrastructure.

The most closely related work to ours is that of Hart and Tirole [1988] and Schmidt [1993], and we have already discussed how our work relates to theirs. Subsequently, many extensions of their models have been studied, such as when consumer preferences vary over time, a monopolist seller selling multiple goods, multile sellers selling the same good who try to poach customers from each other, sellers with multiple versions of the same product, and so on. We refer the reader to Fudenberg and Villas-Boas [2006] for a survey of these results. Another closely related paper is that of Conitzer et al. [2012] who consider a repeated sale game where the buyers have the option of anonymizing themselves at a cost and analyze the effect of varying this cost on the welfare of the buyers. Kanoria and Nazerzadeh [2014] consider a repeated sale setting where in each round two item types are auctioned, to a finite number of agents, with an information structure where agents know their own valuation but not the other agents' valuations. The seller is forced to run a second price auction, but can change the reserve dynamically. Kanoria and Nazerzadeh [2014] show the optimality of static reserve under some assumptions and design an optimal dynamic mechanism when the assumptions fail.

Follow-up work. Immorlica et al. [2017] study the multiple buyers (n buyers) version of our problem, where there is a single fresh copy of a good sold every day via a common posted price for all buyers whose values are all drawn iid from a known distribution F. If more than one buyer is interested in purchasing at the posted price, then the item is awarded to a uniformly random buyer. In this setting, the paper shows that the seller can achieve a revenue that is a constant fraction of the full commitment benchmark in a threshold strategy PBE. The fact that the seller is forced to post a common price for multiple buyers means that the ability to use the reveled preference of the buyers is diminished. This is one of the key differences from the single buyer setting we study. A different but related line of work on repeated auctions is one where the buyer's value is drawn independently from a distribution in every round (unlike our identical value in all rounds but is drawn once initially from a distribution); see Amin et al. [2013], Papadimitriou et al. [2016], Ashlagi et al. [2016], Mirrokni et al. [2016, 2018], Agrawal et al. [2018], Braverman et al. [2018] for more on this line of work. Another line of work on repeated sales with different values across rounds is one where the buyer's value for the good evolves with time/usage: see [Kakade et al., 2013, Chawla et al., 2016].

Organization. In Section 2, we formally define the game, Perfect Bayesian Equilibrium (PBE) and related concepts. In Section 3, we study the finite horizon game with no commitment. In Section 4, we study the finite horizon partial commitment game. In Section 5, we consider the time discounted infinite horizon game with partial commitment and its connections to bargaining and durable goods monopoly. In Section 6 we conclude with some open problems.

2 Preliminaries

Bayesian Nash Equilibrium. The most common notion of equilibrium in a static game of incomplete information is the Bayesian Nash Equilibrium. A profile of strategies is a Bayesian Nash Equilibrium (BNE) if for every agent, given the other agents' strategies, his own strategy maximizes his expected payoff for each of his type. The expected payoff of an agent is computed using the agent's beliefs about the private types of other agents, and all the agents' beliefs are assumed to be consistent with a common prior distribution over all the private types.

History. The game proceeds over n rounds, and each round consists of two stages: round r consists of stages k = 2r - 1 and k = 2r. At k = 2r - 1, the seller sets a price, and at k = 2r, the buyer reacts with a accept or reject. The history after k stages of play is denoted by h^k , and constitutes the prices and accept/reject decisions of all stages k' : 0 < k' < k.

Beliefs and buyer types. In our game, since the buyer's type alone is private, the seller alone has a belief over the buyer's private type. The seller's belief $\mu(\cdot|h^k)$ is a probability density function over the buyer's private type. At the beginning of the game, the buyer types are assumed to be drawn from a publicly known, atomless bounded support distribution in $[\ell, \mathfrak{h}]$.

Strategy spaces. The seller's action space is restricted to posting a non-negative price in every round. Correspondingly, the seller's strategy $\sigma_s(\cdot|h^k)$ is a function that, for every possible history, outputs a probability distribution over his available actions (i.e., non-negative prices). The buyer's action space is restricted to accepting or rejecting a price. Correspondingly, the buyer's strategy $\sigma_b(\cdot|v,h^k)$ is a function that, for every possible private value of the buyer and every possible history, outputs a probability for accepting the item at the posted price.

Perfect Bayesian Equilibrium. Intuitively a Perfect Bayesian Equilibrium combines the notions of subgame perfect equilibrium (used in dynamic games of complete information) and Bayesian update of beliefs (used in games of incomplete information) by requiring that the profile of strategies and beliefs when applied to the continuation game given any history, form a BNE. It is the perfection aspect of PBE that makes commitments non-credible/non-binding: informally, no commitment is credible unless it is a part of a BNE in the continuation game after every possible history that could precede the stage at which the commitment becomes effective. We now formally define Perfect Bayesian Equilibrium for our game, i.e., mention only the restrictions relevant to our game.

A profile of strategies $(\sigma_s^*(\cdot|h^k), \{\sigma_v^*(\cdot|v,h^k)\}_v)$ and beliefs $\mu(\cdot|h^k)$ in the repeated-sale game is a Perfect Bayesian Equilibrium (PBE) when the following conditions are satisfied:

- 1. Bayesian update of seller's beliefs: the seller assumes that the buyer plays the PBE strategy $\sigma_v^*(\cdot|v,h^k)$. If there exists some value v in the support of seller's belief $\mu(\cdot|h^{k-1})$, such that the buyer's action at stage k has a non-zero probability under his equilibrium strategy $\sigma_v^*(\cdot|v,h^k)$ at v, the seller updates his belief $\mu(\cdot|h^k)$ based on Bayes' rule. There are no restrictions on belief updates if the buyer takes an out-of-equilibrium zero-probability action.
- 2. For every k and h^k , the strategies from h^k onwards are a BNE for the remaining game. Formally, conditional on reaching h^k , let $u_s\left(\sigma_s(\cdot|h^k), \{\sigma_v(\cdot|v,h^k)\}_v, \mu(\cdot|h^k)\right)$ denote the expected revenue of seller under strategy profile σ (where the expectation is over both the randomness in σ and the belief $\mu(\cdot|h^k)$), and let $u_v\left(\sigma_s(\cdot|h^k), \{\sigma_v(\cdot|v,h^k)\}_v, \mu(\cdot|h^k)\right)$ denote the expected utility of the buyer type v under strategy profile σ (where the expectation is over the randomness in σ). Then,

$$u_s\left(\sigma_s^*(\cdot|h^k), \{\sigma_v^*(\cdot|v, h^k)\}_v, \mu(\cdot|h^k)\right) \ge u_s\left(\sigma_s(\cdot|h^k), \{\sigma_v^*(\cdot|v, h^k)\}_v, \mu(\cdot|h^k)\right) \qquad \forall k, \forall h^k, \forall \sigma_s$$

$$u_v\left(\sigma_s^*(\cdot|h^k), \{\sigma_v^*(\cdot|v, h^k)\}_v, \mu(\cdot|h^k)\right) \ge u_v\left(\sigma_s^*(\cdot|h^k), \sigma_v(\cdot|v, h^k), \{\sigma_x^*(\cdot|x, h^k)\}_{x \ne v}, \mu(\cdot|h^k)\right) \qquad \forall v, \forall k, \forall h^k, \forall \sigma_v$$

Threshold PBE. A threshold strategy for the buyer computes an accept/reject decision as follows: given history h^k , there exists a deterministic threshold $t(h^k) \ge 0$, and buyer type v accepts the item if $v \ge t(h^k)$ and rejects otherwise. By definition, a threshold strategy is a pure strategy. In this paper, we focus on pure-strategy threshold strategy PBEs, i.e., threshold strategy for the buyer and pure strategy for the seller.

Single buyer vs. continuum of buyers. As discussed in many prior works (e.g. see Section 10.2.1 of Fudenberg and Tirole [1991]), given a distribution of types, one can interpret our problem as a single seller against a single buyer with a private type drawn from this type space, or a continuum of buyer types whose values are given by the type distribution. In the latter case, we assume that the seller cannot distinguish between buyer types and can simply observe the measures of buyer sets that accept or reject. For most of this paper we use the single buyer interpretation.

Simultaneous deviation of several buyer types. Following the convention from previous works (Gul et al. [1986]), we do not specify the equilibrium behavior following simultaneous deviation by several buyers. In the single buyer interpretation, this does not matter because simultaneous deviations are not visible to the seller, but even in the continuum of buyers model where it could be observable if a non-zero measure of buyers deviate we do not specify equilibrium behavior following such deviations.

The no-gap case. We assume that the seller has a fixed marginal cost for producing each copy of the good, and (without loss of generality) that it is 0. Prior work has distinguished two important cases of buyer distributions as a function of the seller cost: whether there is a gap (the gap case) between ℓ and 0 or whether $\ell=0$ (the no-gap case). We focus on the (more natural) no-gap case i.e., $\ell=0$ in Section 3. The threshold PBE existence result in Section 4 applies for both the gap and no-gap case.

PBE specification. As is clear from the definition of a PBE, seller's and buyer's strategies should be specified in off-equilibrium histories as well, for a strategy profile to constitute a PBE. In the theorems we prove, the off-equilibrium behavior can be immediately derived from applying the fact that the threshold buyer is indifferent between buying and rejecting. Thus we skip the excessively long specification of strategies after all off-equilibrium histories. For the 2 round case alone (Section 3.1), we specify strategies in complete detail for the sake of clarity.

Notational convention for thresholds. We use $t = \infty$ to denote the buyer rejecting to buy at all values. Thus, a threshold t lies in $[\ell, \mathfrak{h}] \cup \{\infty\}$.

3 Finite horizon with no commitment

We begin with two simple facts that are useful in analyzing PBEs. Their proofs are immediate.

Fact 1 (Indifference at threshold) In any threshold PBE, the buyer with his value v equals the threshold t is indifferent between accepting and rejecting (except for $t = \infty$).

Fact 2 (Bayesian price update) If the buyer accepts in a given round with threshold t all future round prices are at least t, and if he rejects in a given round with threshold t, all future round prices are at most t.

3.1 Two rounds game

It turns out that for a two rounds game, a threshold PBE is guaranteed to exist and it is essentially unique. Hart and Tirole [1988] and Fudenberg and Villas-Boas [2006] characterize the PBE for the two rounds repeated sales game. For the sake of completeness and for gaining intuition, and because our main result uses the 2 rounds PBE (mildly), we discuss the 2 rounds PBE now.

Notation. We begin with some notation and two quick definitions. Let $F_{[a,b]}$ denote the distribution on v conditioned on the fact that $a \le v \le b$ (and thus $F = F_{[\ell, \mathfrak{h}]}$). Let $p_{[a,b]}^*$ denote an arbitrary element of $\mathop{\mathrm{argmax}}_p p(1 - F_{[a,b]}(p))$ i.e., the set of all single-round revenue maximizing prices or the so called *monopoly prices* for $F_{[a,b]}$. Let $p^* = p_{[\ell, \mathfrak{h}]}^*$. Whenever the monopoly price is not unique $p_{[a,b]}^*$ will denote an arbitrary monopoly price unless specified otherwise.

Revenue Curve. The revenue curve $R_{[a,b]}(p) = p(1 - F_{[a,b]}(p))$ at p gives the expected revenue in a single round game obtained by offering a price p to a buyer whose value is drawn from $F_{[a,b]}$. Let $R(\cdot) = R_{[\ell,\mathfrak{h}]}(\cdot)$ denote the revenue curve for the distribution F.

We prove here a property that any PBE in two rounds game for any atomless bounded support distribution must satisfy.

In the following lemma, p_1 is the price in the first round, p_{20} and p_{21} are the prices in second round, upon buyer's rejection and acceptance respectively in the first round, given that the first round price is p_1 . We will use $t(p_1)$

to denote the threshold used by the buyer in first round. An application of Fact 2 shows that the price p_{21} in the second round is at least $t(p_1)$. A further application of Fact 1 shows that $p_{20} = p_1$ because indifference implies that $t(p_1) - p_{20} = t(p_1) - p_1$.

Lemma 1 For any atomless distribution F of buyer's value supported in $[\ell = 0, \mathfrak{h}]$, every pure strategy threshold PBE of a two rounds repeated sales game will have p_1 such that $\ell < t(p_1) < \mathfrak{h}$.

Proof: First note that $t(p_1)$ is unique. If not, there will be two thresholds $t(p_1) < t'(p_1)$ such that $p_{[\ell,t(p_1)]}^* = p_{[\ell,t'(p_1)]}^* = p_1$. I.e., the virtual values for $F_{[\ell,t]}$ and $F_{[\ell,t']}$, namely $\phi_{F_{[\ell,t']}}$ and $\phi_{F_{[\ell,t']}}$ both become zero at p_1 . This is not possible because for any $x \le t$, we have $F_{[\ell,t']}(x) = F_{[\ell,t]}(x) \cdot \alpha$ for some $\alpha < 1$. This means, $\phi_{F_{[\ell,t']}}(x) < \phi_{F_{[\ell,t]}}(x)$ for all $x \le t$. Therefore both the virtual value functions cannot become zero at the same point.

We will prove the lemma by showing that when $t(p_1) = \ell$ or \mathfrak{h} , the seller's revenue is exactly $R(p^*)$, i.e., the monopolist's single round revenue from F. We show that the seller can do strictly better than $R(p^*)$ in a two round PBE with $\ell < t(p_1) < \mathfrak{h}$. The former statement is immediate: when all buyers reject in the first round, the first round revenue is 0, and the maximum possible revenue from just second round alone is $R(p^*)$. Similarly, when all buyers accept in the first round, the first round price p_1 must have been 0 (otherwise buyers with $v < p_1$ would have incurred negative utility), and here again, all revenue comes from second round, which is at most $R(p^*)$.

Whenever $\ell < t(p_1) < \mathfrak{h}$, note that the seller's revenue is exactly $R(p_{20}) + R(p_{21})$: the buyer buys once when his value exceeds p_{20} and once more when his value exceeds p_{21} . Use this to see that if $\ell < t(p_1) \le p^*$, seller's revenue is strictly larger than $R(p^*)$: because $p_{21} = p^*_{[t(p_1),\mathfrak{h}]} = p^*_{[\ell,\mathfrak{h}]} = p^*$ (the last but one equality holds for all truncated distributions of the form $F_{[x,\mathfrak{h}]}$ where $x \le p^*$: this is true in this case since $t(p_1) \le p^*$), and $p_{20} = p^*_{[\ell,t(p_1)]}$, which yields a revenue of $R(p^*_{[\ell,t(p_1)]}) + R(p^*) > R(p^*)$. This completes the proof.

There is one detail to fill: is it always possible to find a p_1 such that $\ell < t(p_1) \le p^*$? Since $\ell = 0$, it follows that $p^* > \ell = 0$ and clearly setting $t(p_1) = x$ for any x s.t. $0 < x < p^*$ will do, since the distribution $F_{[0,x]}$ will have a non-zero monopoly price, and that will be p_1 .

Solution to the general 2 rounds repeated sales game. The proof of Lemma 1 gives us a recipe for constructing a 2 rounds PBE for an arbitrary atomless bounded support distribution. As discussed in Lemma 1's proof, the seller's goal is to maximize $R(p_{20}) + R(p_{21})$. Because of Fact 1, it follows that $p_1 = p_{20}$ in any PBE. Thus the seller has to compute a first round price of z such that $R(p_{20}) + R(p_{21}) = R(z) + R(p_{[t(z),1]}^*)$ is maximized. This is basically what is captured in the strategies in Algorithms 1 and 2. Note that $p_1 > p^*$ cannot be in the equilibrium path: if $p_1 > p^*$, the equilibrium behavior will be for all buyers to reject; if on the contrary we had $t(p_1) < \mathfrak{h}$, the price p_{20} in the second round is the monopoly price of the distribution $F_{[\ell,t(p_1)]}$ which is strictly lesser than p^* , thereby making the threshold buyer non-best-responding in this case).

For concreteness, we discuss the special case of U[0,1] in Appendix D.

3.2 General n rounds game.

We now move to the main result of this section: in a n rounds repeated sales game for n > 2, a pure strategy threshold PBE never exists.

Theorem 1 For any atomless distribution F of buyer's value supported in $[\ell = 0, \mathfrak{h}]$ and for any n > 2, a pure strategy threshold PBE never exists in a n rounds repeated sales game.

Proof: Consider the three rounds case first. Let p_1 be the first round price. Let $t=t(p_1)$ be the corresponding buyer threshold in the first round. Note that a PBE requires that given any history, the strategies for the continuation game must be mutually best responding. Consider one such history where the first round price is $p_1 > \ell$ (we are not fixing on the equilibrium p_1 , but an arbitrary first round price p_1 — recall that for every first round p_1 a PBE must specify equilibrium behavior in the continuation game). We show in three cases that irrespective of what value $t(p_1)$ takes, we cannot have a threshold PBE with $p_1 > \ell$.

ALGORITHM 1: Seller's strategy in the general 2 rounds game

```
Round-1 strategy: Let t(x) be such that p_{[\ell,t(x)]}^* = x Set p_1 = \operatorname{argmax}_{z \leq p^*} \{R(z) + R(p_{[t(z),1]}^*)\} Round-2 strategy: if p_1 \leq p^* then

| if Buyer rejects in round 1 then
| Set second round price of p_2 = p_1 else if Buyer accepts in round 1 then
| Set second round price of p_{[t(p_1),1]}^* else if p_1 > p^* then
| if Buyer rejects in round 1 then
| Set second round price of p_2 = p^* else if Buyer accepts in round 1 then
| Set second round price of p_2 = p^* else if Buyer accepts in round 1 then
| Set second round price of p_2 = p^* else if Buyer accepts in round 1 then
| Set second round price of p_2 = p^*
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ALGORITHM 2: Buyer's strategy in the general 2 rounds game

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\label{eq:cond-1} \begin{split} & \frac{\textbf{Round-1 strategy:}}{\textbf{if } p_1 \leq p^* \textbf{ then}} \\ & \quad | \quad \text{Let } t(p_1) \textbf{ be such that } p^*_{[\ell,t(p_1)]} = p_1 \\ & \quad | \quad \textbf{if } v \geq t(p_1) \textbf{ then} \\ & \quad | \quad \textbf{ Accept} \\ & \quad \textbf{else} \\ & \quad | \quad \textbf{ Reject} \\ \\ & \quad \textbf{else if } p_1 > p^* \textbf{ then} \\ & \quad | \quad \textbf{ Reject} \\ \\ & \quad \textbf{Round-2 strategy:} \\ & \quad \textbf{if } v \geq p_2 \textbf{ then} \\ & \quad | \quad \textbf{ Accept} \\ & \quad \textbf{else} \\ & \quad | \quad \textbf{ Reject} \\ \end{split}
```

Case 1: When $p_1 > \ell$, $t(p_1) = \ell$ is not possible in PBE. Clearly this is not possible as the buyer with value ℓ gets negative utility in the first round, and 0 utility in the future rounds since price is guaranteed to be at least ℓ in future rounds.

Case 2: When $p_1 > \ell$, $\ell < t(p_1) < \mathfrak{h}$ is not possible in PBE. We show that the threshold buyer is never indifferent between accepting and rejecting in first round, violating Fact 1. Begin by noting that $t(p_1) \ge p_1$.

- 1. If $t(p_1) = p_1$, the threshold buyer makes 0 utility upon acceptance, where as by rejecting he can make a non-negative utility as he will be the largest point in the support of the seller beliefs after rejection, namely $F_{[\ell,t]}$.
- 2. The only remaining case is that $p_1 < t(p_1) < \mathfrak{h}$. In this case, the threshold buyer upon accepting the price p_1 in the first round cannot get any further utility in the future rounds, as the prices are at least $t(p_1)$ in the future given seller beliefs are $[t(p_1),\mathfrak{h}]$ after first round. Thus his total utility is $t(p_1)-p_1$ upon accepting in the first round. We show that by rejecting in the first round, the threshold buyer obtains strictly larger utility which is a contradiction. Let p_{20} be the price in the second round on rejection, and let p_{300} and p_{301} denote the price in the third round upon (reject, reject) and (reject, accept) respectively in the first two rounds. When a buyer with value t rejects in the first round, and accepts in the second and third rounds, he gets a utility of $(t(p_1)-p_{20})+(t(p_1)-p_{301})$. The two claims below show that $p_{20} \leq p_1$ and $p_{301} < t(p_1)$. Therefore the sum $(t(p_1)-p_{20})+(t(p_1)-p_{301})$ is strictly larger than $t(p_1)-p_1$.

Claim 1: $p_{20} \le p_1$. On the contrary suppose that $p_{20} > p_1$. Consider a buyer value with v s.t. $p_1 < v < p_{20}$. Such a buyer gets zero utility upon rejection in the first round because all prices after rejection are strictly larger than his value v (because the second round price $p_{20} > v$ by our choice of v, and a threshold PBE for the remaining two rounds game implies that $p_{20} = p_{300} \le p_{301}$. The equality follows from applying Fact 1 to the threshold buyer in second round, and inequality follows from noting that the third round price of seller is at least second round threshold which is at least second round price of p_{20} . Where as upon acceptance, he would have a gotten a strictly positive utility of $v - p_1$. This says that v accepts in first round, i.e., $v \ge t(p_1)$. But this is a contradiction because $v < p_{20}$ by our choice of v, and, $p_{20} \le t(p_1)$ because the belief after first round is $F_{[\ell,t(p_1)]}$.

Claim 2: $p_{301} < t(p_1)$. We show that all prices after rejection in first round, namely, p_{20}, p_{300}, p_{301} are strictly smaller than $t(p_1)$. Even if the largest among these prices, namely p_{301} , was equal to $t(p_1)$, that would not be a PBE. To see this, consider the threshold t' used by the buyer for the distribution $F_{[\ell,t(p_1)]}$ in the two round continuation game following the first round price of p_1 . By Lemma 1 such a threshold t' is strictly smaller than the largest point in the support $t(p_1)$. Note that p_{301} is simply be the monopoly price for the distribution $F_{[t',t(p_1)]}$. If $t' < t(p_1)$, this monopoly price p_{301} cannot be $t(p_1)$ because that yields a 0 revenue which is not optimal. Thus $p_{301} < t(p_1)$.

Case 3: When $p_1 > \ell$, $t(p_1) = \mathfrak{h}$ is not possible in PBE. For a price $p_1 > \ell$ to be rejected by all buyer types in the first round (i.e., for having $t(p_1) = \mathfrak{h}$), we need $p_{20} \leq p_1$ (for otherwise a buyer with value $\ell < p_1 < v < p_{20}$ would not be best responding by rejecting in the first round). I.e., for a PBE to exist, for every $p_1 > \ell$, there should be a two round threshold PBE for F with a first round price $p_{20} \leq p_1$ for every $p_1 > \ell$. But when p_{20} grows arbitrarily close to 0, the revenue in the last two rounds game becomes arbitrarily close to the last round revenue — on the other hand, from the proof of Lemma 1, we know that the seller in a two rounds PBE can always get a revenue strictly better than and bounded away from the single round revenue, and hence having p_{20} arbitrarily close to 0 cannot be a part of a PBE.

Three rounds to n rounds. If a three rounds threshold PBE cannot exist, neither can an n-rounds threshold PBE.

4 Finite horizon with partial commitment

While a threshold PBE never exists when there is no commitment from the seller's side, things change dramatically if we allow partial commitment. We show that by having the partial commitment from the seller of not raising prices upon purchase, a pure strategy threshold PBE is guaranteed to exist for all distributions (both the gap-case and no-gap case, namely ℓ is not necessarily 0).

Numbering convention. For this section alone we change our convention for numbering rounds compared to what we used in previous sections: the price, and threshold in the first round of the n rounds game are denoted by p_n and t_n (in earlier sections we used p_1 and t_1 for the first round). Similarly the second round's corresponding quantities are p_{n-1} and t_{n-1} and so on.

Theorem 2 For any atomless distribution F of buyer's value supported in $[\ell, \mathfrak{h}]$, and for any n, a pure strategy threshold PBE always exists in a n rounds partial commitment repeated sales game.

Proof: We prove by induction on the number of rounds r that a pure strategy threshold PBE is guaranteed to exist for $F_{[\ell,x]}$ for all $\ell \leq x \leq \mathfrak{h}$.

Base case: When r=1, a pure strategy threshold PBE trivially exists for all $F_{[\ell,x]}$: the seller posts the monopoly price of the distribution $F_{[\ell,x]}$, and buyers with values at least the monopoly price of $F_{[\ell,x]}$ accept that price.

Inductive hypothesis: Assume that for all $r \le n-1$ a threshold PBE exists for $F_{[\ell,x]}$ for all $\ell \le x \le \mathfrak{h}$ in a r rounds partial commitment game.

Inductive step: Consider r = n. Let $p_{r,[a,b]}$ denote the first round PBE price in a r-rounds game for distribution $F_{[a,b]}$ (if more than one PBE exists, fix an arbitrary one). We use p_r to denote $p_{r,[\ell,b]}$.

Price remains fixed upon acceptance. We begin by showing that when a buyer accepts a price of p_n in the first round, the price remains p_n for the remaining n-1 rounds. Clearly by the definition of the game, the price cannot increase beyond p_n . Also, the price will not decrease below p_n because, if t_n is the threshold used by the buyer in the first round $(t_n \ge p_n)$, the future belief of the seller will be $[t_n, \mathfrak{h}]$, i.e., the smallest point t_n in the support of the distribution is larger than p_n , and therefore p_n will be accepted in all future rounds.

Continuation game has a PBE after first round price of p_n . We now show that for each value of the first round price p_n , the continuation game has a PBE. Let $t_n(p_n)$ (abbreviated to just t_n) be the buyer's threshold while facing a price of p_n . Indifference at threshold (Fact 1) implies that

$$n(t_n - p_n) = k \cdot (t_n - p_{k, [\ell, t_n]}) \tag{1}$$

Here the LHS in (1) is the utility of the threshold buyer upon accepting a price of p_n in the first round, and the RHS is his utility upon rejecting p_n in the first round. The LHS utility is clear: once a price of p_n is accepted, it remains so for all the remaining rounds, giving a utility of $n(t_n-p_n)$. For the RHS utility, note that after a threshold of t_n in the first round, the seller's belief is updated to $F_{[\ell,t_n]}$ after rejection. The PBE of $F_{[\ell,t_n]}$ could possibly have several "wasted" rounds where all buyers reject, and finally at some round there is a non-trivial threshold: this is why we have a k in the RHS instead of n-1, and $k \in \{1,2,\ldots,n-1\}$.

As a sanity check, note that we require $t_n \ge p_n$, and this is indeed immediate from (1) because $p_{k,[\ell,t_n]} \le t_n$. The seller now maximizes his revenue R_n from n rounds as

$$R_n = \max_{p_n} \left\{ R_{n-1,[\ell,t_n]} F(t_n) + n p_n (1 - F(t_n)) \right\}$$
 (2)

Given that t_n can be obtained as a function of p_n using (1) and that $R_{n-1,[\ell,t_n]}$ is well defined by induction to be the PBE revenue in n-1 rounds for the distribution $F_{[\ell,t_n]}$, it follows that the seller has a well-defined maximization problem in (2), and picks the price p_n to maximize his revenue.

This proves the existence of a PBE in n rounds for the distribution F. An identical argument will show the existence of n rounds PBE for $F_{[\ell,x]}$ for all $\ell \leq x \leq \mathfrak{h}$.

The U[0,1] case and uniqueness. It is not possible to prove uniqueness at the level of all atomless distributions. For the sake of concreteness, and to illustrate the difference partial commitment can make, we focus on the U[0,1] distribution and compute a threshold PBE that obtains a revenue of $\sqrt{\frac{n}{2} + \frac{\log n}{8} + O(1)}$ in a n rounds game, and also establish its uniqueness. In comparison, the no commitment n rounds game does not even have a threshold PBE.

Theorem 3 For the U[0,1] distribution, the n rounds partial commitment repeated sales game has a unique¹ pure strategy threshold PBE that obtains a revenue of $\sqrt{\frac{n}{2} + \frac{\log n}{8} + O(1)}$.

Proof: We prove by induction on the number of rounds r that for each x in (0,1], the distribution $F_{[0,x]}$ satisfies the following:

- 1. the r rounds partial commitment game on $F_{[0,x]}$ has a pure strategy threshold PBE;
- 2. the threshold PBE is unique;
- 3. the PBE threshold t_r of the buyer in the first round of the r rounds game is non-trivial (i.e., not at the end-points of the support), namely, $0 < t_r < x$.

¹ the uniqueness here refers to the equilibrium path and ignores irrelevant multiplicities that arise in off-equilibrium path. Note that there are trivial ways to have multiplicity in the off equilibrium path. For instance, when the seller posts a price of 0 in a round (an off-equilibrium path), the only possible PBE behavior in the continuation game is for all buyers to accept the price. If some buyer doesn't, that constitutes a zero probability action and the seller is free to update beliefs arbitrarily. At this node, any probability distribution over [0, 1] as belief can still support the equilibrium path behavior discussed in the theorem. We ignore these trivial and irrelevant multiplicities in off-equilibrium path.

Base case: When r = 1, a pure strategy threshold PBE trivially exists for all $F_{[0,x]}$: the seller posts the monopoly price x/2, and buyers with values at least x/2 accept that price. Uniqueness is obvious from the uniqueness of $\underset{n}{\operatorname{argmax}}_n p(x-p)$. The threshold of x/2 is non-trivial, i.e., 0 < x/2 < x.

Inductive hypothesis: Assume that for all $r \le n-1$ a threshold PBE exists for $F_{[0,x]}$ for all $0 < x \le 1$ in a r rounds partial commitment game, and that it is unique, with a non-trivial threshold t_r in the first round.

Inductive step: Consider r = n. Let $p_{r,[a,b]}$ denote the first round PBE price in a r-rounds game for distribution $F_{[a,b]}$. We use p_r to denote $p_{r,[0,1]}$.

Price remains fixed upon acceptance. Identical to our proof in Theorem 2, it holds that when a buyer accepts a price of p_n in the first round, the price remains p_n for the remaining n-1 rounds.

Continuation game has a PBE after first round price of p_n . Just like in Theorem 2, applying indifference at threshold (Fact 1) implies that

$$n(t_n - p_n) = (n-1) \cdot (t_n - p_{n-1,[0,t_n]})$$
(3)

The only difference from (1) is that instead of an arbitrary number $k \le n-1$, the RHS now has exactly n-1: this is because by inductive hypothesis, we have that the threshold is non-trivial in the first round of an n-1 round game. So for the $U[0,t_n]$ distribution that is left after rejecting in the first round, a non-trivial threshold implies that at least the largest point in the support of the distribution, namely t_n , buys in the second round, in which case his utility is $(n-1) \cdot (t_n - p_{n-1,[0,t_n]})$.

To simplify this further, note that $U[0, t_n]$ is simply a scaled version of U[0, 1]. Thus, $p_{n-1,[0,t_n]} = t_n \cdot p_{n-1,[0,1]} = t_n \cdot p_{n-1}$. Thus, we can rewrite (3) as

$$n(t_n - p_n) = (n-1)t_n \cdot (1 - p_{n-1}) \tag{4}$$

Let u_n be the PBE utility of the agent with value 1 in a n rounds game. Clearly,

$$u_n = n(1 - p_n). (5)$$

By this definition of u_n , equation (4) can be rewritten as

$$n(t_n - p_n) = u_{n-1}t_n. (6)$$

The seller's revenue, much like (2) can be written as

$$R_{n} = \max_{p_{n}} \left\{ R_{n-1,[0,t_{n}]} t_{n} + (1 - t_{n}) \cdot n p_{n} \right\}$$

$$= \max_{p_{n}} \left\{ R_{n-1} t_{n}^{2} + (1 - t_{n}) \cdot n p_{n} \right\}.$$
(7)

We now have a four variable recurrence in $\{u_n, t_n, p_n, R_n\}$ to solve, given by equations (5), (6), (7). Substituting for p_n from equation (6) into equation (7) we have

$$R_n = \max_{t_n} \left\{ R_{n-1} t_n^2 + (1 - t_n) t_n (n - u_{n-1}) \right\}.$$
 (8)

This is an expression for revenue that the seller has to maximize. Notice that R_{n-1} and u_{n-1} are fixed quantities that are not to be optimized: these are quantities for the n-1 rounds game for which we assume by induction that there is a unique threshold PBE and hence revenue, utilities etc. are fixed. The only quantity to optimize in this expression is

 t_n . This expression for R_n is maximized at $t_n = \frac{n - u_{n-1}}{2(n - u_{n-1} - R_{n-1})}$. Substituting this value of t_n into equation (6), we get

$$p_n = \frac{(n - u_{n-1})^2}{2n(n - u_{n-1} - R_{n-1})}. (9)$$

Similarly, substituting t_n into equation (8), we get

$$R_n = \frac{(n - u_{n-1})^2}{4(n - u_{n-1} - R_{n-1})}. (10)$$

From equations (9) and (10) it is easy to verify that

$$R_n = \frac{np_n}{2}. (11)$$

Using (11), and combining equations (5) and (10), we eliminate three out of four variables to get

$$R_n = \frac{(1+2R_{n-1})^2}{4(1+R_{n-1})}.$$

To analyze this recursion, substitute $V_n = R_n + 1$. This yields

$$V_n = 1 + \frac{(2V_{n-1} - 1)^2}{4V_{n-1}} = V_{n-1} + \frac{1}{4V_{n-1}}$$

$$\Rightarrow V_n^2 = V_{n-1}^2 + \frac{1}{16V_{n-1}^2} + 1/2$$

To get a precise expression for V_n , we add the differences of $V_i^2 - V_{i-1}^2$.

$$\sum_{k=2}^{n} V_k^2 - V_{k-1}^2 = \frac{n-1}{2} + \sum_{k=1}^{n-1} \frac{1}{16V_{k-1}^2}$$

$$\Rightarrow V_n^2 - V_1^2 = \frac{n-1}{2} + \sum_{k=1}^{n-1} \frac{1}{16V_{k-1}^2}$$
(12)

Note that $V_1^2=(R_1+1)^2=25/16$. This, coupled with (12) shows that the higher order term of V_n is $\sqrt{\frac{n}{2}}$. We use $V_n\sim\sqrt{\frac{n}{2}}$ to substitute $\frac{1}{8(k-1)}$ for the fractional $\frac{1}{16V_{k-1}^2}$ term in the summation. Thus, rewriting (12), we get

$$V_n^2 \sim \frac{n-1}{2} + \frac{H_{n-1}}{8} + \frac{25}{16}$$

$$\sim \frac{n}{2} + \frac{\log n}{8} + O(1)$$

$$\Rightarrow R_n = (V_n - 1) \sim \sqrt{\frac{n}{2} + \frac{\log n}{8} + O(1)}$$

$$\Rightarrow p_n = \frac{2R_n}{n} \sim \sqrt{\frac{2}{n}} \qquad \text{(by Equation (11))}$$

$$\Rightarrow t_n \sim 1 - \frac{1}{\sqrt{2n}} \qquad \text{(by Equation (6))}$$

Note that $t_n < 1$ for all n, i.e., the threshold is non-trivial for all n. Or equivalently, the fact that R_n strictly increases with n already shows that there is a non-zero amount of trade at every round in a PBE.

Uniqueness. Since we have assumed uniqueness for the n-1 round PBE for all $F_{[0,x]}$, uniqueness for the n round game follows from simply the uniqueness of solutions to the set of equations (5), (6), (7). Its summary, given by equation (8) shows that R_n has a quadratic dependence on the optimization variable t_n : clearly the maximum is unique. The second order conditions in (8) can be easily verified to show that optima is indeed a maxima.

Remark 1 Interestingly, although the price starts very low, at $p_n \sim \sqrt{\frac{2}{n}}$, the threshold starts very high at $t_n \sim 1 - \frac{1}{\sqrt{2n}}$. That is, the seller already starts with a very small price, and the buyer still refuses to buy for most of his values, waiting for the price to go down even further.

5 Time discounted infinite horizon

We formally make the connection to bargaining with one-sided information (and equivalently to durable goods monopoly) in this section.

5.1 Bargaining and Repeated Sales

Bargaining with one-sided offer. Recall that in bargaining with one-sided offer, a single seller repeatedly makes price offers to a single buyer for the sale of a single unit of good, till a sale is made. The buyer's private value v for the good is drawn from distribution F and is publicly known. In each round, the seller posts a price, and the buyer can either accept or reject the offer the seller made. Once the buyer accepts at a price, the game ends, and the seller's revenue is that round's price. We consider the infinite horizon bargaining game: the number of rounds in the game is unbounded, but the buyer and seller have a common discount rate of $1-\delta$ on their utilities, i.e., utilities in round i are scaled by $(1-\delta)^{i-1}$. Equivalently, one could think of the there being a $1-\delta$ probability of the game ending after each round.

Theorem 4 For atomless bounded support distribution F, for every pure strategy threshold PBE in the bargaining game with one-sided offer for F, there exists a corresponding pure strategy threshold PBE in the time discounted infinite horizon partial commitment repeated sales game for F, s.t., if the seller's expected revenue and buyer's expected utility are respectively R, U(v) for type v in the bargaining game, they are $\frac{R}{\lambda}$, $\frac{U(v)}{\lambda}$ in the repeated sales game.

Proof: Consider a PBE $(\sigma_s^*, \sigma_b^*, \mu^*)$ for the bargaining game where the σ 's are the strategies of the seller and buyer, and μ is the seller's beliefs. The strategy profile $(\sigma_s^{\dagger}, \sigma_b^{\dagger}, \mu^{\dagger})$ is a PBE for the repeated sales game where

1.
$$\forall k$$
, if $h^k = \text{Reject}^k$, set
$$\begin{cases} \sigma_s^{\dagger}(\cdot|h^k) &= \sigma_s^*(\cdot|h^k) \\ \sigma_b^{\dagger}(\cdot|v,h^k) &= \sigma_b^*(\cdot|v,h^k) \\ \mu^{\dagger}(\cdot|h^k) &= \mu^*(\cdot|h^k) \end{cases} \quad \forall v$$

2. $\forall k$, if $h^k \ni \text{Accept}$, set $\sigma_s^{\dagger}(p|h^k) = 1$ where p is the smallest price for which the buyer accepted in history h^k .

$$3. \ \forall k, \text{if } h^k \ni \text{Accept}, \text{set} \left\{ \begin{array}{ll} \sigma_b^\dagger(\text{Accept}|v,h^k) &= 1 \\ \sigma_b^\dagger(\text{Reject}|v,h^k) &= 1 \end{array} \right. \quad \forall v \ge \text{current-round-price}$$

$$\text{4. } \forall k, \text{if } h^k \ni \text{Accept, set} \left\{ \begin{array}{ll} \mu^\dagger(\cdot|h^k) &= \mu^*(\cdot|h^{k_1+1}) & \text{if } h^k = \text{Reject}^{k_1} \text{Accept}^{k_2} \text{ for } k_1 \geq 0, k_2 > 0 \\ \mu^\dagger(\mathfrak{h}|h^k) &= 1 & \text{if } h^k = \text{Reject}^{k_1} \text{Accept}^{k_2} \text{Reject}^1 \{ \text{Reject,Accept} \}^* \text{ for } k_1 \geq 0, k_2 > 0 \\ \end{array} \right.$$

As described above, the PBE $(\sigma_s^\dagger, \sigma_b^\dagger, \mu^\dagger)$ mimics the PBE $(\sigma_s^*, \sigma_b^*, \mu^*)$ of the bargaining game as long as the buyer has never purchased in history. If the buyer has purchased at least once in the past, the bargaining game ends immediately, but the repeated sales game doesn't. In this case, in the repeated sales game:

1. The seller is bound not to increase price beyond any price at which the buyer accepted in the past, and the seller has no reason to decrease the price either: so σ_s^{\dagger} posts exactly the smallest price accepted in the past, i.e., the "price remains fixed upon acceptance by the buyer" just like the discussion in the proof of Theorem 2.

- 2. The buyer is guaranteed to face prices lower than what he has once accepted, and therefore accepts any price. The condition "v < current-round-price" can never occur in equilibrium-path if buyer has accepted at least once in the past: as that would mean that in the past the buyer accepted in a round v < round-v < round
- 3. As far as beliefs are concerned, the seller never expects to see a buyer who has once accepted in the past to ever reject in the future. If the seller observes this, it is a zero-probability action, and beliefs are (as they can be) arbitrarily updated to a point-mass at the largest point in the support h (i.e., the belief is supported at h with probability 1). If the seller observes no such anomaly, the seller's beliefs are simply borrowed from bargaining game.

The proof for $(\sigma_s^{\dagger}, \sigma_b^{\dagger}, \mu^{\dagger})$ being a PBE in the repeated sales game follows from $(\sigma_s^*, \sigma_b^*, \mu^*)$ being a PBE in the bargaining game. Let $U_{B,s}, U_{B,b}$ be the seller's and buyer's utility in the bargaining game, and let $U_{R,s}, U_{R,b}$ be the same in the repeated sales game. From the description above, it follows that:

1.
$$U_{R,s}(\sigma_s^{\dagger}, \sigma_b^{\dagger}, \mu^{\dagger}) = U_{B,s}(\sigma_s^*, \sigma_b^*, \mu^*)/\delta$$

2.
$$U_{B,b}(\sigma_s^{\dagger}, \sigma_b^{\dagger}, \mu^{\dagger}) = U_{B,b}(\sigma_s^*, \sigma_b^*, \mu^*)/\delta$$

I.e., the buyer, whenever he makes a single purchase at price p in the bargaining game, he makes infinite purchases at the same price, giving the seller a revenue of $\sum_{r=0}^{\infty} p \cdot (1-\delta)^r = p/\delta$. Similarly, the buyer's utility is also scaled by δ .

Given that the utility structure of the two games are identical (scaling doesn't change the utility structure), it is immediate that $(\sigma_s^{\dagger}, \sigma_b^{\dagger}, \mu^{\dagger})$ is a PBE for the repeated sales game.

Remark 2 (**Durable goods monopoly**) An identical connection as in Theorem 4 exists between durable goods monopoly (Coase [1972]) and the partial commitment repeated sales game, given that the math for bargaining and durable goods monopoly is identical.

Remark 3 Note that the partial commitment repeated sales game in the finite horizon setting, discussed in Section 4, cannot be reduced to the bargaining game because, the utility structures are not identical: the number of repeated purchases of an item depends on which round it was first purchased. On the other hand in the infinite horizon setting, the "number" of repeated purchases always appears as a $\frac{1}{\lambda}$ factor facilitating the reduction to the bargaining setting.

5.2 The linear demand case: U[0,1] distribution

For the rest of this section, we focus on the linear demand case, namely, the distribution of buyer's values is U[0,1].

Scale-invariant strategies for seller. A seller's strategy is scale-invariant, if given two different prior beliefs that are scaled versions of one another, the seller's prices given these two beliefs are also scaled versions of one another, with the same ratio as prior beliefs. Concretely, for the uniform distribution, this means that when the belief is U[0,k], there is some constant γ independent of k such that for all k, the seller's price is γk .

Stationary threshold strategies for buyer. A buyer follows a stationary threshold strategy if, given a price p, buyers with $v \ge \lambda p$ purchase, and the rest reject, where λ is independent of p or the round in which it was offered.

5.2.1 Equilibrium selection and Coase conjecture

As discussed in related work (Section 1.1.1), prior work ([Sobel and Takahashi, 1983, Stokey, 1981, Gul et al., 1986]) confirmed the Coase conjecture in the infinite horizon bargaining/durable goods monopoly setting, when we focus on PBEs where the buyer follows stationary threshold strategies, and the seller follows scale-invariant strategies. By the correspondence between bargaining and partial commitment repeated sales that we established in Theorem 4, the PBEs in these models have an equivalent in the partial commitment game, with the seller's revenue being a factor $\frac{1}{\delta}$ larger than in bargaining. For the sake of completeness, we work out that PBE here, adapted to our repeated sales setting. We follow notation much along the lines of Chapter 10 in Fudenberg and Tirole [1991].

Theorem 5 (Sobel and Takahashi [1983], Stokey [1981]) In the time-discounted infinite horizon partial commitment repeated sales game for the U[0,1] distribution, there exists a PBE in which the seller follows a scale-invariant strategy, and the buyer follows a stationary threshold strategy. As $\delta \to 0$, the seller's revenue approaches $\frac{1}{2\sqrt{\delta}}$.

Proof: We look for a PBE where:

- 1. The seller follows a *scale-invariant strategy*: i.e., when his beliefs are in U[0,t], he posts a price of γt where γ is independent of t.
- 2. The buyer follows a *stationary threshold strategy*: if offered a price p, buyers with $v \ge \lambda p$ purchase, and the rest reject, where λ is independent of p or the round in which it was offered.

Let $R_{\delta}(t)$ denote the seller's discounted future revenue, given that his current beliefs are in U[0,t], but the distribution has a total mass of just t instead of 1. Since the seller maximizes his revenue we have the following (note that once the buyer purchases at p the future price remains fixed at p, just like in Theorems 2 and 3).

$$R_{\delta}(t) = \max_{p} \{ (t - \lambda p) \frac{p}{\delta} + (1 - \delta) R_{\delta}(\lambda p) \}$$
(13)

First order conditions applied to (13), differentiating w.r.t. p, yields

$$\frac{1}{\delta}(t - 2\lambda p) + (1 - \delta)\lambda R_{\delta}'(\lambda p) = 0 \tag{14}$$

The envelope theorem, when applied to (13) yields

$$R'_{\delta}(t) = \frac{p(t)}{\delta} = \frac{\gamma t}{\delta} \tag{15}$$

The indifference of threshold buyer yields

$$\frac{\lambda p - p}{\delta} = (1 - \delta) \frac{\lambda p - \gamma \lambda p}{\delta} \tag{16}$$

Combining (14), (15), (16) yields

$$\lambda = \frac{1}{\sqrt{\delta}}$$

$$\gamma = \frac{\sqrt{\delta} - \delta}{1 - \delta}$$

The seller's revenue $R_{\delta}(t)$, is obtained by integrating (15), namely $R_{\delta}(t) = \frac{\gamma t^2}{2\delta}$, thus the revenue for U[0,1] distribution is $\frac{\gamma}{2\delta}$. As $\delta \to 0$, the revenue approaches $\frac{1}{2\sqrt{\delta}}$.

Remark 4 Note that this PBE indeed satisfies Coase's conjecture. As offers are made in very quick succession, namely when the discount factor $1-\delta$ approaches 1, the seller's first round price, γ , approaches 0. The monopolist's revenue approaches 0 in the bargaining game ($\sqrt{\delta}$, to be precise), and in the repeated sales game it approaches $\frac{1}{\sqrt{\delta}}$. The latter is much smaller than the Myerson optimal static revenue of $\frac{1}{4\delta}$.

5.2.2 Folk theorem

While the selection criteria of scale-invariant strategies for seller and stationary threshold policies for buyer helped to confirm Coase conjecture, if we remove these stationarity assumptions, Ausubel and Deneckere [1989] showed a folk theorem that as $\delta \to 0$, any revenue between 0 and monopoly revenue of $\frac{1}{4}$ can be obtained in a PBE for the infinite horizon time discounted bargaining/durable goods monopolist game. By our Theorem 4, this immediately translates to a folk theorem in the partial commitment repeated sales game, with any seller revenue between 0 and $\frac{1}{4\delta}$ possible. We state the folk theorem adapted to our repeated sales game here.

Theorem 6 (Ausubel and Deneckere [1989]) In the time discounted infinite horizon partial commitment repeated sales game for the U[0,1] distribution, for every $\epsilon>0$ there exists a $\delta(\epsilon)>0$ such that for any $\delta<\delta(\epsilon)$, the seller's revenue $R_{\delta}\in [\frac{\epsilon}{\delta},\frac{1/4-\epsilon}{\delta}]$.

Proof idea. We refer the reader to Ausubel and Deneckere [1989] or Chapter 10 of Fudenberg and Tirole [1991] for a proof of this theorem. The main idea is to construct a reputational equilibrium. Let Δ denote the time interval between rounds, and let r be the interest rate, so that the discount factor $1-\delta=e^{-r\Delta}$. Consider the real-time price path of $p_{\tau}=\frac{1}{2}e^{-\eta\tau}$ where τ is real time, and p_{τ} is the price posted by seller at τ . This, price posted at round n is $p_n=\frac{1}{2}e^{-\eta n\Delta}$. When η is very close to 0, the prices are extremely slowly decreasing, and most buyers will be impatient enough to buy very early. In particular, all buyers with value at least $\frac{1}{2}+\epsilon$ for small ϵ buy because $v-\frac{1}{2}=\sup_{\tau}e^{-r\tau}(v-\frac{1}{2}e^{-\eta\tau})$. This means that the seller makes almost his monopoly revenue in this equilibrium. But what's the catch? It is that the seller should not feel tempted at a later point in the path to decrease prices quicker than the announced path to capture more buyers — if the seller ever deviates, the PBE constructed is such that the buyer will immediately switch to the "Coase path" discussed in Theorem 5. The question is if the seller will ever deviate? In particular, given that η is very small, leading to a very slowly decreasing price path, and consequently a very slow rate of sales as time proceeds making him want to switch to the Coase path. This would make the buyers want to wait till this switch to very low prices happen. To ensure that this deviation does not happen, η is fixed and Δ is taken to 0, and a simple calculation is used to show that the seller doesn't have the incentive to deviate from the exponential price path.

Qualitatively, this is a reputational equilibrium, because the seller is forced to maintain his reputation as someone who is determined not to reduce prices too quickly, and always follow the same rate parameter η . The moment the seller deviates from this path, the buyer decides that this seller is not strong and that the seller believes in the Coase conjecture, and hence switches to the "Coase path" with tiny seller revenue.

5.2.3 Exogenous restrictions on strategy space

For the linear demand case (i.e., U[0,1] distribution), Sobel and Takahashi [1983], Stokey [1981] focus on the PBEs where the seller follows scale-invariant strategies. Instead of searching for PBEs that satisfy certain properties (like we did in Section 5.2.1 where seller's strategies are scale-invariant and buyer has stationary threshold strategies), what if we exogenously enforce that the seller follows scale-invariant strategies? I.e., seller's strategy space is restricted to scale-invariant strategies that post a price of pk when his belief is U[0,k] for all k, with p being independent of k. In other words, once the seller announces his initial price of p for the U[0,k] belief, he is immediately committing himself to the price for the entire rejection path (namely, just look at the current belief U[0,k] and post a price of pk). With this exogenous enforcement, we now show that there is a unique PBE in which the seller can extract at least $\frac{4}{3+2\sqrt{2}} \sim 69\%$ of Myerson optimal revenue benchmark, unlike the $O(\frac{1}{\sqrt{\delta}})$ predicted by the Coase-conjecture-confirming Theorem 5 (see Theorem 7 and the following discussion for this result). Further, the price in the first round as $\delta \to 0$ is about 0.586, unlike the $\sqrt{\delta}$ as the first round price predicted by Theorem 5.

Theorem 7 In the time-discounted infinite horizon partial commitment repeated sales game for the U[0,1] distribution, when the seller's strategy space is restricted to scale-invariant strategies, there exists a unique² PBE in which, as $\delta \to 0$, the seller's revenue approaches a $\frac{4}{3+2\sqrt{2}}$ fraction of the Myerson optimal revenue benchmark of $\frac{1}{4\delta}$.

Proof: Let R_{δ} denote the expected revenue. Let t(p) be the buyer's threshold (at times abbreviated to just t) when the seller posts a price of p. Indifference of the threshold buyer yields

$$\frac{t-p}{\delta} = (1-\delta)\frac{t-tp}{\delta}$$

$$t(p) = \frac{p}{\delta + (1-\delta)p}$$
(17)

² as in Theorem 3, the uniqueness refers to equilibrium path and ignores irrelevant multiplicities arising out of several belief updates that are allowed after zero probability actions by the buyer in off-equilibrium paths. Further, we also ignore PBEs (if at all any exist) where a buyer rejects a price that is smaller than the smallest point in the support of seller's belief. It is straightforward to show by backward induction that indeed no such PBEs exist in the finite horizon model, but it is not clear how to rule-out the existence of such PBEs in infinite horizon model.

Since the seller is committed to following scale-invariant strategies, the revenue $R_{\delta,[0,t]}$ for U[0,t] (with a full mass of 1) is simply $t \cdot R_{\delta}$. The expected revenue R_{δ} of the seller can be written as:

$$R_{\delta} = \max_{p} \left\{ (1 - \delta)t R_{\delta,[0,t]} + (1 - t) \cdot p/\delta \right\}$$
$$= \max_{p} \left\{ (1 - \delta) \cdot R_{\delta}t^{2} + (1 - t) \cdot p/\delta \right\}. \tag{18}$$

The first term of (18) is the expected revenue contribution from rejection in the first round: t is the probability of rejection and the expected revenue upon rejection is $R_{\delta,[0,t]}$. The second term of (18) is the expected revenue contribution from acceptance in the first round: 1-t is the probability of acceptance in the first round, and the expected revenue upon acceptance is p in every round appropriately discounted.

Combining equations (17) and (18) gives

$$R_{\delta} = \max_{t} \left\{ \frac{t(1-t)}{(1-(1-\delta)t)(1-(1-\delta)t^2)} \right\}$$
 (19)

First order condition on (19) gives:

$$(1-\delta)^2 t^4 - 2(1-\delta)^2 t^3 + 2(1-\delta)t^2 - 2t + 1 = 0$$
(20)

Substitute $t=1-\theta$ in (20). A bit of introspection hints that $\theta=\Theta(\delta)$. Hence expand equation (20) in θ,δ , ignoring third and higher order terms (i.e., retaining only constants, $\theta,\delta,\theta^2,\delta^2,\theta\delta$ terms), and this gives that as $\delta\to 0$, we have $\theta\sim\frac{\delta}{\sqrt{2}}$. To compute the revenue approximtion, we substitute $t=1-\frac{\delta}{\sqrt{2}}$ in (19) and take the ratio of the expression for R_δ and the Myerson optimal revenue benchmark of $\frac{1}{4\delta}$. As $\delta\to 0$, this ratio approaches from above $\frac{4}{3+2\sqrt{2}}$, which is approximately 0.69. The optimal price p approaches $\frac{\sqrt{2}}{\sqrt{2}+1}$.

Uniqueness. This follows immediately from the uniqueness of the optimization problem in (19) for $t \in [0, 1]$.

Discussion. We remark a few properties of the PBE in Theorem 7:

- 1. The optimal first round price as $\delta \to 0$ is $p = \frac{\sqrt{2}}{\sqrt{2}+1} \approxeq 0.586$. This is very different from the tiny $\sqrt{\delta}$ price found in Theorem 5. So is the revenue approximation factor of 0.69, as compared to the tiny $\Theta(\sqrt{\delta})$ approximation in Theorem 5.
- 2. Equation (17) says that for all p < 1, we have t(p) < 1, i.e., however high the price p in the first round is, there is always a small fraction of buyer population that wants to buy very early. This again is very different from the result in Theorem 5 which, for a first round price of p has the first round threshold $\lambda p = \frac{1}{\sqrt{\delta}}p$, indicating that any first round price $p > \sqrt{\delta}$ is rejected by *all* buyers.

6 Directions for further research

The basic posted-price-for-a-single-buyer setup considered in this paper is a special case of the n buyers setting where one could run auctions with (potentially personalized) reserve prices, as is the common practice with most ad exchanges in the market for display/banner ads. Suppose there are n buyers with values drawn from independent but not necessarily identical distributions, and suppose the seller sells an item in each round via an auction with (personalized) reserve prices. Given the prevalence of this auction format, understanding the equilibrium structure and revenue in this case is an interesting direction for further research.

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A A gentle introduction to Perfect Bayesian Equilibrium

The fishmonger's problem.³ There is a single seller of fish and a single buyer who enjoys consuming a fresh fish every day. The buyer has a private value v for each day's fish, drawn from a publicly known distribution. However, this value is drawn *only once*, i.e., the buyer has the same unknown value on all days. Each day, the seller sets a price for that day's fish, which of course can depend on what happened on previous days. The buyer can then decide whether to buy a fish at that price or to reject. The goal of the buyer is to maximize his total utility (his value minus price on each day he buys and 0 on other days), and the goal of the seller is to maximize profit. How much money can the seller make in n days in equilibrium?

Consider, for example, the case where the distribution of the buyer's value is uniform in [0, 1] (denoted by U[0, 1] for short), and the game lasts for one day. In this case, it is easy to see that the optimal seller price is the monopoly price⁴ of 1/2, resulting in an expected seller profit of 1/4.

What prices should the seller set if the game is to last for two days? A first guess is 1/2 on both days, for an expected profit of 1/4 each day or 1/2 overall. But this is implausible: if the buyer rejects on the first day, the seller might reasonably assume that the buyer's value is U[0,1/2], in which case the seller's best response is to offer a price of 1/4 on the second day. This yields the seller strategy shown in Figure 1a. However, this buyer/seller strategy pair is *not* in equilibrium. This seller strategy is based on the fallacious assumption that the buyer's best response is to buy on both days if his value is above 1/2. Indeed, a buyer with value $1/2 + \epsilon$ gets a utility of 2ϵ for buying both days, whereas his utility is $1/4 + \epsilon$ if he only buys on the second day. Interestingly, if the buyer could be *guaranteed* that the price on the second day was 1/2, then his best response would be to buy both days when v > 1/2. However, since the seller is *unable to commit* to a second day price, the buyer's strategy on the first day must take into account that on the second day the seller will best respond to the buyer's first day strategy. The result, in this case, is that the buyer is incentivized to wait for the lower second day price unless his value is at least 3/4.

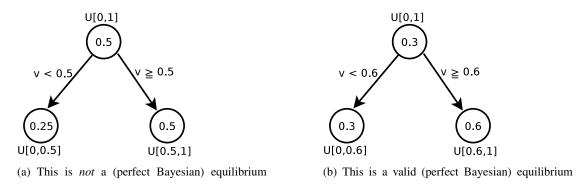


Figure 1: Equilibrium illustration for the 2 days fishmonger's problem. The number in the top circle is the price on the first day. The number in the circle following the left arrow denotes the price on the second day after the buyer rejected on the first day, and the one in the circle following the right arrow denotes the second day price after buyer buys at the posted price on the first day. The distributions are updated depending on whether the buyer purchased or rejected on the first day.

So what is the optimal strategy for this 2-day game (for arbitrary distributions of buyer valuation)? Or for the n-day version? In this paper we study this question: how much can the seller make in n-days in a $Perfect\ Bayesian\ Equilibrium\ (PBE)$?

Commitment and Perfect Bayesian Equilibrium. The lack of commitment in repeated games is the major driver of fundamental differences in outcomes when compared to single-shot games. The absence of commitment in repeated

³We thank Amos Fiat for suggesting this name for the problem.

⁴This is a special case of Myerson [1981]'s theorem which implies that the revenue optimal mechanism for a seller facing a buyer with value drawn from known distribution F is to offer a price of p (monopoly price) that maximizes p(1 - F(p)).

games is captured by the notion of a Perfect Bayesian equilibrium (PBE). Informally, a PBE consists of a seller strategy, describing what price he offers as a function of the history of play at each time, and a (possibly randomized) buyer strategy describing his accept/reject decisions given the history of play and his value. For every possible value the buyer has, and for every possible history of play, his strategy must be a best response to the *subtree* of prices the seller's strategy specifies for that particular history of play. For the seller, for every possible history of play, the subtree of prices offered henceforth must optimize his profit given the buyer's strategy and the induced distribution of values the buyer has (as determined by the history of play). For example, Figure 1b shows the essentially unique PBE strategies for the buyer and the seller in the example discussed above. We refer the reader to Section 3 for a general way to compute PBEs for two round games with arbitrary distributions. For the sake of intuition, we also flesh out in Appendix D the U[0,1] case and provide a complete description of the PBE strategies.

B Broader related work

Ratchet effect. A well known fact about many of these repeated sales settings is the "ratchet effect", that the revelation principle fails to hold [Freixas et al., 1985, Laffont and Tirole, 1988].

Posting prices. Most of the literature assumes that the seller is *restricted* to posting a price, which was justified by Skreta [2006, 2015] who showed (respectively for a single buyer and many buyers) that posting prices is optimal among all mechanisms. Both these papers considered the case where the seller has a single unit.

Other related work in bargaining. While our closest connection in this work is to bargaining with one-sided offer and one-sided incomplete information, there is also work bargaining with two-sided incomplete information and two-sided offers (Fudenberg and Tirole [1983]). We refer the reader to Ausubel et al. [2002] for a survey on bargaining with incomplete information. On the topic of bargaining with complete information, Morgenstern [1973] and Rubinstein [1982] were the first to capture the intrinsically dynamic nature of bargaining and model it as a sequential complete information game (as an alternative to the axiomatic approach to Bargaining [Nash, 1950]). They showed that the process of sequential bargaining yields a unique, Pareto-efficient outcome. While the combination of uniqueness and efficient outcome is remarkable and important, we note that bargaining gains its main interest from incomplete information.

C Lack of commitment can never help even in very general settings

We formally define a very general model of mechanism design here and prove Proposition 1.

Definition 4 General model of mechanism design. An instance of a mechanism design problem is given by a set of m agents \mathcal{A} , a set of outcomes \mathcal{O} and a type space \mathcal{T} for each agent, where each type θ is a function from \mathcal{O} to \mathbb{R} (which is the utility of the agent with type θ for the given outcome). Each agent a has a type θ_a , which is her private information. In the Bayesian setting, additionally, we are given a joint probability distribution F over the types of all the agents, \mathcal{T}^m , from which the type vector $\vec{\theta}$ of the agents is sampled. A mechanism is a multi-party protocol in which the agents participate, as a result of which there is an outcome $o(\vec{\theta})$. The mechanism designer's goal is to maximize his objective $\mathbf{E}_{\vec{\theta} \sim F}[\mathrm{OBJ}(\vec{\theta})]$, where $\mathrm{OBJ}(\cdot)$ is a function from \mathcal{O} to \mathbb{R} .

Note that the above definition includes, in addition to the usual cases of welfare/revenue maximization, constraints such as budget constraints and scenarios such as mechanism design without money, non-linear objectives such as makespan minimization in scheduling and max-min fairness. In order to extend this model to the repeated setting, we need to additionally specify how many times the setting is repeated. We allow the number of repetitions to be a random variable.

Definition 5 General model of repeated mechanism design. An instance of a repeated mechanism design problem is given by an instance of the mechanism design problem, the probabilities $\{q_t\}_{t\in\mathbb{N}}$ with which the t-th repetition is realized, the fractions $\{d_t\}_{t\in\mathbb{N}}$ with which the mechanism designer and the agents discount their t-th round utilities.

We require that $\sum_{t=1}^{\infty} q_t d_t < \infty$ (i.e., the process either doesn't continue infinitely, or if it does, agents discount their future utilities enough to avoid infinite utilities). The buyer types remain the same in every repetition and there are no inter-round constraints except this. The repeated mechanism is now a protocol, which in sequence produces an outcome o_t for each time⁵ t till the process stops (the mechanism designer and the agents know the probabilities q_t that determine this stopping time, but get to know the precise stopping time only when it happens.) The utility of agent a is the sum $\mathbf{E}_{\vec{\theta} \sim F[\theta_a]}[\sum_{t=1}^{\infty} q_t d_t \theta_a(o_t(\vec{\theta}))]$. The objective of the mechanism designer is $\mathbf{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} q_t d_t \Theta \mathbf{B} J(o_t(\vec{\theta}))]$.

It is easy to see that the game defined in Definition 1 is a special case of the above model: $\mathcal{A}=\{1\}, \mathcal{O}=\{\text{accept, reject}\}\times\mathbb{R}, \text{ types of the form }\theta((\text{accept},p))=v-p \text{ for some }v\in\mathbb{R} \text{ and }\theta(o)=0 \text{ otherwise, and objective } \text{OBJ}((\text{accept},p))=p \text{ and } \text{OBJ}(o)=0 \text{ otherwise. For the finite horizon model }q_t=1 \text{ for }t\in\{1,2,\ldots,n\}, \text{ and }q_t=0 \text{ for }t>n, \text{ with }d_t=1 \text{ for all }t. \text{ The time discounted infinite horizon game can be described by setting }q_t=1 \text{ for all }t\in\mathbb{N} \text{ and }d_t=(1-\delta)^{t-1} \text{ for all }t\in\mathbb{N}. \text{ Note that }\sum_{t=1}^{\infty}q_td_t=1/\delta<\infty.$

We restate Proposition 1 formally here and prove it.

Proposition 1 (A simple generalization of the result in Baron and Besanko [1984]) In the general model of repeated mechanism design, the optimal objective value obtained without any commitment is never larger than the optimal objective value obtained when commitment is possible. Formally let OBJ* be the optimal expected objective value for the single round mechanism design problem. Then the optimal expected objective value attainable in any PBE in the repeated mechanism design problem is at most $\mathbf{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} q_t d_t \mathrm{OBJ}^*]$.

Proof: Suppose on the contrary that there was a PBE with expected objective value $\mathbf{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} q_t d_t \mathrm{OBJ}(o_t(\vec{\theta}))] > \mathbf{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} q_t d_t \mathrm{OBJ}^*]$. Consider the following mechanism for the single round game. All agents submit their types to the mechanism designer. The designer chooses day t with probability $\frac{q_t d_t}{\sum_t q_t d_t}$ and runs the said PBE till day t, and the outcome on day t will be the outcome realized for the single round game. Agent a with type θ_a , upon truthful reporting of his type, will get an expected utility of $\mathbf{E}_{\vec{\theta} \sim F|\theta_a}[\sum_{t=1}^{\infty} \frac{q_t d_t}{\sum_t q_t d_t} \theta_a(o_t(\vec{\theta}))]$ which is just a scaled version of his utility in the PBE. Thus the agent has no incentive to deviate in the proposed mechanism because that would mean that the said PBE was not really a PBE. For this mechanism, the expected objective of the designer is $\mathbf{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} \frac{q_t d_t}{\sum_t q_t d_t} \mathrm{OBJ}(o_t(\vec{\theta}))]$. By our assumption, the former quantity is at least $\mathbf{E}_{\vec{\theta} \sim F}[\sum_{t=1}^{\infty} \frac{q_t d_t}{\sum_t q_t d_t} \mathrm{OBJ}^*] > \mathrm{OBJ}^*$. This is a contradiction because in the single round game, it is not possible to get an expected objective value higher than OBJ^* .

D Two Rounds Game

Full solution to the 2 days U[0,1] repeated sales game. We present here the full solution to the 2 days U[0,1] repeated sales game. Note that a PBE has to specify equilibrium behavior at off equilibrium paths as well. In the U[0,1] example, although the first round price is $p_1=0.3$, the seller still has to specify his second round behavior when the first round price is not 0.3.

⁵The mechanism could be randomized and its outcome on day t could depend (apart from $\vec{\theta}$) on the realization of the random coin tosses on days 1 to t-1. To avoid excessively cumbersome notation we avoid spelling this out formally. But our argument and results directly extend to these settings too.

STRATEGY 3: Seller's strategy in the 2 rounds U[0,1] game

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\begin{tabular}{l|l} \hline \textbf{Round-1 pricing:} & Set $p_1=0.3$; \\ \hline \textbf{Round-2 pricing:} \\ \hline \textbf{if } p_1 \leq 0.5 \textbf{ then} \\ \hline \textbf{if } Buyer \ rejects \ in \ round \ 1 \ \textbf{then} \\ \hline \textbf{else if } Buyer \ accepts \ in \ round \ 1 \ \textbf{then} \\ \hline \textbf{Set second round price of } \max(2p_1,0.5) \\ \hline \textbf{else if } p_1 > 0.5 \ \textbf{then} \\ \hline \textbf{if } Buyer \ rejects \ in \ round \ 1 \ \textbf{then} \\ \hline \textbf{Set second round price of } p_2 = 0.5 \\ \hline \textbf{else if } Buyer \ accepts \ in \ round \ 1 \ \textbf{then} \\ \hline \textbf{Set second round price of } p_1 = 0.5 \\ \hline \textbf{else if } Buyer \ accepts \ in \ round \ 1 \ \textbf{then} \\ \hline \textbf{Set second round price of } 1 \\ \hline \end{tabular}
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STRATEGY 4: Buyer's strategy in the 2 rounds U[0,1] game

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\begin{tabular}{ll} \textbf{Round-1 strategy:} \\ \textbf{if } p_1 \leq 0.5 \textbf{ then} \\ \textbf{if } v \geq 2p_1 \textbf{ then} \\ \textbf{Accept} \\ \textbf{else} \\ \textbf{L Reject} \\ \textbf{else if } p_1 > 0.5 \textbf{ then} \\ \textbf{L Reject} \\ \textbf{Round-2 strategy:} \\ \textbf{if } v \geq p_2 \textbf{ then} \\ \textbf{Accept} \\ \textbf{else} \\ \textbf{L Reject} \\ \end{tabular}
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