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We analyze the question of how to distribute the asset value of an insolvent firm among its creditors and the firm itself. Compared to standard bankruptcy games as studied in the game-theoretic literature, we treat the firm as a player and define a new class of transferable utility games called liability games. We show that the core of a liability game is empty. We analyze the nucleolus of the game. The firm always gets a positive payment, at most equal to half of the asset value. Creditors with higher liabilities receive higher payments, but also suffer from higher deficiencies. We provide conditions under which the nucleolus coincides with a generalized proportional rule.

1. Introduction

It is often the case that insolvent agents agree with their creditors to decrease the value of their liabilities by reducing the principal, restructuring the payments, or getting longer maturities. For each liability, the difference between the value of the old liability and the legally binding, but lower level of new liability is called the deficiency on the liability. For sovereign defaults, about 30-40% deficiency is documented by Arslanalp and Henry (2005), D'Erasmo (2011), and Benjamin and Wright (2009).

Insolvent agents can be either countries, states, firms, individuals, or other organizations. Throughout the paper, we will stick to the term firm. A liability problem consists of an insolvent firm and a group of creditors. The question is how to distribute the asset value of the firm among the creditors and the firm itself. A liability rule assigns a vector of payments to the creditors and the firm. Payment vectors should satisfy non-negativity as the firm has limited liability and no creditor should be asked to pay, liabilities boundedness as the firm does not pay in excess of its liabilities, and efficiency meaning

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that the sum of the payments should be equal to the firm's asset value. We introduce generalized proportional rules, assigning a particular non-negative payment to the firm and distributing the remainder in proportion to the liabilities.

Sturzenegger and Zettelmeyer (2007) and Chatterjee and Eyigungor (2015) note that there is no settled theory for the renegotiations concerning the distribution of the asset value of the firm, but creditors may join ad hoc committees in order to be paid first. Our approach is to use cooperative game theory to model the formation of ad hoc committees by creditors. The advantage of this approach is that it does not require us to specify the details of the renegotiation process.

Compared to standard bankruptcy games as studied in the game-theoretical literature, see O'Neill (1982) for a seminal contribution and Thomson (2013, 2015) for recent surveys, we introduce the firm as an explicit player. The worth of a coalition is given by what the members of a coalition can guarantee for themselves, irrespective of what outsiders do. More specifically, given a coalition and its complement, using its asset value the firm first makes payments to the coalition it belongs to, up to the value of the liabilities in the firm's coalition, and then pays to the complementary coalition. The resulting transferable utility game is called a liability game.

We show that the core of a liability game is empty. In this case, a solution concept which minimizes the complaints of the coalitions is a natural candidate for a liability rule. One way of minimizing the complaints is to do so lexicographically, which brings us to the nucleolus (Schmeidler, 1969). A second reason for studying the nucleolus is that it plays a prominent role in bankruptcy games. We want to analyze how the predictions of the nucleolus change as soon as the estate is treated as a player.

We have two results where the nucleolus behaves the same way in liability games as in bankruptcy games. First, we show that the nucleolus satisfies the properties required for a liability rule: efficiency, non-negativity, and liabilities boundedness. Second, creditors with higher claims get higher payments, but there is also higher deficiency on higher liabilities.

In bankruptcy games, the entire estate is allocated to the claimants and the nucleolus coincides with the Talmud rule (Aumann and Maschler, 1985). In liability games, at the nucleolus, the insolvent firm always gets a positive payment and we provide conditions under which the nucleolus coincides with a generalized proportional rule. Treating the bankrupt agent as a player, therefore, makes a significant difference.

The rest of the paper is organized as follows. In Section 2 we define and illustrate liability problems and liability games and show that the core is empty. In Section 3 we analyze the nucleolus. In Section 4 we present conditions under which the nucleolus coincides with a generalized proportional rule. Section 5 contains the conclusion.

2. Liability problems and liability games

Let $N = \{0, 1, \ldots, c\}$ denote the set of agents, where agent 0 is a *firm* having a set of *creditors* $C = \{1, \ldots, c\}$. The firm has *asset value* $A \in \mathbb{R}_+$ and *liabilities* $\ell \in \mathbb{R}_+^C$, with $\ell_i \in \mathbb{R}_+$ the liability to creditor $i \in C$. Given a subset of creditors $S \subseteq C$, we use the notation $\ell(S) = \sum_{i \in S} \ell_i$ for the total liabilities of S.

The question is how to distribute the asset value of the firm among the creditors and the firm itself. In answering it, we construct a particular transferable utility game. Since the definition of this transferable utility game does not change if liabilities larger than the asset value are replaced by the asset value itself, we restrict the analysis to the case where all liabilities are less than or equal to the asset value.[2] Also, to avoid discussing trivial cases, we assume that the asset value is insufficient to honor all the creditors, that is, the firm is insolvent.

Definition 2.1. A *liability problem* is a pair $(A, \ell) \in \mathbb{R}_+ \times \mathbb{R}_+^C$ such that, for every $i \in C$, $\ell_i \leq A$, and $\ell(C) > A$. Let $\mathcal{L}$ be the class of liability problems.

The definition of a liability problem $(A, \ell) \in \mathcal{L}$ implies that $A > 0$ and that there are at least two positive liabilities, that is, $c \geq 2$.

Given a liability problem $(A, \ell) \in \mathcal{L}$, a *payment vector* is $x \in \mathbb{R}_+ \times \mathbb{R}_+^C$ satisfying *liabilities boundedness* and *efficiency*. Liabilities boundedness means that no creditor $i \in C$ receives more than his claim, so $x_i \leq \ell_i$. Efficiency requires that the sum of the payments should be equal to the asset value: $\sum_{i \in N} x_i = A$. Note that non-negativity and efficiency imply that payments to the firm are also bounded from above, $x_0 \leq A$.

A liability rule is a function that assigns a payment vector to each liability problem.

Definition 2.2. A *liability rule* is a function $f : \mathcal{L} \to \mathbb{R}_+^N$ such that, for every $(A, \ell) \in \mathcal{L}$, for every $i \in C$, $f_i(A, \ell) \leq \ell_i$, and $\sum_{i \in N} f_i(A, \ell) = A$.

The agents in our model can form coalitions in bargaining over the eventual payment vector. A coalition is an element of 2^N, the collection of all subsets of N. The worth of a non-empty coalition $S \in 2^N$ is equal to what the members of S can guarantee for themselves, irrespective of what outsiders do. If $0 \in S$, then the worth of coalition S is obtained by having the firm pay its liabilities to its creditors within the coalition, up to the asset value, before paying the liabilities of outsiders. If

[2] For the case where the firm is not treated as a player, truncation of liabilities by the asset value is necessary and sufficient for a division rule to be a game-theoretic decision rule, see Theorem 5 of Curiel et al. (1987).

Table 1
The induced liability game when $N = \{0, 1, 2\}$, $A = 12$, $\ell_1 = 6$, and $\ell_2 = 9$.

S	$\{0\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$	$\{0,1,2\}$
$v(S)$	0	3	6	6	9	12	12

$0 \notin S$, then the members of coalition S can only guarantee for themselves what is left after the firm has paid its liabilities to creditors not in S.

Definition 2.3. Let $(A, \ell) \in \mathcal{L}$ be a liability problem. The induced *liability game* $v : 2^N \to \mathbb{R}$ is defined by setting, for $S \in 2^N$,

$$v(S) = \begin{cases} \min\{A, \ell(S \setminus \{0\})\}, & \text{if } 0 \in S, \\ \max\{0, A - \ell(C \setminus S)\}, & \text{if } 0 \notin S. \end{cases}$$

As an illustration of liability games, consider the following example.

Example 2.4. Consider the liability problem with two creditors, so $N = \{0, 1, 2\}$, $A = 12$, and $\ell = (6, 9)$. The induced liability game v is illustrated in Table 1.

To analyze the properties of liability games, we need the following definitions. For a transferable utility game $v : 2^N \to \mathbb{R}$ and a non-empty coalition $D \in 2^N$, the *subgame* v^D with player set D is obtained by restricting v to subsets of D. The game is *additive* if for all $S \in 2^N$ we have $v(S) = \sum_{i \in S} v(\{i\})$, *constant sum* (von Neumann and Morgenstern, 1944) if for all $S \in 2^N$ we have $v(S) + v(N \setminus S) = v(N)$, *convex* (Shapley, 1971) if for all $S, T \in 2^N$ we have $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, *superadditive* if for all $S, T \in 2^N$ such that $S \cap T = \emptyset$ we have $v(S) + v(T) \leq v(S \cup T)$, and *zero-monotonic* if for all $i \in N$, for all $S \subseteq N \setminus \{i\}$, we have $v(S) + v(\{i\}) \leq v(S \cup \{i\})$.

Remark 2.5. Let $(A, \ell) \in \mathcal{L}$ be a liability problem and let v be the induced liability game. Then we can make the following three observations. By considering player 0 and a creditor $i \in C$ with $\ell_i > 0$, we get that $v(\{0, i\}) > v(\{0\}) + v(\{i\})$, so v is not additive. It is also straightforward to verify that v is constant sum and superadditive, the latter implying that it is also zero-monotonic. For the set of creditors C, the subgame v^C corresponds to the transferable utility game defined by O'Neill (1982) for standard bankruptcy problems, which is shown to be convex by Curiel et al. (1987).

Let a game $v \in \mathcal{G}$ be given. An *allocation* is a vector $x \in \mathbb{R}^N$. The allocation x yields a total payoff of $x(S) = \sum_{i \in S} x_i$ to the members of coalition $S \in 2^N$. The set of *pre-imputations* of a game v is equal to $I^*(v) = \{x \in \mathbb{R}^N | x(N) = v(N)\}$. The addition of individual rationality constraints leads to the set of *imputations* $I(v) = \{x \in I^*(v) |$ for every $i \in N,\ x_i \geq v(\{i\})\}$. The *core* of the game v (Gillies, 1959) is given by $\mathrm{Core}(v) = \{x \in I^*(v) |$ for every $S \in 2^N,\ x(S) \geq v(S)\}$.

In Example 2.4, it is easily seen that the core of the game is empty. The following theorem shows that this is a general phenomenon.

Theorem 2.6. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem and let v be the induced liability game. Then* $\mathrm{Core}(v) = \emptyset$.

Proof. Suppose $\mathrm{Core}(v) \neq \emptyset$ and let $x \in \mathrm{Core}(v)$. Since v is constant sum by Remark 2.5, for all $S \in 2^N$ we have $v(S) + v(N \setminus S) = v(N)$. Since $x(N) = x(S) + x(N \setminus S) = v(N)$, we get that $v(S) - x(S) + v(N \setminus S) - x(N \setminus S) = 0$. Since $x \in \mathrm{Core}(v)$, this implies that $v(S) = x(S)$ for all $S \in 2^N$, which can only hold if v is additive. This contradicts v not being additive as argued in Remark 2.5. $\square$

3. The nucleolus as a liability rule

By Theorem 2.6, the core of a liability game is empty, so for any imputation there is a coalition which objects to it. A solution concept which minimizes the complaints of the coalitions is then a natural candidate for a liability rule. In the analysis of transferable utility games, it is common to minimize the complaints lexicographically, which brings us to the well-known concept of the nucleolus (Schmeidler, 1969). A second reason for studying the nucleolus is that it plays a prominent role in bankruptcy games. We want to verify how the incorporation of the estate as a player influences the properties of the nucleolus.

We define $\mathcal{N} = 2^N \setminus \{\emptyset, N\}$ as the collection of all non-empty coalitions that are proper subsets of the grand coalition. The *excess* $e(S, x)$ of coalition $S \in \mathcal{N}$ at an allocation $x \in \mathbb{R}^N$ is given by $e(S, x) = v(S) - x(S)$. Let $e(x) \in \mathbb{R}^{\mathcal{N}}$ be the vector of excesses at x, indexed by $S \in \mathcal{N}$. The lexicographic order $\preceq_{\text{lex}}$ is the complete order on $\mathbb{R}^{|\mathcal{N}|}$ defined as follows. For $x, y \in \mathbb{R}^{|\mathcal{N}|}$ it holds that $x \preceq_{\text{lex}} y$ if and only if either $x = y$ or there exists a number $k' \in \{1, \ldots, |\mathcal{N}|\}$ such that, for every $k < k'$, $x_k = y_k$, and $x_{k'} < y_{k'}$. We have that $x \prec_{\text{lex}} y$ if and only if $x \preceq_{\text{lex}} y$ and $x \neq y$. The coordinate ordering mapping

Table 2
The excesses at $x = (1, 4, 7)$ in Example 2.4.

S	$\{0\}$	$\{1\}$	$\{2\}$	$\{0,1\}$	$\{0,2\}$	$\{1,2\}$
$e(S,(1,4,7))$	-1	-1	-1	1	1	1

$\theta : \mathbb{R}^{\mathcal{N}} \mapsto \mathbb{R}^{|\mathcal{N}|}$ is defined such that it arranges the coordinates of a vector $x \in \mathbb{R}^{\mathcal{N}}$ in a weakly decreasing order. The *nucleolus* of a game v is given by

$$\mathrm{Nu}(v) = \{x \in I(v) | \text{ for every } y \in I(v),\ \theta(e(x)) \preceq_{\mathrm{lex}} \theta(e(y))\}. \tag{3.1}$$

If we replace the set of imputations $I(v)$ by the set of pre-imputations $I^*(v)$ in (3.1), then we obtain the *pre-nucleolus* $\mathrm{Nu}^*(v)$ of the game. As shown in Schmeidler (1969), the sets $\mathrm{Nu}(v)$ and $\mathrm{Nu}^*(v)$ are singletons. From now on, we will therefore treat them as vectors rather than sets.

Remark 3.1 is used to analyze the nucleolus. It follows easily from Remark 2.5.

Remark 3.1. Let $(A, \ell) \in \mathcal{L}$ be a liability problem and let v be the induced liability game. For every $S \in \mathcal{N}$, for every pre-imputation $x \in I^*(v)$, it holds that $e(S, x) + e(N \setminus S, x) = 0$.

Consider the payment vector $x = (1, 4, 7)$ in Example 2.4, so with a positive payoff of 1 for the firm. The excesses at $(1, 4, 7)$ for coalitions $S \in \mathcal{N}$ are shown in Table 2. As stated in Remark 3.1, the sum of the excesses of a coalition and its complement is equal to zero.

The sum of the excesses of the two-player coalitions at any pre-imputation is $v(\{0, 1\}) + v(\{0, 2\}) + v(\{1, 2\}) - 2x(N) = 27 - 24 = 3$. The highest excess of any of the two-player coalitions is therefore at least equal to 1 and any pre-imputation different from $(1, 4, 7)$ will lead to some two-player coalition having an excess strictly above 1. It follows that the pre-nucleolus $\mathrm{Nu}^*(v) = (1, 4, 7)$. Since the pre-nucleolus is an imputation, it coincides with the nucleolus.

Definition 3.2. Let $(A, \ell) \in \mathcal{L}$ be a liability problem and $x \in \mathbb{R}^N_+$ be a payment vector. Given a creditor $i \in C$, the *deficiency* on liability ℓ_i is given by $d_i(A, \ell, x) = \ell_i - x_i$.

In Example 2.4, the firm, even though it is insolvent, gets a positive payment of 1 at the nucleolus. The deficiency on both liabilities is equal to 2.

Next, we show that the pre-nucleolus satisfies individual rationality and liabilities boundedness.

Theorem 3.3. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem and let v be the induced liability game. Then $v(\{0\}) \leq \mathrm{Nu}^*_0(v)$ and, for every $i \in C$, it holds that $v(\{i\}) \leq \mathrm{Nu}^*_i(v) \leq \ell_i$.*

Proof. The game v is superadditive and therefore zero-monotonic by Remark 2.5. Then it follows from Maschler et al. (1979) that, for every $i \in N$, $v(\{i\}) \leq \mathrm{Nu}^*_i(v)$.

Let $x = \mathrm{Nu}^*(v)$. We show that, for every $i \in C$, $x_i \leq \ell_i$. Suppose $i \in C$ is such that $x_i > \ell_i$. We define $y \in \mathbb{R}^N$ by $y_i = \ell_i$, and, for every $j \in N \setminus \{i\}$, $y_j = x_j + (x_i - \ell_i)/c$. Note that $y \in I(v)$ since $y_i = \ell_i \geq v(\{i\})$, for every $j \in N \setminus \{i\}$, $y_j > x_j \geq v(\{j\})$, and $y(N) = A$.

We show next that $\theta(e(y)) \prec_{\mathrm{lex}} \theta(e(x))$ by partitioning $\mathcal{N}$ in pairs and showing that the ordered excesses of each pair are lexicographically improved when replacing x by y. For every non-empty proper subset S of $N \setminus \{i\}$, it holds that $e(S, x) > e(S \cup \{i\}, x)$ since $v(S \cup \{i\}) - v(S) \leq \ell_i$ and $x_i > \ell_i$. The pair of excesses at y is given by

$$e(S, y) = v(S) - y(S) = v(S) - x(S) - \frac{|S|(x_i - \ell_i)}{c},$$

$$e(S \cup \{i\}, y) = v(S \cup \{i\}) - y(S \cup \{i\}) = v(S \cup \{i\}) - x(S) - \frac{|S|(x_i - \ell_i)}{c} - \ell_i.$$

We have that $e(S, y) < e(S, x)$ and $e(S, y) - e(S \cup \{i\}, y) = v(S) - v(S \cup \{i\}) + \ell_i \geq 0$. Thus for all the pairs of coalitions considered so far, we have lexicographically improved the ordered excesses when replacing the imputation x by the imputation y.

The final pair of coalitions to consider is $\{i\}$ and $N \setminus \{i\}$. We have that

$$e(\{i\}, x) = \max\{0, A - \ell(C \setminus \{i\})\} - x_i < \max\{0, A - \ell(C \setminus \{i\})\} - \ell_i = e(\{i\}, y).$$

Since $\ell(C) > A$, it follows that $e(\{i\}, y) \leq 0$. By Remark 3.1, we now get that $e(N \setminus \{i\}, x) = -e(\{i\}, x) > 0$ and $e(N \setminus \{i\}, x) > e(N \setminus \{i\}, y) \geq 0$. Thus also this pair of ordered excesses is lexicographically improved when x is replaced by y. It follows that $\theta(e(y)) \prec_{\mathrm{lex}} \theta(e(x))$, a contradiction to x being the nucleolus. Consequently, it holds that, for every $i \in C$, $x_i \leq \ell_i$. $\square$

Since the pre-nucleolus satisfies individual rationality by Theorem 3.3, the pre-nucleolus coincides with the nucleolus and from now on we will restrict attention to the nucleolus. By Definition 2.3 it follows immediately that, for every $i \in N$, $v(\{i\}) \geq 0$, and $\sum_{i \in N} \mathrm{Nu}_i(v) = v(N) = A$. Hence it follows from Theorem 3.3 that we can use the nucleolus as a liability rule.

Next, we derive properties of the payment vector generated by the nucleolus. We start by analyzing the asset value that the firm is allowed to keep after making its payments. To do so, we first study the vector of excesses in more detail.

Lemma 3.4. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem, let v be the induced liability game, and let $x = \mathrm{Nu}(v)$. For every $S \subseteq T \subset C$, we have:*

1. *If $\ell(T) \leq A$, then $e(\{0\} \cup S, x) \leq e(\{0\} \cup T, x)$ and $e(C \setminus S, x) \geq e(C \setminus T, x)$.*
2. *If $\ell(S) \geq A$, then $e(\{0\} \cup S, x) \geq e(\{0\} \cup T, x)$ and $e(C \setminus S, x) \leq e(C \setminus T, x)$.*

Proof. Part 1. $\ell(T) \leq A$.

By Definition 2.3, we have that

$$e(\{0\} \cup S, x) = v(\{0\} \cup S) - x(\{0\} \cup S) = \min\{A, \ell(S)\} - x(\{0\} \cup S) = \ell(S) - x(\{0\} \cup S).$$

Similarly, $e(\{0\} \cup T, x) = \ell(T) - x(\{0\} \cup T)$. It follows that

$$e(\{0\} \cup T, x) - e(\{0\} \cup S, x) = \ell(T \setminus S) - x(T \setminus S) \geq 0,$$

where the inequality follows since the nucleolus satisfies liabilities boundedness by Theorem 3.3. Now $e(C \setminus S, x) \geq e(C \setminus T, x)$ follows from Remark 3.1.

Part 2. $\ell(S) \geq A$.

By Definition 2.3, we have that

$$e(\{0\} \cup S, x) = v(\{0\} \cup S) - x(\{0\} \cup S) = \min\{A, \ell(S)\} - x(\{0\} \cup S) = A - x(\{0\} \cup S).$$

Similarly, $e(\{0\} \cup T, x) = A - x(\{0\} \cup T)$. We have that $e(\{0\} \cup S, x) - e(\{0\} \cup T, x) = x(T \setminus S) \geq 0$, where the inequality follows since the nucleolus is a liability rule. Now $e(C \setminus S, x) \leq e(C \setminus T, x)$ follows from Remark 3.1. □

The result of Lemma 3.4 can be used to show that the coalition of all creditors C has the highest excess at the nucleolus.

Theorem 3.5. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem and let v be the induced liability game. For every $S \in \mathcal{N}$ it holds that $e(S, \mathrm{Nu}(v)) \leq e(C, \mathrm{Nu}(v))$.*

Proof. Let $x = \mathrm{Nu}(v) = \mathrm{Nu}^*(v)$. It holds that $e(C, x) = A - (x(N) - x_0) = x_0 \geq v(\{0\}) = 0$.

We show that, for every non-empty $U \subset C$, $e(U, x) \leq e(C, x)$. Let $V = C \setminus U$. If $\ell(V) \leq A$, then by taking $S = \emptyset$ and $T = V$ in Part 1 of Lemma 3.4, we have that $e(C, x) \geq e(C \setminus V, x) = e(U, x)$. Consider next the case $\ell(V) > A$. Take any $i \in U$. By setting $S = V$ and $T = C \setminus \{i\} \supseteq V$ in Part 2 of Lemma 3.4, we get that

$$e(U, x) = e(C \setminus V, x) \leq e(C \setminus (C \setminus \{i\}), x) = e(\{i\}, x) = v(\{i\}) - x_i \leq 0.$$

Since $e(C, x) \geq 0$, it follows that $e(U, x) \leq e(C, x)$.

Let $T \in \mathcal{N}$ be a coalition with the maximal excess at x. Suppose $e(C, x) < e(T, x)$. We have that $0 \in T$ by the first part of the proof. Let $\varepsilon \in (0, e(T, x) - e(C, x))$ and define $y \in I^*(v)$ by $y_0 = x_0 + \varepsilon$ and, for every $i \in C$, $y_i = x_i - \varepsilon/c$. Let some $S \in \mathcal{N}$ be given. If $S \subseteq C$, then $e(S, y) = e(S, x) + \varepsilon|S|/c \leq e(C, x) + \varepsilon < e(T, x)$. If $0 \in S$, then $e(S, y) = e(S, x) + \varepsilon(|S| - 1)/c - \varepsilon < e(S, x)$. It follows that $\max_{S \in \mathcal{N}} e(S, y) < \max_{S \in \mathcal{N}} e(S, x) = e(T, x)$, so $\theta(e(y)) \prec_{\mathrm{lex}} \theta(e(x))$, contradicting that x is equal to the pre-nucleolus. □

Next, we show that at the nucleolus the firm gets a positive payment which is at most equal to half the asset value.

Theorem 3.6. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem and let v be the induced liability game. It holds that $0 < \mathrm{Nu}_0(v) \leq A/2$.*

Proof. Let $x = \mathrm{Nu}(v)$. By Theorem 2.6, $\mathrm{Core}(v) = \emptyset$, so there is a coalition with a positive excess at the nucleolus. Then by Theorem 3.5 it holds that $e(C, x) > 0$. Now $0 < \mathrm{Nu}_0(v)$ follows by observing that $e(C, x) = v(C) - x(C) = A - (A - x_0) = x_0$.

Suppose $x_0 > A/2$. Let $\varepsilon \in (0, x_0 - A/2)$ and define $y \in \mathbb{R}^N$ by $y_0 = x_0 - \varepsilon$, and, for every $i \in C$, $y_i = x_i + \varepsilon/c$. We have that $y(N) = v(N)$, $y_0 > 0 = v(\{0\})$, and, for every $i \in C$, $y_i > x_i \geq v(\{i\})$, so $y \in I(v)$.

Let some $S \in \mathcal{N}$ be given. If $S \subseteq C$, then $e(S, y) = e(S, x) - \varepsilon|S|/c < e(C, x)$, where the inequality follows from Theorem 3.5. If $0 \in S$, then $e(S, x) = v(S) - x(S) < A - A/2 = A/2$ and

Table 3
The liability game when $N = \{0, 1, 2, 3\}$, $A = 10$, $\ell = (1, 8, 8)$. $i \in \{2, 3\}$.

S	$\{0\}$	$\{1\}$	$\{i\}$	$\{0,1\}$	$\{0,i\}$	$\{1,i\}$	$\{2,3\}$	$\{0,1,i\}$	$\{0,2,3\}$	$\{1,2,3\}$
$v(S)$	0	0	1	1	8	2	9	9	10	10

$$e(S, y) = e(S, x) - \varepsilon \tfrac{|S|-1}{c} + \varepsilon \leq e(S, x) + \varepsilon \leq A/2 + \varepsilon < x_0 = e(C, x).$$

It follows that $\max_{S \in \mathcal{N}} e(S, y) < e(C, x) = \max_{S \in \mathcal{N}} e(S, x)$, so $\theta(e(y)) \prec_{\text{lex}} \theta(e(x))$, contradicting that x is equal to the nucleolus. □

Theorem 3.6 highlights an important difference between liability games and bankruptcy games. In liability games, the firm always gets a strictly positive payment at the nucleolus. The intuition for this result is that even though the singleton coalition consisting of the firm has value zero, the firm obtains bargaining power by increasing the value of those coalitions it joins. To put it differently, when there are at least two liabilities, the firm is using the threat to pay others to get debt relief and is able to keep a positive amount of its assets.

We show in Section 4 that the firm can get arbitrarily close to $A/2$ when the nucleolus is used as a liability rule, so the upper bound as provided in Theorem 3.6 is tight.

An interesting question is whether at the nucleolus every creditor with a positive claim faces a loss from the insolvent firm, so whether there is a deficiency on each positive liability. The following example shows this is not always the case.

Example 3.7. Consider the liability problem $(A, \ell) \in \mathcal{L}$ with 3 creditors, where $A = 10$, $\ell_1 = 1$, and $\ell_2 = \ell_3 = 8$. The induced liability game v is depicted in Table 3, where $i = 2$ or $i = 3$.

It is easily verified that $\text{Nu}(v) = x = (7/3, 1, 10/3, 10/3)$, at which $x_1 = \ell_1 = 1$. There is no deficiency on liability 1 even though there is positive deficiency on liability 2 and 3, $d_2(A, \ell, x) = d_3(A, \ell, x) = 14/3$.

The feature in Example 3.7, an insolvent firm paying some of its liabilities in full, can only occur for some of its smaller liabilities. Our last result in this section shows that there is a higher deficiency on higher liabilities at the nucleolus. We also show that creditors with higher claims receive higher payments at the nucleolus. These properties are called order preservation in Aumann and Maschler (1985).

Theorem 3.8. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem and v the induced liability game. Let $i, j \in C$ be such that $\ell_i \leq \ell_j$. At $x = \text{Nu}(v)$ it holds that $x_i \leq x_j$ and $\ell_i - x_i \leq \ell_j - x_j$.*

Proof. Suppose $x_i > x_j$. We define $y \in I(v)$ by $y_i = x_i - \varepsilon$ and $y_j = x_j + \varepsilon$, where $\varepsilon > 0$ is chosen sufficient small such that $y_i > y_j$. The other components of y are set equal to the corresponding components of x. We show next that $\theta(e(y)) \prec_{\text{lex}} \theta(e(x))$ by partitioning the coalitions in $\mathcal{N}$ for which the excess at x is different from the excess at y in pairs and showing that the ordered excesses of each pair are lexicographically improved when replacing x by y.

For coalitions $S \in \mathcal{N}$ such that both i and j belong to S or both i and j belong to $C \setminus S$, it clearly holds that $e(S, x) = e(S, y)$. Take some $S \subseteq N \setminus \{i, j\}$. It holds that

$$\max\{e(S \cup \{i\}, x), e(S \cup \{j\}, x)\} = e(S \cup \{j\}, x),$$
$$\max\{e(S \cup \{i\}, y), e(S \cup \{j\}, y)\} = e(S \cup \{j\}, y) = e(S \cup \{j\}, x) - \varepsilon,$$

where we use that $v(S \cup \{i\}) \leq v(S \cup \{j\})$, $x_i > x_j$, and $y_i > y_j$, so

$$\max\{e(S \cup \{i\}, x), e(S \cup \{j\}, x)\} > \max\{e(S \cup \{i\}, y), e(S \cup \{j\}, y)\}.$$

It follows that $\theta(e(y)) \prec_{\text{lex}} \theta(e(x))$, contradicting that x is equal to the nucleolus. Consequently, it holds that $x_i \leq x_j$.

Suppose $\ell_i - x_i > \ell_j - x_j$. We define $y \in I(v)$ by $y_i = x_i + \varepsilon$ and $y_j = x_j - \varepsilon$, where $\varepsilon > 0$ is chosen sufficient small such that $\ell_i - y_i > \ell_j - y_j$. The other components of y are set equal to the corresponding components of x. For coalitions $S \in \mathcal{N}$ such that both i and j belong to S or both i and j belong to $C \setminus S$, it clearly holds that $e(S, x) = e(S, y)$. Take some $S \subseteq N \setminus \{i, j\}$. It holds that

$$\max\{e(S \cup \{i\}, x), e(S \cup \{j\}, x)\} = e(S \cup \{i\}, x),$$
$$\max\{e(S \cup \{i\}, y), e(S \cup \{j\}, y)\} = e(S \cup \{i\}, y) = e(S \cup \{i\}, x) - \varepsilon,$$

where we use that $v(S \cup \{i\}) - v(S \cup \{j\}) \geq \ell_i - \ell_j$, $\ell_i - x_i > \ell_j - x_j$, and $\ell_i - y_i > \ell_j - y_j$, so

$$\max\{e(S \cup \{i\}, x), e(S \cup \{j\}, x)\} > \max\{e(S \cup \{i\}, y), e(S \cup \{j\}, y)\}.$$

It follows that $\theta(e(y)) \prec_{\text{lex}} \theta(e(x))$, contradicting that x is equal to the nucleolus. Consequently, it holds that $\ell_i - x_i \leq \ell_j - x_j$. □

Theorem 3.8 implies that using the nucleolus as a liability rule leads to the same payment to creditors with identical claims. In terms of deficiency, we have seen in Example 3.7 that there could be no deficiency on low liabilities. Theorem 3.8 generalizes this insight and implies that there is a threshold such that there is no deficiency on liabilities below the threshold and there is positive deficiency above the threshold.

4. Generalized proportional rules

In principle, it is possible to compute the nucleolus by solving a linear programming problem. However, the size of the linear programming problem is related to the number of coalitions, which increases exponentially in the number of players. For instance, in a game with n players, Owen (1974) presents a linear programming formulation to compute the nucleolus that involves $2^{n+1}+n$ variables and 4^n+1 constraints. In this section, we argue that for liability problems there is a relation between the nucleolus and a particular generalized proportional rule and we have an easy to compute closed-form solution for the latter.

A well-known rule in bankruptcy problems is the proportional rule. Under the proportional rule the entire asset value is distributed to the creditors in proportion to their liabilities. The insolvent agent is then left without any assets. A rule is called a generalized proportional rule if the firm receives a non-negative amount and payments to creditors are made in proportion to the liabilities.

Definition 4.1. The liability rule $f : \mathcal{L} \to \mathbb{R}^N_+$ is a *generalized proportional rule* if for every $(A, \ell) \in \mathcal{L}$ there is $\lambda \in [0, 1]$ such that for every $i \in C$ it holds that $f_i(A, \ell) = \lambda \ell_i$.

Let $p : \mathcal{L} \to \mathbb{R}^N_+$ denote the generalized proportional rule obtained by assigning to $(A, \ell) \in \mathcal{L}$ the payment vector

$$p_0(A,\ell) = \frac{(\ell(C)-A)A}{2\ell(C)-A},$$
$$p_i(A,\ell) = \frac{A}{2\ell(C)-A}\ell_i, \quad \text{for all } i \in C.$$

To relate p to the nucleolus, we need some extra assumptions. For a given liability problem $(A, \ell) \in \mathcal{L}$, let $\tilde{C}$ be the set of creditors with a liability strictly in between 0 and A, so $\tilde{C} = \{i \in C \mid 0 < \ell_i < A\}$, and denote the cardinality of $\tilde{C}$ by $\tilde{c}$.[3] For each non-empty $S \subseteq \tilde{C}$, let $\tilde{m}(S) \in \mathbb{R}^{\tilde{C}}$ denote the *membership vector* of S, so $\tilde{m}_i(S) = 1$ if $i \in S$ and $\tilde{m}_i(S) = 0$ if $i \in \tilde{C} \setminus S$. Let $\tilde{\mathcal{S}}$ be the collection of all $S \in \mathcal{N}$ such that $S \subseteq \tilde{C}$. The collection $\tilde{\mathcal{S}}$ is *independent* if $\{\tilde{m}(S) | S \in \tilde{\mathcal{S}}\}$ is a set of independent vectors. The collection $\tilde{\mathcal{S}}$ is *balanced* if there are weights $(\lambda_S)_{S\in\tilde{\mathcal{S}}} \in \mathbb{R}^{\tilde{\mathcal{S}}}_{++}$ such that $\sum_{S\in\tilde{\mathcal{S}}} \lambda_S \tilde{m}(S) = \tilde{m}(\tilde{C})$.

Assumption 4.2. The liability problem $(A, \ell) \in \mathcal{L}$ is such that there exists an independent and balanced collection of coalitions $\tilde{\mathcal{S}}$ with $|\tilde{\mathcal{S}}| = \tilde{c}$ such that for every $S \in \tilde{\mathcal{S}}$ we have that $\ell(S) = A$.

As an illustration, the following example presents a number of cases where Assumption 4.2 is satisfied.

Example 4.3. Case 1. The liability problem $(A, \ell) \in \mathcal{L}$ is such that for every $i \in C$, $\ell_i = 0$ or $\ell_i = A$. It holds that $\tilde{C} = \emptyset$ and Assumption 4.2 is trivially satisfied.

Case 2. The liability problem $(A, \ell) \in \mathcal{L}$ has $\tilde{c} \geq 3$ creditors with a claim in $(0, A)$, without loss of generality, $\tilde{C} = \{1, \ldots, \tilde{c}\}$, and $(\ell_1, \ldots, \ell_{\tilde{c}}) = (A/(\tilde{c}-1), \ldots, A/(\tilde{c}-1))$. Then Assumption 4.2 is satisfied with $\tilde{\mathcal{S}} = \{\{1, \ldots, \tilde{c}-1\}, \{1, \ldots, \tilde{c}-2, \tilde{c}\}, \ldots, \{2, \ldots, \tilde{c}\}\}$ and $\lambda_S = 1/(\tilde{c}-1)$ for all $S \in \tilde{\mathcal{S}}$.

Case 3. The liability problem $(A, \ell) \in \mathcal{L}$ has $\tilde{c} \geq 3$ creditors with a claim in $(0, A)$, without loss of generality, $\tilde{C} = \{1, \ldots, \tilde{c}\}$, $(\ell_1, \ldots, \ell_{\tilde{c}-1}) = (A/(\tilde{c}-1), \ldots, A/(\tilde{c}-1))$, and $\ell_{\tilde{c}} = A(\tilde{c}-2)/(\tilde{c}-1)$. Then Assumption 4.2 is satisfied with $\tilde{\mathcal{S}} = \{\{1, \tilde{c}\}, \ldots, \{\tilde{c}-1, \tilde{c}\}\} \cup \{\{1, \ldots, \tilde{c}-1\}\}$, $\lambda_{\{1,\tilde{c}\}} = \cdots = \lambda_{\{\tilde{c}-1,\tilde{c}\}} = 1/(\tilde{c}-1)$, and $\lambda_{\{1,\ldots,\tilde{c}-1\}} = (\tilde{c}-2)/(\tilde{c}-1)$.

Although Assumption 4.2 is not a generic condition, there are many cases in which it is satisfied, some of which are presented in Example 4.3. One needs to find $\tilde{c}$ coalitions S such that $\ell(S) = A$ and the collection of coalitions is balanced and independent. In general, there are many balanced and independent collections of coalitions, see Peleg (1965). Often Assumption 4.2 is approximately satisfied in the sense that for every element S in some independent and balanced collection of coalitions $\tilde{S}$ it holds that $\ell(S)$ is approximately equal to A. The next result shows that under Assumption 4.2,

[3] The dependence of $\tilde{C}$ and $\tilde{c}$ on the liability problem (A, ℓ) is not made explicit in the notation.

the nucleolus corresponds to the payment vector generated by the generalized proportional rule p. In case Assumption 4.2 only holds approximately, the payment vector generated by p would still be approximately equal to the nucleolus since the associated TU game depends continuously on (A, ℓ) and the Nucleolus depends continuously on TU games (Schmeidler, 1969). Example 4.5 illustrates this by presenting a closed-form solution for the nucleolus in a family of examples where Assumption 4.2 is typically not satisfied and showing that it is approximately equal to the generalized proportional rule p.

Theorem 4.4. *Let $(A, \ell) \in \mathcal{L}$ be a liability problem satisfying Assumption 4.2 and let v be the induced liability game. It holds that* $\mathrm{Nu}(v) = p(A, \ell)$.

Proof. Let $x = p(A, \ell)$. We show that $x = \mathrm{Nu}(v)$. First, we establish that for all coalitions the excesses at x are less than or equal to x_0. Let some $S \in \mathcal{N}$ be given. If $0 \in S$, then we have

$$\begin{aligned} e(S, x) &= \min\{A, \ell(S \setminus \{0\})\} - x_0 - \frac{A}{2\ell(C) - A}\ell(S \setminus \{0\}) \\ &\leq (1 - \frac{A}{2\ell(C) - A}) \min\{A, \ell(S \setminus \{0\})\} - x_0 \leq (1 - \frac{A}{2\ell(C) - A})A - x_0 = x_0. \end{aligned}$$

If $0 \notin S$, then we have $e(S, x) = \max\{0, A - \ell(C \setminus S)\} - (A - x_0 - x(C \setminus S))$. If $A - \ell(C \setminus S) \geq 0$, then the right-hand side equals $-\ell(C \setminus S) + x_0 + x(C \setminus S) \leq x_0$. Otherwise, $A - \ell(C \setminus S) < 0$ and the right-hand side is equal to $-A + x_0 + x(C \setminus S) \leq x_0$.

Let $\tilde{\mathcal{S}}$ be an independent and balanced collection of coalitions with $|\tilde{\mathcal{S}}| = \tilde{c}$ such that for all $S \in \tilde{\mathcal{S}}$ we have that $\ell(S) = A$ and let $(\lambda_S)_{S \in \tilde{\mathcal{S}}} \in \mathbb{R}^{\tilde{\mathcal{S}}}_{++}$ be the corresponding vector of balancing weights. We define $\overline{C} = \{i \in C \mid \ell_i > 0\}$ as the set of creditors with a positive claim and denote its cardinality by $\overline{c} \geq 2$. We define $\overline{\mathcal{S}} = \tilde{\mathcal{S}} \cup \{\{i\} \mid i \in \overline{C} \setminus \tilde{C}\}$. For a non-empty subset S of $\overline{C}$, we define the membership vector $\overline{m}(S) \in \mathbb{R}^{\overline{C}}$ by $\overline{m}_i(S) = 1$ if $i \in S$ and $\overline{m}_i(S) = 0$ if $i \in \overline{C} \setminus S$. We denote $\overline{m}(\overline{C})$ by $\mathbb{1}$.

Since the set $\{\tilde{m}(S) \mid S \in \tilde{\mathcal{S}}\}$ is a set of independent vectors of cardinality $\tilde{c}$ and $\{\overline{m}(\{i\}) \mid i \in \overline{C} \setminus \tilde{C}\}$ is a set of unit vectors with coordinate one at a player not being part of some $S \in \tilde{\mathcal{S}}$, the set $\{\overline{m}(S) \mid S \in \overline{\mathcal{S}}\}$ is a set of independent vectors of cardinality $\overline{c}$. For $i \in \overline{C} \setminus \tilde{C}$, we define $\lambda_{\{i\}} = 1$. We have that $\sum_{S \in \overline{\mathcal{S}}} \lambda_S \overline{m}(S) = \mathbb{1}$, or in matrix notation, with M being the matrix with columns equal to $\overline{m}(S)$ for $S \in \overline{\mathcal{S}}$, $M\lambda = \mathbb{1}$. It holds that $\sum_{S \in \overline{\mathcal{S}}} \lambda_S > 1$, since $\overline{c} \geq 2$ and the set $\overline{\mathcal{S}}$ therefore contains at least two coalitions, whereas the sum of the weights over the members of each coalition is equal to 1.

Let some $S \in \overline{\mathcal{S}}$ be given. We have by Definition 2.3 that $v(\{0\} \cup S) = \min\{A, \ell(S)\} = A$, where the last equality follows from Assumption 4.2 if $S \in \tilde{\mathcal{S}}$ and from $\ell_i = A$ if $S = \{i\}$ for $i \in \overline{C} \setminus \tilde{C}$. It holds that

$$e(\{0\} \cup S, x) = A - x_0 - \frac{A}{2\ell(C) - A}\ell(S) = A - \frac{(\ell(C) - A)A}{2\ell(C) - A} - \frac{A}{2\ell(C) - A}A = \frac{(\ell(C) - A)A}{2\ell(C) - A} = x_0.$$

For the coalition of all creditors, we also have $e(C, x) = v(C) - x(C) = A - (A - x_0) = x_0$.

Let $y = \mathrm{Nu}(v)$ and suppose $y \neq x$. We have that $\theta(e(y)) \prec_{\text{lex}} \theta(e(x))$. From Theorem 3.5 and from $e(C, y) = A - A + y_0 = y_0$ it follows that $y_0 \leq x_0$. For every $S \in \overline{\mathcal{S}}$ it holds that

$$e(\{0\} \cup S, y) = A - y(\{0\} \cup S) \leq y_0 \leq x_0 = e(\{0\} \cup S, x) = A - x(\{0\} \cup S),$$

where the first inequality follows by Theorem 3.5. It follows that, for every $S \in \overline{\mathcal{S}}$, $x_0 + x(S) \leq y_0 + y(S)$, or in matrix notation

$$x_0\mathbb{1} + M^\top x_{\overline{C}} \leq y_0\mathbb{1} + M^\top y_{\overline{C}}, \tag{4.1}$$

where $x_{\overline{C}} = (x_i)_{i \in \overline{C}}$ and $y_{\overline{C}} = (y_i)_{i \in \overline{C}}$. Consider first the case $x_0 = y_0$. Since $x \neq y$ and M has full rank, we have that $M^\top x_{\overline{C}} < M^\top y_{\overline{C}}$. Taking the inner product with the vector of balancing weights $\lambda \in \mathbb{R}^{\tilde{\mathcal{S}}}_{++}$, we obtain

$$\mathbb{1}^\top x_{\overline{C}} = \lambda^\top M^\top x_{\overline{C}} < \lambda^\top M^\top y_{\overline{C}} = \mathbb{1}^\top y_{\overline{C}}. \tag{4.2}$$

For every $i \in C$ such that $\ell_i = 0$, it holds that $x_i = y_i = 0$ since x and y are payment vectors. Also, $x_0 = y_0$ and $x(N) = y(N)$, so we find that $\mathbb{1}^\top x_{\overline{C}} = \mathbb{1}^\top y_{\overline{C}}$, leading to a contradiction with (4.2).

Consider next the case $y_0 < x_0$. We take the inner product of (4.1) with the vector of balancing weights λ. This gives $x_0\lambda^\top\mathbb{1} + A - x_0 \leq y_0\lambda^\top\mathbb{1} + A - y_0$, since $\lambda^\top M^\top x_{\overline{C}} = \mathbb{1}^\top x_{\overline{C}} = A - x_0$ and $\lambda^\top M^\top y_{\overline{C}} = \mathbb{1}^\top y_{\overline{C}} = A - y_0$. Equivalently, we have $(x_0 - y_0)(\lambda^\top\mathbb{1} - 1) \leq 0$. Since $x_0 > y_0$ and $\lambda^\top\mathbb{1} = \sum_{S \in \overline{\mathcal{S}}} \lambda_S > 1$, we obtain a contradiction. Consequently, it holds that $x = y$ as was to be shown. $\square$

Let $(A, \ell) \in \mathcal{L}$ be a liability problem such that $\ell_i = A$ for every $i \in C$. Case 1 in Example 4.3 is satisfied, so the nucleolus $\mathrm{Nu}(v)$ of the induced liability game is equal to $p(A, \ell)$. We have that $p_0(A, \ell) = ((c - 1)A)/(2c - 1)$. Theorem 3.6 states that the payment received by the firm is at most equal to half of the asset value. The expression above shows that the payment received by the firm at the nucleolus can get arbitrarily close to $A/2$ for large values of c.

Our last example, where all liabilities are the same, demonstrates that the generalized proportional rule p is approximately equal to the nucleolus when Assumption 4.2 is approximately satisfied.

Example 4.5. Consider a liability problem $(A, \ell) \in \mathcal{L}$ where all liabilities are identical and equal to $a < A$. The firm is insolvent, so $ca > A$. Let $k \in \mathbb{N}$ be such that $ka \leq A$ and $(k+1)a > A$. If $S \subset C$ is a coalition with k creditors, then $v(\{0\} \cup S) = ka$ and if $T \subseteq C$ is a coalition with $k+1$ creditors than $v(\{0\} \cup T) = A$. We denote the nucleolus $\mathrm{Nu}(v)$ of the induced liability game v by x. We know by Lemma 3.4 that the excess of $\{0\} \cup S$ or $\{0\} \cup T$ is the maximal excess at x among all coalitions containing 0. Since all liabilities are identical, all creditors receive the same payoff, x_1. We have that $e(C, x) = x_0$ and

$$e(\{0\} \cup S, x) = ka - x_0 - kx_1,$$
$$e(\{0\} \cup T, x) = A - x_0 - (k+1)x_1.$$

It holds that $ka - x_0 - kx_1 \geq A - x_0 - (k+1)x_1$ if and only if $x_1 \geq A - ka$.

We solve for $ka - x_0 - kx_1 = x_0$ and $x_0 + cx_1 = A$ to find $x_0 = (cka - kA)/(2c - k)$ and $x_1 = (2A - ka)/(2c - k)$. The condition $x_1 \geq A - ka$ is then equivalent to

$$ka \frac{2c - k - 1}{2c - k - 2} \geq A.$$

This condition is clearly satisfied whenever $ka = A$. In that case, we get exactly the payment vector of the generalized proportional rule $p(A, \ell)$ of Theorem 4.4, so $x_0 = ((c - k)A)/(2c - k)$ and $x_1 = A/(2c - k)$. To the extent that ka is almost equal to A, we get that the nucleolus is almost equal to the generalized proportional rule $p(A, \ell)$.

In case $ka(2c - k - 1) < A(2c - k - 2)$, we find the nucleolus by solving $A - x_0 - (k + 1)x_1 = x_0$ and $x_0 + cx_1 = A$ and find $x_0 = (cA - (k + 1)A)/(2c - k - 1)$ and $x_1 = A/(2c - k - 1)$. Again, to the extent that $(k + 1)a$ is almost equal to A, we get that the nucleolus is almost equal to the payment vector of the generalized proportional rule $p(A, \ell)$.

5. Conclusion

We study the allocation of the asset value of an insolvent firm among creditors using transferable utility games and, contrary to the large body of game-theoretic work on bankruptcy games, treat the firm as a player. In particular, we study the nucleolus of the resulting liability game and prove that it assigns a positive payoff to the firm, at most equal to one half of its asset value. We also show that creditors with higher claims get higher payments, although there is a higher deficiency on higher liabilities.

In general, there is no closed-form solution for the nucleolus. However, we provide conditions under which it coincides with a generalized proportional rule. In the game-theoretic literature on bankruptcy games, the bankrupt agent is not treated as a player, but rather as an exogenous estate. As shown by Aumann and Maschler (1985), the nucleolus corresponds to the Talmud rule. It is striking that the case where the bankrupt agent is a player makes such a significant difference in the allocation of the asset value.

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