# On the Expressive Power of Schemes 

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#### Abstract

We present a calculus, called the scheme-calculus, that permits to express natural deduction proofs in various theories. Unlike $\lambda$-calculus, the syntax of this calculus sticks closely to the syntax of proofs, in particular, no names are introduced for the hypotheses. We show that despite its non-determinism, some typed scheme-calculi have the same expressivity as the corresponding typed $\lambda$-calculi.


## 1 Introduction

We present a calculus, called the scheme-calculus, that permits to express natural deduction proofs without introducing names for the hypotheses.

### 1.1 A scheme calculus

In the algorithmic interpretation of proofs, introduced by Brouwer, Heyting, and Kolmogorov, proofs are expressed by terms of a typed $\lambda$-calculus. In such a calculus, two kinds of variables are often used: those of the logic and those introduced to name the hypotheses. In System $F$, for instance, type variables and proof variables are often distinguished.

When variables are introduced to name the hypotheses, the two occurrences of the proposition $A$ in the context of the sequent $A, A \vdash A$ must be distinguished, and thus the contexts must be multisets of propositions. In contrast, in automated theorem proving, in order to reduce the search space (e.g. to a finite space), the contexts of the sequents are often considered as sets of propositions [1]. Thus, slightly different notions of sequents are used in proof-theory and in automated theorem proving. Moreover, these hypothesis names make the proofs of a given proposition a non-context-free language, even in the minimal propositional logic [2, 3, 4].

In this paper, we introduce a calculus, called the scheme-calculus, that permits to express proofs without introducing names for the hypotheses and where the contexts are just sets of hypotheses. In other words, we keep the variables

[^0]of predicate logic, but do not introduce another category of variables for the hypotheses.

In the scheme-calculus, the proofs of a given proposition in minimal propositional logic and even in the positive fragment of minimal predicate logic form a context-free language. In fact, this scheme-calculus stems from previous works on the grammatical properties of sets of $\lambda$-terms [5, 6, 7, 8, 9, 10, 11, 4, 12].

From the grammar generating the schemes of a given type, we can build an algorithm generating all the $\lambda$-terms of this type, as each scheme corresponds to a finite number of terms that can be computed from it 4]. A scheme containing $n$ abstractions and $p$ variables aggregate up to $p^{n} \lambda$-terms. In this sense, more proofs are identified in the scheme-calculus than in the $\lambda$-calculus, but, unlike in the formalisms based on proof irrelevance, not all the proofs are identified, for instance the terms $\lambda x_{P} \lambda y_{P} x$ and $\lambda x_{P} \lambda y_{P} y$ are identified, but the terms $\lambda x_{P} \lambda f_{P \Rightarrow P} x$ and $\lambda x_{P} \lambda f_{P \Rightarrow P}(f x)$ are not.

Despite its simplicity, we show that this scheme-calculus is as expressive as the dependently-typed $\lambda$-calculus: for some type systems, all the functions that are provably total in impredicative (i.e. second-order) arithmetic can be expressed in the scheme-calculus. In this expressivity result, the determinism does not come from a local property, such as confluence, as for the $\lambda$-calculus, but from the subject-reduction property and the fact that dependent types are powerful enough to specify the value of terms.

### 1.2 The notion of variable

To understand the basic idea of the scheme-calculus, it is useful to go back to the origin of the notion of variable. A term expressing a function is usually built using a function-former, often written as $\lambda$, and a place-holder for the yet unknown argument of the function, sometimes written as $\square$. For instance, the function mapping a number to its double can be expressed by the term

$$
\lambda(2 \times \square)
$$

Applying this term to 4 yields a term that reduces to $2 \times 4$.
But, when applying the term

$$
\lambda \lambda(2 \times \square \times \square \times \square)
$$

that contains several occurrences of the symbol $\lambda$, to the arguments 4 and 5 , for instance, we may get eight different syntactic results by replacing each occurrence of the symbol $\square$ either by 4 or by 5 . Hence arises the need of a pointer associating a function-former occurrence $\lambda$ to each place-holder occurrence $\square$.

In the $\lambda$-calculus, this pointer is expressed by giving a name to each occurrence of a $\lambda$ and to each occurrence of a $\square$. The $\lambda$ associated to a place-holder $\square_{x}$ is then the first $\lambda_{x}$ above it in the term seen as a tree. This way, the function mapping two numbers to the double of the product of the square of the first and of the second is written as

$$
\lambda_{x} \lambda_{y}\left(2 \times \square_{x} \times \square_{x} \times \square_{y}\right)
$$

or, in a simpler way, as

$$
\lambda x \lambda y(2 \times x \times x \times y)
$$

Other solutions have been investigated. A solution related to Bourbaki's is to express the pointer with a directed edge from each $\square$ to the corresponding $\lambda$


While, in the solution proposed by de Bruijn, each $\square$ is assigned the height of its associated $\lambda$ above it. So we get

$$
\lambda \lambda\left(2 \times \square_{2} \times \square_{2} \times \square_{1}\right)
$$

Applying these three terms to the terms 4 and 5 yields terms that reduce, in each formalism, to $2 \times 4 \times 4 \times 5$ only.

In many cases, both $\lambda \mathrm{s}$ and $\square \mathrm{s}$ are typed and the pointers must relate objects of the same type. This identity of types guarantees the subject-reduction property: the reduction of a well-typed term yields a term of the same type. Knowing the type of each $\lambda$ and $\square$ often reduces the possibilities of linking occurrences of the symbols $\square$ and $\lambda$ in a raw term. For instance, in the raw term

$$
\lambda_{\text {scal }} \lambda_{\text {vect }}\left(2 . \square_{\text {scal }} . \square_{\text {scal }} . \square_{\text {vect }}\right)
$$

there is only one way to associate a $\lambda$ to each $\square$, but in the raw term

$$
\lambda_{n a t} \lambda_{n a t}\left(2 \times \square_{n a t} \times \square_{n a t} \times \square_{n a t}\right)
$$

there are still eight ways to associate a $\lambda$ to each $\square$.
In the scheme-calculus, instead of distinguishing eight terms $\lambda x \lambda y(2 \times x \times x \times$ $x), \lambda x \lambda y(2 \times x \times x \times y), \ldots$, we consider a single scheme $\lambda_{\text {nat }} \lambda_{\text {nat }}(2 \times\langle n a t\rangle \times$ $\langle n a t\rangle \times\langle n a t\rangle$ ), where $\langle n a t\rangle$ is the canonical (i.e. only) variable of type nat. In this scheme, each place-holder is possibly associated to any function-former above it, provided they have the same type. The scheme $\left(\lambda_{n a t} \lambda_{n a t}(2 \times\langle n a t\rangle \times\right.$ $\langle n a t\rangle \times\langle n a t\rangle) 45$ ) aggregates eight terms and reduces, in a non-deterministic way, to $(2 \times 4 \times 4 \times 4),(2 \times 4 \times 4 \times 5), \ldots$ The reduction of schemes is therefore non-deterministic, but it does enjoy the subject-reduction property.

In general, the scheme $\left(\left(\lambda_{A} \ldots \lambda_{A}\langle A\rangle\right) t_{1} t_{2} \ldots t_{n}\right)$ reduces, in a non-deterministic way, to each of the $t_{i} \mathrm{~s}$. This is typical of non-deterministic extensions of $\lambda$ calculus, such as G. Boudol's $\lambda$-calculus with multiplicities [13, where the term $(\lambda x x)\left(t_{1}|\ldots| t_{n}\right)$ reduces also in a non-deterministic way to each of $t_{i}$ 's.

In the $\lambda$-calculus, when we apply the substitution $(f y) / x$ to the term $\lambda y$ : $B(g x y)$, we must rename the bound variable $y$ in order to avoid the variable capture. As there is only one variable of type $B$ in the scheme calculus, we are no longer able to rename the variables this way and the variable captures cannot be avoided.

### 1.3 The algorithmic interpretation of proofs

In the algorithmic interpretation of proofs, the subject-reduction property is more important than the uniqueness of results. For instance, consider the natural deduction proof
where $t$ and $u$ are two cut free proofs of the sequent $\vdash A$. This proof can be reduced, in a non-deterministic way, to $t$ or to $u$, but in both cases, we get a cut free proof of $\vdash A$.

When we associate a term of $\lambda$-calculus to this proof, we must associate a variable name to each hypothesis of the sequent $A, A \vdash A$, and we must choose the variable used in the axiom rule. Different choices lead to different proofterms: $((\lambda \alpha \lambda \beta \alpha) t u)$ and $((\lambda \alpha \lambda \beta \beta) t u)$, and each of these terms reduces to a unique normal form.

This example shows that, in some presentations of natural deduction with unnamed hypotheses, proof reduction is non-deterministic, and $\lambda$-calculus introduces determinism in a somewhat artificial way.

As shown by Statman and Leivant (see [14, 15]) the proof reduction process defined directly on natural deduction proofs with unnamed hypotheses is not strongly normalizing, while that of $\lambda$-calculus is. This non-termination can be seen as a consequence of the fact that variable captures are allowed. As, in general, termination is lost in the scheme-calculus, a strategy must be chosen.

### 1.4 Names and specifications

In the cross-fertilization of the theories of proof languages and of programming languages, the expression of natural deduction proofs in $\lambda$-calculus can be seen as the importation in proof theory of the concept of variable name, that is familiar in the theory of programming languages. On the opposite, the schemecalculus can be seen as an importation in the theory of programming languages of the concept of anonymous hypothesis, that is familiar in proof theory.

Yet, this idea of anonymous resource is not completely new in computer science. For instance, when we connect a computer to a local network, we just need to use any unnamed Ethernet cable. Its type "Ethernet cable" is sufficient to guarantee the connection to the network. In the same way, when a type system is strong enough to specify the value returned by a program, there is no need to give names to different programs of the same type: when such a program is needed, any program, that has the right type, goes. Identifying programs by their specification and not by their name may be a way to avoid the proliferation of variable names in programs and other formal objects.

The main calculus we shall introduce in this paper is a scheme-calculus with dependent types (Section 3 ), that permits to express proofs of various theories in minimal predicate logic. We shall prove three properties of this dependentlytyped scheme-calculus, that are subject-reduction (Section 4), normalization (Section 5), and an expressivity result (Section 6). As an introductory example, we start with a simply-typed scheme-calculus.

## 2 A simply-typed scheme-calculus

### 2.1 The calculus

Definition 2.1 (Simple types) Let $\mathcal{P}$ be a set of atomic types. The simple types are inductively defined by

$$
A=P \mid A \Rightarrow A^{\prime}
$$

with $P \in \mathcal{P}$.
Definition 2.2 (Context) $A$ context is a finite set of simple types.
Definition 2.3 (Simply-typed schemes) Schemes are inductively defined by

$$
t=\langle A\rangle\left|\lambda_{A} t\right|\left(t t^{\prime}\right)
$$

The scheme $\langle A\rangle$ is the canonical variable of type $A, \lambda_{A} t$ is the scheme obtained by abstracting the canonical variable $\langle A\rangle$ of type $A$ in $t$, and $\left(t t^{\prime}\right)$ is the application of the scheme $t$ to the scheme $t^{\prime}$.

The typing rules are given in Figure 1. Notice that as contexts are sets, if $A$ is an element of $\Gamma$, then $\Gamma \cup\{A\}$ is just $\Gamma$. For instance, using these rules, the scheme $\lambda_{A} \lambda_{A} \lambda_{A}\langle A\rangle$ can be given the type $A \Rightarrow A \Rightarrow A \Rightarrow A$ with the following derivation.

$$
\begin{gathered}
\overline{A \vdash\langle A\rangle: A} \\
\frac{A \vdash \lambda_{A}\langle A\rangle: A \Rightarrow A}{\overline{A \vdash \lambda_{A} \lambda_{A}\langle A\rangle: A \Rightarrow A \Rightarrow A}} \\
\vdash \lambda_{A} \lambda_{A} \lambda_{A}\langle A\rangle: A \Rightarrow A \Rightarrow A \Rightarrow A
\end{gathered}
$$

Definition 2.4 (Scheme in context) $A$ scheme in context is a pair $t_{\Gamma}$ where $t$ is a scheme and $\Gamma$ is a context such that $t$ is well-typed in $\Gamma$.

We sometimes omit the context $\Gamma$ when there is no ambiguity.

### 2.2 Reduction

When reducing the underlined redex in the scheme

$$
\lambda_{A} \ldots \lambda_{A}\left(\underline{\lambda_{A}\left(\lambda_{A} \ldots \lambda_{A}\langle A\rangle\right) u}\right)
$$

$$
\begin{gathered}
\overline{\Gamma \vdash\langle A\rangle: A} A \in \Gamma \\
\frac{\Gamma \cup\{A\} \vdash t: B}{\Gamma \vdash \lambda_{A} t: A \Rightarrow B} \\
\frac{\Gamma \vdash t: A \Rightarrow B \quad \Gamma \vdash u: A}{\Gamma \vdash(t u): B}
\end{gathered}
$$

Figure 1: Simply-typed schemes
the variable $\langle A\rangle$ may be bound by the $\lambda_{A}$ of the redex, but it may also be bound by another $\lambda_{A}$, either higher or lower in the scheme. So, in general, the scheme $u$ may be substituted for the variable $\langle A\rangle$ or not, hence the non-determinism of the substitution.

However, if the variable $\langle A\rangle$ is bound neither higher nor lower in the term, the only possible binder for $\langle A\rangle$ is that of the redex. In such a case, the variable $\langle A\rangle$ must be substituted. Thus, the fact that the scheme $u$ may or must be substituted for the variable $\langle A\rangle$ depends not only on the reduced redex but also on the position of this redex in the scheme. Therefore, the reduction relation cannot be defined on schemes. Instead, it has to be defined on schemes in contexts.

To define the reduction relation, we must first set up a notion of substitution. A substitution is a function of finite domain, written as $\left[t_{1} / A_{1}, \ldots, t_{n} / A_{n}\right]$, associating schemes $t_{1}, \ldots, t_{n}$ of types $A_{1}, \ldots, A_{n}$, respectively, to the variables $\left\langle A_{1}\right\rangle, \ldots,\left\langle A_{n}\right\rangle$. Applying a substitution to a scheme may produce several results, thus this application produces a set of results. Moreover, this application is always performed with respect to some context $\Gamma$ that specifies the variables for which the substitution may or must be performed. More precisely, when $A$ is in $\Gamma$, we may choose whether we substitute the canonical variable of type $A$ or not and when $A$ is not in $\Gamma$, this substitution is forced. If $\sigma$ is a substitution, $t$ is a scheme and $\Gamma$ a context, we write $\sigma_{\Gamma} t$ for the result of the application of $\sigma$ to $t$, with respect to the context $\Gamma$.

To simplify the notations, if $S$ and $S^{\prime}$ are sets of schemes, we write $\lambda_{A} S$ for the set of schemes of the form $\lambda_{A} t$ for $t$ in $S$ and $\left(S S^{\prime}\right)$ for the set of schemes of the form $\left(t t^{\prime}\right)$ for $t$ in $S$ and $t^{\prime}$ in $S^{\prime}$.

## Definition 2.5 (Substitution)

- $\sigma_{\Gamma}\langle A\rangle=\{\langle A\rangle, \sigma(A)\}$ if $A \in \operatorname{dom}(\sigma)$ and $A \in \Gamma$,
- $\sigma_{\Gamma}\langle A\rangle=\{\sigma(A)\}$ if $A \in \operatorname{dom}(\sigma)$ and $A \notin \Gamma$,
- $\sigma_{\Gamma}\langle A\rangle=\{\langle A\rangle\}$ if $A \notin \operatorname{dom}(\sigma)$,
- $\sigma_{\Gamma}\left(\lambda_{A} t\right)=\lambda_{A} \sigma_{(\Gamma \cup\{A\})} t$,
- $\sigma_{\Gamma}(t u)=\left(\sigma_{\Gamma} t \sigma_{\Gamma} u\right)$.

Definition 2.6 (Reduction) The one step top level $\beta$-reduction, written as $\longrightarrow$, is defined by the rule

$$
\left(\left(\lambda_{A} t\right) u\right)_{\Gamma} \longrightarrow v_{\Gamma}
$$

where $A, t$, $u$, and $\Gamma$ are arbitrary and $v$ is any element of $[u / A]_{\Gamma} t$.
The one step $\beta$-reduction relation $\triangleright$ is the contextual closure of the relation $\longrightarrow$. It is inductively defined by

- if $t_{\Gamma} \longrightarrow t_{\Gamma}^{\prime}$, then $t_{\Gamma} \triangleright t_{\Gamma}^{\prime}$,
- if $t_{\Gamma} \triangleright t_{\Gamma}^{\prime}$, then $(t u)_{\Gamma} \triangleright\left(t^{\prime} u\right)_{\Gamma}$,
- if $u_{\Gamma} \triangleright u_{\Gamma}^{\prime}$, then $(t u)_{\Gamma} \triangleright\left(t u^{\prime}\right)_{\Gamma}$,
- if $t_{\Gamma \cup\{A\}} \triangleright t_{\Gamma \cup\{A\}}^{\prime}$, then $\left(\lambda_{A} t\right)_{\Gamma} \triangleright\left(\lambda_{A} t^{\prime}\right)_{\Gamma}$.

This $\beta$-reduction relation $\triangleright^{*}$ is the reflexive-transitive closure of $\triangleright$.
The reduction relation is not confluent. Indeed, if $A \notin \Gamma$, the scheme $\left(\left(\lambda_{A} \lambda_{A}\langle A\rangle\right) t u\right)_{\Gamma}$ reduces to both $t_{\Gamma}$ and $u_{\Gamma}$, in a non-deterministic way. This formalizes the intuition that, in the scheme $\left(\lambda_{A} \lambda_{A}\langle A\rangle\right)_{\Gamma}$, the variable $\langle A\rangle$ may be bound by either of the occurrences of the symbol $\lambda_{A}$.

More surprisingly this reduction relation is not strongly normalizing.

### 2.3 Counter-examples to strong normalization

Proposition 2.1 (Statman [14], Leivant [15]) The simply-typed scheme-calculus is not strongly normalizing.

Proof. Let $t=\left(\left(\lambda_{A}\langle A\rangle\right)\langle A\rangle\right)_{A}$. The scheme $\left(\left(\lambda_{A} t\right) t\right)_{A}$ reduces to each of the elements of $[t / A]_{A} t$, e.g. to $\left(\left(\lambda_{A} t\right) t\right)_{A}$.

This counter-example shows that, when we express natural deduction with sequents without naming the hypotheses, proof reduction is not strongly normalizing. For instance, the proof

$$
\begin{aligned}
& \frac{\overline{A \vdash A}}{\frac{A \vdash A \Rightarrow A}{A \vdash} \Rightarrow \mathrm{i}} \overline{A \vdash A} \text { ax } \\
& \frac{A \vdash A}{A \vdash A \Rightarrow A} \Rightarrow \mathrm{i}
\end{aligned} \mathrm{e} \quad \frac{\overline{A \vdash A} \mathrm{ax}}{\frac{A \vdash A \Rightarrow A}{A \vdash} \mathrm{i} \quad \overline{A \vdash A}} \mathrm{~A} \Rightarrow \mathrm{e}
$$

contains a cut. If we eliminate this cut, we have to replace in the proof $t$

$$
\frac{\frac{\overline{A \vdash A}^{A \vdash A \Rightarrow A}}{} \Rightarrow \mathrm{i} \quad \overline{A \vdash A}}{A \vdash A} \Rightarrow \mathrm{e}
$$

the axiom rules on the proposition $A$ with the proof $t$ itself. As $A$ was already in the context, before being introduced by the $\Rightarrow \mathrm{i}$ rule of the cut, we may choose to replace each axiom rule or not. If we replace both, we get back the proof we started with.

This counter-example is based on the fact that the scheme-calculus permits the substitution of bound variables. Yet, even if we forbid this substitution of bound variables, the variable captures of the scheme-calculus are sufficient to jeopardize strong normalization. We give here another counter-example.
Example. Consider the context $\Gamma=\{A \Rightarrow B, B \Rightarrow A, A \Rightarrow B \Rightarrow A, A, B\}$, and the schemes in $\Gamma, f=\langle A \Rightarrow B\rangle, g=\langle B \Rightarrow A\rangle, h=\langle A \Rightarrow B \Rightarrow A\rangle$

$$
\begin{gathered}
a=\left(\left(\lambda_{B}\langle A\rangle\right)(f\langle A\rangle)\right) \\
b=\left(\left(\lambda_{A}\langle B\rangle\right)(g\langle B\rangle)\right) \\
u_{0}=\left(\begin{array}{ll}
h & b
\end{array}\right) \\
u_{n+1}=\left(g\left(f u_{n}\right)\right) \\
v_{n}=\left(\left(\lambda_{B} u_{n}\right)\left(f u_{n}\right)\right)
\end{gathered}
$$

Remark that, for each $i$, the schemes $a$ and $b$ are subschemes of the scheme $u_{i}$ and that they do not occur in the scope of any binder.

The scheme $v_{n}$ reduces to $\left[\left(f u_{n}\right) / B\right]_{\Gamma} u_{n}$ that contains a subscheme $\left[\left(f u_{n}\right) / B\right]_{\Gamma} b$, i.e. $\left(\left(\lambda_{A}\left(f u_{n}\right)\right)\left(g\left(f u_{n}\right)\right)\right)$ that, in turn, reduces to $\left[\left(g\left(f u_{n}\right)\right) / A\right]_{\Gamma}\left(f u_{n}\right)$, that contains a subscheme $\left[\left(g\left(f u_{n}\right)\right) / A\right]_{\Gamma} a$, i.e. $\left(\left(\lambda_{B}\left(g\left(f u_{n}\right)\right)\right)\left(f\left(g\left(f u_{n}\right)\right)\right)\right)$ that is $v_{n+1}$. Therefore $v_{n}$ reduces to a scheme that contains $v_{n+1}$ as a subscheme.

### 2.4 Strategies

As with any non-deterministic system, we can restrict the reduction of the scheme-calculus by defining strategies. In the scheme-calculus, non-determinism arises from two different origins. First, as in the $\lambda$-calculus, when a scheme contains several redex occurrences, we may choose to reduce one or another first. Then, once the redex occurrence is chosen, we still have several ways to reduce it, because substitution itself is non-deterministic.

The simplest strategies are obtained by restricting the non-determinism of the substitution.

The definition of the substitution of the minimal strategy is the same as that of the general notion of substitution (Definition 2.5), except for the first clause: here we take the scheme $\langle A\rangle$ only, i.e.

- $\underline{\sigma}_{\Gamma}\langle A\rangle=\{\langle A\rangle\}$, if $A \in \operatorname{dom}(\sigma)$ and $A \in \Gamma$.

Notice that, in this case, $\underline{\sigma}_{\Gamma} t$ is always a singleton. Its only element also is denoted by $\underline{\sigma}_{\Gamma} t$.

For instance, if $A \notin \Gamma$, the scheme $\left(\left(\lambda_{A} \lambda_{A}\langle A\rangle\right) t\right)_{\Gamma}$ reduces to $\left(\lambda_{A}\langle A\rangle\right)_{\Gamma}$, and so does the scheme $\left(\lambda_{A}\left(\left(\lambda_{A}\langle A\rangle\right) t\right)\right)_{\Gamma}$. But the scheme $\left(\left(\lambda_{A}\langle A\rangle\right) t\right)_{\Gamma}$ reduces to $t_{\Gamma}$.

The dual strategy is the maximal strategy. The definition of the substitution of this strategy is the same as that of the general notion of substitution except for the first clause: here we take the scheme $\sigma(A)$ only, i.e.

- $\bar{\sigma}_{\Gamma}\langle A\rangle=\{\sigma(A)\}$, if $A \in \operatorname{dom}(\sigma)$ and $A \in \Gamma$.

The scheme $\left(\left(\lambda_{A} \lambda_{A}\langle A\rangle\right) t\right)_{\Gamma}$ now reduces to $\left(\lambda_{A} t\right)_{\Gamma}$ and so does the scheme $\left(\lambda_{A}\left(\left(\lambda_{A}\langle A\rangle\right) t\right)\right)_{\Gamma}$.

Intuitively, in the minimal strategy, we substitute a variable if we have to, while in the maximal strategy, we substitute a variable if we are able to.

A more complex strategy is the strategy with reference to the closest binder, also known as the total discharge strategy [7, 14]. In this strategy, the variable $\langle A\rangle$ always refers to the closest binder above it. The substitution is the same as that of the minimal strategy, but now the definition of the reduction is modified in such a way that $\left(\left(\lambda_{A} t\right) u\right)_{\Gamma}$ reduces to $\underline{[u / A]_{\varnothing}} t$ instead of $\underline{[u / A]_{\Gamma}} t$. This way, the scheme $\left(\left(\lambda_{A} \lambda_{A}\langle A\rangle\right) t\right)_{\Gamma}$ reduces to $\left.\overline{\left(\lambda_{A}\langle A\rangle\right.}\right)_{\Gamma}$, but the scheme $\left(\lambda_{A}\left(\left(\lambda_{A}\langle A\rangle\right) t\right)\right)_{\Gamma}$ reduces to $\left(\lambda_{A} t\right)_{\Gamma}$.

The dual strategy is the strategy with reference to the furthest binder. The substitution is the same as that of the maximal strategy, but the definition of the reduction is modified in such a way that $\left(\left(\lambda_{A} t\right) u\right)_{\Gamma}$ reduces to $\left(\overline{[u / A]}{ }_{\Gamma} t\right)_{\Gamma}$ when $A \notin \Gamma$ and to $t_{\Gamma}$ when $A \in \Gamma$. This way, the scheme $\left(\left(\lambda_{A} \lambda_{A}\langle A\rangle\right) t\right)_{\Gamma}$ reduces to $\left(\lambda_{A} t\right)_{\Gamma}$, but the scheme $\left(\lambda_{A}\left(\left(\lambda_{A}\langle A\rangle\right) t\right)\right)_{\Gamma}$ reduces to $\left(\lambda_{A}\langle A\rangle\right)_{\Gamma}$.

The counter-examples of Section 2.3 show that the maximal strategy and the strategy with reference to the closest binder do not normalize, even if we restrict to weak reduction, i.e. if we forbid reduction under abstractions. We leave open the problem of the normalization of the minimal strategy and the strategy with reference to the furthest binder. However, we shall prove in Section 5 the normalization of weak reduction for the minimal strategy.

Finally, $\lambda$-calculus is also a strategy of the scheme-calculus. There, in order to reduce the scheme $\left(\left(\lambda_{A} \lambda_{A}\langle A\rangle\right) t\right)_{\Gamma}$, we need to know the history of the reduction, so that we are able to decide which binder the variable $\langle A\rangle$ refers to. In both schemes of type $A \Rightarrow A \Rightarrow A$ in the context $B$ :

$$
\left(\left(\lambda_{A \Rightarrow A} \lambda_{A}\langle A \Rightarrow A\rangle\right)\left(\lambda_{A}\langle A\rangle\right)\right)
$$

and

$$
\lambda_{A}\left(\left(\lambda_{B \Rightarrow A} \lambda_{A}(\langle B \Rightarrow A\rangle\langle B\rangle)\right)\left(\lambda_{B}\langle A\rangle\right)\right)
$$

there is no ambiguity in the reference of the variable $\langle A\rangle$ that appears in the scope of a single binder of type $A$.

When we reduce these schemes, we get the normal scheme $\lambda_{A} \lambda_{A}\langle A\rangle$ in both cases. But to determine the reference of the variable $\langle A\rangle$ in this normal form, we have to know where this scheme is coming from. This is the role of variable names. Calling $x$ the variable $\langle A\rangle, y$ the variable $\langle B\rangle, f$ the variable $\langle A \Rightarrow A\rangle$ and $g$ the variable $\langle B \Rightarrow A\rangle$, the first term $((\lambda f \lambda x f)(\lambda x x))$ reduces to $\lambda x \lambda x x$ and the second $\lambda x((\lambda g \lambda x(g y))(\lambda y x))$ to $\lambda x \lambda x^{\prime} x$, where a new name $x^{\prime}$ has been introduced by substitution to avoid the variable capture and keep the pointer from the occurrence of the variable $x$ to its binder.

In this sense, the scheme-calculus generalizes both the lambda-calculus and the total discharge calculus.

## 3 A dependently-typed scheme-calculus

The simply-typed scheme-calculus is much less expressive than the simply-typed $\lambda$-calculus: with the general $\beta$-reduction the uniqueness of normal forms is lost and if we restrict the calculus to any deterministic strategy, as there is only one normal scheme of type $A \Rightarrow A \Rightarrow A$, it is impossible to express both projections.

In this section, we introduce a scheme-calculus with dependent types and prove that it is as expressive as the corresponding typed $\lambda$-calculus. In particular, we construct a dependent type system that permits to express all the functions that are provably total in impredicative arithmetic. This choice of impredicative arithmetic is just an example and we could construct similar type systems for various theories.

### 3.1 Terms and types

We first define terms and types (or propositions) as usual in many-sorted predicate logic.

We consider a language i.e. a set of sorts, a set of function symbols each of them being equipped with an arity of the form $\left\langle s_{1}, \ldots, s_{n}, s\right\rangle$, where $s_{1}, \ldots, s_{n}, s$ are sorts, and a set of predicate symbols each of them being equipped with an arity $\left\langle s_{1}, \ldots, s_{n}\right\rangle$, where $s_{1}, \ldots, s_{n}$ are sorts. We consider also, for each sort, an infinite set of variables. The terms of sort $s$ are inductively defined by

$$
a=x \mid f\left(a_{1}, \ldots, a_{n}\right)
$$

where $x$ is a variable of sort $s, f$ a function symbol of arity $\left\langle s_{1}, \ldots, s_{n}, s\right\rangle$ and $a_{1}, \ldots, a_{n}$ are terms of sorts $s_{1}, \ldots, s_{n}$, respectively. The types are inductively defined by

$$
A=P\left(a_{1}, \ldots, a_{n}\right)\left|A \Rightarrow A^{\prime}\right| \forall x A
$$

where $P$ is a predicate symbol of arity $\left\langle s_{1}, \ldots, s_{n}\right\rangle$ and $a_{1}, \ldots, a_{n}$ are terms of sorts $s_{1}, \ldots, s_{n}$, respectively.

We could include other connectives and quantifiers and everything would generalize smoothly. However, we prefer to define them in the theory $H A_{2}$ presented in Section 3.2.

Free and bound variables, alphabetic equivalence, as well as substitution are defined as usual on terms and types.

A context is a finite set of types.
To define a theory, such as arithmetic, we do not consider axioms. Instead, we extend the natural deduction rules with a conversion rule

$$
\frac{\Gamma \vdash A}{\Gamma \vdash B} A \equiv B \mathrm{conv}
$$

$$
\begin{gathered}
x_{1}, \ldots, x_{p} \epsilon_{p} f_{\left\langle x_{1}, \ldots, x_{p}\right\rangle,\left\langle y_{1}, \ldots, y_{n}\right\rangle, A}\left(y_{1}, \ldots, y_{n}\right) \longrightarrow A \\
x=y \longrightarrow \forall c\left(x \epsilon_{1} c \Rightarrow y \epsilon_{1} c\right) \\
N(x) \longrightarrow \forall c\left(0 \epsilon_{1} c \Rightarrow \forall y\left(N(y) \Rightarrow y \epsilon_{1} c \Rightarrow S(y) \epsilon_{1} c\right) \Rightarrow x \epsilon_{1} c\right) \\
N u l l(0) \longrightarrow \forall c\left(\epsilon_{0}(c) \Rightarrow \epsilon_{0}(c)\right) \\
\operatorname{Null}(S(x)) \longrightarrow \forall c \epsilon_{0}(c) \\
\operatorname{Pred}(0) \longrightarrow 0 \\
\operatorname{Pred}(S(x)) \longrightarrow x
\end{gathered}
$$

Figure 2: The rewrite system $H A_{2}$
allowing us to replace a proposition by an equivalent one for a given congruence, at any time in a proof, like in Deduction modulo [16. The congruence has to be non-confusing, that is, if $A \equiv B$ then either at least one of the propositions $A, B$ is atomic, or both are implications or both are universal quantifications, if $\left(A \Rightarrow A^{\prime}\right) \equiv\left(B \Rightarrow B^{\prime}\right)$ then $A \equiv B$ and $A^{\prime} \equiv B^{\prime}$, and if $(\forall x A) \equiv(\forall x B)$ then $A \equiv B$.

### 3.2 Impredicative arithmetic

Following [17, we can express predicative (i.e. first-order) and impredicative (i.e. second-order) arithmetic in Deduction modulo, hence the proofs of these theories can be expressed in the scheme-calculus.

We introduce a sort $\iota$ for natural numbers and a sort $\kappa_{n}(n=0,1,2, \ldots)$ for $n$-ary classes of natural numbers. The function symbols are 0 (of sort $\iota$ ), $S$ and Pred (of arity $\langle\iota, \iota\rangle$ ). The predicate symbols are $=$ of arity $\langle\iota, \iota\rangle, N$ and Null of arity $\langle\iota\rangle$ and $\epsilon_{n}$ of arity $\left\langle\iota, \ldots, \iota, \kappa_{n}\right\rangle$. We write $p \epsilon_{1} c$ to express that the number $p$ is an element of the (unary) class $c$, and $p_{1}, \ldots, p_{n} \epsilon_{n} c$ to express that the sequence $p_{1}, \ldots, p_{n}$ is an element of the $n$-ary class $c$. Thus, $\epsilon_{0}(c)$ is the proposition corresponding to the nullary class $c$. Moreover, for each proposition $A$, and sequences of variables $\left\langle x_{1}, \ldots, x_{p}\right\rangle,\left\langle y_{1}, \ldots, y_{n}\right\rangle$, such that the free variables of $A$ are among $x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{n}$, we introduce a function symbol $f_{\left\langle x_{1}, \ldots, x_{p}\right\rangle,\left\langle y_{1}, \ldots, y_{n}\right\rangle, A}$ which is, informally speaking, obtained by Skolemizing the instance of the comprehension scheme corresponding to $A$ with $x_{1}, \ldots, x_{p}$ as arguments of the class of arity $p$ and $y_{1}, \ldots, y_{n}$ as parameters. Such symbols exist for all propositions not containing Skolem symbols themselves, in particular for propositions containing quantifiers on classes (hence the impredicativity).

The meaning of these symbols is not expressed by axioms but by the rewrite rules in Figure 2. These rules define a congruence on terms and propositions.

As is well known, the connectives and quantifiers $\top, \perp, \neg, \wedge, \vee, \Leftrightarrow$, and $\exists$
can be defined in $H A_{2}$.

$$
\begin{aligned}
\top & =\forall c\left(\epsilon_{0}(c) \Rightarrow \epsilon_{0}(c)\right) \\
\perp & =\forall c \epsilon_{0}(c) \\
\neg A & =A \Rightarrow \perp \\
A \wedge B & =\forall c\left(\left(A \Rightarrow B \Rightarrow \epsilon_{0}(c)\right) \Rightarrow \epsilon_{0}(c)\right) \\
A \vee B & =\forall c\left(\left(A \Rightarrow \epsilon_{0}(c)\right) \Rightarrow\left(B \Rightarrow \epsilon_{0}(c)\right) \Rightarrow \epsilon_{0}(c)\right) \\
A \Leftrightarrow B & =(A \Rightarrow B) \wedge(B \Rightarrow A) \\
\exists x A & =\forall c\left(\left(\forall x\left(A \Rightarrow \epsilon_{0}(c)\right)\right) \Rightarrow \epsilon_{0}(c)\right)
\end{aligned}
$$

Using the congruence defined by the rules in Figure 2 and the conversion rule, the usual axioms of impredicative arithmetic, can easily be proven.

$$
\begin{gathered}
\forall x(x=x) \\
\forall x \forall y \forall c\left(x=y \Rightarrow x \epsilon_{1} c \Rightarrow y \epsilon_{1} c\right) \\
N(0) \\
\forall x(N(x) \Rightarrow N(S(x))) \\
\forall x \forall y(S(x)=S(y) \Rightarrow x=y) \\
\forall x \neg(0=S(x)) \\
\forall c\left(0 \epsilon_{1} c \Rightarrow \forall y\left(N(y) \Rightarrow y \epsilon_{1} c \Rightarrow S(y) \epsilon_{1} c\right) \Rightarrow \forall x\left(N(x) \Rightarrow x \epsilon_{1} c\right)\right. \\
\forall y_{1} \ldots \forall y_{n} \exists c \forall x_{1} \ldots \forall x_{p}\left(\left(x_{1} \ldots x_{p} \epsilon_{p} c\right) \Leftrightarrow A\right)
\end{gathered}
$$

where $A$ is any proposition not containing Skolem symbols, and whose free variables are among $y_{1}, \ldots, y_{n}, x_{1}, \ldots, x_{p}$.

### 3.3 Schemes

Definition 3.1 (Schemes) Schemes are inductively defined as follows.

$$
t=\langle A\rangle\left|\lambda_{A} t\right|\left(t t^{\prime}\right)|\Lambda x t|(t a)
$$

Each construct corresponds to a natural deduction rule. Typing rules are given in Figure 3. They are the rules of natural deduction.

Definition 3.2 (Scheme in context) $A$ scheme in context is a pair $t_{\Gamma}$ where $t$ is a scheme and $\Gamma$ is a context such that $t$ is well-typed in $\Gamma$.

We sometimes omit the context $\Gamma$ when there is no ambiguity.
We now define the reduction relation on schemes. Before that, we define the application of a substitution of term variables and that of scheme variables to a scheme.

Definition 3.3 Let $\theta$ be a substitution of term variables and $t$ be a scheme. The scheme $\theta t$ is inductively defined by

$$
\begin{gathered}
\overline{\Gamma \vdash\langle A\rangle: A} A \in \Gamma \mathrm{ax} \\
\frac{\Gamma \cup\{A\} \vdash t: B}{\Gamma \vdash \lambda_{A} t: A \Rightarrow B} \Rightarrow \mathrm{i} \\
\frac{\Gamma \vdash t: A \Rightarrow B \quad \Gamma \vdash u: A}{\Gamma \vdash(t u): B} \Rightarrow \mathrm{e} \\
\frac{\Gamma \vdash t: A}{\Gamma \vdash \Lambda x t: \forall x A} x \notin F V(\Gamma) \forall \mathrm{i} \\
\frac{\Gamma \vdash t: \forall x A}{\Gamma \vdash(t a):[a / x] A} \forall \mathrm{e} \\
\frac{\Gamma \vdash t: A}{\Gamma \vdash t: B} A \equiv B \text { conv }
\end{gathered}
$$

Figure 3: Dependently-typed schemes

- $\theta\langle A\rangle=\langle\theta A\rangle$,
- $\theta\left(\lambda_{A} t\right)=\lambda_{\theta A} \theta t$,
- $\theta(u v)=(\theta u \theta v)$,
- $\theta(\Lambda x t)=\Lambda x^{\prime}\left(\theta\left[x^{\prime} / x\right] t\right)$, where $x^{\prime}$ is a variable which occurs neither in $\Lambda x t$ nor in $\theta$,
- $\theta(t a)=(\theta t \theta a)$.

Remark that this substitution, as usual, avoids variable capture by renaming bound term variables.

A substitution of scheme variables is a function of finite domain associating schemes to types. The application of a substitution to a scheme with respect to a context is defined as follows.

## Definition 3.4 (Substitution)

- $\sigma_{\Gamma}\langle A\rangle=\{\langle A\rangle, \sigma(A)\}$, if $A \in \operatorname{dom}(\sigma)$ and $A \in \Gamma$,
- $\sigma_{\Gamma}\langle A\rangle=\{\sigma(A)\}$, if $A \in \operatorname{dom}(\sigma)$ and $A \notin \Gamma$,
- $\sigma_{\Gamma}\langle A\rangle=\{\langle A\rangle\}$, if $A \notin \operatorname{dom}(\sigma)$,
- $\sigma_{\Gamma} \lambda_{A} t=\lambda_{A} \sigma_{(\Gamma \cup\{A\})} t$,
- $\sigma_{\Gamma}(u v)=\left(\sigma_{\Gamma} u \sigma_{\Gamma} v\right)$,
- $\sigma_{\Gamma} \Lambda x t=\Lambda x^{\prime} \sigma_{\Gamma}\left[x^{\prime} / x\right] t$, where $x^{\prime}$ is a variable that occurs neither in $\Lambda x t$ nor in $\sigma$,
- $\sigma_{\Gamma}(t a)=\left(\sigma_{\Gamma} t a\right)$.

Definition 3.5 (Reduction) The one step top level $\beta$-reduction is defined by the rules

- $\left(\left(\lambda_{A} t\right) u\right)_{\Gamma} \longrightarrow v_{\Gamma}$, for all $v \in[u / A]_{\Gamma} t$,
- $((\Lambda x t) a)_{\Gamma} \longrightarrow([a / x] t)_{\Gamma}$.

The one step $\beta$-reduction relation $\triangleright$ is the contextual closure of this relation and the $\beta$-reduction relation $\triangleright^{*}$ is the reflexive-transitive closure of the relation -

## 4 Subject-reduction

## Proposition 4.1 (Substitution)

- If $\Gamma \vdash t: B$, then $[a / x] \Gamma \vdash[a / x] t:[a / x] B$.
- If $\Gamma \cup\{A\} \vdash t: B$ and $\Gamma \vdash u: A$, then $\Gamma \vdash v: B$, for all $v$ in $[u / A]_{\Gamma} t$.

Proof. By induction over the structure of $t$.
Remark that this substitution lemma holds although bound variables may be substituted and variable capture is allowed. That is because the captured variables have the same type as the binder that captures them.

Proposition 4.2 (Inversion) Let $\Gamma \vdash t: A$.

1. If $t$ is some variable $\langle B\rangle$, then $\Gamma$ contains the proposition $B$ and $A \equiv B$,
2. If $t=\lambda_{B} u$, then there exists a type $C$ such that $\Gamma \cup\{B\} \vdash u: C$ and $A \equiv(B \Rightarrow C)$.
3. If $t=\left(\begin{array}{ll}u & v\end{array}\right)$, where $u$ and $v$ are schemes, then there exist types $B$ and $C$ such that $\Gamma \vdash u: B \Rightarrow C$ and $\Gamma \vdash v: B$, and $A \equiv C$.
4. If $t=\Lambda x u$, then there exists a variable $x$ and a type $B$ such that $\Gamma \vdash u: B$ and $A \equiv(\forall x B)$ and $x \notin F V(\Gamma)$.
5. If $t=\left(\begin{array}{ll}u & a\end{array}\right)$, where $u$ is a scheme and a a term, then there exists a type $B$ such that $\Gamma \vdash u: \forall x B$ and $A \equiv[a / x] B$.

Proof. By induction on the typing derivation. If the last rule is conversion, we apply the induction hypothesis and the transitivity of $\equiv$. Otherwise the premises of the rule yield the result.

We are now ready to prove the subject-reduction property. Before that, we need to prove the proposition below.

Proposition 4.3 If $\Gamma \vdash t: A$ and $t_{\Gamma} \longrightarrow u_{\Gamma}$, then $\Gamma \vdash u: A$.

Proof. If $t=\left(\left(\lambda_{B} t_{1}\right) t_{2}\right)$ and $u \in\left[t_{2} / B\right]_{\Gamma} t_{1}$, then by Proposition $4.2(3)$, there exist types $B^{\prime}$ and $C^{\prime}$ such that $\Gamma \vdash \lambda_{B} t_{1}: B^{\prime} \Rightarrow C^{\prime}, \Gamma \vdash t_{2}: B^{\prime}$ and $A \equiv C^{\prime}$ and by Proposition 4.2(2), there exists a type $C$ such that $\Gamma \cup\{B\} \vdash t_{1}: C$ and $B^{\prime} \Rightarrow C^{\prime} \equiv B \Rightarrow C$. As the congruence is non-confusing, we have $B \equiv B^{\prime}$ and $C \equiv C^{\prime}$. Using the conversion rule, we have $\Gamma \vdash t_{2}: B$. By Proposition 4.1, we get $\Gamma \vdash u: C$ and using the conversion rule, $\Gamma \vdash u: A$.

If $t=\left(\left(\Lambda x t_{1}\right) a\right)$ and $u=[a / x] t_{1}$, we choose $x$ not occurring in $\Gamma$. Using Proposition 4.2 non-confusion and conversion, we get a type $B^{\prime}$ such that $\Gamma \vdash$ $t_{1}: B^{\prime}$ and $A \equiv[a / x] B^{\prime}$. We conclude with Proposition 4.1 and conversion.

Theorem 4.1 (Subject-reduction) If $\Gamma \vdash t: A$ and $t_{\Gamma} \triangleright^{*} u_{\Gamma}$, then $\Gamma \vdash u: A$.
Proof. We show, by induction on the derivation of $t_{\Gamma} \triangleright u_{\Gamma}$, that if $\Gamma \vdash t: A$ and $t_{\Gamma} \triangleright u_{\Gamma}$ then $\Gamma \vdash u: A$ and we conclude by induction on the length of reduction sequences.

## 5 Weak normalization of weak reduction

We now prove that each scheme can be reduced to a normal form. Because of the counter-examples given in Section 2, we cannot expect to prove strong normalization for the reduction of the scheme-calculus. Of course, it is possible to prove weak normalization by mimicking the reductions of $\lambda$-calculus. But the scheme reduction strategy provided by the proof of the normalization theorem is as important as the theorem itself and the strategy provided by this trivial proof would require to introduce variable names, which is precisely what we want to avoid. Thus, we shall give another normalization proof which provides a strategy that can be defined without introducing variable names.

The first step towards a normalization result is to restrict substitution to minimal substitution, i.e. to modify the first clause of Definition 3.4 instead of taking the clause

- $\sigma_{\Gamma}\langle A\rangle=\{\langle A\rangle, \sigma(A)\}$, if $A \in \operatorname{dom}(\sigma)$ and $A \in \Gamma$
we take the following one
- $\underline{\sigma}_{\Gamma}\langle A\rangle=\{\langle A\rangle\}$, if $A \in \operatorname{dom}(\sigma)$ and $A \in \Gamma$.

Restricting substitution to minimal substitution rules out the counter-examples of Section 2. Moreover, minimal substitution enjoys several properties of substitution of $\lambda$-calculus. In particular, bound variables are never substituted. Thus, we conjecture this minimal reduction to be strongly normalizing. However, we shall leave this problem open and prove a slightly weaker result: the normalization of weak reduction, i.e. of the reduction where reduction is not performed under abstractions. Indeed, the minimal reduction lacks one property of the reduction of $\lambda$-calculus: the commutation of reduction and substitution, i.e. that whenever $t_{\Gamma \cup\{A\}}$ reduces to $u_{\Gamma \cup\{A\}}$ and $v$ is a scheme
of type $A$ in the context $\Gamma$, then $\left([v / A]_{\Gamma} t\right)_{\Gamma}$ reduces to $\left(\underline{[v / A]} \Gamma_{\Gamma} u\right)_{\Gamma}$. For instance, if $\Gamma=\{A \Rightarrow A, B \Rightarrow A, B\}, t=\left(\left(\lambda_{A}\langle A\rangle\right)(\langle A \Rightarrow A\rangle\langle A\rangle)\right), u=\langle A\rangle$ and $v=(\langle B \Rightarrow A\rangle\langle B\rangle)$, then $t_{\Gamma \cup\{A\}}$ reduces to $u_{\Gamma \cup\{A\}}$. But $\left([v / A]_{\Gamma} t\right)_{\Gamma}=$ $\left(\left(\lambda_{A}\langle A\rangle\right)(\langle A \Rightarrow A\rangle(\langle B \Rightarrow A\rangle\langle B\rangle))\right)_{\Gamma}$ reduces to $(\langle A \Rightarrow A\rangle(\langle B \Rightarrow A\rangle\langle B\rangle))$ and not to $\left([v / A]_{\Gamma} u\right)_{\Gamma}=(\langle B \Rightarrow A\rangle\langle B\rangle)$.

This property is unfortunately needed in normalization proofs for strong reduction based on reducibility candidates. But, it is not needed, if we restrict to weak reduction.

On the other hand, the normalization of weak reduction is sufficient to prove the existence of weak head normal forms, which is itself sufficient to extract witnesses from existential proofs.

The proof presented in this section is based on ideas similar to those of [18]. The main difference is that we take into account that reduction does not commute with substitution.

### 5.1 Reduction

The one step minimal top level reduction $\longrightarrow_{\min }$ is defined as in Definition 3.5 except that substitution is minimal substitution. Instead of considering the contextual closure of this relation, we define the one step weak minimal reduction as follows.

Definition 5.1 (Weak minimal reduction) The one step weak minimal reduction $\rightarrow$ is defined by considering any abstraction and any application whose left-hand side is normal, as a normal form, otherwise by reducing the leftmost reduct. It is inductively defined as follows.

Let $t$ and $u$ be schemes and $a$ be a term,

- if $t_{\Gamma} \longrightarrow$ min $u_{\Gamma}$ then $t_{\Gamma} \rightarrow u_{\Gamma}$,
- if $t_{\Gamma} \rightarrow t_{\Gamma}^{\prime}$, then $(t u)_{\Gamma} \rightarrow\left(t^{\prime} u\right)_{\Gamma}$,
- if $t_{\Gamma} \rightarrow t_{\Gamma}^{\prime}$, then $(t a)_{\Gamma} \rightarrow\left(t^{\prime} a\right)_{\Gamma}$.

The weak minimal reduction relation $\rightarrow{ }^{*}$ is the reflexive-transitive closure of the relation $\rightarrow$.

Notice that the relation $\rightarrow$ is functional (i.e. deterministic) in the sense that, for each scheme $t$, there is at most one scheme $t^{\prime}$ such that $t \rightarrow t^{\prime}$.

The reduction sequence issued from $t_{\Gamma}$ is the (finite or infinite) sequence $t_{0, \Gamma}, t_{1, \Gamma}, t_{2, \Gamma}, \ldots$ such that $t_{0, \Gamma}=t_{\Gamma}$, and for all $i$, if there exists a $t^{\prime}$ such that $t_{i, \Gamma} \rightarrow t_{\Gamma}^{\prime}$, then the sequence is defined at $i+1$ and $t_{i+1, \Gamma}=t_{\Gamma}^{\prime}$, otherwise $t_{i, \Gamma}$ is the last element of the sequence. A scheme in context $t_{\Gamma}$ is said to be normalizing if its reduction sequence is finite. Hereafter, we write $\mathcal{N}$ for the set of normalizing schemes in contexts.

## Proposition 5.1 (Properties of minimal substitution)

1. If $t$ is well-typed in $\Gamma$, then $\underline{[w / A]} \Gamma \Gamma=t$.
2. If $A \in \Gamma$, then $\underline{[w / A]}{ }_{\Gamma} t=t$.
3. If $B \neq A$, then $\underline{[w / A]_{\Gamma \cup\{B\}}} \underline{[w / A]}_{\Gamma} t$.

Proof.

1. By induction on the structure of $t$. The only non-trivial case is when $t=\langle B\rangle$. In this case, $B \in \Gamma$ and both schemes are equal to $\langle B\rangle$.
2. By induction on the structure of $t$. The only non-trivial case is when $t=\langle A\rangle$. In this case $A \in \Gamma$ and thus both schemes are equal to $\langle A\rangle$.
3. By induction on the structure of $t$. The only non-trivial case is when $t=\langle A\rangle$. In this case, either $A \in \Gamma$ in which case both schemes are equal to $\langle A\rangle$ or $A \notin \Gamma$, in which case $A \notin(\Gamma \cup\{B\})$ and both schemes are equal to $w$.

### 5.2 Girard's reducibility candidates

Definition 5.2 (Operations on sets of schemes) If $E$ and $F$ are sets of schemes in contexts, we define the set

$$
E \Rightarrow F=\left\{t_{\Gamma} \in \mathcal{N} \mid \forall t^{\prime} \forall u\left(\left(t_{\Gamma} \rightarrow^{*}\left(\lambda_{A} t^{\prime}\right)_{\Gamma} \text { and } u_{\Gamma} \in E\right) \Rightarrow\left(\underline{[u / A]_{\Gamma}} t^{\prime}\right)_{\Gamma} \in F\right)\right\}
$$

If $S$ is a set of sets of schemes in contexts, we define the set

$$
\tilde{\forall} S=\left\{t_{\Gamma} \in \mathcal{N} \mid \forall t^{\prime} \forall a \forall E\left(\left(t_{\Gamma} \rightarrow^{*}\left(\Lambda x t^{\prime}\right)_{\Gamma} \text { and } E \in S\right) \Rightarrow\left([a / x] t^{\prime}\right)_{\Gamma} \in E\right)\right\}
$$

Definition 5.3 (Reducibility candidate [19]) A scheme is said to be neutral if it corresponds to an axiom rule or an elimination rule, but not to an introduction rule. A set $R$ of schemes in contexts is said to be a reducibility candidate, if the following conditions are satisfied:

- if $t_{\Gamma} \in R$, then $t_{\Gamma}$ is normalizing,
- if $t_{\Gamma} \in R$ and $t_{\Gamma} \rightarrow^{*} t_{\Gamma}^{\prime}$, then $t_{\Gamma}^{\prime} \in R$,
- if $t_{\Gamma}$ is neutral, and for every $t_{\Gamma}^{\prime}$ such that $t_{\Gamma} \rightarrow t_{\Gamma}^{\prime}$, we have $t_{\Gamma}^{\prime} \in R$, then $t_{\Gamma} \in R$.

We write $\mathcal{C}$ for the set of reducibility candidates.
Remark that, as the reduction relation is deterministic, the third condition can be rephrased as: (1) if $t_{\Gamma}$ is neutral and normal, then $t_{\Gamma} \in R$, and (2) if $t_{\Gamma}$ is neutral, has a one-step reduct $t_{\Gamma}^{\prime}$ and this reduct is in $R$, then $t_{\Gamma}$ is in $R$.

Proposition 5.2 If $E$ and $F$ are sets of schemes in contexts, then $E \underset{\sim}{\Rightarrow} F$ is a reducibility candidate. If $S$ is a set of sets of schemes in contexts, then $\tilde{\forall} S$ is a reducibility candidate.

Proof. By definition, all the schemes in the sets $E \underset{\Rightarrow}{F}$ and $\tilde{\forall} S$ are normalizing.
For closure by reduction, just remark that if $t_{\Gamma} \rightarrow^{*} t_{\Gamma}^{\prime}$ and $t_{\Gamma}$ is normalizing, then so is $t_{\Gamma}^{\prime}$ and that if $t_{\Gamma} \rightarrow^{*} t_{\Gamma}^{\prime}$ and $t_{\Gamma}^{\prime} \rightarrow^{*} u_{\Gamma}$, then $t \rightarrow^{*} u_{\Gamma}$.

For the third property, remark that if $t_{\Gamma}$ is a scheme in context and for all $t_{\Gamma}^{\prime}$ such that $t_{\Gamma} \rightarrow t_{\Gamma}^{\prime}, t_{\Gamma}^{\prime}$ is normalizing then $t_{\Gamma}$ is normalizing and that if $t_{\Gamma}$ is a neutral scheme in context and $t_{\Gamma} \rightarrow^{*} u_{\Gamma}$ where $u$ is an introduction, then the reduction sequence is not empty, thus there exists a scheme $t_{\Gamma}^{\prime}$ such that $t_{\Gamma} \rightarrow t_{\Gamma}^{\prime} \rightarrow^{*} u_{\Gamma}$.

## $5.3 \quad \mathcal{C}$-models

A model valued in the algebra of reducibility candidates, or $\mathcal{C}$-model, is defined as a classical model except that propositions are interpreted in the algebra $\mathcal{C}$ of reducibility candidates. Thus it consists of a set $M_{s}$, for each sort $s$, a function $\hat{f}$ from $M_{s_{1}} \times \cdots \times M_{s_{n}}$ to $M_{s}$, for each function symbol $f$ of arity $\left\langle s_{1}, \ldots, s_{n}, s\right\rangle$, and a function $\hat{P}$ from $M_{s_{1}} \times \cdots \times M_{s_{n}}$ to $\mathcal{C}$, for each predicate symbol $P$ of arity $\left\langle s_{1}, \ldots, s_{n}\right\rangle$. The denotation of terms in a valuation is defined as usual. The denotation of propositions is defined by

- $\llbracket P\left(a_{1}, \ldots, a_{n}\right) \rrbracket_{\phi}=\hat{P}\left(\llbracket a_{1} \rrbracket_{\phi}, \ldots, \llbracket a_{n} \rrbracket_{\phi}\right)$,
- $\llbracket A \Rightarrow B \rrbracket_{\phi}=\llbracket A \rrbracket_{\phi} \xlongequal[\Rightarrow]{\Rightarrow}\left[B \rrbracket_{\phi}\right.$,
- $\llbracket \forall x A \rrbracket_{\phi}=\tilde{\forall}\left\{\llbracket A \rrbracket_{\phi+x=e} \mid e \in M_{s}\right\}$, where $s$ is the sort of the variable $x$ and $\phi+x=e$ is the valuation coinciding with $\phi$ everywhere except in $x$ where it takes the value $e$.

Definition 5.4 $A$ congruence $\equiv$ is said to be valid in a $\mathcal{C}$-model $\mathcal{M}$ if for all types $A$ and $B$, and every valuation $\phi, A \equiv B$ implies $\llbracket A \rrbracket_{\phi}=\llbracket B \rrbracket_{\phi}$.

### 5.4 Weak normalization of weak reduction

As variable captures are allowed in the scheme calculus, the substitutions cannot be composed as usual. For instance if $\Gamma=\{B \Rightarrow A, C \Rightarrow B, C\}, f$ is the variable $\langle B \Rightarrow A\rangle$ and $u$ is the term $(\langle C \Rightarrow B\rangle\langle C\rangle)$, we have

$$
[u / B]_{\Gamma}[(f\langle B\rangle) / A]_{\Gamma \cup\{B\}}\left(\lambda_{B}\langle A\rangle\right)=\left(\lambda_{B}(f\langle B\rangle)\right)
$$

and

$$
[u / B]_{\Gamma}[(f\langle B\rangle) / A]_{\Gamma \cup\{B\}}\langle A\rangle=(f u)
$$

but there is no substitution $\sigma$ and context $\Delta$ such that

$$
\sigma_{\Delta}\left(\lambda_{B}\langle A\rangle\right)=\left(\lambda_{B}(f\langle B\rangle)\right)
$$

and

$$
\sigma_{\Delta}\langle A\rangle=(f u)
$$

because we cannot have at the same time $\sigma_{\Delta \cup\{B\}}\langle A\rangle=(f\langle B\rangle)$ and $\sigma_{\Delta}\langle A\rangle=$ ( $f u$ ). Thus arises the need for the notion of free sequence of substitutions.

Definition 5.5 (Free sequence of substitutions) Let $\Gamma$ be a context, and $\phi$ be a valuation, the free sequences of substitutions in $\Gamma, \phi$ are inductively defined as follows.

- The empty sequence is a free sequence of substitutions.
- If $\rho$ is a free sequence of substitutions, $C$ is a type, $w$ is a scheme in the context $\Gamma$, such that $w_{\Gamma} \in \llbracket C \rrbracket_{\phi}$, then $\left(\underline{[w / C]_{\Gamma}}, \rho\right)$ is a free sequence of substitutions.
- If $\rho$ is a free sequence of substitutions, $x$ is a term variable that does not occur in $\rho$, and $a$ is a term, then $([a / x], \rho)$ is a free sequence of substitutions.

Definition 5.6 Let $\rho$ be a free sequence of substitutions in $\Gamma, \phi$ and a be a term. The term $\rho a$ is defined as follows.

- If $\rho$ is the empty sequence, then $\rho a=a$,
- If $\rho=\left(\underline{[w / C]_{\Gamma}}, \rho^{\prime}\right)$, then $\rho a=\rho^{\prime} a$,
- If $\rho=\left([b / x], \rho^{\prime}\right)$, then $\rho a=[b / x]\left(\rho^{\prime} a\right)$.

Let $A$ be a type, the type $\rho A$ is defined as follows.

- If $\rho$ is the empty sequence, then $\rho A=A$,
- If $\rho=\left(\underline{[w / C]} \Gamma, \rho^{\prime}\right)$, then $\rho A=\rho^{\prime} A$,
- If $\rho=\left([b / x], \rho^{\prime}\right)$, then $\rho A=[b / x]\left(\rho^{\prime} A\right)$.

Let $t$ be a scheme, the scheme $\rho t$ is defined as follows.

- If $\rho$ is the empty sequence, then $\rho t=t$,
- If $\rho=\left(\underline{[w / C]_{\Gamma}}, \rho^{\prime}\right)$, then $\rho t=\underline{\left[w / \rho^{\prime} C\right]_{\Gamma}}\left(\rho^{\prime} t\right)$,
- If $\rho=\left([b / x], \rho^{\prime}\right)$, then $\rho t=[b / x]\left(\rho^{\prime} t\right)$.

In the proposition below, we prove, as usual, that if a scheme has type $A$ then it is an element of the interpretation of $A$ (hence we shall be able to deduce that it is normalizing).

Proposition 5.3 Let $\equiv$ be a congruence, $\mathcal{M}$ be a $\mathcal{C}$-model of $\equiv, \Gamma$ and $\Delta$ be contexts, $\phi$ be a valuation, $t$ be a scheme of type $A$ modulo $\equiv$ in $\Delta$, and $\rho$ be a free sequence of substitutions in $\Gamma, \phi$ such that $\rho t$ is a scheme well-typed in $\Gamma$. Then $(\rho t)_{\Gamma} \in \llbracket A \rrbracket_{\phi}$.

Proof. By induction on the typing derivation of $t$.

- ax. The scheme $t$ is equal to $\langle A\rangle$. If $A$ is not in the domain of any substitution of $\rho$ or $A \in \Gamma$, then $(\rho t)_{\Gamma}=\langle\rho A\rangle_{\Gamma}$. Thus, as the candidate $\llbracket A \rrbracket_{\phi}$ contains all normal neutral schemes, $(\rho t)_{\Gamma} \in \llbracket A \rrbracket_{\phi}$. Otherwise, let $[w / A]_{\Gamma}$ be the rightmost substitution of $\rho$ binding $A$. We have $\rho=\rho_{2},[w / A]_{\Gamma}, \rho_{1}$ and $\left(\left([w / A]_{\Gamma}, \rho_{1}\right)\langle A\rangle\right)_{\Gamma}=\left(\left[w / \rho_{1} A\right]_{\Gamma},\left\langle\rho_{1} A\right\rangle\right)_{\Gamma}=w_{\Gamma}$. The sequence $\rho$ is a free sequence of substitutions, the scheme $w$ is well-typed in $\Gamma$, and it does not contain any term variable bound in $\rho_{2}$, thus, using Proposition $5.1(1),(\rho t)_{\Gamma}=w_{\Gamma} \in \llbracket A \rrbracket_{\phi}$.
- $\Rightarrow i$. The scheme $t$ has the form $\lambda_{B} u, A=\left(B \Rightarrow B^{\prime}\right)$ and $(\rho t)_{\Gamma}=$ $\left(\rho\left(\lambda_{B} u\right)\right)_{\Gamma}$. Traversing the abstraction, the substitutions of $\rho$ have their context extended to $\Gamma \cup\{C\}$ for some type $C$. Using Proposition 5.1(2), we drop those substitutions in $\rho$ that bind the type $C$ and using Proposition $5.1(3)$, we erase $C$ from the context of the remaining ones. We get this way another free sequence of substitutions $\rho^{\prime}$ in $\Gamma, \phi$ and $(\rho t)_{\Gamma}=$ $\left(\lambda_{\rho B}\left(\rho^{\prime} u\right)\right)_{\Gamma}=\left(\lambda_{\rho^{\prime} B}\left(\rho^{\prime} u\right)\right)_{\Gamma}$. This scheme is normal, hence it is normalizing and it only reduces to itself. To prove that it is in $\llbracket A \rrbracket_{\phi}=\llbracket B \Rightarrow B^{\prime} \rrbracket_{\phi}$, we need to prove that for all schemes $v$ in $\Gamma$ such that $v_{\Gamma} \in \llbracket B \rrbracket_{\phi}$, the scheme $\left(\left[v / \rho^{\prime} B\right]_{\Gamma}\left(\rho^{\prime} u\right)\right)_{\Gamma}=\left(\left([v / B]_{\Gamma}, \rho^{\prime}\right) u\right)_{\Gamma}$ is in $\llbracket B^{\prime} \rrbracket_{\phi}$. This follows from induction hypothesis and the fact that $\left([v / B]_{\Gamma}, \rho^{\prime}\right)$ is a free sequence of substitutions.
- $\forall i$. The scheme $t$ has the form $\Lambda x u$, we can assume, without loss of generality, that $x$ does not occur in $\rho$. We have $A=(\forall x B)$ and $\rho t=$ $\Lambda x \rho u$. This scheme is normal, hence it is normalizing and it only reduces to itself. To prove that it is in $\llbracket A \rrbracket_{\phi}=\llbracket \forall x B \rrbracket_{\phi}$, we need to prove that for all terms $a$, and $e$ in $M_{s}$, where $s$ is the sort of the variable $x$, the scheme $([a / x](\rho u))_{\Gamma}=(([a / x], \rho) u)_{\Gamma}$ is in $\llbracket B \rrbracket_{\phi+x=e}$. As $x$ does not occur in $\rho$, the sequence $([a / x], \rho)$ is a free sequence of substitutions for $\Gamma,(\phi+x=e)$. Thus, this scheme is in $\llbracket B \rrbracket_{\phi+x=e}$ by induction hypothesis.
- $\Rightarrow e$. The scheme $t$ has the form $(u v)$. Thus, $\rho t=\left(u^{\prime} \rho v\right)$, where $u^{\prime}=\rho u$. By induction hypothesis, $u_{\Gamma}^{\prime} \in \llbracket B \Rightarrow A \rrbracket_{\phi}$ and $(\rho v)_{\Gamma} \in \llbracket B \rrbracket_{\phi}$. Thus, the scheme $u_{\Gamma}^{\prime}$ is normalizing. Let $n$ be the length of the reduction sequence starting from $u_{\Gamma}^{\prime}$. We prove, by induction on $n$ that if $u_{\Gamma}^{\prime} \in \llbracket B \Rightarrow A \rrbracket_{\phi}$ and the length of the reduction sequence starting from $u$ is $n$, and $v_{\Gamma}^{\prime} \in \llbracket B \rrbracket_{\phi}$ then $\left(u^{\prime} v^{\prime}\right)_{\Gamma} \in \llbracket A \rrbracket_{\phi}$. As $\left(u^{\prime} v^{\prime}\right)_{\Gamma}$ is neutral, all we need to prove is that its potential one-step reduct is in $\llbracket A \rrbracket_{\phi}$. If the reduction takes place in $u^{\prime}$, we just apply the induction hypothesis. Otherwise, the reduction takes place at top level. We have $u_{\Gamma}^{\prime}=\left(\lambda_{\rho B} u^{\prime \prime}\right)_{\Gamma}$ and the reduct is $\left(\left[v^{\prime} / \rho B\right]_{\Gamma} u^{\prime \prime}\right)_{\Gamma}$ which is in $\llbracket A \rrbracket_{\phi}$ by definition of $\llbracket B \Rightarrow A \rrbracket_{\phi}$.
- $\forall e$. The scheme $t$ has the form $(u a)$, where $u$ has type $\forall x B, A=[a / x] B$, $\rho t=\left(u^{\prime} \rho a\right)$, where $u^{\prime}=\rho u$. By induction hypothesis, $u_{\Gamma}^{\prime} \in \llbracket \forall x B \rrbracket_{\phi}$. Thus, the scheme $u_{\Gamma}^{\prime}$ is normalizing. Let $n$ be the length of the reduction sequence starting from this scheme. We prove, by induction on $n$ that if $u_{\Gamma}^{\prime} \in \llbracket \forall x B \rrbracket_{\phi}$, the length of the reduction sequence starting from $u$ is
$n$, and $a^{\prime}$ is a term, then $\left(u^{\prime} a^{\prime}\right)_{\Gamma} \in \llbracket[a / x] B \rrbracket_{\phi}=\llbracket B \rrbracket_{\phi+x=\llbracket a \rrbracket_{\phi}}$. As this scheme is neutral, all we need to prove is that its potential one-step reduct is in $\llbracket B \rrbracket_{\phi+x=\llbracket a \rrbracket_{\phi}}$. If the reduction takes place in $u^{\prime}$, we just apply the induction hypothesis. Otherwise, the reduction takes place at top level. We have $u_{\Gamma}^{\prime}=\left(\Lambda x u^{\prime \prime}\right)_{\Gamma}$ and the reduct is $\left(\left[a^{\prime} / x\right] u^{\prime \prime}\right)_{\Gamma}$ which, by definition of $\llbracket \forall x B \rrbracket_{\phi}$, is in $\llbracket B \rrbracket_{\phi+x=\llbracket a \rrbracket_{\phi}}$.
- conv. If the last rule is a conversion rule, by induction hypothesis, we have $(\rho t)_{\Gamma} \in \llbracket B \rrbracket_{\phi}$ for some $B \equiv A$, and we have $\llbracket B \rrbracket_{\phi}=\llbracket A \rrbracket_{\phi}$. Thus $(\rho t)_{\Gamma} \in \llbracket A \rrbracket_{\phi}$.

Theorem 5.1 (Normalization) Let $\equiv$ be a congruence that has a $\mathcal{C}$-model $\mathcal{M}$. Let $\Gamma$ be a context and $t$ a scheme of type $A$ modulo $\equiv$ in $\Gamma$. Then $t_{\Gamma}$ is normalizing.

Proof. By Proposition 5.3, for all $\phi, t_{\Gamma} \in \llbracket A \rrbracket_{\phi}$, thus it is normalizing.

### 5.5 Normalization in $H A_{2}$

Proposition 5.4 All schemes well-typed in $H A_{2}$ are normalizing.
Proof. We construct a $\mathcal{C}$-model as follows. Let $M_{\iota}=\mathbb{N}$ and $M_{\kappa_{n}}=\mathbb{N}^{n} \rightarrow \mathcal{C}$. The symbols $0, S$, and Pred are interpreted in the standard way. The function $\hat{\epsilon}_{n}$ maps $k_{1}, \ldots, k_{n}$ and $f$ to $f\left(k_{1}, \ldots, k_{n}\right)$, $\hat{=}$ maps $n$ and $m$ to $\llbracket \forall c\left(x \epsilon_{1} c \Rightarrow\right.$ $\left.y \epsilon_{1} c\right) \rrbracket_{n / x, m / y}$ and Null maps 0 to $\llbracket \forall c\left(\epsilon_{0}(c) \Rightarrow \epsilon_{0}(c)\right) \rrbracket$ and the other numbers to $\llbracket \forall c \epsilon_{0}(c) \rrbracket$.

To define $\hat{N}$, we first define the function $\Phi$ that maps any function $\alpha$ of $\mathbb{N} \rightarrow \mathcal{C}$ to the function that maps $n$ to the interpretation of the proposition $\forall c\left(0 \epsilon_{1} c \Rightarrow \forall y\left(N(y) \Rightarrow y \epsilon_{1} c \Rightarrow S(y) \epsilon_{1} c\right) \Rightarrow x \epsilon_{1} c\right)$, for the valuation $n / x$, in the model of domains $M_{\iota}$ and $M_{\kappa_{n}}$, and where 0 and $S$ are interpreted in the standard way, $\epsilon_{n}$ is interpreted by $\hat{\epsilon}_{n}$, but $N$ is interpreted by $\alpha$. The set $\mathbb{N} \rightarrow \mathcal{C}$ ordered by pointwise inclusion is complete and the function $\Phi$ is monotonous, thus it has a fixed point $\beta$. We let $\hat{N}=\beta$.

This way we can interpret every proposition $A$ that does not contain Skolem symbols. Finally, we interpret the symbols $f_{x_{1}, \ldots, x_{p}, y_{1}, \ldots, y_{n}, A}$ as the functions mapping $a_{1}, \ldots, a_{p}$ to the function mapping $b_{1}, \ldots, b_{n}$ to $\llbracket A \rrbracket a_{1} / x_{1}, \ldots, a_{p} / x_{p}, b_{1} / y_{1}, \ldots, b_{n} / y_{n}$.

## 6 Expressivity

We shall now see that, despite the non-determinism of the reduction, given in Definition [3.5, the uniqueness of results may be guaranteed for some schemes, and that every function that is provably total in $H A_{2}$ can be expressed by such a scheme.

If $n$ is a natural number, we write $\underline{n}$ for the term $S^{n}(0)$.
Proposition 6.1 (Parigot's numerals [20]) Let $n$ be a natural number, then there exists a scheme $\rho_{n}$ of type $N(\underline{n})$.

Proof. Let $A=\left(0 \epsilon_{1} c\right)$ and $B=\left(\forall y\left(N(y) \Rightarrow y \epsilon_{1} c \Rightarrow S(y) \epsilon_{1} c\right)\right)$. Take

$$
\rho_{0}=\Lambda c \lambda_{A} \lambda_{B}\langle A\rangle
$$

and

$$
\rho_{n+1}=\Lambda c \lambda_{A} \lambda_{B}\left(\langle B\rangle \underline{n} \rho_{n}\left(\rho_{n} c\langle A\rangle\langle B\rangle\right)\right)
$$

Proposition 6.2 (Witness property) Let $\exists x$ A be a closed proposition. From a scheme $t$ of type $\exists x A$ i.e. $\forall c\left(\left(\forall x\left(A \Rightarrow \epsilon_{0}(c)\right)\right) \Rightarrow \epsilon_{0}(c)\right)$ in the empty context, we can extract $a$ term $b$ and a scheme of type $[b / x] A$ in the empty context.

Proof. Consider a term variable $c$ of sort $\kappa_{0}$ and $g=\left\langle\forall x\left(A \Rightarrow \epsilon_{0}(c)\right)\right\rangle$. The scheme $(t c g)$ has type $\epsilon_{0}(c)$ in the context $\left\{\forall x\left(A \Rightarrow \epsilon_{0}(c)\right)\right\}$, thus its weak normal form has the form ( $g a u$ ) where $a$ is a term of sort $\iota$ and $u$ a scheme of type $[a / x] A$ in the context $\left\{\forall x\left(A \Rightarrow \epsilon_{0}(c)\right)\right\}$. Let $e=f_{\exists x A}$ and $w$ be a closed
 The scheme $v$ has type $[b / x] A$ in the empty context.

From the witness property we get the expressibility of all functions that are provable in $H A_{2}$. We need first to use the following result of elementary logic.

Proposition 6.3 For every computable function $f$ from $\mathbb{N}^{n}$ to $\mathbb{N}$, there exists a proposition $A$ such that $\left[\underline{p}_{1} / x_{1}, \ldots, \underline{p}_{n} / x_{n}, \underline{q} / y\right] A$ is provable in $H A_{2}$ if and only if $q=f\left(p_{1}, \ldots, p_{n}\right)$.

Definition 6.1 (Provably total function) The function $f$ is said to be provably total in $H A_{2}$ if

$$
\forall x_{1}\left(N\left(x_{1}\right) \Rightarrow \ldots \Rightarrow \forall x_{n}\left(N\left(x_{n}\right) \Rightarrow \exists y(N(y) \wedge A)\right) \ldots\right)
$$

is provable in $\mathrm{HA}_{2}$.
Theorem 6.1 For every computable function $f$ provably total in $H A_{2}$, there exists a scheme $t$ such that for all $p_{1}, \ldots, p_{n}$, the normal form of the witness extracted from the scheme ( $\left.t \underline{p}_{1} \rho_{p_{1}} \underline{p}_{2} \rho_{p_{2}} \ldots \underline{p}_{n} \rho_{p_{n}}\right)$ is $\underline{f\left(p_{1}, \ldots, p_{n}\right) .}$

Proof. Take any scheme of type $\forall x_{1}\left(N\left(x_{1}\right) \Rightarrow \ldots \Rightarrow \forall x_{n}\left(N\left(x_{n}\right) \Rightarrow \exists y(N(y) \wedge\right.\right.$ A))...).

Whether the set of functions provably total in $\mathrm{HA}_{2}$ is equal or a strict subset of the set of functions that can be expressed in the scheme calculus, is left as an open problem.

## 7 Future Work

Besides $H A_{2}$, Theorem 5.1 applies to many theories e.g. simple type theory and some variants of set theory. When they cannot be defined in the theory, all
connectives and quantifiers must be taken as primitive, like in 18. Although tedious, the normalization proof generalizes smoothly.

A more challenging problem is to prove normalization for other reduction strategies than weak minimal reduction. This probably requires to generalize proofs by reducibility to cases where reduction and substitution do not commute.

## 8 Acknowledgments

The authors want to thank the anonymous referees who helped them to improve the paper in many respects. This work is partially supported by NSFC 60673045, NSFC 60833001 and NSFC 60721061.

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