

LEARNING FAMILIES OF ALGEBRAIC STRUCTURES FROM INFORMANT

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ABSTRACT. We combine computable structure theory and algorithmic learning theory to study learning of families of algebraic structures. Our main result is a model-theoretic characterization of the learning type \mathbf{InfEx}_{\cong} , consisting of the structures whose isomorphism types can be learned in the limit. We show that a family of structures is \mathbf{InfEx}_{\cong} -learnable if and only if the structures can be distinguished in terms of their Σ_2^{inf} -theories. We apply this characterization to familiar cases and we show the following: there is an infinite learnable family of distributive lattices; no pair of Boolean algebras is learnable; no infinite family of linear orders is learnable.

1. INTRODUCTION

In this paper we combine computable structure theory and algorithmic learning theory to study the question of extracting semantic knowledge from a finite amount of structured data.

Computable structures can be regarded as structures output by a Turing machine (with no input) step by step, where the number of steps is potentially infinite (but at most countable). At each step we observe larger and larger finite pieces of the structure: as soon as the algorithm outputs an element, it also reveals the relations between this element and all the elements that appeared at previous stages. The algorithm can never change its mind whether a relation holds on particular elements or not. We refer the reader to Section 2 for a formal definition.

Looking at computable structures as described above is well-suited for an application in inductive inference as initiated by Gold [9]. Here a learner receives step by step more and more data (finite amount at each step) on an object to be learned, and outputs a sequence of hypotheses that converges to a finite description of the target object. In general, learning can be viewed as a dialogue between a teacher and a learner, where the learner must succeed in learning, provided the teacher satisfies a certain protocol.

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The formalization of this idea has two aspects: convergence behavior and teacher constraints. Again, formal definitions follow below.

Most work in inductive inference concerns either learning of formal languages or learning of general recursive functions [13, 24, 16]. The case of learning other structures has first been considered by Glymour [8] and is surveyed by Martin and Osherson [17]. More recently, in [11, 18, 23] Stephan and co-authors considered learnable ideals of rings, subgroups and submonoids of groups, subspaces of vector spaces and isolated branches on uniformly computable sequences of trees. They showed that different types of learnability of various classes of computable or computably enumerable structures have strong connections to their algebraic characterizations (see, e.g., [11, Theorem 3.1]). The fact of such correspondence between learnability from different types of information and algebraic properties of structures is of big interest from a mathematical point of view. In a sense, it is a way to study the interplay between algorithmic and algebraic properties of structures.

In this paper, we employ an approach that can be applied to an arbitrary class of computable structures. The main idea is the following. Suppose we have a class of computable structures. And suppose we step by step get finite amounts of data about one of them. Then we learn the class, if after finitely many steps we correctly identify the structure we are observing. This is why, in this setting, we consider learning of a class of computable structures as a task of extracting semantic knowledge from finite amount of data.

In a recent paper [6] Fokina, Kötzing and San Mauro considered learnable classes of equivalence structures. They reworked and extended the results, which appeared in Glymour [8]. In this paper we continue this line of investigation by applying the setup to other classes of structures. Our results (see Theorem 3.1) are similar to Martin and Osherson's approach [17], but by using Turing computable embeddings, we can extract more information: in particular, we offer an upper bound to the computational power needed to learn a given family of structures (see Corollary 4.1).

The paper is organized as follows. In Section 2 we give all the necessary definitions and useful facts from computable structure theory and learning theory. In Section 3 we prove our main result: a model-theoretic characterization of learnable families of structures. In Section 4 we apply the characterization from the previous section to get examples of learnable and non-learnable classes of natural computable structures.

2. PRELIMINARIES

In this section we review the necessary definitions about computable structures (Section 2.1), infinitary formulas (Section 2.5), and locking sequences (Section 2.4). In Section 2.2, we offer a gentle exposition to our learning paradigm, which is formally defined in Section 2.3.

Our computability theoretic terminology is standard and as in [22]. In particular, we denote by $\{\varphi_e\}_{e \in \omega}$ a uniformly computable list of all computable functions, and by $\{\Phi_e^X\}_{e \in \omega}$ a uniformly computable list of all Turing operators with oracle X .

2.1. Computable structures. A signature is a collection of function symbols and relation symbols that characterize an algebraic structure; a signature with no function symbol is *relational*. An L -structure \mathcal{M} consists of a domain M with an interpretation of the symbols of L : it is common to denote the interpretation of a function f (resp. a relation R) in an L -structure as $f^{\mathcal{M}}$ ($R^{\mathcal{M}}$). Two L -structures \mathcal{M}, \mathcal{N} are isomorphic if there is a bijection $F: \text{dom}(\mathcal{M}) \rightarrow \text{dom}(\mathcal{N})$ such that:

- For every function symbol g in L of arity n , for all a_1, \dots, a_n in $\text{dom}(\mathcal{M})^n$, $F(g^{\mathcal{M}}(a_1, \dots, a_n)) = g^{\mathcal{N}}(F(a_1), \dots, F(a_n))$.
- For every relation symbol R in L of some arity m , for all a_1, \dots, a_m in $\text{dom}(\mathcal{M})$, $R^{\mathcal{M}}(a_1, \dots, a_m)$ if and only if $R^{\mathcal{N}}(F(a_1), \dots, F(a_m))$.

We write $\mathcal{M} \cong \mathcal{N}$ to denote that \mathcal{M} and \mathcal{N} are isomorphic. The isomorphism is an equivalence relation on L -structures. The equivalence classes with respect to the relation \cong are called *isomorphism types*. We denote the isomorphism type of a structure \mathcal{M} (i.e., the family of structures isomorphic to \mathcal{M}) as $[\mathcal{M}]_{\cong}$.

In the paper, we consider only finite signatures. When we talk about learnable families of L -structures, we assume that the domain of any countably infinite structure is equal to the set ω of the natural numbers. This allows us to effectively identify, through a fixed Gödel numbering, any sentence about such an L -structure with a natural number. We can then define the *atomic diagram* $D(\mathcal{M})$ of such an L -structure \mathcal{M} to be the set of $n \in \omega$ such that n represents an atomic L_M -sentence true in \mathcal{M} or the negation of an atomic L_M -sentence that is false in \mathcal{M} . To measure the complexity of a structure, we identify it to its atomic diagram: we say that a structure \mathcal{M} is **d**-computable if $D(\mathcal{M})$ is a **d**-computable subset of ω , where **d** is a Turing degree. A *presentation* of a countable algebraic structure is an arbitrary isomorphic copy $\mathcal{M}' \cong \mathcal{M}$ with the universe a subset of ω . We call a structure \mathcal{M} *computably presentable* if it has a presentation \mathcal{M}' which is computable. A structure is called **d**-computably presentable if for some $\mathbf{d}_0 \leq \mathbf{d}$ there exists a presentation $\mathcal{M}' \cong \mathcal{M}$ which is \mathbf{d}_0 -computable.

Any computable structure \mathcal{A} in a relational signature can be presented as an increasing union of its finite substructures

$$\mathcal{A}^0 \subseteq \mathcal{A}^1 \subseteq \dots \subseteq \mathcal{A}^i \subseteq \dots,$$

where \mathcal{A}^n is the restriction of \mathcal{A} to the domain $\{0, 1, \dots, n\}$ and $\mathcal{A} = \bigcup_i \mathcal{A}^i$.

By \mathbb{K}_L we denote the class of all L -structures with domain ω . Since the goal of our learning paradigm, as described below, is to identify the isomorphism type of structure from any of its presentations, we assume that

every considered class of L -structures is closed under isomorphisms (modulo the restriction of the domain).

For additional background on computable structures, the reader is referred to [2].

2.2. Informal discussion of our learning paradigm. Fokina, Kötzing, and San Mauro [6] introduced the paradigm of informant learning for families of computably presentable structures. Before delving into the formal details, we illustrate the paradigm by considering two simple learning problems, by which we specify the following six items that characterize our paradigm: the learning domain, the hypothesis space, the information source, the prior knowledge, the criterion of success, and the learner. The first problem, denoted as \mathcal{P}_1 , consists in learning the family \mathfrak{C} , which consists of two countably infinite, undirected graphs:

- (1) G_1 which contains only cycles of size two, and
- (2) G_2 containing only 3-cycles.

The learner. The learner is always assumed to be an algorithm.

The learning domain. Our paradigm aims at capturing the ability, or lack thereof, of learning a given structure independently of the way in which such a structure is presented. This approach is analogous with the idea, common in computable structure theory, of characterizing the sets X that are *coded* in a structure \mathcal{S} as the sets that can be computed from any presentation of \mathcal{S} . Hence, the learning domain of \mathcal{P}_1 consists of the family \mathfrak{C}^* of all possible presentations of G_1 and G_2 , i.e., $\mathfrak{C}^* = \{H : H \cong G_1 \text{ or } H \cong G_2\}$. Observe that \mathfrak{C}^* coincides with the union of the isomorphism types of G_1 and G_2 .

The hypothesis space. The hypothesis space of \mathcal{P}_1 is the set $\{1, 2, ?\}$, where the symbols “1” and “2” means that the learner conjectures that the target graph is isomorphic, respectively, to G_1 and G_2 , and the symbol “?” means that the learner has no clue about the isomorphism type of the target graph. Notice that, since our paradigm deals with learning *up to isomorphism*, it is sufficient to specify the symbols to refer to the nonisomorphic structures in \mathfrak{C} (i.e., G_1 and G_2) and there is no need to extend the hypothesis space with other symbols for denoting all structures of \mathfrak{C}^* .

The information source. An informant I for a graph H in \mathfrak{C}^* is an infinite list of pairs containing: all pairs (x, y) of natural numbers, as the first component; and either 0 or 1, as the second component, where this second component is 1 if and only if x and y are adjacent in H . So, each entry provided by I can be regarded as a triple $(x, y, z) \in \omega \times \omega \times \{0, 1\}$. We assume that, at any stage s , the learner receives the first s triples of the informant I .

This style of learning in which the learner receives both positive and negative information about the target object is called, after Gold [9], **Inf**-learning. Learning without negative information is called learning from text

(as opposed to learning from informant) and is denoted by **Txt** instead of **Inf**. In [6], the authors considered **Txt**-learning of equivalence structures. In the present paper we focus only on **Inf**-learning, postponing a systematic analysis of learning algebraic structures from text to a future work.

The prior knowledge. The prior knowledge of \mathcal{P}_1 consists of the knowledge that the target graph is isomorphic to either G_1 or G_2 .

The criterion of success. Finally, the learning problem \mathcal{P}_1 is positively solved, if there is a learner that, receiving larger and larger pieces of any graph G in \mathfrak{C}^* , eventually stabilize to a correct guess about whether G is isomorphic to G_1 or G_2 .

So, our learning paradigm is an instance of *limit learning*: we allow the learner to have an arbitrary (but finite) number of mind changes before stabilizing on a correct conjecture. This style of learning, which dates back to Gold [9], is often called *explanatory learning* (e.g., in [5]) and denoted as **Ex**.

Having informally specified the key items of our learning paradigm, one can easily design an algorithm for learning the family \mathfrak{C} :

- Given a graph H as input, we search for a cycle of size $n \in \{2, 3\}$ inside H . If $n = 2$, then $A_{\mathfrak{C}}$ conjectures that H is a copy of G_1 . If $n = 3$, then $A_{\mathfrak{C}}$ thinks that $H \cong G_2$.

More formally, the algorithm $A_{\mathfrak{C}}$ is arranged as follows:

- We define $A_{\mathfrak{C}}(I[0]) := ?$. At a stage $s + 1$, proceed as follows:
 - If $A_{\mathfrak{C}}(I[s]) \neq ?$, then just set $A_{\mathfrak{C}}(I[s + 1]) := A_{\mathfrak{C}}(I[s])$.
 - Otherwise, search for the least tuple \bar{a} from ω such that the string $I[s + 1]$ contains the following data: the tuple \bar{a} forms a cycle of size n , where $n \in \{2, 3\}$.
 - * If $n = 2$, then set $A_{\mathfrak{C}}(I[s + 1]) := 1$.
 - * If $n = 3$, then $A_{\mathfrak{C}}(I[s + 1]) := 2$.
 - * If there is no such \bar{a} , then define $A_{\mathfrak{C}}(I[s + 1]) := ?$.

The described algorithm $A_{\mathfrak{C}}$ learns the family \mathfrak{C} : Suppose that an input I encodes a structure M , which is isomorphic to either G_1 or G_2 . Then there is a stage s_0 such that for any $s \geq s_0$, we have $A_{\mathfrak{C}}(I[s]) = A_{\mathfrak{C}}(I[s_0])$. Moreover, the conjecture $A_{\mathfrak{C}}(I[s_0])$ correctly identifies the isomorphism type of the graph M .

Our second learning problem, denoted as \mathcal{P}_2 , is a generalization of the first one. Consider an *infinite* family \mathfrak{D} , which consists of the following undirected graphs: for each $i \geq 1$, the graph G_i contains infinitely many $(i + 1)$ -cycles and no other cycles.

The main features of \mathcal{P}_2 resemble those of \mathcal{P}_1 : the learning domain of \mathcal{P}_2 is the family \mathfrak{D}^* of *all* presentations of the graphs in \mathfrak{D} ; each informant I provides both positive and negative information about any given graph in \mathfrak{D}^* ; every conjecture is an element of the set $\omega \cup \{?\}$; a learner for \mathcal{P}_2 is an

algorithm that learns, up to isomorphism, any graph in \mathfrak{D}^* ; the prior knowledge of \mathcal{P}_2 consists of the knowledge that the target graph is isomorphic to some graph from the family \mathfrak{D} .

The intuition behind the desired learning algorithm $A_{\mathfrak{D}}$ is pretty straightforward:

- Given a graph H , search for a cycle of some size $l + 1$ inside it. When the first such cycle is found, start outputting the conjecture “ H is a copy of G_l .”

The only technical problem of the algorithm $A_{\mathfrak{D}}$ is how to specify the hypothesis space of \mathcal{P}_2 . Or, in other words:

How does one formally define the set of possible conjectures?

We discuss two possible solutions of the problem, as they both seem to be pretty natural.

First Solution. One can assume that, for any $m \in \omega$, the conjecture “ m ” means that “ $H \cong G_{m+1}$.”

This solution is similar to the so-called *exact learning*, considered in the setting of computably enumerable (c.e.) languages (see, e.g., [15, 12]), where one assumes that the hypothesis space of the problem is precisely the class being learned with the corresponding indexing. The exact learning algorithm $A_{\mathfrak{D}}^e$ is a straightforward modification of the algorithm $A_{\mathcal{E}}$:

- At a stage $s + 1$, $A_{\mathfrak{D}}^e$ searches for the least tuple \bar{a} such that the string $I[s + 1]$ encodes the following data: the tuple \bar{a} forms a cycle of some size $n \geq 2$. When such \bar{a} is found, the algorithm starts outputting the conjecture “ $n - 1$.”

One drawback of exact learning is that it can be computationally very hard to enumerate certain familiar families of computable structures, up to isomorphism: e.g., Goncharov and Knight [10] proved that for the classes of computable Boolean algebras, linear orders, and Abelian p -groups (we explore all such classes in Section 4) one cannot even hyperarithmetically enumerate their isomorphism types. This fact motivates the next solution.

Second Solution. Fix a uniformly computable sequence $(\mathcal{M}_e)_{e \in \omega}$ of all computable undirected graphs. W.l.o.g., one may assume that $\mathcal{M}_0 \notin \mathfrak{D}$ and $\mathcal{M}_{\langle i, 0 \rangle} \cong G_i$ for all $i \geq 1$. We assume that the conjecture “ m ” means that “ $H \cong \mathcal{M}_m$.”

This solution is similar to the so-called *class-comprising learning* (see, e.g., [15, 12]), where one assumes that the hypothesis space of the problem should only contain the class being learned.

The class-comprising learning algorithm $A_{\mathfrak{D}}^{cc}$ works on an input I as follows:

- First, as in the honest $A_{\mathfrak{D}}^{cc}$, we search for a cycle of some size $n \geq 2$. When the cycle is found, start outputting the conjecture “ $\langle n - 1, 0 \rangle$.”
- After that stage, assume that we find a finite piece of evidence (provided by I) showing that $G(I) \not\cong G_{n-1}$: e.g., we see that

- $G(I)$ contains a component of size at least $n + 1$, or
- $G(I)$ contains a vertex of degree at least 3, or
- $G(I)$ contains a cycle of size at most $n - 1$.

Then we start outputting the conjecture “0.”

The learning algorithms $A_{\mathfrak{D}}^e$ and $A_{\mathfrak{D}}^{cc}$ can be unified in a general framework as follows. One can consider an arbitrary superclass $\mathfrak{K} \supseteq \mathfrak{D}$. We assume that the class \mathfrak{K} is *uniformly enumerable*, i.e., there is a uniformly computable sequence of structures $(\mathcal{N}_e)_{e \in \omega}$ such that:

- (1) Any structure from \mathfrak{K} is isomorphic to some \mathcal{N}_e .
- (2) For every e , \mathcal{N}_e belongs to \mathfrak{K} .

Then for a number $e \in \omega$, the conjecture “ e ” is interpreted as “the input structure is isomorphic to \mathcal{N}_e .”

2.3. Learning families of structures: Formal details. We are now in a position of offering the formal definition of our learning paradigm: see Definition 2.4 for the definition of the learning type \mathbf{InfEx}_{\cong} .

We begin with the necessary formal preliminaries.

Let $L = \{P_0^{n_0}, P_1^{n_1}, \dots, P_k^{n_k}\}$ be a relational signature. An L -informant is a function

$$I: \omega \rightarrow (\omega^{n_0} \times \{0, 1\}) \times (\omega^{n_1} \times \{0, 1\}) \times \dots \times (\omega^{n_k} \times \{0, 1\}).$$

For a number m , the value $I(m)$ is treated as a $(k + 1)$ -tuple

$$I(m) = (I_0(m), I_1(m), \dots, I_k(m)),$$

where $I_j(m) \in \omega^{n_j} \times \{0, 1\}$. Let $\text{content}_j^+(I) := \{\bar{a} \in \omega^{n_j} : (\bar{a}, 1) \in \text{range}(I_j)\}$.

That is, $\text{content}_j^+(I)$ is the set of all positive examples of predicate P_j .

The *positive content* of the informant I is the tuple

$$\text{content}^+(I) = (\text{content}_0^+(I), \text{content}_1^+(I), \dots, \text{content}_k^+(I)).$$

Henceforth, for the sake of readability, we will often omit the arities of predicates. For an L -informant I and an L -structure $\mathcal{S} = (\omega; P_0, P_1, \dots, P_k)$, we say that I is an *informant for \mathcal{S}* if for every $i \leq k$, $\text{content}_i^+(I) = P_i$. By $\mathbf{Inf}(\mathcal{S})$ we denote the set of all informants for the structure \mathcal{S} . Observe that each informant, so defined, offers all positive, as well as all negative, data of the target structure.

If a signature L contains functional symbols and/or constants, then one can use a standard convention from computable structure theory: by replacing functions with their graphs, we can treat any L -structure as a relational one. If a signature L is clear from the context, then we will talk about informants without specifying their prefix L -.

For a number n and a function f with $\text{dom}(f) = \omega$, by $f[n]$ we denote the finite sequence $f(0), f(1), \dots, f(n - 1)$.

A *learner* is a function M mapping initial segments of informants to conjectures (elements of $\omega \cup \{?\}$). The *learning sequence* of a learner M on

an informant I is the function $p: \omega \rightarrow \omega \cup \{?\}$ such that $p(n) = M(I[n])$ for every n .

Let $\sigma = (\sigma_1, \dots, \sigma_j, \dots, \sigma_k)$ be an initial part of an L -informant. By \mathcal{A}_σ we denote the finite structure which is defined as follows: The domain of \mathcal{A}_σ is the greatest (under set-theoretic inclusion) set $D \subset \omega$ with the following properties:

- (a) Every $x \in D$ is mentioned in σ , i.e., there are numbers $m < |\sigma|$, $j \leq k$, and a tuple \bar{a} such that x occurs in \bar{a} and $\sigma_j(m)$ is equal to either $(\bar{a}, 0)$ or $(\bar{a}, 1)$.
- (b) If $j \leq k$ and \bar{b} is a tuple from D such that $|\bar{b}| = n_j$, then there is (the least) $m < |\sigma|$ with $\sigma_j(m) \in \{(\bar{b}, 0), (\bar{b}, 1)\}$.

The predicates on \mathcal{A}_σ are recovered from the string σ in a natural way: If $\sigma_j(m) = (\bar{b}, 1)$, then we set $\mathcal{A}_\sigma \models P_j(\bar{b})$. Otherwise, we define $\mathcal{A}_\sigma \models \neg P_j(\bar{b})$.

Informally speaking, the structure \mathcal{A}_σ is constructed according to the following principle: We want to mine as much information from σ as possible, but this information must induce a *complete* diagram (of a finite structure).

Note that \mathcal{A}_σ is allowed to be an empty L -structure. Nevertheless, if I is an L -informant for a non-empty structure \mathcal{B} , then there is a stage s_0 such that for all $s \geq s_0$, we have $\mathcal{A}_{I[s]} \neq \emptyset$. Furthermore, it is clear that

$$\mathcal{A}_{I[s]} \subseteq \mathcal{A}_{I[s+1]} \text{ and } \mathcal{B} = \bigcup_{s \in \omega} \mathcal{A}_{I[s]}.$$

Definition 2.1. Let \mathfrak{K} be a class of L -structures. An *effective enumeration* of the class \mathfrak{K} is a function $\nu: \omega \rightarrow \mathfrak{K}$ with the following properties:

- (1) The sequence of L -structures $(\nu(e))_{e \in \omega}$ is uniformly computable.
- (2) For any $\mathcal{A} \in \mathfrak{K}$, there is an index e such that the structures \mathcal{A} and $\nu(e)$ are isomorphic.

In other words, the function ν effectively lists all isomorphism types from the class \mathfrak{K} (possibly listing also other L -structures).

Sometimes we abuse our notations: we assume that the notions “enumeration” and “effective enumeration” are synonymous. If ν and μ are two enumerations, then a new enumeration $\nu \oplus \mu$ is defined as follows.

$$(\nu \oplus \mu)(2n) := \nu(n), \text{ and } (\nu \oplus \mu)(2n + 1) := \mu(n).$$

Definition 2.2. Let ν be an effective enumeration of a class \mathfrak{K} , and let \mathcal{A} be a structure from \mathfrak{K} . The *index set of the structure \mathcal{A} w.r.t. ν* is defined as follows:

$$\text{Ind}(\mathcal{A}; \nu) = \{e \in \omega : \nu(e) \cong \mathcal{A}\}.$$

We say that an effective enumeration ν is *decidable* if the set

$$\{(i, j) : \nu(i) \cong \nu(j)\}$$

is computable. An effective enumeration ν is *Friedberg* if $\nu(i) \not\cong \nu(j)$ for all $i \neq j$ (Friedberg [7] proved that there is an effective enumeration of all c.e. sets with no repetitions).

Remark 2.3. *Note that any Friedberg enumeration is decidable. Moreover, if ν is a decidable enumeration of a class \mathfrak{K} , then for any $\mathcal{A} \in \mathfrak{K}$, its index set $\text{Ind}(\mathcal{A}; \nu)$ is computable.*

Now we are ready to give the notion of informant learning:

Definition 2.4. Let \mathfrak{K} be a class of L -structures, and let ν be an effective enumeration of \mathfrak{K} . Suppose that \mathfrak{C} is a subclass of \mathfrak{K} . We say that \mathfrak{C} is **InfEx $_{\cong}[\nu]$ -learnable** if there is a learner M with the following property: If I is an informant for a structure $\mathcal{A} \in \mathfrak{C}$, then there are e and s_0 such that $\nu(e) \cong \mathcal{A}$ and $M(I[s]) = e$ for all $s \geq s_0$. In other words, in the limit, the learner M learns all isomorphism types from \mathfrak{C} .

Recall that the classes \mathfrak{K} and \mathfrak{C} are closed under isomorphisms. Hence, we emphasize that every structure $\mathcal{A} \in \mathfrak{C}$ has a computable copy, *but* both the atomic diagram of \mathcal{A} and an informant I can have *arbitrary* Turing degree.

We say that an L -structure \mathcal{A} is **InfEx $_{\cong}[\nu]$ -learnable** if the class $\{\mathcal{A}\}$ (or more formally, the class containing all isomorphic copies of \mathcal{A}) is **InfEx $_{\cong}[\nu]$ -learnable**. Observe that every family \mathfrak{C} consisting of a single isomorphism type $[\mathcal{A}]_{\cong}$ is **InfEx $_{\cong}[\nu]$ -learnable**: a learner just constantly outputs \mathcal{A} .

In this paper, we concentrate only on learning the isomorphism types of structures. Note that in [6], the learning notions were given for an arbitrary equivalence relation \sim on a class \mathfrak{K} .

Remark 2.5. *It might be natural to regard the classical setting of learning c.e. languages as a special case of our paradigm for learning computable structures. Yet, let us stress again that our framework is designed for modelling learning up to isomorphism (as opposed to the learning of a given presentation of data, in Gold-style [9]). So, since we assume that each structure considered has domain ω , the only set that can be a target structure in our framework is ω .*

2.4. Locking sequences. The paper [6] is focused on different versions of learning for various classes of equivalence structures. Here we briefly recap the results of [6] on locking sequences, but now we formulate them for arbitrary classes of structures. The notion of a locking sequence was introduced by Blum and Blum [3].

We say that a finite sequence σ *describes a finite part* of an L -structure \mathcal{A} if σ is an initial segment of some L -informant for the structure \mathcal{A} . Note that since we are working with informant learning, σ contains both positive and negative data about the structure \mathcal{A} .

Definition 2.6 ([6, Definition 17]). Suppose that M is a learner and \mathcal{A} is an L -structure. A sequence σ describing a finite part of \mathcal{A} is a *weak informant locking sequence of M on \mathcal{A}* if for every $\tau \supseteq \sigma$ describing a finite part of \mathcal{A} , we have $M(\tau) = M(\sigma)$.

Theorem 2.7 ([6, Theorem 18]). *Let ν be an effective enumeration of a class \mathfrak{K} , and let \mathcal{A} be a structure from \mathfrak{K} . Suppose that a learner M $\mathbf{InfEx}_{\cong}[\nu]$ -learns the structure \mathcal{A} . Let σ_0 be a sequence which describes a finite part of \mathcal{A} . Then there is a finite sequence $\sigma \supseteq \sigma_0$ such that σ is a weak informant locking sequence of M on \mathcal{A} . Furthermore, $\nu(M(\sigma)) \cong \mathcal{A}$.*

Proof Sketch. Towards a contradiction, suppose that there is σ_0 with no weak locking sequence $\sigma \supseteq \sigma_0$. Then for any $\sigma \supseteq \sigma_0$ describing a finite part of \mathcal{A} , there is a string $ext(\sigma) \supset \sigma$ such that $ext(\sigma)$ also describes a finite part of \mathcal{A} , and $M(ext(\sigma)) \neq M(\sigma)$.

Fix an informant I for \mathcal{A} . Then one can produce a new informant I' for \mathcal{A} such that the learner M does not correctly converge on I' : Just “alternate” between the data given by I and “bad” extensions $ext(\sigma)$, in an appropriate way. \square

Definition 2.8 ([6, Definition 19]). Let M be a learner and \mathcal{A} be an L -structure. We say that M is *informant locking* on \mathcal{A} if for every informant I for \mathcal{A} , there is an n such that $I[n]$ is a weak informant locking sequence for M on \mathcal{A} . Assume that a class \mathfrak{A} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable. A learner M which $\mathbf{InfEx}_{\cong}[\nu]$ -learns \mathfrak{A} is *informant locking* if it is informant locking for every $\mathcal{A} \in \mathfrak{A}$.

Theorem 2.9 (see Theorem 20 in [6]). *If a class \mathfrak{A} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable, then there is an informant locking learner M which $\mathbf{InfEx}_{\cong}[\nu]$ -learns \mathfrak{A} .*

2.5. Infinitary formulas. Suppose that $X \subseteq \omega$ is an oracle, and α is an X -computable non-zero ordinal. Following Chapter 7 of [2], we describe the class of X -computable infinitary Σ_α formulas (or $\Sigma_\alpha^c(X)$ formulas, for short) in a signature L .

- (a) $\Sigma_0^c(X)$ and $\Pi_0^c(X)$ formulas are quantifier-free first-order L -formulas.
- (b) A $\Sigma_\alpha^c(X)$ formula $\psi(x_0, \dots, x_m)$ is an X -computably enumerable (X -c.e.) disjunction

$$\bigvee_{i \in I} \exists \bar{y}_i \xi_i(\bar{x}, \bar{y}_i),$$

where each ξ_i is a $\Pi_{\beta_i}^c(X)$ formula, for some $\beta_i < \alpha$.

- (c) A $\Pi_\alpha^c(X)$ formula $\psi(\bar{x})$ is an X -c.e. conjunction

$$\bigwedge_{i \in I} \forall \bar{y}_i \xi_i(\bar{x}, \bar{y}_i),$$

where each ξ_i is a $\Sigma_{\beta_i}^c(X)$ formula, for some $\beta_i < \alpha$.

In the paper, we mainly work with $\Sigma_\alpha^c(X)$ formulas for finite ordinals α (even more, for $\alpha \leq 2$). Henceforth, in this section we assume that $\alpha = n$ is a natural number.

Infinitary Σ_n formulas (or Σ_n^{inf} formulas, for short) are defined in the same way as above, modulo the following modification: infinite disjunctions and conjunctions are not required to be X -c.e. It is clear that a formula ψ is

logically equivalent to a Σ_n^{inf} formula iff ψ is equivalent to a $\Sigma_n^c(X)$ formula for some oracle X . A similar fact holds for Π_n^{inf} formulas. For more details on infinitary formulas, we refer the reader to [2].

As usual, the Σ_n^{inf} -theory of an L -structure \mathcal{S} is the set

$$\Sigma_n^{\text{inf}}\text{-Th}(\mathcal{S}) = \{\psi : \psi \text{ is a } \Sigma_n^{\text{inf}} \text{ sentence true in } \mathcal{S}\}.$$

3. LEARNING FROM INFORMANT, AND INFINITARY Σ_2 -THEORIES

In this section, we offer a model-theoretic characterization of what families of structures are $\mathbf{InfEx}_{\cong}[\nu]$ -learnable: Informally speaking, we show that a family of structures \mathfrak{K} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable if and only if the (isomorphism types of) structures from \mathfrak{K} can be distinguished in terms of their Σ_2^{inf} -theories.

Suppose that \mathfrak{K}_0 is a class of L -structures, and ν is an effective enumeration of the class \mathfrak{K}_0 .

Theorem 3.1. *Let $\mathfrak{K} = \{\mathcal{B}_i : i \in \omega\}$ be a family of structures such that $\mathfrak{K} \subseteq \mathfrak{K}_0$, and the structures \mathcal{B}_i are infinite and pairwise non-isomorphic. Then the following conditions are equivalent:*

- (1) *The class \mathfrak{K} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable.*
- (2) *There is a sequence of Σ_2^{inf} sentences $\{\psi_i : i \in \omega\}$ such that for all i and j , we have $\mathcal{B}_j \models \psi_i$ if and only if $i = j$.*

Theorem 3.1 talks about classes \mathfrak{K} which contain infinitely many isomorphism types. Nevertheless, one can easily formulate (and prove) an analogous result for classes with only finitely many isomorphism types: Just work with a family $\mathfrak{K} = \{\mathcal{B}_0, \mathcal{B}_1, \dots, \mathcal{B}_n\}$ and the corresponding finite sequence of Σ_2^{inf} sentences $\{\psi_0, \psi_1, \dots, \psi_n\}$.

Remark 3.2. *The statement of Theorem 3.1 is similar to a result due to Martin and Osherson [17, p. 79, Corollary (52)]. Yet, our proof is novel and based on a technique introduced by Knight, Miller, and Vanden Boom [14] in the context of Turing computable embeddings. A main upshot of our approach is that it provides an upper bound for the Turing complexity of the learners (Corollary 4.1), which will be crucial, in Section 4, for analyzing the learnability of familiar classes of structures.*

The proof of Theorem 3.1 is organized as follows. Section 3.1 discusses the necessary preliminaries on Turing computable embeddings, which constitute one of the main ingredients of the proof. In Section 3.2, we give a result (Proposition 3.6) which provides a connection between \mathbf{InfEx}_{\cong} -learnability and Turing computable embeddings. Section 3.3 finishes the proof. Section 3.4 discusses some further questions related to the proof.

3.1. Turing computable embeddings. When we are working with Turing computable embeddings, we consider structures \mathcal{S} such that the domain

of \mathcal{S} is an *arbitrary* subset of ω . In contrast, recall that our learning paradigm applies only to structures with domain equal to ω . As before, any considered class of structures is closed under isomorphisms, modulo the domain restrictions.

Let \mathfrak{K}_0 be a class of L_0 -structures, and \mathfrak{K}_1 be a class of L_1 -structures.

Definition 3.3 ([4, 14]). A Turing operator $\Phi = \Phi_e$ is a *Turing computable embedding* of \mathfrak{K}_0 into \mathfrak{K}_1 , denoted by $\Phi: \mathfrak{K}_0 \leq_{tc} \mathfrak{K}_1$, if Φ satisfies the following:

- (1) For any $\mathcal{A} \in \mathfrak{K}_0$, the function $\Phi_e^{D(\mathcal{A})}$ is the characteristic function of the atomic diagram of a structure from \mathfrak{K}_1 . This structure is denoted by $\Phi(\mathcal{A})$.
- (2) For any $\mathcal{A}, \mathcal{B} \in \mathfrak{K}_0$, we have $\mathcal{A} \cong \mathcal{B}$ if and only if $\Phi(\mathcal{A}) \cong \Phi(\mathcal{B})$.

The term ‘‘Turing computable embedding’’ is often abbreviated as *tc-embedding*. One of the important results in the theory of *tc-embeddings* is the following. Recall that ω_1^{CK} denotes the smallest ordinal which is noncomputable.

Theorem 3.4 (Pullback Theorem; Knight, Miller, and Vanden Boom [14]). *Suppose that $\mathfrak{K}_0 \leq_{tc} \mathfrak{K}_1$ via a Turing operator Φ . Then for any computable infinitary sentence ψ in the signature of \mathfrak{K}_1 , one can effectively find a computable infinitary sentence ψ^* in the signature of \mathfrak{K}_0 such that for all $\mathcal{A} \in \mathfrak{K}_0$, we have $\mathcal{A} \models \psi^*$ if and only if $\Phi(\mathcal{A}) \models \psi$. Moreover, for a non-zero $\alpha < \omega_1^{CK}$, if ψ is a Σ_α^c formula (Π_α^c formula), then so is ψ^* .*

An analysis of the proof of Theorem 3.4 shows that this result admits a full relativization as follows.

Fix an oracle $X \subseteq \omega$. In a natural way, a *Turing X -relativized operator* $\varphi_{e,X}$ can be defined as follows: for a set $Z \subseteq \omega$ and a natural number k , let

$$\Phi_{e,X}^Z(k) := \Phi_e^{Z \oplus X}(k),$$

where $Z \oplus X$ denotes the usual join of Z and X , i.e., $Z \oplus X = \{2x : x \in Z\} \cup \{2x + 1 : x \in X\}$.

We often denote a Turing X -relativized operator as $\Phi_{[X]}$. Informally speaking, one can identify a Turing X -relativized operator with a Turing machine which has three tapes: the input tape (on which the machine is allowed to work), the output tape, and the oracle tape, where the oracle tape *always* contains the characteristic function of X .

In a straightforward way, one can use the notion of a Turing X -relativized operator to introduce *Turing X -computable embeddings*. If there is a Turing X -computable embedding from \mathfrak{K}_0 into \mathfrak{K}_1 , then we write $\mathfrak{K}_0 \leq_{tc}^X \mathfrak{K}_1$.

One can obtain the following consequence of Theorem 3.4.

Corollary 3.5 (Relativized Pullback Theorem). *Suppose that $X \subseteq \omega$, and $\mathfrak{K}_0 \leq_{tc}^X \mathfrak{K}_1$ via an operator $\Phi_{[X]}$. Then for any X -computable infinitary sentence ψ in the signature of \mathfrak{K}_1 , one can find, effectively with respect to X , an X -computable infinitary sentence ψ^* in the signature of \mathfrak{K}_0 such that*

for all $\mathcal{A} \in \mathfrak{K}_0$, we have $\mathcal{A} \models \psi^*$ if and only if $\Phi_{[X]}(\mathcal{A}) \models \psi$. Furthermore, for a non-zero $\alpha < \omega_1^X$, if ψ is a $\Sigma_\alpha^c(X)$ formula ($\Pi_\alpha^c(X)$ formula), then so is ψ^* .

3.2. Connecting InfEx $_{\cong}$ -learnability and tc -embeddings. Let L be a finite signature, and \mathfrak{K}_0 be a class of L -structures. Let ν be an effective enumeration of the class \mathfrak{K}_0 .

Suppose that $\mathfrak{K} = \{\mathcal{B}_i : i \in \omega\}$ is a family of L -structures with the following properties:

- (a) \mathfrak{K} is a subclass of \mathfrak{K}_0 . All \mathcal{B}_i are infinite and pairwise non-isomorphic.
- (b) There is a learner M which **InfEx $_{\cong}$** $[\nu]$ -learns the class \mathfrak{K} .

We choose the oracle X as follows:

$$(1) \quad X := M \oplus \{\langle i, k \rangle : i \in \omega, k \in \text{Ind}(\mathcal{B}_i; \nu)\} \oplus \{j : \exists i (j \in \text{Ind}(\mathcal{B}_i; \nu))\}.$$

Consider a signature

$$L_{st} := \{\leq\} \cup \{P_i : i \in \omega\},$$

where every P_i is a unary relation. For $i \in \omega$, we define an L_{st} -structure \mathcal{S}_i as follows: All P_j are disjoint. For $j \neq k$, if $x \in P_j$ and $y \in P_k$, then x and y are incomparable under \leq . Every P_j , $j \neq i$, contains a \leq -structure isomorphic to the order type η of the rationals. The relation P_i contains a copy of $1 + \eta$.

Let \mathfrak{K}_{st} denote the class $\{\mathcal{S}_i : i \in \omega\}$.

Proposition 3.6. *There is a Turing X -computable embedding $\Phi_{[X]}$ from \mathfrak{K} into \mathfrak{K}_{st} such that for any $i \in \omega$, we have $\Phi_{[X]}(\mathcal{B}_i) \cong \mathcal{S}_i$.*

Proof. Let \mathcal{C} be a structure such that \mathcal{C} is isomorphic to some \mathcal{B}_i , and $\text{dom}(\mathcal{C}) \subseteq \omega$.

It is not hard to show that there is a Turing operator Ψ with the following property: If \mathcal{E} is a countably infinite L -structure with $\text{dom}(\mathcal{E}) \subseteq \omega$, then $\Psi^{D(\mathcal{E})}$ is the atomic diagram of a structure \mathcal{E}_1 such that $\text{dom}(\mathcal{E}_1) = \omega$ and \mathcal{E}_1 is $D(\mathcal{E})$ -computably isomorphic to \mathcal{E} .

The existence of the operator Ψ implies that w.l.o.g., we may assume that the domain of our \mathcal{C} is equal to ω . For simplicity, we assume that $L = \{Q_0, Q_1, \dots, Q_l\}$, where each Q_i has arity $i + 1$. For $i \leq l$, fix a computable bijection $\gamma_i : \omega \rightarrow \omega^{i+1}$.

We describe the construction of the L_{st} -structure $\Phi_{[X]}(\mathcal{C})$. First, define an L -informant $I^{\mathcal{C}}$ as follows. For $i \leq l$ and $m \in \omega$, set:

$$I_i^{\mathcal{C}}(m) = \begin{cases} (\gamma_i(m), 1), & \text{if } \mathcal{C} \models Q_i(\gamma_i(m)), \\ (\gamma_i(m), 0), & \text{if } \mathcal{C} \models \neg Q_i(\gamma_i(m)). \end{cases}$$

Fix a computable copy \mathcal{M} of the ordering η , and choose a computable descending sequence $q_0 >_{\mathcal{M}} q_1 >_{\mathcal{M}} q_2 >_{\mathcal{M}} \dots$.

The construction of the structure $\mathcal{E} = \Phi_{[X]}(\mathcal{C})$ proceeds in stages.

Stage 0. Put inside every $P_j^\mathcal{E}$, $j \in \omega$, a computable copy of the interval $(q_0; \infty)_\mathcal{M}$.

Stage $s + 1$. Recall that the learner M $\mathbf{InfEx}_{\cong}[\nu]$ -learns the class \mathfrak{K} . Compute the value $t := M(I^C[s + 1])$. Using the oracle X , one can find whether the number t is a ν -index for some \mathcal{B}_j , $j \in \omega$.

If t is not a ν -index for any \mathcal{B}_j , then extend every $P_k^\mathcal{E}$, $k \in \omega$, to a copy of $(q_{s+1}; \infty)_\mathcal{M}$.

Otherwise, assume that t is an index for \mathcal{B}_j . If $P_j^\mathcal{E}[s]$ has the least element, then do not change $P_j^\mathcal{E}[s]$. If $P_j^\mathcal{E}[s]$ has no least element, then define $P_j^\mathcal{E}[s+1]$ as a copy of the interval $[q_{s+1}; \infty)_\mathcal{M}$. Note that this interval is isomorphic to $1 + \eta$. In any case, extend every $P_k^\mathcal{E}[s]$, $k \neq j$, to a copy of the open interval $(q_{s+1}; \infty)_\mathcal{M}$.

This concludes the description of the construction. It is not hard to show that the construction gives a Turing X -computable operator $\Phi_{[X]}$. Moreover, if the input structure \mathcal{C} is isomorphic to \mathcal{B}_i , then there is a stage s_0 such that for any $s \geq s_0$, we have $M(I^C[s]) = M(I^C[s_0])$ is a ν -index of the structure \mathcal{B}_i . Hence, $P_i^{\Phi_{[X]}(\mathcal{C})}$ contains a copy of $1 + \eta$, and for every $j \neq i$, $P_j^{\Phi_{[X]}(\mathcal{C})}$ copies η . Thus, $\Phi_{[X]}(\mathcal{C})$ is isomorphic to \mathcal{S}_i .

Proposition 3.6 is proved. \square

3.3. Proof of Theorem 3.1.

Proof. (1) \Rightarrow (2): Choose an oracle X according to Equation (1). By Proposition 3.6, there is a Turing X -computable embedding

$$\Phi_{[X]}: \mathfrak{K} \leq_{tc}^X \mathfrak{K}_{st}$$

such that $\Phi_{[X]}(\mathcal{B}_i)$ is a copy of \mathcal{S}_i .

Consider an $\exists\forall$ -sentence in the signature L_{st}

$$\xi_i := \exists x \forall y [P_i(y) \rightarrow (x \leq y)].$$

Note that $\mathcal{S}_j \models \xi_i$ if and only if $i = j$. By Corollary 3.5, we obtain a sequence of X -computable infinitary Σ_2 sentences $(\xi_i^*)_{i \in \omega}$. Clearly, this sequence has the desired properties.

(2) \Rightarrow (1): W.l.o.g., for all i , assume that

$$\psi_i := \exists x_1, \dots, x_{n_i} \bigwedge_{j \in J_i} \forall y_1, \dots, y_{m_{i,j}} \varphi_{i,j}(x_1, \dots, x_{n_i}, y_1, \dots, y_{m_{i,j}}),$$

where every $\varphi_{i,j}$ is a quantifier-free formula.

Let \mathcal{C} be a finite structure, and $i \in \omega$. We say that the formula ψ_i is \mathcal{C} -compatible via a tuple $\bar{a} \in \omega^{n_i}$ if within $\text{dom}(\mathcal{C})$ there is no pair (j, \bar{b}) , with $j \in J_i$ and $\bar{b} \in \omega^{m_{i,j}}$, such that $\mathcal{C} \models \neg \varphi_{i,j}(\bar{a}, \bar{b})$.

We fix a sequence $(e_i)_{i \in \omega}$ such that for every i , the structure $\nu(e_i)$ is a copy of \mathcal{B}_i .

A learner M for the class \mathfrak{K} can be arranged as follows: Suppose that M reads a string σ , which is an initial part of some L -informant. Then we

search for the least pair $\langle i, \bar{a} \rangle$ such that the formula ψ_i is \mathcal{A}_σ -compatible via the tuple \bar{a} . If the pair $\langle i, \bar{a} \rangle$ is found, then set $M(\sigma) := e_i$. Otherwise, define $M(\sigma) := 0$.

Verification. Fix $j \in \omega$. Let I be an informant for the structure \mathcal{B}_j . Recall that $\mathcal{B}_j = \bigcup_{s \in \omega} \mathcal{A}_{I[s]}$ and $\mathcal{A}_{I[s]} \subseteq \mathcal{A}_{I[s+1]}$.

We note the following simple fact: Suppose that a formula ψ_i is not $\mathcal{A}_{I[t_0]}$ -compatible via a tuple \bar{d} . Then for any $t \geq t_0$, ψ_i also cannot be $\mathcal{A}_{I[t]}$ -compatible via \bar{d} .

Recall that $\mathcal{B}_j \models \psi_i$ if and only if $i = j$. Hence, there exists the least tuple $\bar{a} \in \omega^{n_j}$ with the following property: there is a stage s_0 such that for every $s \geq s_0$, the formula ψ_j is $\mathcal{A}_{I[s]}$ -compatible via the tuple \bar{a} . Furthermore, it is not difficult to see that

$$\mathcal{B}_j \models \neg \psi_i \Leftrightarrow (\forall \bar{c} \in \omega^{n_i})(\exists s_1)(\psi_i \text{ is not } \mathcal{A}_{I[s_1]}\text{-compatible via } \bar{c}).$$

Hence, for every number $\langle k, \bar{c} \rangle < \langle j, \bar{a} \rangle$, there is a stage t_1 such that for any $t \geq t_1$, the formula ψ_k is not $\mathcal{A}_{I[t]}$ -compatible via \bar{c} . This means that there is t^* , such that the current conjecture $M(I[t^*])$ is correct (i.e., $\nu(M(I[t^*]))$ is a copy of \mathcal{B}_j), and our learner M does not change its mind after the stage t^* .

Therefore, the class \mathfrak{K} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable by the learner M . This concludes the proof of Theorem 3.1. \square

3.4. Further discussion. We note that it would be interesting to attack the following question: If a class $\mathfrak{K} = \{\mathcal{B}_i : i \in \omega\}$ is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable, could one construct *explicitly* some sequence $\{\psi_i : i \in \omega\}$ of Σ_2^{inf} -sentences distinguishing the structures \mathcal{B}_i ?

To our best knowledge, it seems that our proof of Theorem 3.1 does not provide such a construction. Furthermore, even for the case when a class \mathfrak{K} is learnable by a computable learner, it is quite hard to give a nice description of the properties of \mathcal{B}_i expressed by our formulas ψ_i .

Indeed, suppose that $\{\mathcal{B}_i : i \in \omega\}$ is learnable by a computable learner. Then in general, the oracle X from Eq. (1) can be noncomputable. A simple example of a noncomputable X is provided by the family containing two isomorphism types of linear orders: $\mathcal{B}_0 = \omega$ and $\mathcal{B}_1 = \omega^*$. Suppose that we consider the standard effective enumeration ν , which enumerates all computable structures in the signature $\{\leq\}$. Then it is not hard to show that both index sets $\text{Ind}(\mathcal{B}_0; \nu)$ and $\text{Ind}(\mathcal{B}_1; \nu)$ are Π_3^0 -complete. Therefore, the corresponding oracle X is not even $\mathbf{0}^{(2)}$ -c.e., let alone computable.

For the sake of simplicity, assume that for a particular class \mathfrak{K} , the obtained oracle X is computable. Even in this case, there are further complications. We illustrate these problems by an informal “toy” example. The example is, in a sense, a simplified version of the proof (1) \Rightarrow (2) of Theorem 3.1.

Consider a class \mathfrak{K} consisting of two computable undirected graphs:

- (1) The graph G_1 contains infinitely many isolated nodes and one cycle of size $2k + 3$ for each $k \in \omega$.
- (2) The graph G_2 has infinitely many isolated nodes and one cycle of size $2k + 4$ for each k .

We define a computable learner M , which acts according to the following rules:

- Given an input graph H , M searches for the least natural numbers $k_0 < l_0$ such that H contains the edge (k_0, l_0) .
- When k_0 and l_0 are found, M searches for a cycle of size $n \in \{2k_0 + 3, 2k_0 + 4\}$ inside H . If $n = 2k_0 + 3$, then M says that H is isomorphic to G_1 . If $n = 2k_0 + 4$, then M says $H \cong G_2$.

Consider finite undirected graphs F_1 and F_2 such that $\text{dom}(F_1) = \text{dom}(F_2) = \{0, 1\}$, the two nodes of F_1 are isolated, and F_2 contains an edge between 0 and 1. By employing the learner M , one can proceed similarly to Proposition 3.6 and construct a Turing computable embedding

$$\Phi: \{G_1, G_2\} \leq_{tc} \{F_1, F_2\}$$

such that Φ satisfies a stronger condition: for each $i \in \{1, 2\}$, if H is an isomorphic copy of G_i , then $\Phi(H)$ equals F_i .

Consider two existential sentences in the signature of graphs:

$$\xi_1 = \exists x \exists y [x \neq y \ \& \ \neg \text{Edge}(x, y)] \text{ and } \xi_2 = \exists x \exists y [x \neq y \ \& \ \text{Edge}(x, y)].$$

One can apply the proof of the Pullback Theorem for Σ_1^c -sentences (see Special Case on p. 905 of [14]). The tc -embedding Φ induces Σ_1^c -sentences ξ_1^* and ξ_2^* such that

$$(2) \quad G_1 \models \xi_1^* \ \& \ \neg \xi_2^* \text{ and } G_2 \models \neg \xi_1^* \ \& \ \xi_2^*.$$

An analysis of the proof of [14] shows that for *this particular* tc -embedding Φ , we have:

- (1) ξ_1^* is an infinite disjunction, which includes formulas

$$\theta_{2k+3} = \exists x_1 \exists x_2 \dots \exists x_{2k+3} [x_i\text{-s form a cycle of size } 2k + 3]$$

(and possibly some other \exists -formulas).

- (2) Similarly, ξ_2^* includes a disjunction of formulas

$$\theta_{2k+4} = \exists x_1 \exists x_2 \dots \exists x_{2k+4} [x_i\text{-s form a cycle of size } 2k + 4].$$

On the other hand, it is clear that one can replace these formulas ξ_1^* and ξ_2^* with $\xi_1^\# = \theta_3$ and $\xi_2^\# = \theta_4$, while preserving the property (2).

The described example shows that in general, the concrete formulas ξ_i^* , built in Theorem 3.1, depend on the choice of tc -embedding Φ . Thus, it is hard to say how the formulas are related to familiar algebraic properties of the original structures \mathcal{B}_i .

In conclusion, we note that Theorem 3.1 *does not use* the full strength of the Pullback Theorem: it is sufficient to employ Pullback only for finitary

formulas of the form $\xi = \exists \bar{x} \forall \bar{y} \theta(\bar{x}, \bar{y})$, where θ is quantifier-free. Nevertheless, it seems that the proof of such restricted version of the Pullback Theorem still requires developing essentially the same forcing machinery as for the general form.

4. APPLICATIONS OF THE MAIN RESULT

The first application gives an upper bound for the Turing complexity of learners. A straightforward analysis of the proof of Theorem 3.1 provides us with the following:

Corollary 4.1. *Let $X \subseteq \omega$ be an oracle. Let \mathfrak{K}_0 be a class of countably infinite L -structures, and ν be an effective enumeration of \mathfrak{K}_0 . Assume that either $I = \omega$, or I is a finite initial segment of ω . Consider a subclass $\mathfrak{K} = \{\mathcal{B}_i : i \in I\}$ inside \mathfrak{K}_0 . Assume that*

- (i) *There is uniformly X -computable sequence of $\Sigma_2^c(X)$ sentences $(\psi_i)_{i \in I}$ such that:*

$$\mathcal{B}_j \models \psi_i \Leftrightarrow i = j.$$

- (ii) *There is an X -computable sequence $(e_i)_{i \in I}$ such that $\nu(e_i) \cong \mathcal{B}_i$ for all i . Note that if the set I is finite, then one can always choose this sequence in a computable way.*

Then the class \mathfrak{K} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable via an X -computable learner.

The rest of the section discusses applications of Theorem 3.1 and Corollary 4.1 to some familiar classes of algebraic structures.

4.1. Simple examples of learnable classes. Here we give two examples of learnable infinite families.

The first one deals with distributive lattices. We treat lattices as structures in the signature $L_{\text{lat}} := \{\vee, \wedge\}$.

Selivanov [21] constructed a uniformly computable family $\{\mathcal{D}_i : i \in \omega\}$ of finite distributive lattices with the following property: If $i \neq j$, then there is no isomorphic embedding from \mathcal{D}_i into \mathcal{D}_j (see Figure 1).

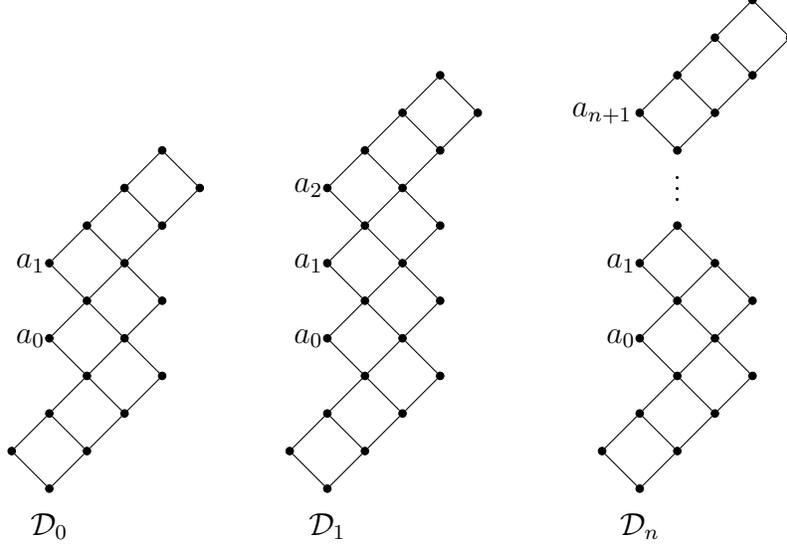
For $i \in \omega$, we define a countably infinite poset \mathcal{B}_i . Informally speaking, \mathcal{B}_i is a direct sum of the lattice \mathcal{D}_i and the linear order ω . More formally, we set:

- $\text{dom}(\mathcal{B}_i) = \{\langle x, 0 \rangle : x \in \mathcal{D}_i\} \cup \{\langle y, 1 \rangle : y \in \omega\}$.
- We always assume that $\langle x, 0 \rangle \leq \langle y, 1 \rangle$. The ordering of the elements $\langle x, 0 \rangle$ is induced by \mathcal{D}_i . We have $\langle y, 1 \rangle \leq \langle z, 1 \rangle$ if and only if $y \leq_{\omega} z$.

It is not hard to show that \mathcal{B}_i is a distributive lattice, thus, we will treat \mathcal{B}_i as an L_{lat} -structure.

Let $\mathfrak{K}_{\text{lat}}$ denote the class $\{\mathcal{B}_i : i \in \omega\}$. It is clear that one can build a Friedberg effective enumeration ν_{lat} as follows: just define $\nu_{\text{lat}}(i)$ as a natural computable copy of \mathcal{B}_i .

Proposition 4.2. *The class $\mathfrak{K}_{\text{lat}}$ is $\mathbf{InfEx}_{\cong}[\nu_{\text{lat}}]$ -learnable via a computable learner.*

FIGURE 1. Finite lattices \mathcal{D}_i , $i \in \omega$.

Proof. For $i \in \omega$, one can easily define a first-order \exists -sentence ψ_i which fully describes the finite lattice \mathcal{D}_i . We have that: for a structure \mathcal{S} , $\mathcal{S} \models \psi_i$ iff the finite lattice \mathcal{D}_i can be isomorphically embedded into \mathcal{S} .

Note the following properties of the considered objects:

- \mathcal{D}_i embeds into \mathcal{B}_j if and only if $i = j$.
- The sequence $\{\psi_i\}_{i \in \omega}$ is uniformly computable (this follows from the fact that the family $\{\mathcal{D}_i : i \in \omega\}$ is uniformly computable).
- For every i , $\nu_{lat}(i) \cong \mathcal{B}_i$.

Therefore, one can apply Corollary 4.1 with a computable oracle X . Proposition 4.2 is proved. \square

Recall that $\mathbb{K}_{L_{lat}}$ is the class of all countably infinite L_{lat} -structures.

Corollary 4.3. *Suppose that ν is an arbitrary effective enumeration of the class $\mathbb{K}_{L_{lat}}$. Then the following holds:*

- The class \mathfrak{K}_{lat} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable. Note that here the complexity of the learner depends only on the complexity of the sequence $(e_i)_{i \in \omega}$ from Corollary 4.1.
- \mathfrak{K}_{lat} is $\mathbf{InfEx}_{\cong}[\nu \oplus \nu_{lat}]$ -learnable by a computable learner.

Our second example deals with abelian p -groups. We treat abelian groups as structures in the signature $L_{ag} := \{+, 0\}$.

For a number $i \in \omega$, define the group

$$\mathcal{A}_i := \bigoplus_{j \in \omega} \mathbb{Z}(p^{i+1}).$$

We set $\mathfrak{K}_{ag} := \{\mathcal{A}_i : i \in \omega\}$, and we construct a Friedberg effective enumeration ν_{ag} as follows: just define $\nu_{ag}(i)$ as a natural computable copy of \mathcal{A}_i .

Proposition 4.4. *The class \mathfrak{K}_{ag} is $\mathbf{InfEx}_{\cong}[\nu_{ag}]$ -learnable by a computable learner.*

Proof. For $i \in \omega$, one can define a first-order sentence ψ_i which means the following: $\mathcal{S} \models \psi_i$ if and only if $\mathbb{Z}(p^{i+1})$ is a subgroup of \mathcal{S} , but $\mathbb{Z}(p^{i+2})$ is not a subgroup of \mathcal{S} . Clearly, ψ_i is logically equivalent to a conjunction of an \forall -formula (saying that for any element x , the condition $p^{i+2}x = 0$ implies $p^{i+1}x = 0$) and an \exists -formula (saying that there is an element y such that $p^{i+1}y = 0$ and $p^i y \neq 0$). The rest of the proof is similar to Proposition 4.2. Indeed, observe that:

- $\mathcal{A}_i \models \psi_j$ if and only if $i = j$.
- The sequence $\{\psi_i\}_{i \in \omega}$ is uniformly computable.
- For every i , $\nu_{ag}(i) \cong \mathcal{A}_i$.

Therefore, one can apply Corollary 4.1 to conclude that \mathfrak{K}_{ag} is $\mathbf{InfEx}_{\cong}[\nu_{ag}]$ -learnable. \square

Corollary 4.5. *Suppose that ν is an arbitrary effective enumeration of the class $\mathbb{K}_{L_{ag}}$. Then the following holds:*

- (a) *The class \mathfrak{K}_{ag} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable.*
- (b) *\mathfrak{K}_{ag} is $\mathbf{InfEx}_{\cong}[\nu \oplus \nu_{ag}]$ -learnable by a computable learner.*

4.2. Boolean algebras. Proposition 4.2 provides us with an example of an infinite learnable family of distributive lattices. Here we show that in the realm of Boolean algebras, the situation is dramatically different: informally speaking, one cannot learn even two different isomorphism types of infinite Boolean algebras.

Let \mathcal{A} and \mathcal{B} be structures in the same signature, and n be a non-zero natural number. We write $\mathcal{A} \leq_n \mathcal{B}$ if every infinitary Π_n sentence true in \mathcal{A} is also true in \mathcal{B} . The relation \leq_n is usually called the *n -th back-and-forth relation*.

For a Boolean algebra \mathcal{C} , let $\#_{atom}(\mathcal{C})$ denote the cardinality of the set of atoms of \mathcal{C} .

Proposition 4.6. *Let \mathfrak{K} be some class of infinite Boolean algebras, and let ν be an effective enumeration of \mathfrak{K} . Suppose that \mathfrak{C} is a subclass of \mathfrak{K} such that \mathfrak{C} contains at least two non-isomorphic members. Then the class \mathfrak{C} is not $\mathbf{InfEx}_{\cong}[\nu]$ -learnable.*

Proof. Suppose that \mathcal{A} and \mathcal{B} are structures from the class \mathfrak{C} such that $\mathcal{A} \not\cong \mathcal{B}$.

Using the description of the back-and-forth relations on Boolean algebras [2, § 15.3.4], one can prove the following fact: The condition $\mathcal{A} \leq_2 \mathcal{B}$ holds if and only if $\#_{atom}(\mathcal{A}) \geq \#_{atom}(\mathcal{B})$ (see, e.g., Lemma 11 in [1] for more details).

This fact implies that at least one of the following two conditions must be true:

$$\Sigma_2^{\text{inf}}\text{-Th}(\mathcal{A}) \subseteq \Sigma_2^{\text{inf}}\text{-Th}(\mathcal{B}) \text{ or } \Sigma_2^{\text{inf}}\text{-Th}(\mathcal{B}) \subseteq \Sigma_2^{\text{inf}}\text{-Th}(\mathcal{A}).$$

Therefore, by Theorem 3.1, we deduce that the class \mathfrak{C} is not $\mathbf{InfEx}_{\cong}[\nu]$ -learnable. \square

4.3. Linear orders. First, we show that linear orders exhibit learning properties, which cannot be witnessed by Boolean algebras.

Proposition 4.7. *Let $n \geq 2$ be a natural number. Then there is a class of computable infinite linear orders \mathfrak{C} with the following properties:*

- (a) \mathfrak{C} contains precisely n isomorphism types.
- (b) Suppose that \mathfrak{K} is a superclass of \mathfrak{C} , and ν is an effective enumeration of \mathfrak{K} . Then the class \mathfrak{C} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable by a computable learner.

Proof. We show how to build a family \mathfrak{C} containing precisely k non-isomorphic structures. We set

$$\mathfrak{C} = \{k + \eta + 1; (k - 1) + \eta + 2; (k - 2) + \eta + 3; \dots; 2 + \eta + (k - 1); 1 + \eta + k\}.$$

We also define first-order $\exists\forall$ -sentences ψ_i as follows: for a linear order \mathcal{L} ,

- (1) The sentence ψ_1 says that \mathcal{L} has k consecutive elements in the beginning, i.e., there are elements $a_0 < a_1 < \dots < a_{k-1}$ such that a_0 is the least element and a_{i+1} is the immediate successor of a_i , for every $i \leq k - 1$.
- (2) For $1 < i < k$, ψ_i says that \mathcal{L} has $k - i + 1$ consecutive elements in the beginning and i consecutive elements in the end (i.e., there are $b_{i-1} < b_{i-2} < \dots < b_0$ such that b_0 is the greatest and b_{j+1} is the immediate predecessor of b_j).
- (3) ψ_k says that \mathcal{L} has k consecutive elements in the end.

We apply Corollary 4.1 to the class \mathfrak{C} and the sequence $\{\psi_i\}_{1 \leq i \leq k}$. Thus, we obtain the desired learnability via a computable learner. Proposition 4.7 is proved. \square

On the other hand, the next result shows that one still cannot learn *infinite* families of linear orders.

Theorem 4.8. *Let \mathfrak{K} be some class of infinite linear orders, and let ν be an effective enumeration of \mathfrak{K} . Suppose that \mathfrak{C} is a subclass of \mathfrak{K} such that \mathfrak{C} contains infinitely many pairwise non-isomorphic members. Then the class \mathfrak{C} is not $\mathbf{InfEx}_{\cong}[\nu]$ -learnable.*

Proof. The key ingredient of the proof is an analysis of Σ_2^{inf} formulas for linear orders \mathcal{L} . First, we define the following auxiliary relations on \mathcal{L} :

- A first-order \forall -formula $\text{First}(x)$ says that x is the least element of \mathcal{L} .
- An \forall -formula $\text{Last}(x)$ says that x is the greatest element of \mathcal{L} .

- An \forall -formula $\text{Succ}(x, y)$ says that x and y are consecutive elements, i.e., $(x < y) \ \& \ \neg\exists z(x < z < y)$.
- A Σ_2^{\exists} formula $\text{Block}(x, y)$ says the following: either $x = y$, or there are only finitely many elements z between x and y in \mathcal{L} . The *block* of an element $x \in \mathcal{L}$ is the set

$$\text{Block}_{\mathcal{L}}[x] := \{y : \mathcal{L} \models \text{Block}(x, y)\}.$$

Lemma 4.9 ([20]). (1) *In the class of countably infinite linear orders, every Π_1^{inf} formula in the signature $\{\leq\}$ is logically equivalent to a Σ_1^{inf} formula in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$.*

(2) *Let \mathcal{A} and \mathcal{B} be countably infinite linear orders. Then we have:*

$$\mathcal{A} \leq_2 \mathcal{B} \Leftrightarrow (\mathcal{A}, \text{First}, \text{Last}, \text{Succ}) \leq_1 (\mathcal{B}, \text{First}, \text{Last}, \text{Succ}).$$

Proof. The proof of (1) can be recovered from [20, p. 871], see also Lemma II.43 in [19].

(2): Recall that the relations First , Last , and Succ are definable by \forall -formulas in the signature $\{\leq\}$. This implies that every first-order \exists -formula $\psi(\bar{x})$ in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$ is logically equivalent to a first-order $\exists\forall$ -formula $\psi^{[1]}(\bar{x})$ in the signature $\{\leq\}$.

For a linear order \mathcal{L} , let $\mathcal{L}^{\#}$ denote the structure $(\mathcal{L}, \text{First}, \text{Last}, \text{Succ})$. Suppose that $\mathcal{A}^{\#} \not\leq_1 \mathcal{B}^{\#}$. Then there is a Σ_1^{inf} -sentence

$$\xi = \bigvee_{i \in I} \exists \bar{x}_i \psi_i(\bar{x}_i),$$

where ψ_i are quantifier-free, such that $\mathcal{B}^{\#} \models \xi$ and $\mathcal{A}^{\#} \not\models \xi$. We choose an index $i_0 \in I$ such that the \exists -sentence $\theta := \exists \bar{x}_{i_0} \psi_{i_0}(\bar{x}_{i_0})$ is true in $\mathcal{B}^{\#}$. Clearly, $\mathcal{A}^{\#} \not\models \theta$. Hence, the $\exists\forall$ -sentence $\theta^{[1]}$ is true in \mathcal{B} and false in \mathcal{A} . Therefore, $\mathcal{A} \not\leq_2 \mathcal{B}$.

Suppose that $\mathcal{A} \leq_2 \mathcal{B}$. Then there is a Σ_2^{inf} -sentence

$$\xi = \bigvee_{j \in J} \exists \bar{y}_j \psi_j(\bar{y}_j),$$

where ψ_j are Π_1^{inf} -formulas, such that $\mathcal{B} \models \xi$ and $\mathcal{A} \not\models \xi$. Choose an index $j_0 \in J$ such that the formula $\exists \bar{y}_{j_0} \psi_{j_0}(\bar{y}_{j_0})$ is true in \mathcal{B} . By item (1), there is a Σ_1^{inf} -formula $\lambda(\bar{y}_{j_0})$ in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$, which is logically equivalent to ψ_{j_0} . In turn, the formula $\exists \bar{y}_{j_0} \lambda(\bar{y}_{j_0})$ is logically equivalent to a Σ_1^{inf} -sentence δ in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$. It is not hard to show that $\mathcal{B}^{\#} \models \delta$ and $\mathcal{A}^{\#} \not\models \delta$. Therefore, $\mathcal{A}^{\#} \not\leq_1 \mathcal{B}^{\#}$. \square

Towards a contradiction, we suppose that there is a family of infinite linear orders $\mathfrak{C} = \{\mathcal{C}_i : i \in \omega\}$ such that \mathfrak{C} is $\mathbf{InfEx}_{\cong}[\nu]$ -learnable and the structures \mathcal{C}_i are pairwise non-isomorphic. Then by Theorem 3.1, there is a sequence of Σ_2^{inf} sentences $(\psi_i)_{i \in \omega}$ such that

$$\mathcal{C}_i \models \psi_j \Leftrightarrow i = j.$$

We apply Lemma 4.9.(1), and for every i , we obtain a Σ_1^{inf} sentence ξ_i in the signature $\{\leq, \text{First}, \text{Last}, \text{Succ}\}$, which is equivalent to ψ_i . W.l.o.g., one can choose ξ_i as a finitary \exists -sentence: this is because any Σ_1^{inf} sentence φ is a countable disjunction of finitary \exists -formulas, and therefore φ is true if and only if there is at least one of such \exists -formulas which is true. Thus, the intuition behind ξ_i can be explained as follows. The sentence ξ_i describes a finite substructure $\mathcal{F}_i \subset (\mathcal{C}_i, \text{First}, \text{Last}, \text{Succ})$ such that \mathcal{F}_i cannot be isomorphically embedded into \mathcal{C}_j , for $j \neq i$.

Clearly, at least one of the following four cases is satisfied by infinitely many \mathcal{C}_i :

- (1) \mathcal{C}_i has neither least nor greatest elements;
- (2) \mathcal{C}_i has the least element, but there is no greatest one;
- (3) \mathcal{C}_i has the greatest element, but there is no least;
- (4) \mathcal{C}_i has both.

Thus, w.l.o.g., one may assume that every \mathcal{C}_i has both least and greatest elements. All other cases can be treated in a way similar to the exposition below.

We give an excerpt from the description [20, p. 872] of the relation \leq_2 for linear orders.

Let \mathcal{A} be a countably infinite linear order. We define:

- Let $t_0(\mathcal{A}) = n$ if $\mathcal{A} = n + \mathcal{A}_1$, where $n \in \omega$ and the order \mathcal{A}_1 has no least element. Set $t_0(\mathcal{A}) = \infty$ if $\mathcal{A} = \omega + \mathcal{A}_1$, where \mathcal{A}_1 has no least element.
- Define $t_2(\mathcal{A}) = m$ if $\mathcal{A} = \mathcal{A}_2 + m$, where $m \in \omega$ and \mathcal{A}_2 has no greatest element. Let $t_2(\mathcal{A}) = \infty$ if $\mathcal{A} = \mathcal{A}_2 + \omega^*$, where \mathcal{A}_2 has no greatest element.

As per usual, we assume that ∞ is greater than every natural number. We write $\mathcal{A} \equiv_2 \mathcal{B}$ if $\mathcal{A} \leq_2 \mathcal{B}$ and $\mathcal{B} \leq_2 \mathcal{A}$.

Lemma 4.10 ([20]). *Let \mathcal{A} and \mathcal{B} be countably infinite linear orders.*

- (1) *Suppose that $\max(t_0(\mathcal{A}), t_2(\mathcal{A})) = \infty$. Then, independently of \mathcal{B} , we have*

$$\mathcal{A} \leq_2 \mathcal{B} \Leftrightarrow t_0(\mathcal{A}) \geq t_0(\mathcal{B}) \text{ and } t_2(\mathcal{A}) \geq t_2(\mathcal{B}).$$

- (2) *Suppose that $\mathcal{A} = n_0 + \mathcal{A}_1 + n_2$ and $\mathcal{B} = m_0 + \mathcal{B}_1 + m_2$, where $n_0, n_2, m_0, m_2 \in \omega$, and both \mathcal{A}_1 and \mathcal{B}_1 have no endpoints. Then*

$$\mathcal{A} \leq_2 \mathcal{B} \Leftrightarrow (n_0 \geq m_0) \text{ and } (\mathcal{A}_1 \leq_2 \mathcal{B}_1) \text{ and } (n_2 \geq m_2).$$

- (3) *Suppose that both \mathcal{A} and \mathcal{B} have no endpoints. Then:*

- (3.1) *If for every non-zero $n \in \omega$, \mathcal{A} has a tuple of n consecutive elements, then $\mathcal{A} \leq_2 \mathcal{B}$.*
- (3.2) *Suppose that m is a non-zero natural number, and both \mathcal{A} and \mathcal{B} do not have tuples of $m + 1$ consecutive elements. If \mathcal{A} has infinitely many tuples of m consecutive elements, then $\mathcal{A} \leq_2 \mathcal{B}$.*

Lemma 4.10.(1) implies the following: if $t_0(\mathcal{A}) = t_0(\mathcal{B}) = \infty$, then we always have either $\mathcal{A} \leq_2 \mathcal{B}$ or $\mathcal{B} \leq_2 \mathcal{A}$. Hence, we deduce that there is at most one structure \mathcal{C}_i with $t_0(\mathcal{C}_i) = \infty$.

A similar argument shows that there is at most one \mathcal{C}_i with $t_2(\mathcal{C}_i) = \infty$. Therefore, w.l.o.g., one can assume that for every $i \in \omega$, both values $t_0(\mathcal{C}_i)$ and $t_2(\mathcal{C}_i)$ are finite. Let

$$\mathcal{C}_i = m_i + \mathcal{D}_i + n_i,$$

where $m_i, n_i \in \omega$, and the order \mathcal{D}_i has no endpoints. For $i \in \omega$, we define

$$q_i := \sup\{\text{card}(\text{Block}_{\mathcal{D}_i}[x]) : x \in \mathcal{D}_i\}.$$

Claim 4.11. *There are only finitely many i with $q_i = \infty$.*

Proof. For simplicity of exposition, towards a contradiction, suppose that every q_i is infinite. Note that Lemma 4.10.(3.1) shows that $\mathcal{D}_i \equiv_2 \mathcal{D}_j$ for all i and j .

Since for every $j \neq 0$, we have $\mathcal{C}_j \not\leq_2 \mathcal{C}_0$, by Lemma 4.10.(2), we obtain that \mathcal{C}_j satisfies at least one of the following two conditions: $m_j < m_0$ or $n_j < n_0$. W.l.o.g., we assume that there are infinitely many j with $m_j < m_0$. Then there is a number $m^* < m_0$ and an infinite sequence $j[0] < j[1] < j[2] < \dots$ such that $m_{j[k]} = m^*$ for all k .

Recall that $\mathcal{C}_{j[k]} \not\leq_2 \mathcal{C}_{j[0]}$ for all $k \neq 0$. By Lemma 4.10.(2), we have $n_{j[k]} < n_{j[0]}$ for every non-zero k . Hence, there is a number $n^* < n_{j[0]}$ such that $n_{j[k]} = n^*$ for infinitely many k . Clearly, if $k \neq k'$ are such numbers, then $\mathcal{C}_{j[k]} \equiv_2 \mathcal{C}_{j[k']}$, which gives a contradiction. \square

By Claim 4.11, one can assume that $q_i < \infty$ for every i .

Claim 4.12. *There is a number $r \in \omega$ such that $q_i \leq r$ for every i .*

Proof. Again, for simplicity of exposition, assume that $q_0 < q_1 < q_2 < \dots$. Recall that $\mathcal{C}_j \not\equiv \xi_0$ for all $j \neq 0$. Suppose that the finite structure \mathcal{F}_0 associated with the \exists -sentence ξ_0 contains precisely l_0 elements.

Choose j^* such that $q_{j^*} \geq 2l_0$. Clearly, for every $j \geq j^*$, the order \mathcal{D}_j contains at least one block of size at least $2l_0$. Thus, \mathcal{F}_0 cannot be embedded into \mathcal{C}_j only because of one of the following two obstacles:

- $m_j < m_0$, i.e., the size of the first (under $\leq_{\mathcal{C}_j}$) block in \mathcal{C}_j is too small for an appropriate embedding; or
- $n_j < n_0$, i.e., the size of the last block in \mathcal{C}_j is too small.

The relation $\text{Succ}^{\mathcal{C}_j}$ won't give us any problems, since one can embed all the \mathcal{F}_0 -blocks (except the first one and the last one) inside a \mathcal{D}_j -block of size $\geq 2l_0$.

As in Claim 4.11, we can assume that there is a number $m^* < m_0$ such that $m_j = m^*$ for infinitely many $j \geq j^*$. Form an increasing sequence $j[0] < j[1] < j[2] < \dots$ of these j . Recall that $q_{j[l]} < q_{j[l+1]}$ for all $l \in \omega$. Re-iterating the argument above, we obtain that there is a number $n^* < n_{j[0]}$ such that there are infinitely many l with $n_{j[l]} = n^*$. Choose

a sequence $l[0] < l[1] < l[2] < \dots$ of these l . Suppose that the structure $\mathcal{F}_{j[l[0]]}$ contains precisely t_1 elements.

Find the least $l^* = l[s^*]$ with $q_{j[l^*]} \geq 2t_1$. Recall that we have $m_{j[l^*]} = m_{j[l[0]]} = m^*$ and $n_{j[l^*]} = n_{j[l[0]]} = n^*$. Thus, as before, it is not hard to show that the structure $\mathcal{F}_{j[l[0]]}$ can be embedded into $\mathcal{C}_{j[l^*]}$. This shows that $\mathcal{C}_{j[l^*]} \models \xi_{j[l[0]]}$, which gives a contradiction. \square

By Claim 4.12, we obtain that

$$r := \sup\{q_i : i \in \omega\} < \infty.$$

Moreover, we will assume that $q_i = r$ for all $i \in \omega$: indeed,

- If there are only finitely many i with $q_i = r$, then we just delete the corresponding structures \mathcal{C}_i . After that the value r goes down.
- If there are already infinitely many i with $q_i = r$, then we delete all \mathcal{C}_j with $q_j < r$.

Claim 4.13. *There are only finitely many i such that the order \mathcal{D}_i has infinitely many blocks of size r .*

Proof. Again, for simplicity, assume that every \mathcal{D}_i has infinitely many blocks of size r . Since $q_i = r$ for all i , Lemma 4.10.(3.2) implies that $\mathcal{D}_i \equiv_2 \mathcal{D}_j$ for all i and j .

As in Claim 4.12, \mathcal{F}_0 is not embeddable into \mathcal{C}_j , $j \neq 0$, and this is witnessed by one of the following: either $m_j < m_0$ or $n_j < n_0$. We recover a number $m^* < m_0$ and a sequence $j[0] < j[1] < j[2] < \dots$ such that $m_{j[l]} = m^*$ for all l .

The finite structure $\mathcal{F}_{j[0]}$ is not embeddable into $\mathcal{C}_{j[l]}$, $l \neq 0$. By Lemma 4.10.(2), this implies that $n_{j[l]} < n_{j[0]}$ for non-zero l . Again, there is a number $n^* < n_{j[0]}$ and a sequence $l[0] < l[1] < l[2] < \dots$ such that $n_{j[l[s]]} = n^*$ for all s . This shows that $\mathcal{C}_{j[l[1]]} \models \xi_{j[l[0]]}$, and this yields a contradiction. \square

Claim 4.13 implies that one may assume the following: each \mathcal{D}_i has only finitely many blocks of size $r = q_i$.

The rest of the proof is only sketched, since all the key ideas are already present. Let $\#(r; i)$ denote the number of blocks of size r inside \mathcal{D}_i .

Claim 4.14. *There is a number N such that $\#(r; i) \leq N$ for all i .*

Proof. Assume that $\#(r; i) < \#(r; i + 1)$ for all i . As before, the finite structure \mathcal{F}_0 cannot be embedded into \mathcal{C}_j , where j is large enough, and this can be witnessed only by one of the following conditions: $m_j < m_0$ or $n_j < n_0$ for such j . Hence, we assume that there is a sequence $j[0] < j[1] < j[2] < \dots$ with $m_{j[l]} = m^* < m_0$ for all l . By considering possible embeddings of the finite structure $\mathcal{F}_{j[0]}$, we recover a sequence $l[0] < l[1] < l[2] < \dots$ with $n_{j[l[s]]} = n^* < n_{j[0]}$ for all l . Clearly, $\mathcal{F}_{j[l[0]]}$ can be embedded into any $\mathcal{C}_{j[l[s]]}$, where s is large enough, and this produces a contradiction. \square

By Claim 4.14, one can assume that $\#(r; i) = N < \infty$ for all i . For simplicity, consider $N = 2$. Then every \mathcal{D}_i can be presented in the following form:

$$\mathcal{D}_i = \mathcal{D}_{i,0} + r + \mathcal{D}_{i,1} + r + \mathcal{D}_{i,2}, \text{ where}$$

- every $\mathcal{D}_{i,j}$ does not have endpoints, and
- every block inside $\mathcal{D}_{i,j}$ has size at most $r - 1$.

After that, one needs to write a cumbersome proof by recursion in r . The arrangement of this recursion can be recovered from the ideas from [20, p. 872].

In our case, the first stage of recursion will roughly consist of the following claims:

- (a) We say that a block of size $(r - 1)$ is *large*. Then one can prove that there are only finitely many i such that every $\mathcal{D}_{i,j}$ contains infinitely many large blocks.
- (b) If there are infinitely many i such that, say, both $\mathcal{D}_{i,0}$ and $\mathcal{D}_{i,1}$ contain infinitely many large blocks, then one can assume that there is a number N_1 such that every $\mathcal{D}_{i,2}$ has at most N_1 large blocks. In this case, the next stage of recursion will play essentially only with $\mathcal{D}_{i,2}$.
- (c) Assume that there are infinitely many i such that $\mathcal{D}_{i,0}$ has infinitely many large blocks, but every $\mathcal{D}_{i,1}$ and $\mathcal{D}_{i,2}$ has only finitely many large blocks. Then there are three main variants:
 - (c.1) There are a number N_2 and a sequence $i_0 < i_1 < i_2 < \dots$ such that for every k , $\mathcal{D}_{i_k,1}$ has precisely N_2 large blocks and $\mathcal{D}_{i_k,2}$ contains, say, at least k large blocks. Then one needs to invoke recursion for $\mathcal{D}_{i,1}$.
 - (c.2) A case similar to the previous one, but here we require that every $\mathcal{D}_{i_k,1}$ has at least k large blocks. Then one can obtain a contradiction.
 - (c.3) There is a number N_3 such that every $\mathcal{D}_{i,1}$ or $\mathcal{D}_{i,2}$ has at most N_3 large blocks. Then proceed to the next recursion stage by considering both $\mathcal{D}_{i,1}$ and $\mathcal{D}_{i,2}$ simultaneously.
- (d) Assume that each $\mathcal{D}_{i,j}$ has only finitely many large blocks. The main cases are as follows:
 - (d.1) There are a number N_4 and a sequence $i_0 < i_1 < i_2 < \dots$ such that for every k , $\mathcal{D}_{i_k,0}$ contains precisely N_4 large blocks and each of $\mathcal{D}_{i_k,1}$ and $\mathcal{D}_{i_k,2}$ has at least k large blocks. Then one calls recursion for $\mathcal{D}_{i,0}$.
 - (d.2) A case similar to the previous one, but now we require that $\mathcal{D}_{i_k,1}$ always keeps precisely N_5 large blocks. Then the next recursion stage will work with $\mathcal{D}_{i,0}$ and $\mathcal{D}_{i,1}$ simultaneously.
 - (d.3) For all k , every block $\mathcal{D}_{i_k,j}$ contains at least k large blocks. This leads to a contradiction.

- (d.4) There is a number N_6 such that each $\mathcal{D}_{i,j}$ contains at most N_6 large blocks. Then we go to the next stage of recursion, and we have to consider all $\mathcal{D}_{i,j}$ simultaneously.

When the outlined recursion procedure finishes, we will get a contradiction in all considered cases. This implies that the class \mathfrak{C} cannot be $\mathbf{InfEx}_{\cong}[\nu]$ -learnable. Theorem 4.8 is proved. \square

5. CONCLUSIONS AND OPEN PROBLEMS

In this paper, we investigated the problem of learning computable structures up to isomorphism. We used infinitary logic to offer a model-theoretic characterization of which families of structures are \mathbf{InfEx} -learnable. Applying such a characterization, we proved that our learning paradigm is very sensitive to the algebraic properties of the structures to be learned: e.g., while there is an infinite learnable family of distributive lattices, no infinite family of linear orders is learnable.

Many questions remain open. In particular, one shall ask which families of structures can be learned when only positive data of the target structure is available. The ideal goal would be to obtain an analogue of Theorem 3.1 for the learning type \mathbf{TxtEx}_{\cong} (already introduced in [6]). Moreover, one obtains natural variants of the learning problems considered in this paper by replacing isomorphism with weaker notions (such as the bi-embeddability relation discussed in [6]) or with stronger ones (such as computable isomorphism).

Finally, in this paper we only marginally considered the complexity of the learners described. We still have a limited understanding of which families of structures can be learned by a learner of a given fixed complexity. In this direction, the following question looks particularly intriguing: is there a pair of two (non-isomorphic) structures which is \mathbf{InfEx}_{\cong} -learnable, but no computable learner can learn it?

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