

A Self-Stabilizing Hashed Patricia Trie*

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Abstract

While a lot of research in distributed computing has covered solutions for self-stabilizing computing and topologies, there is far less work on self-stabilization for distributed data structures. Considering crashing peers in peer-to-peer networks, it should not be taken for granted that a distributed data structure remains intact. In this work, we present a self-stabilizing protocol for a distributed data structure called the *hashed Patricia Trie* (Kniesburges and Scheideler WALCOM'11) that enables efficient prefix search on a set of keys. The data structure has a wide area of applications including string matching problems while offering low overhead and efficient operations when embedded on top of a distributed hash table. Especially, longest prefix matching for x can be done in $\mathcal{O}(\log |x|)$ hash table read accesses. We show how to maintain the structure in a self-stabilizing way. Our protocol assures low overhead in a legal state and a total (asymptotically optimal) memory demand of $\Theta(d)$ bits, where d is the number of bits needed for storing all keys.

Keywords and phrases Self-Stabilizing, Prefix Search, Distributed Data Structure

1 Introduction

We consider the problem of maintaining a distributed data structure for efficient *Longest Prefix Matching* in peer-to-peer (P2P) systems. We focus on the *hashed Patricia Trie* (HPT) introduced in [14] and present an algorithm rendering a self-stabilizing version of this data structure when applied on top of any reliable *distributed hash table* (DHT).

► **Definition 1.** (Longest Prefix Matching) Consider a set of binary strings called *keys* and a binary string x . The task of Longest Prefix Matching is to find a key y sharing the longest common prefix with x . A prefix of a binary string is a substring beginning with the first bit. We denote the longest common prefix of x and y by $\ellcp(x, y)$.

We denote a prefix p of x by $p \sqsubseteq x$. p is a *proper prefix* of x ($p \sqsubset x$) if p is a prefix of x and $|p| < |x|$, where $|p|$ is the length of p . Longest Prefix Matching is an old problem with applications in various areas including string matching problems and IP lookup in Internet routers. To solve it efficiently in a distributed P2P system, the HPT has been introduced [14]. The HPT is a distributed data structure applied to any common DHT which allows efficient prefix search for x in $\mathcal{O}(\log |x|)$ read accesses to the hash table, i.e., solely based on the length of the search word x . The costs for an insertion of x is in $\mathcal{O}(\log |x|)$ read accesses

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and $\mathcal{O}(1)$ write accesses, while deletion can be done in $\mathcal{O}(1)$ accesses. The memory space used is asymptotically optimal in $\Theta(\text{sum of all key lengths})$. Moreover, *Suffix Trees* can be implemented efficiently using Patricia Tries and thus also hashed Patricia Tries (called *PAT Trees* [10]). This allows us to efficiently decide if a given string x is a substring of a text in a runtime only depending on the length of x .

The usefulness of Patricia Tries motivates us to investigate how a HPT can be maintained in a P2P system where nodes may enter/leave or even fail. While a lot of research has considered the design of self-stabilizing computation or topologies (See Section 1.2), to the best of our knowledge there are far fewer results concerning self-stabilizing distributed data structures. However, considering failures of peers, the stability of any distributed data structure can also be affected. Therefore, we consider the problem of finding an efficient distributed protocol to rebuild the HPT in a self-stabilizing way.

1.1 Model

We assume the existence of a self-stabilizing *distributed hash table* (DHT) which provides the operations $\text{DHT-INSERT}(x)$ to insert data and $\text{DHT-SEARCH}(x)$ to retrieve data. These operations are carried out reliably on the stored data, i.e., no operation is ever canceled. We assume the existence of a collision-free hash function which maps binary strings to positions in $[0, 1)$ to store data in the DHT. The function is available locally at every peer. Each peer has a unique identifier, manages local variables and maintains a *channel*. When a peer sends a message m to peer p , it puts m in the channel of p . A channel has unbounded capacity and messages never get lost. If a peer processes a message in its channel, the message is removed from the channel afterwards.

We distinguish between two types of *actions*: The first one is for standard procedures and has the form $\langle \text{label} \rangle (\langle \text{parameters} \rangle) : \langle \text{command} \rangle$ where *label* is the name of the action, *parameters* define the set of parameters and *command* defines the statements that are executed when calling the action. It may be executed locally or remotely. The second type has the form $\langle \text{label} \rangle : (\langle \text{guard} \rangle) \rightarrow \langle \text{command} \rangle$ where *label* and *command* are defined as above and *guard* is a predicate over local variables. An action at peer p can only be executed if its guard is *true* or a message in the channel of p requests to call it. We call such an action *enabled*. The guard of our protocol routine `TIMEOUT` is always *true*.

A *state* of the system is defined by the assignment of variables at every peer, the data items and their values stored at every peer and all messages in channels of peers. The system can transform from a state s to another state s' by execution of an enabled *action* at a peer. An infinite sequence of states (s_1, s_2, \dots) is a *computation* if s_{i+1} can be reached by execution of an action enabled in s_i for all $i \geq 1$. The state s_1 is called *initial state*. We assume *fair message receipt*, i.e., every message contained in a channel is eventually processed. Also, we assume *weakly fair action execution* such that any action that is enabled in all but finitely many states is executed infinitely often. This especially applies to the `TIMEOUT` procedure. Most importantly, our protocol is *self-stabilizing*. We call a protocol self-stabilizing, if it fulfills *Convergence* and *Closure*. Convergence means that starting from an arbitrary initial state, the protocol transforms the system to a legal state in finite time. Closure denotes that starting from a legal state, the protocol only transforms the system to consecutive legal states. Our goal is to reach a state in which the HPT provided by the system is in a *legal state*. We define the legal state of a HPT later in Section 4.1.

1.2 Related Work

The basic data structure we consider here is the Patricia Trie. This compressed tree structure has been introduced by Morrison in [15]. It was extended to the hashed Patricia Trie by Kniesburges and Scheideler in [14]. In [10], Gonnet et al. presented PAT Trees which are essentially Patricia Tries for special suffixes (*sistrings*) of a text. This widens the applications of Patricia Tries to general string problems such as deciding if a word or sentence is contained in a text [10]. The work on self-stabilization started with the research of Dijkstra in [7] where he analyzed self-stabilization in a token ring scenario. Since then, research has covered wide areas including self-stabilizing computation [3, 5] and coordination [1, 2, 7, 9]. Furthermore, with the rise of P2P systems [17, 19], self-stabilizing topologies in the sense of overlay networks gained attraction [4, 6, 8, 11, 12, 13, 18]. We use approaches originally presented for topological self-stabilization. This includes a technique called *Linearization* presented by Onus et al. in [16]. A common approach for storing data in overlay networks is a distributed hash table (DHT) like Chord [19]. Using hashing, data items, as well as network peers, are mapped to the $[0, 1)$ interval such that a mapping between them is established. There are various results on self-stabilizing DHTs in the literature (for example [13]). Further, most (self-stabilizing) overlay networks can easily be extended to a DHT given sortable unique identifiers for the peers which is a common assumption.

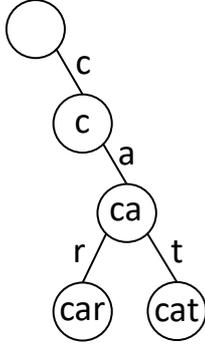
1.3 Our Contribution

We present a self-stabilizing protocol called SHPT to maintain a slightly modified version of the HPT as presented in [14]. Whenever we refer to HPT, we implicitly mean the modified version. The HPT and our modification are briefly introduced in Section 2. Afterwards, Section 3 gives a high-level description of the most important mechanisms of our protocol. We only require for an initial state that the underlying DHT is in a legal state and that a set of unique keys is stored at DHT nodes. In Section 4, we show that our protocol stabilizes a HPT in finite time out of any initial state. When the HPT is in a legal state, our protocol guarantees a low overhead of a constant amount of hash table read accesses and messages generated at each DHT node per call of the protocol routine. Furthermore, we can bound the total memory consumption in a legal state to $\Theta(d)$ bits if d is the number of bits needed to store all keys. To improve readability, we deferred the Pseudocode to Appendix A and the full proofs concerning correctness and overhead to Appendix B and Appendix C.

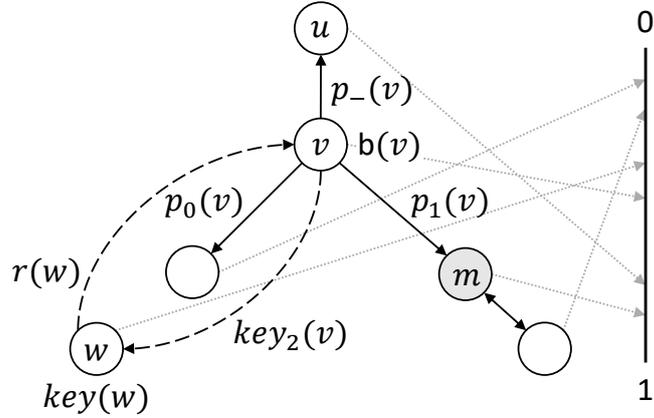
2 Hashed Patricia Trie

We consider a data structure called the *hashed Patricia Trie* (HPT) as presented in [14]. The HPT is an extended Patricia Trie that is distributed in a P2P System by using a DHT. We briefly describe the construction. For details, we refer to [14]. The Patricia Trie is a compressed trie which was proposed by Morrison in [15]. Suppose we are given a key set KEYS consisting of strings. A trie is a tree structure that consists of labeled nodes and labeled edges. The root node is labeled by the empty string and every edge is labeled by one character. The label of a node is the concatenation of all edge labels of edges traversed on the unique path from the root to the node. For each $k \in \text{KEYS}$ there is a node labeled by k (see Figure 1). The Patricia Trie introduces compression by allowing edge labels to be strings such that inner nodes with a single child, which do not represent a key, can be avoided. Similar to [14], we restrict ourselves to keys represented by binary strings. We store the Patricia Trie in a DHT by hashing all nodes by their label resulting in the hashed

Patricia Trie. Our notation is close to the one of [14] and can be seen in Figure 2.



■ **Figure 1** Example of a classical Trie containing the keys "car" and "cat".



■ **Figure 2** Values stored at nodes of the HPT from the perspective of v . Nodes are stored by hashing their label to $[0, 1]$ in combination with a DHT. White nodes denote Patricia nodes while Msd nodes are depicted in gray.

Every Patricia node v has a label denoted by $b(v)$ and stores three edges. The *root* node stores the empty string $b(\text{root}) = \varepsilon$. $p_-(v)$ is the parent edge of v pointing to the parent node u such that $b(u) \circ p_-(v) = b(v)$. We denote by \circ the concatenation of strings. By $p_x(v)$ we denote the child edge of v starting with the value x for $x \in \{0, 1\}$. If $b(w) \in \text{KEYS}$ for a Patricia node w , we set $\text{key}(w) = b(w)$. Additionally, an inner Patricia node stores a $\text{key}_2(v) = k$, where k is a key with $b(v) \sqsubseteq k$. For efficient updates, the node w storing k has a field $r(w) = b(v)$. These key_2 values allow returning a valid result for a prefix search when stopping at any Patricia node. It is possible to assure that every inner Patricia node with two children has a key_2 pointing to a leaf node in its subtree.

To allow efficient prefix search, the Patricia Trie has been extended in [14]. Between every pair of directly connected Patricia nodes, Msd nodes (from Most Significant Digit) are added. Their length is chosen in a way that those nodes are hit by a binary search first. More specifically, Msd nodes are inserted between Patricia nodes such that their length is considered first by the binary search before the Patricia nodes around them are considered. We only give a short definition of the calculation of an Msd label in Theorem 2. In the special case that an Msd label equals the label of a surrounding Patricia node, no Msd node is needed at that position. For details on how Msd nodes improve the prefix search operation, see [14].

► **Definition 2 (Msd Label).** Let $a = (a_m, \dots, a_0)$ and $b = (b_m, \dots, b_0)$ be two binary strings of the same length. Possibly, one of them is filled up with leading zeros to have length $m+1$. We define $\text{msd}(a, b)$ to be the position j where $a_j \neq b_j$ and $a_i = b_i$ for all $i > j$. That means, $\text{msd}(a, b)$ is the most significant bit (digit) at which a and b differ.

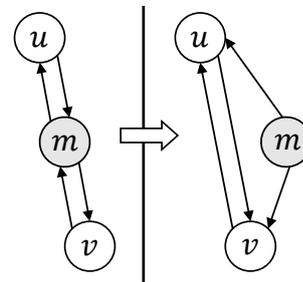
Consider the binary labels $b(u)$ and $b(v)$ of two nodes u, v . Let $\ell_u = |b(u)|$ and $\ell_v = |b(v)|$ and without loss of generality let $\ell_u < \ell_v$. We define the Msd label $b(m)$ between u and v to be the prefix of v of length $\sum_{i=\text{msd}(\ell_u, \ell_v)}^{\lceil \log \ell_v \rceil + 1} (\ell_v)_i \cdot 2^i$.

For example, consider u, v with $b(u) = 10$ and $b(v) = 100101$, where $\ell_u = |b(u)| = (10)_2$ and $\ell_v = |b(v)| = (110)_2$. Then $\text{msd}(\ell_u, \ell_v) = \text{msd}((010)_2, (110)_2) = 2$, such that an Msd node m between u and v has label $b(m) = 1001 \sqsubseteq b(v)$ of length $2^2 = 4$.

The HPT supports operations $\text{PREFIXSEARCH}(x)$ and $\text{INSERT}(x)$ for a binary string x in $\mathcal{O}(\log|x|)$ read accesses on the hash table. Insertion takes additional $\mathcal{O}(1)$ write accesses and $\text{DELETE}(x)$ is supported in constant hash table accesses. Furthermore, the memory space usage is in $\Theta(\sum_{k \in \text{KEYS}} |k|)$.

Modification. We modify the HPT to simplify the stabilization technique. Consider Figure 3. The original HPT has a structure as shown on the left side. The Msd node m is in between the Patricia nodes u and w such that u and w point to m and m points to u (parent) and w (child). We modify this structure by having u and w point to each other and not to m . By this, deletions of Msd nodes do not concern the connectivity between Patricia nodes while the advantages of Msd nodes are still present. The crucial property of Msd nodes is that they point to Patricia nodes. Edges towards Msd nodes are not needed for the efficient operations introduced in [14]. For the rest of this paper, when we refer to the HPT, we mean the HPT with this small modification.

Next, we introduce some common terms that are used throughout the paper. HPT is the set of all data nodes of the HPT. This includes PAT as the set of nodes used in the original Patricia Trie and MSD which are the Msd nodes. By definition $\text{HPT} = \text{PAT} \cup \text{MSD}$. We denote by KEYS the set of keys stored by the HPT and by MSG the set of all messages currently existing. Let $u, v \in \text{HPT}$ with $b(u) \sqsubset b(v)$. In this case we say, u is *above* v while v is *below* u . Let $w \in \text{HPT}$ such that $b(u) \sqsubset b(w) \sqsubset b(v)$. Then w is in *between* u and v . If for two $u, v \in \text{HPT}$ with $b(u) \sqsubset b(v)$ there is no $w \in \text{HPT}$ with $b(u) \sqsubset b(w) \sqsubset b(v)$, then u and v are *closest* to each other. We say a child edge e of $v \in \text{HPT}$ is *valid*, if there exists a node $w \in \text{HPT}$ with $b(v) \circ e = b(w)$. Similar, a parent edge e of $v \in \text{HPT}$ is valid, if there exists a node $w \in \text{HPT}$ with $b(w) \circ e = b(v)$. Consider two nodes $v, u \in \text{HPT}$, where u has an edge pointing to v and vice versa. We then speak of a *bidirectional* edge.



■ **Figure 3** Modified HPT

3 The SHPT Protocol

In the following, we present SHPT, our self-stabilizing protocol for maintaining a HPT. The corrections of SHPT can be divided into several parts. We present our assumptions concerning the underlying DHT first. Afterwards, we give an intuition on the different types of repairs our protocol performs. We often speak about actions executed by a HPT node v . This translates to actions that are executed by the corresponding DHT node storing v . For detailed Pseudocode, we refer to Appendix A.

3.1 Properties of the DHT

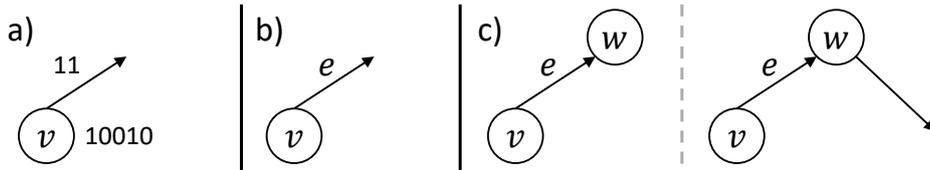
We assume that the underlying DHT is in a legal state, i.e., it provides the actions $\text{DHT-SEARCH}(x)$ and $\text{DHT-INSERT}(x)$ which are carried out reliably on the stored data. Deletion of data is only done locally by our protocol. Stability of the DHT is crucial as our protocol relies on finding/manipulating nodes of the HPT solely based on their hash value given by their label. There are a lot of different self-stabilizing DHTs presented in the literature. Some of them are mentioned in Section 1.2.

Our main demand on the DHT is that at some point nodes are stored such that they can always be retrieved by their labels. HPT nodes are essentially data-items. Every DHT node regularly checks if all its stored data is at the correct peer based on the hashing. If data is stored incorrectly, it is sent towards the correct DHT node. When a data item i is inserted at a DHT node n , n checks if i is already present. If yes, i is only inserted if it does not collide with an already stored Patricia node that stores a key. If a HPT node v has been inserted, a presentation method is triggered for v and v is directly presented to the nodes referred to by $p_-(v)$, $p_0(v)$ and $p_1(v)$. The presentation mechanism is presented later. This assumption assures that keys are preserved while insertion is not blocked and every HPT node is presented at least once.

3.2 Correcting Edge Information

One general problem for self-stabilizing solutions is that every stored information can be corrupted. Thus, our protocol regularly checks information stored in a HPT node. Consider a node $v \in \text{HPT}$. We refer to the information provided by the fields $p_-(v)$, $p_1(v)$ and $p_0(v)$ as well as $key_2(v)$ and $r(v)$ as *edge information*. Edge information can be checked rather simply as it allows reconstruction of a node's label $b(w)$. The label can be used to query the DHT for an (incomplete) copy of w . v can then compare the information stored at w with its own and decide for corrections. Some inconsistencies in the local structure can also be checked without querying the DHT. In general, when checking an edge e at node v , we distinguish three cases:

- a) e has a wrong form. For example, if $p_1(v) = (0\dots)$ or $p_-(v)$ is not a suffix of $b(v)$. In this case, the edge is considered corrupted and is cleared.
- b) The node w that e points to does not exist. Again, e is not correct and is cleared.
- c) The node $w \in \text{HPT}$ that e points to does exist, but the edge provided by w which should point to v does not match e . Several sub-cases arise here. The protocol may have to simply present v to w , or a new node may need to be inserted.



■ **Figure 4** Examples for the cases of wrong edge information.

Additionally, every node avoids edges pointing to Msd nodes. Such edges are treated as if they pointed to a non-existing node. A node v can check the values of $p_-(v)$, $key_2(v)$ and $r(v)$ by calculating if the prefix relation between itself and the respective nodes fulfills the definition of the hashed Patricia Trie. To prevent the spreading of incorrect information, new edges are only stored if they comply with the definition of the hashed Patricia Trie from the local perspective of v . We will go into detail on the creation of new edges and the insertion of nodes later.

3.3 Maintaining Connections

Our goal to stabilize the Patricia nodes of a HPT can also be formulated using *Branch Sets* as described in Theorem 3. A Branch Set consists of all Patricia nodes on a branch from

the root to a leaf node (see Figure 5). When the HPT is in a legal state, there are as many Branch Sets as there are leaf nodes.

► **Definition 3** (Branch Set). Consider a set of Patricia nodes with maximum cardinality S such that $u, w \in S$ implies $b(u) \sqsubset b(w)$ or $b(w) \sqsubset b(u)$ and the Patricia node $v \in S$ with maximum label length stores a key k . We call this set the *Branch Set* of k .

We apply a technique called Linearization [16] to all Patricia nodes to create a list sorted by label length for all Branch Sets in finite time. It is important to exclude Msd nodes from the Linearization. Msd nodes are not presented nor do they delegate presentation messages. Due to deletion of a Patricia node, an Msd node might still be presented accidentally. However, we limit this problem by carefully handling deletions and insertions as described later. For the Linearization to work, we need to make sure that all nodes in a Branch Set are brought into and kept in a weakly connected state.

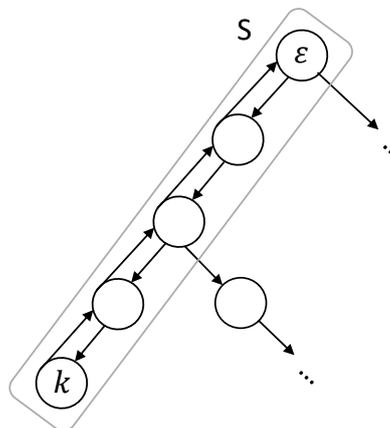
A Patricia node v with an empty parent edge tries to recreate connectivity by doing a modified PREFIXSEARCH($b(v)$) similar to the one presented in [14]. The procedure we call BINARYPREFIXSEARCH($b(v)$) does not search for $b(v)$ itself and only consists of the binary search phase of the PrefixSearch(x) of [14], returning a copy of a Patricia node w with $b(w) \sqsubset b(v)$. If no such node exists, we conclude that the root node is non-existent and trigger a construction of it.

Further, we let every Patricia node present its own label to its parent and its two children using a *presentation message*. A message presenting v is delegated to the Patricia node w closest to v . Delegation happens only by using edges and intermediate nodes sharing a Branch Set with v . All nodes maintain connections to labels which are closest to them while delegating presentations of other labels. This behavior resembles the Linearization approach presented in [16], allowing our protocol to form a sorted list for all branches of the HPT.

There is still an important issue we need to resolve. Consider a Branch Set S of nodes. We can end up in situations where nodes exist that do not contribute to the hashed Patricia Trie. Such nodes can be Patricia nodes not storing a key. To reduce memory demands, we are interested in removing unneeded nodes. In principle, deletion without harming connectivity can be done since the root node is always known implicitly. However, deletion increases distances. In addition, our protocol must provide the ability to create and integrate new Patricia nodes. When inserting and deleting nodes, we need to make sure that no loops are possible in which the system may take forever to stabilize. We will explain how to avoid such loops in the following.

3.4 Removal/Creation of Nodes

Due to the implicitly known root node, deletion is possible and should be considered to reduce memory demands. We distinguish between Msd nodes and Patricia nodes. Our modification allows us to handle Msd nodes in a simple and efficient way. We try to avoid any edges pointing to Msd nodes such that eventually, deletion and creation of Msd nodes



■ **Figure 5** Branch Set S from the root (ε) to a leaf node (k) is the set of nodes in a branch of the hashed Patricia Trie in a legal state.

does not influence the Patricia nodes and their structure. Only if there are two Patricia nodes u, w connected via a bidirectional edge, an Msd node between them might be inserted. Fortunately, Msd labels can be calculated locally and a corresponding Msd node can easily be accessed by querying the DHT. Any Msd node which is not between such two Patricia nodes, or has an incorrect label, is deleted.

A Patricia node v (except for the root) is *unnecessary* if $key(v) = nil$ and there are no two Patricia nodes u, w , both storing a key, such that $b(v) = lcp(b(u), b(w))$, i.e., u should be in a different subtree than w below v . From a global point of view, we can easily decide if v is unnecessary solely based on information about the situation below v . From a local perspective, v cannot decide but only assume to be unnecessary if it lacks child edges. We make the local protocol aggressive by deleting any node that lacks child edges and assumes to be unnecessary. This also introduces deletion of necessary Patricia nodes. Therefore, we always trigger a creation of new HPT nodes by Patricia nodes below the new ones. This avoids loops of creation and deletion of nodes, because newly created nodes inherently have valid children and, thus, do not assume to be unnecessary. Patricia nodes storing a key essentially form a stable starting point, because they are never deleted. The need to insert a Patricia node is detected by comparing a node's parent edge with the corresponding edge provided by the parent.

3.5 Distribution of References to Keys

In addition, SHPT tries to achieve the following. Every inner Patricia node v with two children should store a $key_2(v) = b(w)$ which points to a leaf node w storing a key such that $b(v) \sqsubset b(w)$. The respective leaf node w stores an $r(w)$ value pointing to v . This property is helpful for efficient prefix search. No matter at which Patricia node the prefix search stops, there is a key referenced having the node's label as a prefix. This key is a valid result for the search query. We call all inner Patricia nodes with two children and the root node *key₂ nodes*. Due to the resemblance of the hashed Patricia Trie with a binary tree, Fact 1 holds.

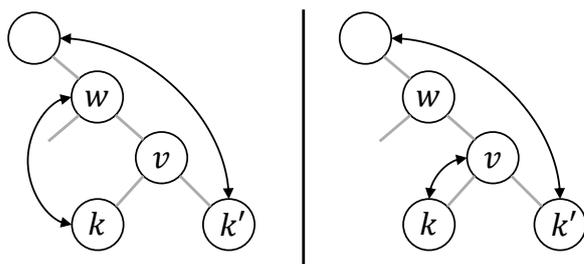
► **Fact 1.** Let L be the number of leaf nodes. Let I be the number of key₂ nodes. When the HPT is in a legal state, it holds $I \leq L \leq I + 1$. $L = I$, if the root has one child and $L = I + 1$ if it has two.

To assure that every leaf node is referenced by a key₂ node, we allow the root to store up to two key₂ values. This reduces the number of hash table accesses created by our protocol, when the HPT is in a legal state.

If we naively assign leaf nodes to key₂ nodes, this may lead to situations in which a key₂ node cannot get a key₂ value. For an example, consider Figure 6. The critical observation is that key₂ nodes with a shorter label, in general, have more possible leaf nodes they can point to than key₂ nodes with a longer label. Therefore, our protocol aims at prioritizing key₂ nodes which are closer to leaf nodes.

We divide the protocol into three parts. First, all nodes continuously check if they should store a key₂ or r value and whether such a value points to a leaf node, respectively key₂ node. Second, if a leaf node v does not store a value in $r(v)$, it presents its label upwards in the HPT by sending a message crossing only parent edges. The first key₂ node w without a key₂ receiving the message sets $key_2(w) = b(v)$. Third, a key₂ node v repairs in the following way. If $key_2(v)$ points to leaf node w with $b(v) \sqsubset b(w)$, there are two cases.

- a) $b(v) \sqsubset r(w)$: Then $key_2(v)$ is set to *nil* since there may already be some key₂ node with longer label pointing at w .



■ **Figure 6** Example where v cannot get a key_2 (left). The leaf nodes k and k' storing a key are already associated to Patricia nodes above v . The blocking of v is resolved as v takes over the key_2 of w (right).

b) Else, v has either longer label than $r(w)$ or $r(w) = \text{nil}$. The protocol sets $r(w) = b(v)$.

If $\text{key}_2(v) = \text{nil}$, a message is sent upwards in the HPT and the first key_2 node w with $b(v) \sqsubset \text{key}_2(w)$ responds to v . Then, $\text{key}_2(v)$ is set to $\text{key}_2(w)$. Eventually, v takes over the key_2 value of w , because w executes case a).

Intuitively, key_2 nodes without a key_2 pull values from nodes with shorter label. Simultaneously, leaf nodes without an r value present their label towards the root.

4 Protocol Analysis

In this section, we show that SHPT is self-stabilizing and transforms the HPT in finite time to a legal state. Furthermore, we present results concerning memory usage and the number of hash table accesses and messages when the HPT is in a legal state.

4.1 Correctness

We begin by showing the correctness of our self-stabilizing protocol. We use a commonly known technique introduced by Dijkstra in [7]. Our goal is to show Theorem 4. For that we consider a sequence of intermediate states that are reached consecutively until the HPT is in a legal state. For every state we show *convergence* towards the state and *closure* within it, i.e., the properties of the state are kept by our protocol.

► **Theorem 4.** *The algorithm creates in finite time a hashed Patricia Trie in a legal state out of any initial state in which the DHT is in a legal state and there is a set of unique keys stored at DHT nodes.*

In the following, we briefly sketch the main proof by presenting a sequence of main lemmas that roughly reflect the states the system reaches. Each main lemma thereby consists of multiple properties that are proven by a set of lemmas on its own. The full proof consisting of all lemmas, their respective proofs, and the complete definition of a legal state of the HPT can be found in Appendix B.

To prove the correctness captured in Theorem 4, we first need to formally define a legal state of the HPT. At this point, we only give an intuitive definition. For the complete definition, see Appendix B Theorem 7. Intuitively, the HPT is in a legal state if we have as few HPT nodes as possible in the system, all keys are stored correctly, the structure is consistent to the (modified) definition presented in Section 2, and the references to keys in key_2 nodes are existing and stored at correct nodes.

Initially, we only assume that a set of unique keys is stored at DHT nodes. The first lemma states that general repair mechanisms assure correctly stored keys and Patricia nodes.

► **Lemma 1.** *In finite time it holds: Every key k is stored in a node $v \in PAT$ with $b(v) = k$. Furthermore, every node is stored at the DHT node responsible for it. Consider any $v \in HPT$ that is deleted. As long as v is not reconstructed, in finite time it holds:*

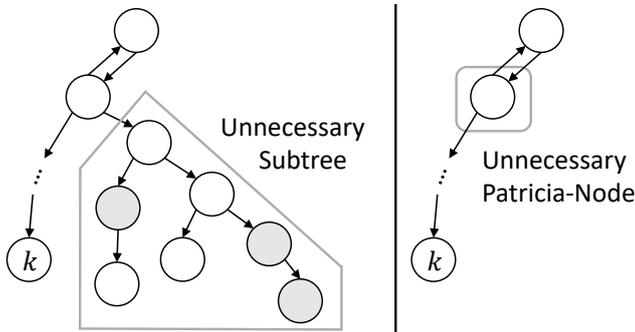
- a) *There is no presentation message for $b(v)$.*
- b) *There is no edge pointing towards $b(v)$ in the system.*

From now on, the proof consists of three phases. In a first phase, all Patricia nodes which are not needed for the final structure are removed. The second phase considers the reconstruction of the binary tree structure of the HPT and corrects the sets of Patricia nodes and Msd nodes. In the third and last phase, information stored in key_2 and r fields is made consistent.

4.1.1 Phase I – Deletion of Patricia nodes

In this phase, the protocol makes sure that all Patricia nodes which are not needed in the final structure are removed. Initially, information stored at HPT nodes that directly contradicts the definition of the HPT is cleared. This can be information such as a parent edge at $v \in HPT$ that is no suffix of $b(v)$. After that, Patricia nodes and Msd nodes in unnecessary subtrees, i.e., subtrees not containing a key, and unnecessary inner Patricia nodes are gradually removed. Every leaf node in an unnecessary subtree detects in finite time that it has no valid children and is deleted.

► **Lemma 2.** *In finite time, every unnecessary Patricia node is removed. A Patricia node v is unnecessary if there are no two keys k_1 and k_2 with $b(v) = lcp(k_1, k_2)$.*



■ **Figure 7** Node k stores a key. Msd nodes are sketched in grey. First, unnecessary subtrees are deleted (left), then remaining unnecessary Patricia nodes are removed (right).

Patricia nodes which are necessary may still be deleted because of their local perspective. However, this deletion is limited and stops after finitely many deletions. This holds, because Patricia nodes are only deleted due to incorrect child edges. If a new Patricia node with a long label is inserted, its child edges are initially valid and stay valid. There cannot be infinitely many deletions triggered, because the structure stabilizes bottom-up.

► **Lemma 3.** *In finite time, every Patricia node has valid child edges pointing to Patricia nodes and no further Patricia node is deleted.*

4.1.2 Phase II – Reconstruction

In the second phase, SHPT reconstructs the HPT by rebuilding missing Patricia nodes and repairing connections. Since every node tries to create a parent edge pointing to a Patricia node with shorter label, eventually all missing Patricia nodes are detected and can be inserted. The process works in a bottom-up fashion, i.e., Patricia nodes with longer labels

reconstruct missing nodes with shorter ones. The Patricia nodes storing a key as well as the root node act as fixed points in this case, because they are never deleted once constructed.

► **Lemma 4.** *In finite time, the root node exists and no Patricia node points to an Msd node. Furthermore, missing Patricia nodes are reconstructed. Also, every Patricia node has valid edges pointing only to existing Patricia nodes, i.e., there is a path from every Patricia node to the root and there is a path from the root to every Patricia node.*

It is crucial that no Patricia node points to an Msd node, because edges to Msd nodes are effectively treated as corrupt ones. This property assures that Msd nodes are eventually excluded from the Linearization procedure. Linearization then allows us to show that every Branch Set (see Theorem 3) of Patricia nodes eventually forms a stable sorted list. Incorrect Msd nodes are removed without affecting the rest of the HPT and missing Msd nodes are inserted. Further, correct Msd nodes are not deleted, because the two Patricia nodes closest to a correct Msd node are not deleted and do not change their edges any more. All these properties are reflected in Lemma 5. For completeness, we refer to the definition of incorrect and missing Msd nodes in the full proof in Appendix B.

► **Lemma 5.** *In finite time for every Branch Set S it holds: Between every pair of closest Patricia nodes $u, w \in S$ there is a bidirectional edge. Furthermore, every incorrect Msd node is removed and all missing Msd nodes are inserted.*

4.1.3 Phase III – Consistency

In the final phase the information stored in key_2 and r fields is corrected to be consistent. Due to Fact 1, we know that this can be achieved. The root is allowed to store up to two key_2 values. Therefore, there is always a way to store all keys of leaf nodes in key_2 nodes. First, we show that nodes which should not store a key_2 value remove any such stored value. Further, references in key_2 and r fields are deleted when they contradict the relationship $r(key_2(v)) = b(v)$, where v is a key_2 nodes and $key_2(v)$ references a leaf node.

► **Lemma 6.** *In finite time, only key_2 nodes store a key_2 and only leaf nodes store an r value. Every key_2 value stored at a Patricia node v points to a leaf w with $b(v) \sqsubset b(w)$ and every r value stored at a Patricia node w points to a key_2 node v with $b(v) \sqsubset b(w)$.*

From now on, key_2 nodes not storing a key_2 try to acquire the key_2 of a key_2 node above them. Leaf nodes lacking a reference in r present themselves to key_2 nodes above them. Therefore, the length of the longest label of a key_2 node not storing a staying key_2 reduces over time. As this length is finite, the process terminates. Thereafter, the r values of leaf nodes are corrected, because the key_2 values do not change any more.

► **Lemma 7.** *In finite time, all key_2 nodes store a stable key_2 and all leaf nodes store a stable r value. For every key_2 node v , the node w with $b(w) = key_2(v)$ is a leaf node with $r(w) = b(v)$.*

Finally, our protocol is correct as all unnecessary nodes are removed, missing nodes are inserted, Patricia nodes are connected by bidirectional edges, and the information stored in key_2 and r fields is consistent such that the HPT is in a legal state in finite time.

4.2 Overhead

Assume, the HPT is in a legal state. We give results for the complexity in terms of hash table accesses and messages and the memory overhead of our solution. We refer to Appendix C

for the proofs of the following theorems. When a DHT node executes SHPT by calling its TIMEOUT Method, exactly one HPT node is checked. Thereby, at most a constant number of other HPT nodes may be partially acquired or notified and Theorem 5 holds.

► **Theorem 5.** *When the HPT is in a legal state, SHPT creates a constant number of hash table (read) accesses and messages per call of TIMEOUT at each DHT node.*

Unnecessary Patricia nodes and incorrect Msd nodes are removed by SHPT. Therefore, the HPT nodes are the same as presented in the construction in Section 2 and Theorem 6 holds.

► **Theorem 6.** *Let d be the number of bits needed to store all keys. The total memory used by a HPT in a legal state is in $\Theta(d)$ bits.*

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A Pseudocode

In the following, we present the Pseudocode for SHPT. For simplicity, we assume the following functions to exist.

Function Name	Effect when called
DHT-SEARCH($x \in \{0, 1\}^*$)	Returns a copy of the HPT node v with $b(v) = x$.
DHT-INSERT(x)	Stores the data-item x in the DHT.
PARENT($v \in \text{HPT}$)	Provides the bit string x such that $x \circ p_-(v) = b(v)$.
CHILDREN($v \in \text{HPT}$)	Returns the number of children of v .
EDGE($w \in \text{HPT}, v \in \text{HPT}$)	Provides the bit string of the edge e of w that should point to v if v was w 's child node.
MSDMISSING($w \in \text{PAT}, v \in \text{PAT}$)	Returns <i>true</i> if there is a missing Msd node between w and v .
BINARYPREFIXSEARCH($x \in \{0, 1\}^*$)	Executes the binary search phase of the PrefixSearch(x) algorithm provided by [14] and returns a copy of the found Patricia node w with $b(w) \sqsubset x$.
MSDCHILDEGE($v \in \text{MSD}$)	Provides the only child edge of an Msd node. Returns <i>nil</i> if v has more than one child. An ordinary Msd node only has one child edge.
MSDLABEL($x \in \{0, 1\}^*, y \in \{0, 1\}^*$)	Returns a valid Msd label between the labels x and y .
BIDIRECTIONAL($v \in \text{HPT}, w \in \text{HPT}$)	Returns <i>true</i> if there is a bidirectional edge between v and u .

■ **Table 1** Assumed existing functions

Algorithm 1 The DHT protocol \rightarrow executed at DHT node n

```

1: procedure TIMEOUT
2:    $K \leftarrow$  set of keys not in Patricia nodes stored at  $n$ 
3:   if  $K \neq \emptyset$  then ▷ Integrate keys
4:     | Insert Patricia node  $v$  for every key  $k \in K$  with  $b(v) = k$ 
5:    $v \leftarrow$  next HPT node stored at  $n$ 
6:   if  $n$  not responsible for  $b(v)$  then ▷ Check position in DHT
7:     | delegate  $v$  towards the correct DHT node
8:   CHECKNODEINFO( $v$ ) ▷ Consistency of stored information
9:   CHECKPARENTEGEINFO( $v$ ) ▷ Check parent edge
10:  CHECKCHILDEGEINFO( $v$ ) ▷ Check child edges
11:  CHECKVALIDITY( $v$ ) ▷ Check if node is needed
12:  CHECKKEY2INFO( $v$ ) ▷ Check key2 information
13:  LINEARIZETIMEOUT( $v$ ) ▷ Do Linearization

```

Algorithm 1 calls different checks for one HPT node v on a regular basis.

Line(s)	Affected Lemmas
2 – 4	Lemma 8
6 – 7	Lemma 9

■ **Table 2** Lemmas affected by Algorithm 1

Algorithm 2 Checking information stored directly at a HPT node

```

1: procedure CHECKNODEINFO( $v \in \text{HPT}$ )
2:   if  $p_-(v)$  is not suffix of  $b(v)$  then                                ▷ Parent edge violates definition
3:      $p_-(v) \leftarrow \text{nil}$ 
4:   if  $p_0(v)$  does not comply  $0(0|1)^*$  then                            ▷ Child edge violates definition
5:      $p_0(v) \leftarrow \text{nil}$ 
6:   if  $p_1(v)$  does not comply  $1(0|1)^*$  then                            ▷ Child edge violates definition
7:      $p_1(v) \leftarrow \text{nil}$ 
8:   if  $v \notin \text{MSD}$  then
9:     if  $\text{key}(v) \neq \text{nil}$  and  $b(v) \neq \text{key}(v)$  then                    ▷ Wrong label
10:       $w \leftarrow \text{new Patricia node}$ 
11:       $\text{key}(w) \leftarrow \text{key}(v)$ 
12:      delete  $v$ 
13:      DHT-INSERT( $w$ )
14:     else
15:       if  $\text{key}_2 \neq \text{nil}$  then
16:         if  $b(v) \not\subseteq \text{key}_2(v)$  or ( $\text{CHILDREN}(v) < 2$  and  $v \neq \text{root}$ ) then
17:            $\text{key}_2(v) \leftarrow \text{nil}$                                        ▷  $v$  not a  $\text{key}_2$  node
18:         if  $r(v) \neq \text{nil}$  and  $r(v) \not\subseteq b(v)$  then
19:            $r(v) \leftarrow \text{nil}$                                            ▷ Wrong reference
20:         if  $r(v) \neq \text{nil}$  and ( $p_0(v) \neq \text{nil}$  or  $p_1(v) \neq \text{nil}$ ) then
21:            $r(v) \leftarrow \text{nil}$                                            ▷  $v$  not a leaf node

```

Algorithm 2 locally checks if the information stored at HPT node v is consistent. The algorithm removes information at v which cannot be correct.

Line(s)	Affected Lemmas
2 – 3	Lemma 11
4 – 5	Lemma 11
6 – 7	Lemma 11
8 – 13	Lemma 8, Lemma 15
15 – 17	Lemma 11, Lemma 24, Lemma 25
18 – 19	Lemma 11, Lemma 24, Lemma 25
20 – 21	Lemma 24

■ **Table 3** Lemmas affected by Algorithm 2

Algorithm 3 Checking parent edge of $v \in \text{HPT}$

```

1: procedure CHECKPARENTEGEINFO( $v \in \text{HPT}$ )
2:   if  $p_-(v) = \text{nil}$  and  $b(v) \neq \varepsilon$  and  $v \notin \text{MSD}$  then
3:      $w \leftarrow \text{BINARYPREFIXSEARCH}(b(v))$ 
4:     if  $w = \text{nil}$  then ▷ Root does not exist
5:        $\text{root} \leftarrow \text{new Patricia node}$ 
6:        $b(\text{root}) \leftarrow \varepsilon$ 
7:        $\text{DHT-INSERT}(\text{root})$ 
8:     else
9:        $p_-(v) \leftarrow x$  [s.t.  $b(w) \circ x = b(v)$ ] ▷ New parent
10:  else if  $b(v) \neq \varepsilon$  and  $v \notin \text{MSD}$  then
11:     $\text{par} \leftarrow \text{DHT-SEARCH}(\text{PARENT}(v))$  ▷ Acquire parent
12:    if  $\text{par} = \text{nil}$  or  $\text{par} \in \text{MSD}$  then
13:       $p_-(v) \leftarrow \text{nil}$ 
14:    else ▷ Compare edges
15:       $e_{\text{par}} \leftarrow \text{EDGE}(\text{par}, v)$ 
16:      if  $e_{\text{par}} \neq p_-(v)$  and  $e_{\text{par}} \not\sqsubseteq p_-(v)$  and  $p_-(v) \not\sqsubseteq e_{\text{par}}$  then
17:         $n \leftarrow \text{new Patricia node}$ 
18:         $b(n) \leftarrow \text{lcp}(b(v), b(\text{par}) \circ e_{\text{par}})$  ▷ Node between  $v$  and its parent
19:         $n' \leftarrow \text{DHT-SEARCH}(b(n))$ 
20:        if  $n \neq \text{nil}$  and  $n \notin \text{MSD}$  then
21:           $n' \leftarrow \text{LINEARIZE}(v)$  ▷ Present  $v$ 
22:        else
23:           $n \leftarrow \text{MULTILINEARIZE}(\{v, \text{par}, \text{par} \circ e_{\text{par}}\})$  ▷ Create  $n$  in DHT
24:           $\text{DHT-INSERT}(n)$ 
25:        else
26:          if  $v, \text{par} \in \text{PAT}$  and  $e_{\text{par}} = p_-(v)$  then
27:            if  $\text{MSDMISSING}(v, \text{par})$  then
28:               $m \leftarrow \text{new Msd node}$  ▷ Insert Msd node
29:               $b(m) \leftarrow \text{MSDLABEL}(\text{par}, v)$ 
30:               $m \leftarrow \text{MULTILINEARIZE}(\{v, \text{par}\})$ 
31:               $m' \leftarrow \text{DHT-SEARCH}(b(m))$ 
32:              if  $m' = \text{nil}$  or ( $m'$  has different edges than  $m$ 
33:                and  $m' \in \text{MSD}$ ) then
34:                 $\text{DHT-INSERT}(m)$  ▷ Overwrite with correct Msd node

```

Algorithm 3 checks if the parent edge of the HPT node v is correct. If necessary, the edge is corrected, a root is created or new nodes are inserted.

Line(s)	Affected Lemmas
4 – 7	Lemma 16
8 – 9	Lemma 17
12 – 13	Lemma 10, Lemma 16, Lemma 17, Lemma 18
16 – 24	Lemma 13, Lemma 17, Lemma 19
26 – 34	Lemma 22, Lemma 23

■ **Table 4** Lemmas affected by Algorithm 3

Algorithm 4 Checking child edges of $v \in \text{HPT}$

```

1: procedure CHECKCHILDEGEINFO( $v \in \text{HPT}$ )                                ▷ Check each child
2:   if  $v \in \text{PAT}$  then
3:     CHECKCHILD( $v$ , DHT-SEARCH( $b(v) \circ p_0(v)$ ), 0)
4:     CHECKCHILD( $v$ , DHT-SEARCH( $b(v) \circ p_1(v)$ ), 1)
5: procedure CHECKCHILD( $v$ ,  $c \in \text{HPT}$ ,  $x \in \{0, 1\}$ )
6:   if  $c = \text{nil}$  or  $c \in \text{MSD}$  then
7:      $p_x(v) = \text{nil}$                                                        ▷ Child non-existing or Msd node

```

Algorithm 4 checks if the child edges of v point to valid Patricia nodes.

Line(s)	Affected Lemmas
6 – 7	Lemma 10, Lemma 14, Lemma 18

■ **Table 5** Lemmas affected by Algorithm 4

Algorithm 5 Checking whether $v \in \text{HPT}$ should exist

```

1: procedure CHECKVALIDITY( $v \in \text{HPT}$ )
2:   if  $v \in \text{MSD}$  then
3:     if  $p_-(v) \neq \text{nil}$  and  $\text{MSDCHILDEGE}(v) \neq \text{nil}$  then
4:        $p \leftarrow \text{DHT-SEARCH}(\text{PARENT}(v))$ 
5:        $c \leftarrow \text{DHT-SEARCH}(b(v) \circ \text{MSDCHILDEGE}(v))$ 
6:       if not ( $\text{BIDIRECTIONAL}(p, c)$  and  $p, c \in \text{PAT}$ 
7:         and  $b(v) = \text{MSDLABEL}(b(p), b(c))$ ) then
8:         delete  $v$  ▷ Incorrect Msd node
9:       else
10:      delete  $v$  ▷ No parent edge or not one child edge
11:   else
12:     if  $\text{key}(v) = \text{nil}$  and  $\text{CHILDREN}(v) < 2$  and  $v \neq \text{root}$  then
13:       delete  $v$  ▷ Unnecessary Patricia node

```

Algorithm 5 checks if v should exist at all. In case that v lacks child edges, it is removed.

Line(s)	Affected Lemmas
3 – 8	Lemma 15, Lemma 22, Lemma 23
9 – 10	Lemma 12, Lemma 15, Lemma 22, Lemma 23
12 – 13	Lemma 12, Lemma 13, Lemma 14, Lemma 15, Lemma 16

■ **Table 6** Lemmas affected by Algorithm 5

Algorithm 6 Checking if $v \in \text{HPT}$ has valid key_2 information

```

1: procedure CHECKKEY2INFO( $v \in \text{HPT}$ )
2:   if  $v \in \text{PAT}$  then
3:     if  $\text{CHILDREN}(v) = 2$  then ▷ Inner node
4:       if  $\text{key}_2(v) \neq \text{nil}$  then
5:          $k \leftarrow \text{DHT-SEARCH}(\text{key}_2(v))$ 
6:         if  $k = \text{nil}$  or  $k \in \text{MSD}$  or  $\text{CHILDREN}(k) > 0$  or  $b(v) \sqsubset r(k)$  then
7:            $\text{key}_2(v) \leftarrow \text{nil}$  ▷  $v$  should not store  $k$ 
8:         else if  $r(k) = \text{nil}$  or  $r(k) \sqsubset b(v)$  then
9:            $r(k) \leftarrow b(v)$  ▷  $v$  references  $k$ 
10:        if  $\text{key}_2(v) = \text{nil}$  then
11:           $k \leftarrow$  next Patricia node above  $v$  such that  $b(v) \sqsubset \text{key}_2(k)$ 
12:           $\text{key}_2(v) \leftarrow \text{key}_2(k)$ 
13:        else if  $\text{CHILDREN}(v) = 0$  then ▷ leaf node
14:          if  $r(v) \neq \text{nil}$  then
15:             $k \leftarrow \text{DHT-SEARCH}(r(v))$ 
16:            if  $k = \text{nil}$  or  $\text{CHILDREN}(k) < 2$  or  $\text{key}_2(k) \neq b(v)$  then
17:               $r(v) \leftarrow \text{nil}$  ▷  $k$  not suitable
18:            else if  $\text{key}_2(k) = \text{nil}$  then
19:               $\text{key}_2(k) \leftarrow b(v)$  ▷ Repair reference
20:          else
21:             $k \leftarrow w \in \text{PAT}$  with  $\text{CHILDREN}(w) = 2$ ,  $\text{key}_2(w) \in \{\text{nil}, b(v)\}$ 
22:            above  $v$  with  $|b(w)|$  maximal
23:             $r(v) \leftarrow k$  ▷ Find node for  $r(v)$ 
24:             $\text{key}_2(k) \leftarrow b(v)$ 

```

Algorithm 6 checks if the key_2/r information stored at v is valid. If necessary, a new key_2 or r value is obtained.

Line(s)	Affected Lemmas
6 – 7	Lemma 25, Lemma 26
8 – 9	Lemma 24, Lemma 26
10 – 12	Lemma 25, Lemma 26
16 – 17	Lemma 25, Lemma 27
18 – 19	Lemma 24, Lemma 25
20 – 24	Lemma 24, Lemma 26

■ **Table 7** Lemmas affected by Algorithm 6

For better readability, we denote by $v \leftarrow \text{LINEARIZE}(u)$ a call of $\text{LINEARIZE}(v, u)$ at the DHT node responsible for v . Similar, $v \leftarrow \text{MULTILINEARIZE}(U)$ translates to $\text{MULTILINEARIZE}(v, U)$ at the DHT node responsible for v .

Algorithm 7 Linearization methods at DHT node n

```

1: procedure LINEARIZETIMEOUT( $v \in \text{HPT}$ )
2:   if  $v \in \text{PAT}$  then
3:     PARENT( $v$ )  $\leftarrow$  LINEARIZE( $b(v)$ )
4:      $b(v) \circ p_0(v) \leftarrow$  LINEARIZE( $v$ )
5:      $b(v) \circ p_1(v) \leftarrow$  LINEARIZE( $v$ )
6: procedure MULTILINEARIZE( $v \in \text{HPT}, U \in \text{HPT}^*$ )
7:   for each  $u \in U$  do
8:     LINEARIZE( $v, u$ )
9: procedure LINEARIZE( $v, u \in \text{HPT}$ ) ▷ Present  $u$  to  $v$ 
10:  if not ( $v \in \text{MSD}$  or  $u \in \text{MSD}$ ) and  $v \neq u$  then
11:    if  $|\ell_{cp}(b(u), b(v))| < |b(v)|$  then
12:      if  $p_-(v) = \text{nil}$  and  $b(u) \sqsubseteq b(v)$  then ▷  $u$  above  $v$ 
13:         $p_-(v) \leftarrow x$  [where  $b(u) \circ x = b(v)$ ]
14:      else
15:        if PARENT( $v$ )  $\neq b(u)$  then
16:          PARENT( $v$ )  $\leftarrow$  LINEARIZE( $u$ )
17:        if PARENT( $v$ )  $\sqsubset b(u)$  and  $b(u) \sqsubset b(v)$  then ▷  $u$  in between
18:           $p_-(v) \leftarrow x$  [where  $b(u) \circ x = b(v)$ ]
19:      else if  $b(v) \sqsubset b(u)$  then ▷  $v$  above  $u$ 
20:         $x \leftarrow b(u)$  at position  $|b(v)| + 1$ 
21:         $c \leftarrow b(v) \circ p_x(v)$  ▷ Respective child edge
22:        if  $p_x(v) = \text{nil}$  then
23:           $p_x(v) \leftarrow y$  [where  $b(v) \circ y = b(u)$ ]
24:        else
25:          if  $b(c) \sqsubset b(u)$  then ▷  $c$  above  $u$ 
26:             $c \leftarrow$  LINEARIZE( $u$ )
27:          else if  $b(u) \sqsubset b(c)$  then ▷  $c$  below  $u$ 
28:             $p_x(v) \leftarrow y$  [where  $b(v) \circ y = b(u)$ ]
29:             $c \leftarrow$  LINEARIZE( $u$ )
30:          else ▷ Common parent for  $c, u$  needed
31:             $u \leftarrow$  LINEARIZE( $v$ )

```

Algorithm 7 denotes the Linearization procedure of our protocol. A Patricia node v tries to maintain connections to the closest Patricia nodes of the form w where either $b(v) \sqsubset b(w)$ or $b(w) \sqsubset b(v)$. Presentations of other Patricia nodes are delegated and Msd nodes are not presented nor do they delegate messages.

B Correctness Proof

In the following, a complete correctness proof follows. Assuming that the underlying DHT is in a legal state, we show that our protocol converges the system to a state in which the hashed Patricia Trie is in a legal state with respect to Theorem 7 and our modification as presented in Section 2. Further, once such a legal state for the HPT is reached, closure holds and the HPT stays legal in consecutive states. For this, we show a sequence of lemmas, each representing properties that the system converges to and that are preserved.

In the proofs, we often define *potential functions*. We denote by a potential function a function depending on the current system state of which the value decreases over time, i.e., in following states, and is not increased. Often, our potential functions depend on a set of elements. For convenience, we assume that as soon as the corresponding set is empty, the function is defined to be zero. In some cases, there may be messages in the system that present a non-existent HPT node. These messages may temporarily increase our potential functions. However, on the long term, they are removed from the system. Therefore, we often implicitly assume in our arguments that this has happened and ignore temporary increases of potential functions. We aim at proving Theorem 4 with respect to the complete definition of a legal state of the HPT below.

► **Theorem 4.** *The algorithm creates in finite time a hashed Patricia Trie in a legal state out of any initial state in which the DHT is in a legal state and there is a set of unique keys stored at DHT nodes.*

► **Definition 7 (Legal state of the HPT).** We say a HPT is in a legal state, if the following holds. Only Patricia nodes given by the following conditions exist. Every $k \in \text{KEYS}$ is stored in a Patricia node v with $b(v) = k$. For every pair of keys k_1, k_2 , there exists an inner Patricia node v with $b(v) = \text{lcp}(k_1, k_2)$. Only Msd nodes given by the following exist: Between every pair u, v of closest Patricia nodes, there is exactly one Msd node m (if needed) with $b(m)$ as given by Theorem 2, if $b(u) \sqsubset b(m) \sqsubset b(v)$. m has a parent edge to u and exactly one child edge to v . In general, no HPT node v exists with $\text{key}(v) = \text{nil}$ and there is no $k \in \text{KEYS}$ such that $b(v) \sqsubseteq k$. Also, every HPT node is stored at the DHT node responsible for it.

The edges of the HPT need to be correct as defined in the following. For every Branch Set S , there are bidirectional valid edges between every pair of closest Patricia nodes of S . Every leaf node $v \in \text{PAT}$ stores $r(v) = w$ for a $w \in \text{PAT}$ with two children, such that $\text{key}_2(w) = v$ and $b(w) \sqsubseteq b(v)$. No other nodes than those affected by this statement store a value in r or key_2 .

► **Lemma 1.** *In finite time it holds: Every key k is stored in a node $v \in \text{PAT}$ with $b(v) = k$. Furthermore, every node is stored at the DHT node responsible for it. Consider any $v \in \text{HPT}$ that is deleted. As long as v is not reconstructed, in finite time it holds:*

- a) *There is no presentation message for $b(v)$.*
- b) *There is no edge pointing towards $b(v)$ in the system.*

Proof. The proof of the lemma is given by the proofs of Lemma 8, Lemma 9, Theorem 8 and Lemma 10. ◀

► **Lemma 8.** *Let $\phi_1 = |\{k \in \text{KEYS} \mid k \text{ is not stored in a } v \in \text{PAT}, \text{ or } b(v) \neq k\}|$ be the number of keys which are stored by a Patricia node of wrong label. ϕ_1 is a potential function.*

Proof. Consider any key k not stored in a Patricia node. In finite time, k is detected and stored as a Patricia node by the respective DHT node n (see Algorithm 1 Line 4). Let $k \in \text{KEYS}$ be a key stored at $v \in \text{PAT}$ with $b(v) \neq k$. In finite time our algorithm performs a check of v detecting that $b(v) \neq k$. As a result, k is reinserted in a node $w \in \text{PAT}$ with $b(w) = k$ such that ϕ_1 is reduced (see Algorithm 2 Line 9). Our protocol does not insert a key k at node v where $b(v) \neq k$. Hence, ϕ_1 cannot increase. ◀

We call a HPT node *searchable* if it is stored at the DHT node responsible for it and can be retrieved by DHT-SEARCH.

► **Lemma 9.** *In finite time every HPT node is searchable.*

Proof. Consider any node $v \in \text{HPT}$ stored at a DHT node n which is not responsible for v . In finite time, our protocol checks if v is stored correctly in the DHT (see Algorithm 1 Line 7). As this is not the case, v is moved to the corresponding DHT node and becomes searchable. ◀

From Lemma 8 and Lemma 9 it directly follows:

► **Corollary 8.** *In finite time, every $k \in \text{KEYS}$ is searchable.*

► **Lemma 10.** *Consider any $v \in \text{HPT}$ that is deleted. As long as v is not reconstructed, in finite time it holds:*

- a) *There is no presentation message for $b(v)$.*
- b) *There is no edge pointing towards $b(v)$ in the system.*

Proof.

a) Since v does not exist and is not reconstructed, no new presentation messages for $b(v)$ are generated. Since the DHT is in a legal state, every message containing $b(v)$ eventually arrives at its destination DHT node. Furthermore, a delegation in the HPT only happens finitely often, because there is only a finite number of nodes and a presentation is not delegated in circles. This directly follows from the applied Linearization procedure. For details on Linearization, we refer to [16]. Eventually, a presentation is processed and perhaps a new edge towards v in the HPT is created. In any case the number of messages is reduced.

b) From a) it follows that in finite time no presentation messages concerning $b(v)$ are present. Let $w \in \text{HPT}$ be a node pointing to v . In finite time, w checks the corresponding edge and determines that v is non-existing (see Algorithm 3 Line 12 and Algorithm 4 Line 6). The edge is deleted and will never be rebuilt, because there are no more presentation messages concerning v . ◀

Phase I – Deletion of Patricia nodes

► **Lemma 2.** *In finite time, every unnecessary Patricia node is removed. A Patricia node v is unnecessary if there are no two keys k_1 and k_2 with $b(v) = \text{lcp}(k_1, k_2)$.*

Proof. The lemma is correct by Lemma 11, Lemma 12 and Lemma 13. ◀

► **Lemma 11.** *Consider the number of initially wrong reference information*

$$\begin{aligned} \phi_2 = & |\{v \in \text{HPT} \mid \nexists x \in \{0, 1\}^* \text{ such that } x \circ p_-(v) = b(v)\}| \\ & + |\{v \in \text{HPT} \mid p_0(v) \text{ does not comply } 0(0|1)^*\}| \\ & + |\{v \in \text{HPT} \mid p_1(v) \text{ does not comply } 1(0|1)^*\}| \\ & + |\{v \in \text{PAT} \mid r(v) \neq \text{nil and } r(v) \not\sqsubseteq b(v)\}| \\ & + |\{v \in \text{PAT} \mid \text{key}_2 \neq \text{nil and } b(v) \not\sqsubseteq \text{key}_2(v)\}|. \end{aligned}$$

ϕ_2 is a potential function.

Proof. All the information types gathered in ϕ_2 are checked locally and without acquiring other nodes. When some of the types appear, the corresponding edges are set to *nil*, reducing ϕ_2 (see Algorithm 2 Line 2, Line 4, Line 6, Line 16, Line 18). The protocol never stores information in a way as defined by ϕ_2 , such that closure holds. ◀

► **Lemma 12.** *Let $\phi_3 = \max_{v \in \mathcal{U}} |b(v)|$, where:*

$$\begin{aligned} \mathcal{U} = & \{v \in \text{HPT} \mid \text{key}(v) = \text{nil and } \nexists w \in \text{PAT} \\ & \text{with } \text{key}(w) \neq \text{nil and } b(v) \sqsubseteq b(w)\}. \end{aligned}$$

\mathcal{U} is the set of all HPT nodes in subtrees that can safely be deleted. ϕ_3 is a potential function.

Proof. Let $\mathcal{T} = \{v \in \mathcal{U} \mid |b(v)| = \phi_3\}$. We show that $|\mathcal{T}|$ is reduced and never increased as long as ϕ_3 stays the same. When $|\mathcal{T}|$ reaches zero, ϕ_3 is reduced and \mathcal{T} changes. We show that ϕ_3 does not increase and follow that ϕ_3 is a potential function.

Consider a $v \in \mathcal{T}$. If $v \in \text{MSD}$, then v has no existing child. Else, the child of v would have a longer label than v , such that $v \notin \mathcal{T}$. v will be deleted in finite time by the protocol (see Algorithm 5 Line 10). In the other case, let $v \in \text{PAT}$. Again, v has no existing child since $|b(v)| = \phi_3$ and any child would be a member of \mathcal{U} with a longer label. According to Lemma 10, any child edge of v will be deleted in finite time. Afterwards, v deletes itself since $\text{key}(v) = \text{nil}$ (see Algorithm 5 Line 12). In any case, $|\mathcal{T}|$ is reduced. Also, $|\mathcal{T}|$ and ϕ_3 will never be increased, because a node u can only be created by a node w where $b(u) \sqsubset b(w)$. The existence of w would contradict the maximality of \mathcal{T} concerning ϕ_3 . ◀

► **Lemma 13.** *Let $\phi_4 = |\mathcal{F}|$, where*

$$\begin{aligned} \mathcal{F} = & \{v \in \text{PAT} \mid \text{key}(v) = \text{nil and } \nexists u, w \in \text{PAT} \\ & \text{with } \text{key}(u), \text{key}(w) \neq \text{nil} \\ & \text{and } b(v) = \text{lcp}(b(u), b(w))\}. \end{aligned}$$

\mathcal{F} is the set of all remaining Patricia nodes that are not needed in the HPT. ϕ_4 is a potential function.

Proof. Consider a node $v \in \mathcal{F}$. We have that $v \notin \mathcal{U}$ as \mathcal{U} is empty in finite time due to Lemma 12. Hence, v has at least one non-existing child node w which will never be created. This holds, because by definition there is no $u \in \text{PAT}$ with $\text{key}(u) \neq \text{nil}$ and $b(w) \sqsubseteq b(u)$. According to Lemma 10, the edge to w will be deleted at v in finite time such that v deletes itself resulting in a reduction of ϕ_4 (see Algorithm 5 Line 12). A new Patricia node v is only created if there are two Patricia nodes u, w with $\text{key}(u), \text{key}(w) \neq \text{nil}$ and $b(v) = \text{lcp}(b(u), b(w))$ (see Algorithm 3). This results in $v \notin \mathcal{F}$ and therefore, ϕ_4 is not increased. ◀

► **Lemma 3.** *In finite time, every Patricia node has valid child edges pointing to Patricia nodes and no further Patricia node is deleted.*

Proof. The lemma holds due to the correctness of Lemma 14 and Lemma 15. ◀

► **Lemma 14.** *Let \mathcal{N} be the set of nodes, defined by their labels, which do not exist in the HPT. Let $\phi_5 = \max_{v \in \mathcal{D}} |b(v)|$, where*

$$\begin{aligned} \mathcal{D} = & \{v \in \text{MSD} \cup \mathcal{N} \mid \text{there is a } v \in \text{PAT} \text{ with } b(w) \sqsubseteq b(v) \text{ that points to } v\} \\ & \cup \{m \in \text{MSG} \mid b(m) = b(v), v \in \text{MSD} \cup \mathcal{N} \cup \mathcal{L}\} \cup \mathcal{L} \\ \mathcal{L} = & \{v \in \text{PAT} \mid \text{key}(v) = \text{nil} \text{ and } p_0(v) = \text{nil} \text{ or } p_1(v) = \text{nil}\} \end{aligned}$$

\mathcal{L} is the set of Patricia nodes that are unnecessary in their local perspective. \mathcal{D} is the set of nodes and messages that may result in a deletion of a Patricia node. ϕ_5 is a potential function.

Proof. Similar to the proof of Lemma 12, we define $\mathcal{T} = \{e \in \mathcal{D} \mid |b(e)| = \phi_5\}$ to be the set of all elements in \mathcal{D} of longest label. We will show that as long as ϕ_5 stays the same, $|\mathcal{T}|$ is reduced and never increases. When $|\mathcal{T}|$ reaches zero, ϕ_5 is decreased and \mathcal{T} contains a new set of elements. We also show that ϕ_5 is never increased and follow that ϕ_5 is a potential function. For $e \in \mathcal{S}$, we distinguish the following cases:

- a) $e \in \text{MSD} \cup \mathcal{N}$ and there is a finite set \mathcal{A} of Patricia nodes which have a child edge pointing to e . Consider any Patricia node v out of this set. Either e is overwritten by a Patricia node, v is presented a Patricia node (see Algorithm 7) or v 's respective child edge $p_x(v)$ is checked in finite time. In the first case, e is no longer in \mathcal{T} such that $|\mathcal{T}|$ reduced. In the other two cases, $p_x(v)$ is set to *nil* (see Algorithm 4 Line 6). As $|b(v)| < |b(e)|$ and $|\mathcal{A}|$ is finite, this implies a reduction of $|\mathcal{T}|$ in finite time.
- b) $e \in \text{MSG}$ with $b(e) = b(v)$ for some node v which does not exist or is an Msd node. If a new Patricia node v is created, we argue below that this node is not in \mathcal{T} . Further, e is no longer in \mathcal{T} in this case such that $|\mathcal{T}|$ reduced. Else, due to Lemma 10, e vanishes in finite time and may only result in one element in \mathcal{T} for which case a) applies.
- c) $e \in \text{MSG}$ with $b(e) = b(v)$ and v is a Patricia node in \mathcal{L} . Then $v \in \mathcal{T}$ and case d) applies to v .
- d) $e \in \mathcal{L}$. Either e is presented an existing Patricia node (see Algorithm 7), or e does not contribute to the system in its local view. If e is presented an existing Patricia node, $|\mathcal{T}|$ reduces and all messages presenting e are no longer in \mathcal{T} . If e does not contribute to the system in its local view, it will be deleted in finite time (see Algorithm 5 Line 12) and $|\mathcal{T}|$ reduces.

In all cases $|\mathcal{T}|$ is eventually reduced. It is left to show that ϕ_5 is not increased and $|\mathcal{T}|$ is not increased. Both could be increased if a Patricia node changes its child edge, if a node is inserted or if a Patricia node is deleted.

Assume a node $v \in \text{PAT}$ changes its child edge such that ϕ_5 or $|\mathcal{T}|$ increases. If v deletes its child edge, the edge pointed to a node n which was either non-existing or an Msd node. As the deletion increased ϕ_5 or $|\mathcal{T}|$, $|b(n)|$ was greater than ϕ_5 which is a contradiction because $n \in \mathcal{D}$. If v changes its edge, let m be the responsible message presenting $|b(w)| \geq \phi_5$. It holds $m \in \mathcal{D}$. As m was processed when v 's child edge was created, $|\mathcal{T}|$ did not increase. An increase of ϕ_5 poses a contradiction, because m was already in \mathcal{D} .

Consider the insertion of a node. Assume the creation of an Msd node m with $|b(m)| \geq \phi_5$. Msd nodes are not presented and no Patricia node is overwritten by an Msd node.

Hence, the existence of a node $v \in \text{PAT}$ above m that has a child edge to m implies that v previously had a child edge to a non-existing node n with $|b(n)| = |b(m)|$. If ϕ_5 increased due to m , this poses a contradiction. Also, $|\mathcal{T}|$ was not increased because n does not count towards $|\mathcal{T}|$ after insertion of m .

Assume the creation of a Patricia node v with $|b(v)| \geq \phi_5$. v has been created, because there are two Patricia nodes u, w with $b(v) = \text{lcp}(b(u), b(w))$. By definition $u, w \notin \mathcal{T}$ since their labels are longer than $|b(v)| \geq \phi_5$. Assume w created v . Then w is a Patricia node. v could only be in \mathcal{D} if there was a Patricia node above v having a child edge pointing to a node $m \in \text{MSD} \cup \mathcal{N}$. v has in this case both m and w as initial children such that $|b(m)| > |b(v)|$. Since $m \in \mathcal{D}$ this implies that $|b(v)| < \phi_5$ resulting in a contradiction.

Consider the deletion of a node v . The deletion of an Msd node does not influence ϕ_5 or $|\mathcal{T}|$, therefore assume $v \in \text{PAT}$. If $v \in \mathcal{D}$, then $|b(v)| \leq \phi_5$ and the deletion of v does not increase ϕ_5 or $|\mathcal{T}|$. Assuming that ϕ_5 or $|\mathcal{T}|$ increased, $v \notin \mathcal{D}$. Thus, v either stores a key or has two existing nodes as children. This implies that v is not deleted posing a contradiction (see Algorithm 5 Line 12).

Hence, ϕ_5 and $|\mathcal{T}|$ do not increase and decreases over time which proves the lemma. ◀

► **Lemma 15.** *In finite time no further Patricia nodes will be deleted during stabilization and all messages concerning non-existing Patricia nodes vanished.*

Proof. Due to Lemma 14, in finite time every Patricia node falls in one of the following categories:

- a) Leaf nodes storing a key.
- b) Inner Patricia nodes storing a key.
- c) Inner Patricia nodes with two existing Patricia nodes as child nodes.

For each of these categories it holds, that the conditions leading to deletion are not fulfilled (deletions happen in Algorithm 2 Line 9 and Algorithm 5 Line 8, Line 10, Line 12). Furthermore, no Patricia nodes which do not fall into one of the categories will be created by the protocol, because we proved closure of the state of Lemma 14. Lemma 10 assures that all messages concerning non-existing nodes vanish in finite time. ◀

Phase II – Reconstruction

► **Lemma 4.** *In finite time, the root node exists and no Patricia node points to an Msd node. Furthermore, missing Patricia nodes are reconstructed. Also, every Patricia node has valid edges pointing only to existing Patricia nodes, i.e., there is a path from every Patricia node to the root and there is a path from the root to every Patricia node.*

Proof. We prove the lemma by proving Lemma 16, Lemma 17, Theorem 9, Lemma 18, Lemma 19 and Lemma 20. ◀

► **Lemma 16.** *In finite time, a root node exists.*

Proof. Assume that there is no root node. Our first observation is, that there must be a node $v \in \text{PAT}$ with shortest label. Hence, either $p_-(v) = \text{nil}$ or v has a parent node w which is non-existent or an Msd node. In the latter case, the edge $p_-(v)$ is checked and set to nil in finite time (see Algorithm 3 Line 12). If $p_-(v) = \text{nil}$, v checks in finite time for a parent using `BINARYPREFIXSEARCH`. The resulting node is nil , such that v restores the root node (see Algorithm 3 Line 4). Furthermore, the root node is never deleted (see Algorithm 5 Line 12). ◀

► **Lemma 17.** *Let $\phi_6 = \max_{k \in KEYS} |k| - \min_{v \in \mathcal{P}} |b(v)|$, where:*

$$\mathcal{P} = \{v \in PAT \mid p_-(v) = nil \text{ or } \nexists u \in PAT, b(u) \circ p_-(v) = b(v)\}.$$

\mathcal{P} is the set of all Patricia nodes which do not have an existing Patricia node as parent node. ϕ_6 is a potential function, i.e., in finite time every Patricia node has a parent edge to an existing Patricia node.

Proof. First we observe that $\max_{k \in KEYS} |k| \geq |b(v)|$ for all $v \in PAT$ such that $\phi_6 \geq 0$. We define $\mathcal{T} = \{v \in \mathcal{P} \mid \max_{k \in KEYS} |k| - |b(v)| = \phi_6\}$. We will show that $|\mathcal{T}|$ decreases and never increases as long as ϕ_6 stays the same. When $|\mathcal{T}|$ reaches zero, ϕ_6 is reduced. ϕ_6 is never increased and it follows that ϕ_6 is a potential function.

Consider a Patricia node $v \in \mathcal{T}$. If the node u with $b(u) \circ p_-(v) = b(v)$ does not exist or is an Msd node, the value of $p_-(v)$ is deleted in finite time by the protocol (see Algorithm 3 Line 12). If $p_-(v) = nil$, v will be presented a Patricia node w with $b(w) \sqsubset b(v)$ either due to a presentation message or due to a check of v (see Algorithm 7 and Algorithm 3 Line 9), because at least the root node exists as stated in Lemma 16. In any case, $|\mathcal{T}|$ has reduced.

Furthermore, $|\mathcal{T}|$ and ϕ_6 do not increase. Consider any Patricia node v with $\max_{k \in KEYS} |k| - |b(v)| \geq \phi_6$. If the edge $p_-(v)$ changes, v received a presentation message from an existing Patricia node w with $b(w) \sqsubset b(v)$. This holds as non-existing nodes and Msd nodes are not presented and, due to Lemma 15, no Patricia node is deleted any more.

If a new Patricia node v is created it node is initialized with edges provided by a Patricia node w with $b(v) \sqsubset b(w)$ (see Algorithm 3 Line 16). $v \notin \mathcal{P}$ holds, because w only creates a node if it has an existing Patricia node as parent such that $w \notin \mathcal{P}$ holds.

Hence, ϕ_6 does not increase and is reduced over time. ◀

► **Corollary 9.** *In finite time, there exists a path from every node to the root.*

The corollary follows from Lemma 17 and Lemma 15.

► **Lemma 18.** *In finite time, no Patricia node has an edge pointing to an Msd node.*

Proof. Due to Lemma 15, no Patricia node v not storing a key which has a child edge pointing to an Msd node exists. Else, v would delete itself which is a contradiction. Lemma 17 assures that no Patricia node has a parent edge pointing to an Msd node. Furthermore, every Patricia node storing a key which points to an Msd node will delete the respective edge in finite time (see Algorithm 3 Line 12 and Algorithm 4 Line 6). Patricia nodes do not create edges pointing to Msd nodes (see Algorithm 7). When a Patricia node v is created, it is inserted between $u \in PAT$ and $w \in PAT$ with initial edges to u and w . Additionally, v has another child node p which was formerly a child of u and is thus no Msd node. Therefore, newly inserted Patricia nodes do not point to Msd nodes as well. ◀

► **Lemma 19.** *In finite time, every Patricia node can be reached over a path starting at the root.*

Proof. We already know that every Patricia node has a path to the root node in finite time. The correctness of Lemma 19 follows as we use a technique called Linearization [16]. In [16], the authors show that their technique creates in finite time a sorted list. We apply Linearization for every path consisting of Patricia nodes from the root to a Patricia node storing a key. If Patricia nodes are missing at positions where two Branch Sets collide, such nodes are inserted (see Algorithm 3 Line 16). As stated in Lemma 18, no Patricia node

points to an Msd node, so we can ignore Msd nodes when considering Linearization. It is left to show that insertion of Patricia nodes does not harm the Linearization process. Consider the case when a Patricia node v is inserted between $u \in \text{PAT}$ and $w \in \text{PAT}$. Initially, v has valid edges to u and w such that the connectivity between them is not destroyed. ◀

► **Lemma 20.** *Let $\phi_7 = |\mathcal{M}|$ where:*

$$\begin{aligned} \mathcal{M} = \{v \in \text{PAT} \mid \nexists v \text{ and } \exists u, w \in \text{PAT} \\ \text{with } \text{key}(u), \text{key}(w) \neq \text{nil} \\ \text{and } b(v) = \text{lcp}(b(u), b(w))\}. \end{aligned}$$

\mathcal{M} denotes the set of all Patricia nodes that are needed in the HPT but currently non-existing. ϕ_7 is a potential function.

Proof. Assume that there are paths from every node to the root and from the root to every node as stated in Theorem 9 and Lemma 19. Additionally, let $\phi_7 > 0$. Consider a node $v \in \mathcal{M}$. For the nodes $u, w \in \text{PAT}$ with $\text{key}(u), \text{key}(w) \neq \text{nil}$ and $b(v) = \text{lcp}(b(u), b(w))$ it holds that they can be reached over a path starting from the root and the root can be reached over a path starting at u or w . This implies either the existence of v or an edge at a Patricia node pointing to a non-existent node or an Msd node. In all cases we have a contradiction.

According to Lemma 15, no deletions happen any more. Hence, ϕ_7 is not increased. ◀

► **Lemma 5.** *In finite time for every Branch Set S it holds: Between every pair of closest Patricia nodes $u, w \in S$ there is a bidirectional edge. Furthermore, every incorrect Msd node is removed and all missing Msd nodes are inserted.*

Proof. The lemma holds by the correctness of Lemma 21, Lemma 22, Lemma 23 and Theorem 11. ◀

► **Lemma 21.** *In finite time for every Branch Set S it holds: Between every pair of closest Patricia nodes $u, w \in S$ there is a bidirectional edge.*

Proof. Consider any Branch Set S . Lemma 19 and Theorem 9 assure that a weak connectivity is given for the set of Patricia nodes in S . Observe that due to Lemma 18, Msd nodes do not influence the stabilization of S . Furthermore, Lemma 15 and Lemma 20 assure that no more Patricia nodes are deleted or created such that S does not change. All Patricia nodes in S perform a Linearization procedure derived from the proposition in [16]. Therefore, in finite time, a sorted list is created for all nodes in S . This implies the lemma. ◀

► **Lemma 22.** *We call an Msd node m incorrect, if m does not have a parent edge to $u \in \text{PAT}$, m does not have a child edge to $w \in \text{PAT}$, u and w are not connected by a bidirectional edge, or $b(m)$ has incorrect length (see Theorem 2). Let \mathcal{I} be the set of incorrect Msd nodes in the system. $\phi_8 = |\mathcal{I}|$ is a potential function.*

Proof. We observe that the set of incorrect Msd nodes does not increase after every Branch Set consists of bidirectional edges. This holds as new Msd nodes are only created if there is a bidirectional edge between two Patricia nodes u, w (see Algorithm 3 Line 27). If such an edge exists, then an inserted Msd node m is correct by the correctness of the creation. If m became incorrect, this would mean a Patricia node was inserted between u and v . This

is a contradiction to the assumption that the Branch Set of u and w only had bidirectional edges when m was created.

Consider an Msd node $m \in \mathcal{I}$. m is checked in finite time and is determined to be incorrect (see Algorithm 5 Line 8, Line 10) and deleted. Thus, $|\mathcal{I}|$ is reduced.

As the set of incorrect Msd nodes is finite, it follows that all incorrect Msd nodes are removed in finite time. \blacktriangleleft

► **Definition 10.** Consider $u, w \in \text{PAT}$ with $b(u) \sqsubset b(w)$ and $\nexists v \in \text{PAT}$ such that $b(u) \sqsubset b(v) \sqsubset b(w)$. We call an Msd node m of correct form (as given by Theorem 2) between u and w *missing* if m does not exist.

► **Lemma 23.** *In finite time all missing Msd nodes are created.*

Proof. Consider two Patricia nodes $u, w \in \text{PAT}$ with $b(u) \sqsubset b(w)$ where an Msd node m is missing. This means that there is no $v \in \text{PAT}$ between u and w . Due to Lemma 21, there is a bidirectional edge between u and w . When w is checked, it inserts m in finite time (see Algorithm 3 Line 27). According to Lemma 15, no more deletions happen. In addition, m has a parent edge to $u \in \text{PAT}$ and a child edge to $w \in \text{PAT}$, the edge between u and w is bidirectional and $b(m)$ has correct length. Therefore, m is not deleted (see Algorithm 5 Line 8, Line 10), the number of missing Msd nodes does not increase and Lemma 23 follows. \blacktriangleleft

► **Corollary 11.** *In finite time the structure of the HPT is rebuilt.*

Theorem 11 follows directly as Patricia nodes are connected by bidirectional edges, no incorrect Msd nodes or unnecessary Patricia nodes exist, and all missing Msd nodes have been created.

Phase III – Consistency

► **Lemma 6.** *In finite time, only key_2 nodes store a key_2 and only leaf nodes store an r value. Every key_2 value stored at a Patricia node v points to a leaf w with $b(v) \sqsubset b(w)$ and every r value stored at a Patricia node w points to a key_2 node v with $b(v) \sqsubset b(w)$.*

Proof. The lemma is correct by the correctness of Lemma 24 and Lemma 25. \blacktriangleleft

► **Lemma 24.** *In finite time, only key_2 nodes store a key_2 and only leaf nodes store an r value.*

Proof. Consider a Patricia node v which is no key_2 node but stores a key_2 . In finite time, v is checked and sets $\text{key}_2(v) = \text{nil}$ (see Algorithm 2 Line 16). No $v \in \text{PAT}$ which is not a key_2 node starts to store a key_2 due to our protocol (see Algorithm 6 Line 18, Line 22).

Consider a Patricia node v which is not a leaf node but stores an r value. v sets $r(v) = \text{nil}$ in finite time (see Algorithm 2 Line 18, Line 20). No inner node stores an r value based on the protocol (see Algorithm 6 Line 8, Line 13). \blacktriangleleft

► **Lemma 25.** *In finite time, every key_2 value stored at $v \in \text{PAT}$ refers to a leaf node below v and every r value stored at $w \in \text{PAT}$ refers to a key_2 node above w .*

Proof. According to Lemma 24, in finite time only inner nodes with two children store key_2 values and only leaf nodes store r values. When a node $v \in PAT$ detects that $key_2(v)$ is not the label of a leaf node below it, such a reference is deleted (see Algorithm 2 Line 16 and Algorithm 6 Line 6). Also, when a leaf node v detects that $r(v)$ is not the label of a key_2 node above it, $r(v)$ is set to nil (see Algorithm 2 Line 18 and Algorithm 6 Line 16). Nodes which are not key_2 nodes are not considered when propagating keys and only labels of leaf nodes are propagated using parent edges only (see Algorithm 6 Line 13, Line 18). Also, key_2 nodes only acquire key_2 values that point to leaf nodes below them (see Algorithm 6 Line 10). Therefore, the state described in the lemma is reached. ◀

► **Lemma 7.** *In finite time, all key_2 nodes store a stable key_2 and all leaf nodes store a stable r value. For every key_2 node v , the node w with $b(w) = key_2(v)$ is a leaf node with $r(w) = b(v)$.*

Proof. The lemma follows from Lemma 26 and Lemma 27. ◀

► **Lemma 26.** *Let $\phi_9 = \max_{v \in \mathcal{K}} |b(v)|$, where:*

$$\begin{aligned} \mathcal{K} = & \{v \in PAT \mid \text{CHILDREN}(v) = 2 \text{ or } v = \text{root}\} \\ & \cap \{v \in PAT \mid key_2(v) = nil \text{ or } \exists w \in PAT : key_2(w) = key_2(v), b(v) \sqsubset b(w)\}. \end{aligned}$$

\mathcal{K} is the set of all key_2 nodes not storing a key_2 that stays. ϕ_9 is a potential function.

Proof. Let $\mathcal{T} = \{v \in \mathcal{K} \mid |b(v)| = \phi_9\}$. We will show that $|\mathcal{T}|$ is reduced and never increases as long as ϕ_9 stays the same. Further, ϕ_9 does not increase. When $|\mathcal{T}|$ reaches zero, ϕ_9 is reduced. As a result, the lemma follows.

Consider $v \in \mathcal{T}$. If $key_2(v) \neq nil$, then there has to be a node w with $key_2(w) = key_2(v)$ and $b(v) \sqsubset b(w)$. Since $v \in \mathcal{T}$, exactly one such node w exists. In finite time, w checks $u := key_2(w)$ and sets $r(u) = w$ (see Algorithm 6 Line 8). Afterwards, v detects that $b(v) \sqsubset r(u)$ and sets $key_2(v) = nil$ (see Algorithm 6 Line 6). So, assume $key_2(v) = nil$. Due to Fact 1, there is a leaf node u below v , such that either (a) no inner node stores $b(u)$ as a key_2 , or (b) some key_2 node w with $b(w) \sqsubset b(v)$ has $key_2(w) = b(u)$. Let w be maximal considering $|b(w)|$. In (a) it holds that $b(u)$ is in finite time presented to v , because $b(v) \sqsubset b(u)$ and every node between v and u already stores a key_2 (see Algorithm 6 Line 22). In (b) $b(w) \sqsubset b(v) \sqsubset b(u)$ holds and a message from v reaches w in finite time such that v sets $b(u)$ as key_2 (see Algorithm 6 Line 10). In any case, $|\mathcal{T}|$ reduces.

No node v with $|b(v)| > \phi_9$ deletes its $key_2(v)$, because this only happens if there is a node w with $b(v) \sqsubset b(w)$ and $key_2(w) = key_2(v)$ which is not the case by definition. Therefore, ϕ_9 is never increased. Since no node with a legitimate key_2 drops this value, no deletions happen (Lemma 15), and there is no node $v \in \mathcal{K}$ with $|b(v)| > \phi_9$, $|\mathcal{T}|$ is never increased until it reaches zero. Hence, the lemma follows. ◀

► **Lemma 27.** *Let $\phi_{10} = |\mathcal{R}|$, where:*

$$\mathcal{R} = \{v \in PAT \mid \text{CHILDREN}(v) = 0, r(v) = nil \text{ or } key_2(r(v)) \neq b(v)\} \setminus \{\text{root}\}.$$

\mathcal{R} denotes the set of leaf nodes without an r value that stays stable. ϕ_{10} is a potential function.

Proof. Assume that we reached a state where every inner node has a stable key_2 (Lemma 26). Consider a node $v \in \mathcal{R}$. If $r(v) \neq \text{nil}$ and $\text{key}_2(r(v)) \neq b(v)$, then v sets $r(v) = \text{nil}$ in finite time (see Algorithm 6 Line 16). If $r(v) = \text{nil}$, then there must be a node w with $\text{key}_2(w) = b(v)$. This node will correct $r(v)$ in finite time to $r(v) = w$ leading to a reduction of ϕ_{10} (see Algorithm 6 Line 16). Furthermore, no node $v \in \text{PAT}$ with $\text{CHILDREN}(v) = 0$ and $\text{key}_2(r(v)) = b(v)$ will delete $r(v)$ and the key_2 values stay fixed according to the closure of Lemma 26. Hence, ϕ_{10} is never increases. Fact 1 assures that \mathcal{R} is empty in finite time. ◀

C Overhead Proof

► **Theorem 5.** *When the HPT is in a legal state, SHPT creates a constant number of hash table (read) accesses and messages per call of TIMEOUT at each DHT node.*

Proof. On a call of TIMEOUT, exactly one HPT node is checked per DHT node. This HPT node v has at most three edges provided by $p_-(v)$, $p_0(v)$ and $p_1(v)$, and one reference provided by either $key_2(v)$ or $r(v)$. In total, no more than four nodes may be (partially) acquired using the DHT. Furthermore, Linearization presents v to at most three other nodes. When the HPT is in a legal state, no reinsertion, no presenting of $b(v)$ as a key_2 value and no searching of other key_2 values is done. Hence, we have $\Theta(1)$ created hash table accesses and messages at a DHT node. ◀

► **Theorem 6.** *Let d be the number of bits needed to store all keys. The total memory used by a HPT in a legal state is in $\Theta(d)$ bits.*

Proof. In [14] it was mentioned that the HPT needs $\Theta(d)$ memory space if $d = \sum_{k \in \text{KEYS}} |k|$ is the number of bits needed to store all keys. The modifications we made do not change the number of Patricia nodes and Msd nodes. Our protocol rebuilds the structure of the HPT and deletes every unnecessary Patricia node as well as every incorrect Msd node (see Lemma 13 and Lemma 22). Thus, the rebuilt HPT has an asymptotically optimal memory demand of $\Theta(d)$ bits. ◀