# Hennessy-Milner Properties via Topological Compactness 

Jim de Groot ( $\boxtimes$ ), Dirk Pattinson<br>School of Computing<br>The Australian National University<br>Canberra, Australia


#### Abstract

We give Hennessy-Milner classes for intuitionistic, dual-intuitionistic and biintuitionistic logic interpreted in intuitionistic Kripke models, and generalise these results to modal (dual- and bi-)intuitionistic logics. Our main technical tools are image-compact and pre-image-compact relations that provide a semantical description of modal saturation properties.


Keywords: Bisimulation, Hennessy-Milner property, Intuitionistic logic, Bi-intuitionistic logic, Modal logic

## 1. Introduction

Bisimulations play a crucial role in the model theory of modal logic as the canonical notion of semantic equivalence: bisimilar worlds necessarily satisfy precisely the same formulae. If the converse is also true, the (usually finitary) logical language is powerful enough to describe the (typically infinitary) semantics: this is the so-called Hennessy-Milner property [1].

Bisimulations were introduced in [2] to characterise normal modal logic over a classical base as the bisimulation-invariant fragment of first-order logic. Independently, they arose in the field of computer science as an equivalence relation between process graphs [3, 4], and as extensional equality in non-wellfounded set theory [5].

By and large, the Hennessy-Milner property is well understood for normal modal logic over a classical base, where it is known to hold for all modally saturated models, see Section 2 of [6]. In the realm of (dual- and bi-)intuitionistic logic and their modal extensions, much less is known. Some explorations are made in (7] where the Hennessy-Milner property is established for intuitionistic propositional logic, interpreted over intuitionistic Kripke models [8], and in [9], where a Hennessy-Milner property is given for tense intuitionistic logic where

[^0]all modalities are interpreted using a single additional relation. Besides, 10] contains Hennessy-Milner results for strict-weak languages, and 11] discusses a Hennessy-Milner result for unimodal extensions of positive, intuitionistic and bi-intuitionistic logic.

In this paper we aim to derive Hennessy-Milner properties for a large variety of logics using the notion of image-compactness. A relation is image-compact if its successor sets of a single points are compact in a topology that includes all truth sets of formulae as clopens. Similar methods have previously been used in the setting of normal modal logic over a classical base 12 and unimodal logic over a positive, intuitionistic and bi-intuitionistic base 11. Our results apply to intuitionistic, dual-intuitionistic, and bi-intuitionistic propositional logic, as well as their extension with normal modal operators. Moreover, we can use them to obtain new Hennessy-Milner type results for various logics previously studied, notably modal intuitionistic, and tense bi-intuitionistic logic.

Technically, we show that logical equivalence and bisimulations coincide for image-compact Kripke models, and obtain a (known) characterisation for intuitionistic propositional logic. We then dualise the semantics to obtain the same result for dual-intuitionistic logic, which is the extension of positive logic with a binary subtraction arrow < residuated with respect to disjunction. While this may seem like a mathematical curiosity at first, subtraction has found multiple applications. In computer science it can be used to describe control mechanisms such as co-routines [13] and in philosophy the subtraction arrow provides a tool to reason about refutation [14, 15].

Thereafter, we merge the results for intuitionistic propositional logic and its dual to obtain a characterisation of bisimulation for bi-intuitionistic logic (which can be viewed as the union of intuitionistic and dual-intuitionistic logic) in terms of logical equivalence. Bi-intuitionistic logic is also known as subtractive logic [13] and Heyting-Brouwer logic [16], and was introduced by Rauszer with Kripke semantics and a Hilbert calculus [17]. We refer to 18] for an excellent overview of the logic, that moreover clarifies some of Rauszer's confusions.

In a second step, we extend the underlying propositional languages with modal operators that are interpreted like Božić and Došen did in [19], where $\square$ and $\diamond$ are a priori unrelated modalities. Our approach is similar to the propositional case: a Hennessy-Milner theorem for intuitionistic propositional logic augmented with $\square$ gives, by duality, an analogous theorem for dual-intuitionistic logic with $\diamond$, and both can be combined to get the same for bi-intuitionistic logic, extended with an arbitrary number of $\square$ and $\diamond$-operators.

Finally, we apply our results to obtain new Hennessy-Milner theorems for a large variety of logics studied in the literature. These fall into two classes: various flavours of intiutionistic modal logic 20, 21, 22, 23] and various flavours of tense bi-intuitionistic logic [24, 25, 26].

Structure of the Paper. In Section 2 we recall intuitionistic Kripke frames and models as semantics for intuitionistic, dual-intuitionistic and bi-intuitionistic logic. We give the definition of general frames and use these to define the notions of image-compactness and pre-image-compactness. Subsequently, in Section 3 ,
we show how one can relate the relations of logical equivalence for different languages, borrowing a simple observation from the theory of institutions.

Bisimulations between intuitionistic Kripke models are defined in Section 4 and the notions of (pre-)image-compactness are shown to give rise to HennessyMilner type results for (bi- and dual-)intuitionistic logic.

In Section 5 we extend our scope to modal extensions of the previously studied logics. We give a suitable notion of frame and model and define bisimulations between them. Again, the notions of (pre-)image-compactness give rise to Hennessy-Milner results. We then specialise these results to obtain HennessyMilner theorems for a number of logics studied in the literature in Section 6

Finally, in Section 7 we detail how in some cases image-compactness coincides with notions of saturation, and in Section 8 we suggest several avenues for further research.

Related Work. As mentioned above, in [10] the author proves Hennessy-Milner type theorems for strict-weak languages. Amongst such languages are intuitionistic logic, where implication is viewed as a strict arrow, dual-intuitionistic logic, modelling subtraction as a weak arrow, and bi-intuitionistic logic. In fact, the framework in op. cit. allows one to add as many such arrows as desired. The strict and weak arrows are interpreted using a relation in the same way implication and subtraction are interpreted (see Section 2 below). Moreover, every arrow gives rise to a box- or diamond-like modality via $\square \phi:=\top \rightarrow \phi$ and $\diamond \phi:=\phi \leftharpoondown \perp$, where $\leftharpoondown$ denotes a weak arrow. However, boxes and diamonds are not defined separately. This means that, when proving that some relation satisfies the back-and-forth conditions of a bisimulation, one can always make use of the arrows interpreted via each relation in the frame. This simplifies the proof of Hennessy-Milner results, because each clause resembles the proof of [7, Theorem 21] or Theorem 4.3 below, or its dual. In Section 5 of the current paper, dealing with normal modal extensions of (bi- and dual-)intuitionistic logic, we do not have this luxury.

In (11] the author considers modal extensions of positive, intuitionistic and bi-intuitionistic logic. Moreover, the relation used to interpret the modalities is not required to interact with the underlying partial order at all. The level of generality forces to author to obtain a Hennessy-Milner theorem via a duality, because the potential absence of implication or subtraction arrow frustrates a more direct approach like in [7, Theorem 21] or [6, Proposition 2.54]. By cleverly extending the duality to a dual adjunction, a slightly larger HennessyMilner class is derived. However, the models it contains are still based on prePriestley spaces. In our setting we begin with (bi- or dual-)intuitionistic logic, so that we always have an arrow in our language. Furthermore, the relations we use to interpret additional modal operators are required to satisfy certain coherence conditions with respect to the pre-order underlying a frame. These extra constraints allow us to derive a stronger Hennessy-Milner result.

Finally, in [9] the author derives a Hennessy-Milner theorem for tense intuitionistic logic. This is a bit farther removed from our research, because the underlying intuitionistic logic is interpreted in topological spaces, rather than
the more restrictive intuitionistic Kripke frames (= Alexandrov spaces) used here. We discuss this setting as a potential avenue for further research in the conclusion.

Relation to Predecessor Paper. The current paper is an extension of preliminary results reported in 27]. Conceptually, we identify the core notion of image compactness as the key stepping stone in establishing Hennessy-Milner type theorems. Technically, this yields stronger results: in op. cit., we have established Hennessy-Miler type theorems for descriptive and finite models of bi-intuitionistic logic. Both are special cases of (pre-)image-compact models. Moreover, (pre-)image compact models are closed under disjoint unions whence closure under disjoint unions, reported in op. cit., is automatic, and all results follow from Theorem 4.10 below. Similarly, the results from Section 5 of [27] about descriptive and finite $\mathrm{Bi}-$ int $_{\square \diamond}$-models are subsumed by Theorem 5.15, again noting that image compactness subsumes both finiteness and being descriptive. Finally, the treatment of bisimulations for modal and epistemic intuitionistic, and tense bi-intuitionistic logic is new.

## 2. Intuitionistic Kripke Models and Image Compactness

We recall the Kripke semantics of intuitionistic, dual-intuitionistic and bi-intuitionistic propositional logic, and introduce the semantic notion at the heart of our results: image-compact relations. Throughout the paper, we write Prop for a (possibly infinite) set of propositional variables.
Definition 2.1. The language Bi-int(Prop) of bi-intuitionistic propositional logic over the set Prop of propositional variables is given by the grammar

$$
\phi::=\top|\perp| p|\phi \wedge \phi| \phi \vee \phi|\phi \rightarrow \psi| \phi \multimap \phi
$$

where $\rightarrow$ is intuitionistic implication and $\prec$ its dual, sometimes called subtraction.

The language $\operatorname{lnt}($ Prop $)$ of intuitionistic propositional logic is the set of $\prec-$ free bi-intuitionistic formulae, and the language Int $^{\partial}$ (Prop) consists of all implication-free formulae.

All three languages can be interpreted over intuitionistic Kripke models. These are simply pre-ordered sets, i.e., sets with a reflexive and transitive relation on them. If $(X, \leq)$ is a pre-order and $a \subseteq X$ then we write $\uparrow a=\{y \in X \mid x \leq$ $y$ for some $x \in a\}$ for the upwards closure of $a$, and for $x \in X$ we abbreviate $\uparrow x:=\uparrow\{x\}$. The set $a$ is called an upset if $\uparrow a=a$, and we write $\mathcal{U} p(X, \leq)$ for the collection of upsets of $(X, \leq)$.

Definition 2.2. An intuitionistic Kripke frame is a pre-ordered set ( $X, \leq$ ). An intuitionistic Kripke model is a triple $(X, \leq, V)$ where $(X, \leq)$ is a pre-order, and $V: \operatorname{Prop} \rightarrow \mathcal{U} p(X, \leq)$ is an upset-valued valuation.

The truth of bi-intuitionistic formulae in an intuitionistic Kripke model $\mathfrak{M}=$ $(X, \leq, V)$ at a world $x \in X$ is defined inductively by

$$
\begin{array}{rll}
\mathfrak{M}, x \Vdash \top & \text { always } \\
\mathfrak{M}, x \Vdash \perp & \text { never } \\
\mathfrak{M}, x \Vdash p & \text { iff } & x \in V(p) \\
\mathfrak{M}, x \Vdash \phi \wedge \psi & \text { iff } & x \Vdash \phi \text { and } x \Vdash \psi \\
\mathfrak{M}, x \Vdash \phi \vee \psi & \text { iff } & x \Vdash \phi \text { or } x \Vdash \psi \\
\mathfrak{M}, x \Vdash \phi \rightarrow \psi & \text { iff } & \text { for all } y \geq x, \text { if } y \Vdash \phi \text { then } y \Vdash \psi \\
\mathfrak{M}, x \Vdash \phi \prec \psi & \text { iff } & \text { there exists } y \leq x \text { such that } y \Vdash \phi \text { xand } y \Vdash \psi .
\end{array}
$$

We write $x{ }^{\longrightarrow \longrightarrow}$ Bi-int $x^{\prime}$ to denote that two states $x \in X$ and $x^{\prime} \in X^{\prime}$ of two intuitionistic Kripke models $\mathfrak{M}=(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$ are logically equivalent with respect to bi-intuitionistic propositional logic, i.e.,

$$
\mathfrak{M}, x \Vdash \phi \quad \text { iff } \quad \mathfrak{M}^{\prime}, x^{\prime} \Vdash \phi
$$

for all $\phi \in \operatorname{Int}$. The relations $\stackrel{r r}{ } \operatorname{lnt}^{2}$ and ${ }^{\mu m} \operatorname{lnt}^{a}$ are the relations of logical equivalence with respect to Int and $\operatorname{Int}^{\partial}$ are defined analogously. In an intuitionistic Kripke model $\mathfrak{M}=(X, \leq, V)$, we write $\llbracket \phi \rrbracket^{\mathfrak{M}}=\{x \in X \mid x \Vdash \phi\}$ for the truth set of $\phi$ in $\mathfrak{M}$.

If we define the operators $\rightarrow$, $\preceq: \mathcal{U} p(X, \leq) \times \mathcal{U} p(X, \leq) \rightarrow \mathcal{U} p(X, \leq)$ by
$a \longrightarrow b=\{x \in X \mid$ for all $y \in X$, if $x \leq y$ and $y \in a$ then $y \in b\}$
$a \preceq b=\{x \in X \mid$ there exists $y \leq x$ such that $y \in a$ and $y \notin b\}$
then evidently $\llbracket \phi \rightarrow \psi \rrbracket^{\mathfrak{M}}=\llbracket \phi \rrbracket^{\mathfrak{M}} \rightrightarrows \llbracket \psi \rrbracket^{\mathfrak{M}}$ and $\llbracket \phi \prec \psi \rrbracket^{\mathfrak{M}}=\llbracket \phi \rrbracket^{\mathfrak{M}} \preceq \llbracket \psi \rrbracket^{\mathfrak{M}}$ for any intuitionistic Kripke model $\mathfrak{M}$.

The logics Int, $\mathrm{Int}^{2}$, Bi -int are sometimes interpreted over posets (rather than pre-orders), for example in the predecessor paper of this one [27] and in 28]. Here, we choose the more general semantics.

The relationship between intuitionistic and dual-intuitionistic logic is best clarified in terms of dual models (with reversed order).

Definition 2.3. The dual of an intuitionistic Kripke model $\mathfrak{M}=(X, \leq, V)$ is the model $\mathfrak{M}^{\partial}=\left(X, \geq, V^{\partial}\right)$, where $V^{\partial}$ is defined by $V^{\partial}(p)=X \backslash V(p)$.

The notion of dual model is well defined, as the complement $X \backslash a$ of an upset $a$ in a pre-order $(X, \leq)$ is a downset, and hence an upset for the dual pre-order $(X, \geq)$. On the level of languages, we have a translation $(\cdot)^{t}: \operatorname{Int} \rightarrow \operatorname{Int}{ }^{\partial}$ such that $\phi$ is true at a state $x$ in a model $(X, \leq, V)$ if and only if its translation $\phi^{t}$ is false in the dual model. We define this inductively via

$$
\begin{aligned}
& \perp^{t}=\top \\
& \mathrm{T}^{t}=\perp \\
& p^{t}=p \\
& (\phi \wedge \psi)^{t}=\phi^{t} \vee \psi^{t} \\
& (\phi \vee \psi)^{t}=\phi^{t} \wedge \psi^{t} \\
& (\phi \rightarrow \psi)^{t}=\psi^{t} \prec \phi^{t} \\
& (\phi \prec \psi)^{t}=\psi^{t} \rightarrow \phi^{t}
\end{aligned}
$$

Clearly, $(\cdot)^{t}$ is an involution of Bi -int which restricts to translations $\operatorname{Int} \rightarrow \operatorname{Int}^{\partial}$ and Int ${ }^{\partial} \rightarrow$ Int.

Lemma 2.4. Let $\mathfrak{M}=(X, \leq, V)$ be an intuitionistic Kripke model and $\phi \in$ Bi-int be a formula. Then we have

$$
\mathfrak{M}, x \Vdash \phi \quad \text { iff } \quad \mathfrak{M}^{\partial}, x \Vdash \phi^{t} .
$$

Proof. This follows from a straightforward induction. We showcase one of the inductive steps:

$$
\begin{array}{lll}
\mathfrak{M}, x \Vdash \phi \rightarrow \psi & \text { iff } & \text { for all } y \geq x \text { either } \mathfrak{M}, y \Vdash \phi \text { or } M, y \Vdash \psi \\
& \text { iff } & \text { for all } y \geq x \text { either } \mathfrak{M}^{\partial}, y \Vdash \phi^{t} \text { or } \mathfrak{M}^{\partial}, y \Vdash \psi^{t} \\
& \text { iff } & \text { there is no } y \geq x \text { such that } \mathfrak{M}^{\partial}, y \Vdash \psi^{t} \text { and } \mathfrak{M}^{\partial}, y \Vdash \phi^{t} \\
& \text { iff } & \mathfrak{M}^{\partial}, x \Vdash \psi^{t} \prec \phi^{t}=(\phi \rightarrow \psi)^{t}
\end{array}
$$

All other cases are similar.
We now define image-compactness, the main technical vehicle that we use to establish Hennessy-Milner results in this paper. For this, we augment models with a collection of admissible subsets, that is, a selection of subsets of the carrier that includes all truth sets. This allows us to topologise the model using the patch topology, and use compactness to get a finitary handle on the successors of any given world.

Definition 2.5. A general model is a tuple $\mathfrak{M}=(X, \leq, V, A)$ such that $(X, \leq, V)$ is an intuitionistic Kripke model, $A \subseteq \mathcal{U} p(X, \leq)$ is a collection of up-closed subsets of ( $X, \leq$ ) that (i) is closed under finite union and finite intersection, and (ii) contains $\emptyset, X$ and $V(p)$ for every $p \in$ Prop.

We call $\mathfrak{M}$ a general Int-model (resp. Int ${ }^{\partial}$-model) if $A$ is moreover closed under $\rightarrow$ (resp. $\preceq$ ), and a general Bi-int-model if $A$ is closed under both $\rightarrow$ and $\preceq$.

The patch topology on a general model $\mathfrak{M}=(X, \leq, V, A)$ is the topology $\tau_{A}$ on $X$ generated by the (clopen) subbase $A \cup-A$, where $-A=\{X \backslash a \mid a \in A\}$.

What will be of special interest later are the compact subsets of a general model $\mathfrak{M}=(X, \leq, V, A)$. Recall that a subset $U \subseteq X$ is compact if every open cover $\left(O_{i}\right)_{i \in I}$ of $U$ (that is, $U \subseteq \bigcup\left\{O_{i} \mid i \in I\right\}$ and $O_{i} \in \tau_{A}$ for all $i \in I$ ) has a finite subcover (that is, there exists a finite $J \subseteq I$ such that $U \subseteq \bigcup\left\{O_{j} \mid j \in J\right\}$ ).

In particular, if $x \in X$ is a world in a model $(X, \leq, V, A)$, then bisimulation requires us to establish a property for all successors in $\mathfrak{M}$, i.e., for the set $\uparrow \leq x=\{y \in X \mid x \leq y\}$. If $\uparrow \leq x$ is compact, this can be achieved in a finitary way. This motivates the following definition of image-compactness.
Definition 2.6. An intuitionistic Kripke model $(X, \leq, V)$ is called (pre-) imagecompact for L (where $\mathrm{L} \in\left\{\operatorname{lnt}, \operatorname{lnt}{ }^{\partial}, \mathrm{Bi}-\mathrm{int}\right\}$ ) if there exists a set $A$ of admissibles such that $(X, \leq, A, V)$ is a general L-model and for all $x \in X$ the set $\uparrow \leq x$ (resp. $\downarrow \leq x$ ) is compact in the patch topology $\tau_{A}$.

Observe that, like saturation, (pre-)image-compactness is a property of models, rather than a property of frames. Furthermore, note that by definition of the patch topology, proposition letters are interpreted as clopen sets in this topology. We conclude the section with the following examples.

Example 2.7. 1. A Kripke model $\mathfrak{M}=(X, \leq, V)$ is image-finite if the set $\{y \in X \mid x \leq y\}$ is finite for every $x \in X$. Clearly every image-finite Kripke model is image-compact: take $A$ to be the collection of all upward closed subsets of $W$.
2. Image-compact is strictly more general than image-finite. Consider for example $X=\mathbb{N} \cup\{\infty\}$ where $n \leq \infty$ for all $n \in \mathbb{N} \cup\{\infty\}$ (and $\leq$ is as usual otherwise), with the valuation $V\left(p_{i}\right)=\{x \in X \mid i \leq x\}$, for $i \in \mathbb{N}$. Clearly, this is not image-finite. If we take $A$ to consist of all sets of the form $\{x \in X \mid x \geq n\}$ where $n$ ranges over $\mathbb{N}$, then this is easily seen to be image-compact.
3. Every descriptive intuitionistic Kripke frame [28, Section 8.4] is automatically image-compact. This follows because descriptive frames are precisely Esakia spaces [29], hence topologically compact, and upsets of single points are closed in this topology.
4. If $\mathfrak{M}=(X, \leq, V, A)$ is a general model, and $\mathfrak{M}^{\partial}=\left(X, \geq, V^{\partial}, A^{\partial}\right)$ is its dual where $A^{\partial}=\{X \backslash a \mid a \in A\}$, then $\mathfrak{M}$ is image-compact if and only if $\mathfrak{M}^{\partial}$ is pre-image-compact.

## 3. Relating Logical Equivalence for Different Logics

As this paper is concerned with many different logics, it is useful to structure the relationships between them. More precisely, we will often show that the relation of logical equivalence between two models is a bisimulation for a certain logic. The following simple fact, borrowed from the theory of institutions [30], allows us to transfer such results from one logic to another.

Let us abstractly define a semantics for a language $L$ to be a class of models M such that:

- Each $\mathfrak{M} \in \mathbb{M}$ has an underlying set, denoted by $\mathcal{U M}$; and
- Each model $\mathfrak{M} \in \mathbb{M}$ comes with a theory map $t_{\mathfrak{M}}: \mathcal{U} \mathfrak{M} \rightarrow \mathcal{P}$ that sends a state $x \in \mathcal{U M}$ to the collection of L-formulae true at that state. $(\mathcal{P L}$ denotes the powerset of L.)

The collection $\mathbb{M}$ may be regarded as a category and $\mathcal{U}$ as a functor $\mathbb{M} \rightarrow$ Set from $\mathbb{M}$ to the category of sets. However, we do not need this categorical perspective for our purposes.
Example 3.1. One can think of $L=\operatorname{lnt}$, with $\mathbb{M}$ the collection of intuitionistic Kripke models from Definition 2.2. Then for $\mathfrak{M}=(X, \leq, V) \in \mathbb{M}$, the underlying set is given by $\mathcal{U M}=X$ and the theory map is induced by the interpretation from Definition 2.2 via

$$
\text { th }_{\mathfrak{M}}: X \rightarrow \mathcal{P} \text { Int }: x \mapsto\{\phi \in \operatorname{Int} \mid x \Vdash \phi\} .
$$

It is easy to see that, mutatis mutandis, this yields semantics for $\mathrm{Int}^{\partial}$ and $\mathrm{Bi}-\mathrm{int}$ as well.

If we have sufficient coherence between two such logic, then logical equivalence of one implies logical equivalence of the other. The next lemma describes this in detail.

Lemma 3.2. Let $L_{1}$ and $L_{2}$ be two languages with semantics $\mathbb{M}_{1}$ and $\mathbb{M}_{2}$. Denote the underlying set of a model $\mathfrak{M} \in \mathbb{M}_{i}$ by $\mathcal{U}_{i} \mathfrak{M}$, and the theory of $x \in$ $\mathcal{U}_{i} \mathfrak{M}$ by $\operatorname{th}_{i}(\mathfrak{M})(x)$. Let

- $t: \mathrm{L}_{1} \rightarrow \mathrm{~L}_{2}$ is a surjective translation from $\mathrm{L}_{1}$ to $\mathrm{L}_{2}$; and
- $r: \mathbb{M}_{2} \rightarrow \mathbb{M}_{1}$ a transformation of models such that $\mathcal{U}_{1}(r \mathfrak{M})=\mathcal{U}_{2} \mathfrak{M}$ for all $\mathfrak{M} \in \mathbb{M}_{2}$.

Moreover, suppose that

$$
\begin{equation*}
t(\phi) \in t h_{2}(\mathfrak{M})(x) \quad \text { iff } \quad \phi \in \operatorname{th}_{1}(r \mathfrak{M})(x) \tag{1}
\end{equation*}
$$

for all $\phi \in \mathrm{L}_{1}$ and $\mathfrak{M} \in \mathbb{M}_{2}$ and $x \in \mathcal{U}_{2} \mathfrak{M}$. Then we have

$$
\begin{equation*}
\operatorname{th}_{2}(\mathfrak{M})(x)=\operatorname{th}_{2}\left(\mathfrak{M}^{\prime}\right)(y) \quad \text { iff } \quad \operatorname{th}_{1}(r \mathfrak{M})(x)=\operatorname{th}_{1}\left(r \mathfrak{M}^{\prime}\right)(y) \tag{2}
\end{equation*}
$$

for all $\mathfrak{M}, \mathfrak{M}^{\prime} \in \mathbb{M}_{2}$ and $x \in \mathcal{U}_{2} \mathfrak{M}$ and $y \in \mathcal{U}_{2} \mathfrak{M}^{\prime}$.
We omit the obvious proof. Observe that (22) simply says that two worlds are $L_{1}$-logically equivalent if and only if they are $L_{2}$-logically equivalent. Let us have a look at an example.

Example 3.3. Let $\mathrm{L}_{1}=$ Int and $\mathrm{L}_{2}=\mathrm{Bi}-\mathrm{int}$, both generated by the same set Prop of proposition letters. Since both can be interpreted in intuitionistic Kripke frames, there is an evident transformation $r: \mathbb{M}_{2} \rightarrow \mathbb{M}_{1}$, namely the identity on the class of intuitionistic Kripke models. If we let $t$ : Int $\rightarrow \mathrm{Bi}$-int be the obvious translation, then clearly (1) is satisfied. However, the translation $t$ is not surjective.

To overcome this, we can enrich Int with an additional proposition letter $p_{\phi}$ for every formula $\phi \in \mathrm{Bi}$-int that is not in Int. These can be interpreted by extending the valuation $V$ of an intuitionistic Kripke model $(X, \leq, V)$ via $V\left(p_{\phi}\right)=\llbracket \phi \rrbracket$, where the latter interpretation is given by the clauses in Definition 2.2. Denote this collection of additional proposition letters by Prop ${ }^{\prime}$. Then clearly the translation $t: \operatorname{Int}($ Prop $) \rightarrow$ Bi-int(Prop) extends to a surjective translation $t: \operatorname{Int}\left(\operatorname{Prop} \cup\right.$ Prop $\left.^{\prime}\right) \rightarrow$ Bi-int(Prop). Moreover, we still have an obvious transformation of models and (11) is satisfied. It follows that the relation of Bi-int(Prop)-logical equivalence between two intuitionistic Kripke models coincides with $\operatorname{Int}\left(\operatorname{Prop} \cup\right.$ Prop $\left.^{\prime}\right)$-logical equivalence.

More generally, if $L_{2}$ freely extends $L_{1}$ with one or more operators, then we can use this method to transfer properties of $L_{1}$-logical equivalence to $L_{2}$,
achieving surjectivity by adding a proposition letter $p_{\phi}$ to $L_{1}$ for every $L_{2}$-formula that is not already in $L_{1}$.

We use this as follows: Suppose we know that logical equivalence between certain models for $L_{1}$ is a bisimulation relation, and hence implies certain back-and-forth conditions. Then by the lemma the logical equivalence relation between models for $L_{2}$ coincides with $L_{1}$-logical equivalence, and therefore allows us to inherit the back-and-forth conditions.

## 4. Bisimulations

We begin this section by recalling the definition of bisimulation between Kripke models given in [7], and prove a Hennessy-Milner result. We then dualise this to obtain a corresponding result for dual intuitionistic logic. Taken together, both results imply the Hennessy-Milner property for bi-intuitionistic logic.

### 4.1. Bisimulations for Intuitionistic Logic

Definition 4.1. Let $\mathfrak{M}=(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$ be two intuitionistic Kripke models. An intuitionistic bisimulation or Int-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ is a relation $B \subseteq X \times X^{\prime}$ such that for all $\left(x, x^{\prime}\right) \in B$ we have:
$\left(B_{1}\right)$ For all $p \in \operatorname{Prop}, x \in V(p)$ iff $x^{\prime} \in V^{\prime}(p)$;
$\left(B_{2}\right)$ If $x \leq y$ then there exists $y^{\prime} \in X^{\prime}$ such that $x^{\prime} \leq^{\prime} y^{\prime}$ and $y B y^{\prime}$;
$\left(B_{3}\right) \quad$ If $x^{\prime} \leq^{\prime} y^{\prime}$ then there exists $y \in X$ such that $x \leq y$ and $y B y^{\prime}$.
Two states $x$ and $x^{\prime}$ are called Int-bisimilar if there is an Int-bisimulation linking them, notation: $x \rightleftharpoons_{\mathrm{Int}} x^{\prime}$.

A straightforward inductive argument proves that bisimilar states satisfy the same formulae.

Proposition 4.2. If $x \rightleftharpoons_{\operatorname{lnt}} x^{\prime}$ then $x$ rm $\operatorname{lnt} x^{\prime}$.
Furthermore, it is easy to see (but of no relevance for us in the sequel) that intuitionistic bisimulations are closed under composition, and the graph of a bounded morphism is an intuitionistic bisimulation. We prove a HennessyMilner property for image-compact models.

Theorem 4.3. Let $x, x^{\prime}$ be worlds in two image-compact models $\mathfrak{M}=(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$. Then

$$
x \rightleftharpoons_{\mathrm{Int}} x^{\prime} \quad \text { iff } \quad x \longleftrightarrow{ }_{\mathrm{Int}} x^{\prime} .
$$

Proof. Since we assume $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ to be image-compact, they both carry a general model structure, i.e., we can find a collection $A$ of up-closed subsets of $(X, \leq)$ such that $(X, \leq, V, A)$ is a general model and $\uparrow_{\leq} x$ is compact in $\tau_{A}$ for all $x \in X$, and similarly for $\mathfrak{M}^{\prime}$. Suppose we have chosen such $A$ and $A^{\prime}$.

The direction from left to right is soundness of the notion of bisimulation (Proposition4.2). For the converse direction, we show that the relation of logical equivalence is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$.

Clearly, if $x \rightarrow{ }^{4} \rightarrow{ }^{\prime}$ nt $x^{\prime}$ we have $x \in V(p)$ iff $x^{\prime} \in V^{\prime}(p)$, so item $\left(B_{1}\right)$ is satisfied. We now prove that $\left(B_{2}\right)$ holds. Let $x \rightsquigarrow \operatorname{lnt}^{\prime} x^{\prime}$ and $x \leq y$. Then we need to find $y^{\prime} \in X^{\prime}$ such that $x^{\prime} \leq^{\prime} y^{\prime}$ and $y{ }^{4} \varliminf_{\operatorname{lnt}} y^{\prime}$. Suppose towards a contradiction that such a $y^{\prime}$ does not exist. Then for each $\leq^{\prime}$-successor $z^{\prime}$ of $x^{\prime}$ we can either find a separating formula $\phi_{z^{\prime}}$ such that $\mathfrak{M}, y \Vdash \phi_{z^{\prime}}$ and $\mathfrak{M}^{\prime}, z^{\prime} \Vdash \phi_{z^{\prime}}$, or a separating formula $\psi_{z^{\prime}}$ such that $\mathfrak{M}, y \Vdash \psi_{z^{\prime}}$ and $\mathfrak{M}^{\prime}, z^{\prime} \Vdash \psi_{z^{\prime}}$. Pick such a separating formula for each $z^{\prime}$. Let $\Phi$ be the collection of such formulae that are not satisfied at $z^{\prime}$, and $\Psi$ the collection of separating formulae that are satisfied at $z^{\prime}$.

Since the interpretants of the formulae are clopen in the topology on $X^{\prime}$ generated by $A^{\prime} \cup-A^{\prime}$, the collection

$$
\left\{X \backslash \llbracket \phi \rrbracket^{\mathfrak{M}^{\prime}} \mid \phi \in \Phi\right\} \cup\left\{\llbracket \psi \rrbracket^{\mathfrak{M}^{\prime}} \mid \psi \in \Psi\right\}
$$

is an open cover of $\uparrow \leq^{\prime} x^{\prime}$. As the latter is assumed to be compact, we get finite subsets $\Phi^{\prime} \subseteq \Phi$ and $\bar{\Psi}^{\prime} \subseteq \Psi$ such that

$$
\left\{X \backslash \llbracket \phi \rrbracket^{\mathfrak{M}^{\prime}} \mid \phi \in \Phi^{\prime}\right\} \cup\left\{\llbracket \psi \rrbracket^{\mathfrak{M}^{\prime}} \mid \psi \in \Psi^{\prime}\right\}
$$

covers $\uparrow_{\leq \prime} x^{\prime}$. As a consequence, for every successor $z^{\prime}$ of $x^{\prime}$ there either exists a $\phi \in \Phi^{\prime}$ such that $z^{\prime} \Vdash \phi$, or a $\psi \in \Psi^{\prime}$ such that $z^{\prime} \Vdash \psi$. Therefore,

$$
x^{\prime} \Vdash \bigwedge \Phi^{\prime} \rightarrow \bigvee \Psi^{\prime}
$$

Since the disjunction and conjunction are taken over finite sets, this is a formula in Int. Furthermore, $y$ satisfies all $\phi \in \Phi^{\prime}$ and none of the $\psi \in \Psi^{\prime}$, and hence

$$
x \Vdash \bigwedge \Phi^{\prime} \rightarrow \bigvee \Psi^{\prime}
$$

This is a contradiction with the assumption that $x$ and $x^{\prime}$ are logically equivalent. Therefore there must exist $y^{\prime} \in X^{\prime}$ which is logically equivalent to $y$ and satisfies $x^{\prime} \leq^{\prime} y^{\prime}$. Item $\left(B_{3}\right)$ is proven symmetrically.

Theorem 4.3 does not give a strict characterisation of models where logical equivalence coincides with bisimilarity. This is witnessed by the following example, which gives a model that is not image-compact while logical equivalence (between the model and itself) does imply bisimilarity.

Example 4.4. Consider the intuitionistic Kripke frame consisting of the rational numbers ordered as usual. Let $\operatorname{Prop}=\left\{p_{q} \mid q \in \mathbb{Q}\right\}$ be a countable set of proposition letters and define a valuation $V: \operatorname{Prop} \rightarrow \mathcal{U} p(\mathbb{Q}, \leq)$ by $V\left(p_{q}\right)=\{x \in \mathbb{Q} \mid q<x\}$. Then $\mathfrak{Q}=(\mathbb{Q}, \leq, V)$ is an intuitionistic Kripke model.

We claim that $\mathfrak{Q}$ is not image-compact. To see this, let $A$ be any general frame structure such that $(\mathbb{Q}, \leq, A, V)$ is a general model. By definition $\llbracket p_{q} \rrbracket \in$ $A \cup-A$ and $\mathbb{Q} \backslash \llbracket p_{q} \rrbracket \in A \cup-A$ for all $p_{q} \in$ Prop. We note that $\uparrow \leq 0$ is covered by

$$
\left(\mathbb{Q} \backslash \llbracket p_{0} \rrbracket\right) \cup \bigcup\left\{\llbracket p_{1 / n} \rrbracket \mid n \in \mathbb{N}\right\}
$$

and clearly this cover does not have a finite subcover. However, the relation of logical equivalence between $\mathfrak{Q}$ and itself is the identity, and hence is automatically an Int-bisimulation.

Also, it is not in general true that logical equivalence implies bisimilarity. In [7, Proposition 27] the author gives an example of two intuitionistic Kripke models such that logical equivalence does not imply bisimilarity. (The notion of image-finiteness used in loc. cit. is not the usual one.) Alternatively, one can give a counterexample using "porcupine models" similar to Example 4.11below.

### 4.2. Bisimulations for Dual- and Bi-Intuitionistic Logic

Definition 4.5. A dual-intuitionistic bisimulation or Int ${ }^{\partial}$-bisimulation between intuitionistic Kripke models $\mathfrak{M}=(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$ is a relation $B \subseteq X \times X^{\prime}$ such that for all $\left(x, x^{\prime}\right) \in B$ we have:
$\left(B_{1}\right)$ For all $p \in \operatorname{Prop}, x \in V(p)$ iff $x^{\prime} \in V^{\prime}(p)$;
$\left(B_{4}\right)$ If $y \leq x$ then there exists $y^{\prime} \in X^{\prime}$ such that $y^{\prime} \leq^{\prime} x^{\prime}$ and $y B y^{\prime}$;
$\left(B_{5}\right) \quad$ If $y^{\prime} \leq^{\prime} x^{\prime}$ then there exists $y \in X$ such that $y \leq x$ and $y B y^{\prime}$.
If moreover $B$ satisfies $\left(B_{2}\right)$ and $\left(B_{3}\right)$ (from Definition 4.1) then we call $B$ a bi-intuitionistic bisimulation, or Bi -int-bisimulation. We define $\mathrm{Int}^{2}$-bisimilarity and Bi -int-bisimilarity as usual, and write these as $x \rightleftharpoons_{\mathrm{Int}^{\partial}} x^{\prime}$ and $x \rightleftharpoons_{\mathrm{Bi}-\mathrm{int}} x^{\prime}$.

Remark 4.6. Directed Bi-int-bisimulations 31, Definition 4] between intuitionistic Kripke models are pairs $\left(Z_{1}, Z_{2}\right)$ of simulations, i.e., pairs $\left(Z_{1}, Z_{2}\right)$ of two relations $Z_{1} \subseteq X \times X^{\prime}$ and $Z_{2} \subseteq X^{\prime} \times X$ satisfying certain back-and-forth conditions. This is closely related to Bi -int-bisimulation as just introduced: if $B$ is a Bi -int-bisimulation then $\left(B, B^{-1}\right)$ is a directed Bi -int-bisimulation, and conversely if $\left(Z_{1}, Z_{2}\right)$ is a directed Bi -int-bisimulation, then $Z_{1} \cap Z_{2}^{-1}$ is a Bi -intbisimulation.

Although not carried out in op. cit., one could define $x$ and $x^{\prime}$ to be directed Bi -int-bisimilar if there is a directed Bi -int-bisimulation $\left(Z_{1}, Z_{2}\right)$ with $\left(x, x^{\prime}\right) \in Z_{1}$ and $\left(x^{\prime}, x\right) \in Z_{2}$. Directed Bi-int-bisimilarity and Bi-int-bisimilarity as defined in Definition 4.5 above are then easily seen to coincide.


Figure 1: The zigs and zags of a Bi-int-bisimulation.

Proposition 4.7. Let $(X, \leq, V)$ and $\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$ be two intuitionistic Kripke models and $x \in X, x^{\prime} \in X^{\prime}$. Then $x \rightleftharpoons_{\operatorname{lnt}^{\partial}} x^{\prime}$ implies $x{ }^{\rightsquigarrow} \operatorname{lnt}^{\partial}$ and $x \rightleftharpoons_{\mathrm{Bi}-\mathrm{int}} x^{\prime}$ implies $x \xrightarrow{m i-i n t} x^{\prime}$.

The following lemma allows us to view an Int ${ }^{\partial}$-bisimulation between two models $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ as an Int-bisimulation between the corresponding dual models.

Lemma 4.8. Let $\mathfrak{M}=(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$ be two intuitionistic Kripke models. Then $B \subseteq X \times X^{\prime}$ is a $\operatorname{Int}^{\partial}$-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ if and only if it is an Int-bisimulation between $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$.

Using this lemma we can convert the result from Theorem 4.3 to a HennessyMilner theorem for dual-intuitionistic logic.

Theorem 4.9. Let $x, x^{\prime}$ be worlds in two pre-image-compact intuitionistic Kripke models $\mathfrak{M}=(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$. Then

$$
x \rightleftharpoons_{\operatorname{lnt}^{2} \partial} x^{\prime} \quad \text { iff } \quad x \longleftrightarrow \operatorname{lnt}^{2} x^{\prime} .
$$

Proof. Let $B$ be the relation of logical equivalence between $\mathfrak{M}$ and $\mathfrak{M}$. We show that it is an $\operatorname{Int}^{\partial}$-bisimulation. Alternatively, it suffices to show that it is an Int-bisimulation between $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$.

By Lemma 2.4, two states $x, x^{\prime}$ in $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$ satisfy the same Intformulae if and only if they satisfy the same Int ${ }^{2}$-formulae in $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$. Therefore the relation $B$ coincides with logical equivalence between $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$. Furthermore, $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$ are image-compact because $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ are pre-image-compact. So it follows from Theorem 4.3 that $B$ is an Int-bisimulation between $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$, hence an Int $^{\partial}$-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$.

Combining Lemma 3.2 and Theorems 4.3 and 4.9 yields:
Theorem 4.10. Let $x, x^{\prime}$ be worlds in two intuitionistic Kripke models $\mathfrak{M}=$ $(X, \leq, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, V^{\prime}\right)$ that are both image-compact and pre-imagecompact. Then

$$
x \rightleftharpoons_{\text {Bi-int }} x^{\prime} \quad \text { iff } \quad x{ }^{m} \boldsymbol{B i}_{\text {Bi-int }} x^{\prime} .
$$

Proof. The direction from left to right follows is Proposition 4.7 For the converse, we will show that the relation $B$ of logical equivalence between them is a bisimulation. $\left(B_{1}\right)$ follows immediately from the fact that $B$ is logical equivalence.

To show that $\left(B_{2}\right)$ and $\left(B_{3}\right)$ hold, we use Lemma 3.2. Let Prop' be defined as in Example 3.3 and extend the valuations $V$ and $V^{\prime}$ of $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ to $\hat{V}$ and $\hat{V}^{\prime}$ by setting $\hat{V}\left(p_{\phi}\right)=\llbracket \phi \rrbracket^{\mathfrak{M}}$, and similar for $\hat{V}^{\prime}$. Then as a consequence of Lemma 3.2, $B$ coincides with Int-logical equivalence between $(X, \leq, \hat{V})$ and $\left(X^{\prime}, \leq^{\prime}, \hat{V}^{\prime}\right)$. Furthermore, these new models are image-compact, and therefore properties $\left(B_{2}\right)$ and $\left(B_{3}\right)$ follow from Theorem 4.3 .

One can similarly obtain $\left(B_{4}\right)$ and $\left(B_{5}\right)$ from Theorem 4.9 ,
We complete this section with a detailed example showing that logical equivalence for bi-intuitionistic formulae does not in general imply Bi-int-bisimilarity.

Example 4.11. Let $W=\{(n, k) \in(\mathbb{N} \cup\{\infty\}) \times \mathbb{N} \mid k<n\} \cup\{x\}$ and define an order $\preccurlyeq$ by: $(n, k) \preccurlyeq x$ for all $(n, k) \in W$ and $(n, k) \preccurlyeq(m, \ell)$ iff $n=m$ and $k \leq \ell$. For Prop $=\left\{p_{i} \mid i \in \mathbb{N}\right\} \cup\{q\}$ define the valuation $V$ by $V(q)=\{x\}$ and $V\left(p_{i}\right)=\{(n, k) \in W \mid i \leq k\} \cup\{x\}$. Then the triple $(W, \preccurlyeq, V)$ is a Kripke model.
 $\left\{\left(n^{\prime}, k^{\prime}\right) \in \mathbb{N} \times \mathbb{N} \mid k^{\prime}<n^{\prime}\right\} \cup\left\{x^{\prime}\right\}$. Note that $\mathfrak{W}^{\prime}$ does not have an infinite branch. (We use primes to distinguish the two models.) See Figure 2 for pictorial presentations of the two models.

We claim that $x$ and $x^{\prime}$ are logically equivalent but not bisimilar. Suppose towards a contradiction that there exists a bisimulation $B$ linking $x$ and $x^{\prime}$. Since $(\infty, 0) \preccurlyeq x$ in $W$ there must be some $y^{\prime} \in W^{\prime}$ such that $(\infty, 0) B y^{\prime}$ and $y^{\prime} \preccurlyeq^{\prime} x^{\prime}$. Then $y^{\prime}$ cannot be $x^{\prime}$, because $W,(\infty, 0) \Vdash q$, hence $W^{\prime}, y^{\prime} \Vdash q$, whereas $W^{\prime}, x^{\prime} \Vdash q$. So $y^{\prime}$ is of the form $\left(n^{\prime}, k^{\prime}\right)$ for some $n^{\prime}, k^{\prime} \in \mathbb{N}$ with $k^{\prime}<n^{\prime}$. But then $W^{\prime},\left(n^{\prime}, k^{\prime}\right) \Vdash p_{n^{\prime}+1} \rightarrow q$, while $W,(\infty, 0) \Vdash p_{n^{\prime}+1} \rightarrow q$. Therefore $(\infty, 0)$ and $\left(n^{\prime}, k^{\prime}\right)$ are not logically equivalent, hence by Proposition 4.7 they cannot be bisimilar. This contradicts the assumption that there exists a bisimulation $B$ linking $x$ and $x^{\prime}$, thus $x$ and $x^{\prime}$ are not bisimilar.

Next we show that $x \in W$ and $x^{\prime} \in W^{\prime}$ are logically equivalent. For $m \in \mathbb{N}$, let $\operatorname{Prop}_{m}=\left\{p_{i} \mid i \in \mathbb{N}, i \leq m\right\} \cup\{q\}$. Then $\operatorname{Int}($ Prop $)=\bigcup_{m \in \mathbb{N}} \operatorname{Int}\left(\operatorname{Prop}_{m}\right)$. Define $B_{m} \subseteq W \times W^{\prime}$ by

$$
\begin{aligned}
B_{m}=\left\{\left(x, x^{\prime}\right)\right\} \cup\left\{\left((n, k),\left(n^{\prime}, k^{\prime}\right)\right) \mid\right. & \text { either }\left[n=n^{\prime} \text { and } k=k^{\prime}\right] \\
& \text { or }\left[k, k^{\prime} \geq m\right] \\
& \text { or } \left.\left[n, n^{\prime}>m \text { and } k=k^{\prime}<m\right]\right\} .
\end{aligned}
$$

It can be shown by induction that whenever $\left(z, z^{\prime}\right) \in B_{m}$, we have $W, z \Vdash \phi$ iff $W^{\prime}, z^{\prime} \Vdash \phi$ for all $\phi \in \mathbb{L}\left(\operatorname{Prop}_{m}\right)$. It follows that $x$ and $x^{\prime}$ are logically equivalent because $\left(x, x^{\prime}\right) \in B_{m}$ for all $m \in \mathbb{N}$. As we have already established that $x$ and $x^{\prime}$ are not bisimilar, we conclude that logical equivalence cannot imply bisimilarity in general.

## 5. Modal Bi-/Dual-/Intuitionistic Logics

In this section we enrich the logics from Section 4 with (several copies of) the unary modal operators $\square$ and $\diamond$. Following [19], we shall treat $\square$ and $\diamond$ as two different modalities that a priori are not related via axioms. Semantically, $\square$ and $\diamond$ are interpreted via distinct relations, so that boxes and diamonds do not necessarily come in pairs. For $L \in\left\{\operatorname{lnt}, \operatorname{lnt}{ }^{\partial}\right.$, Bi -int $\}$, we write $\mathrm{L}_{n, m}$ for the languages that arises from adjoining $L$ with boxes $\square_{1}, \ldots, \square_{n}$ diamonds $\diamond_{1}, \ldots, \diamond_{m}$. In the special case where $n=1$ and $m=0$ we write $\mathrm{L}_{\square}:=\mathrm{L}_{1,0}$, and similarly we sometimes use $\mathrm{L}_{\diamond}:=\mathrm{L}_{0,1}$ and $\mathrm{L}_{\square \diamond}:=\mathrm{L}_{1,1}$.

Since we do not assume any axioms relating boxes and diamonds, each modality is interpreted via its own relation in the same way as in classical modal


Figure 2: The figures depicts the models $\mathfrak{W}$ and $\mathfrak{W}^{\prime}$ from Example 4.11 The coordinates indicate the names of some of the states. The $p_{i}$ denote the lowest occurrence of a proposition letter in each branch of the models. That is, if $p_{i}$ is true in some state, then it is also true in all states above.
logic. As such, a model for $\mathrm{L}_{n, m}$ is an intuitionistic Kripke model with an additional relation $R_{i}$ for each box and $S_{j}$ for each diamond, satisfying certain coherence conditions with respect to the order $\leq$ to ensure that the interpretation of every formula is an upset. This approach resembles that of $H \square$ - and $H \diamond$-models introduced in [19].

The main objective of this section is to obtain a Hennessy-Milner type theorem for the modal bi-intuitionistic logic $\mathrm{L}_{n, m}$ interpreted in the models sketched above. We shall prove intermediate results for $\operatorname{Int}_{\square}=\operatorname{Int}_{1,0}$ and $\operatorname{Int}_{\diamond}^{\partial}=\operatorname{Int}_{0,1}^{\partial}$, which we then combine for the desired result using Lemma 3.2

### 5.1. Semantics for Modal Bi-/Dual-/Intuitionistic Logics

If $Z$ and $Z^{\prime}$ are two relations on a set $X$, then we denote by $Z \circ Z^{\prime}$ the relation $\left\{(x, y) \in X \times X \mid \exists u \in X\right.$ s.t. $x Z u$ and $\left.u Z^{\prime} y\right\}$.

Definition 5.1. A (modal) $\mathrm{L}_{n, m}$-frame is a tuple $\left(X, \leq, R_{1}, \ldots, R_{n}, S_{1}, \ldots, S_{m}\right)$ that consists of an intuitionistic Kripke frame ( $X, \leq$ ) and relations $R_{i}, S_{j} \subseteq$ $X \times X$ satisfying

$$
\left(\leq \circ R_{i}\right) \subseteq\left(R_{i} \circ \leq\right), \quad\left(\geq \circ S_{j}\right) \subseteq\left(S_{j} \circ \geq\right)
$$

It is called strictly condensed if $\left(\leq \circ R_{i} \circ \leq\right) \subseteq R_{i}$ and $\left(\geq \circ S_{j} \circ \geq\right) \subseteq S_{j}$ for all $i \in\{1, \ldots, n\}$ and $j \in\{1, \ldots, m\}$. The corresponding notion of an $\mathrm{L}_{n, m}$-model arises from adding a valuation.

Note that, since $\leq$ is reflexive, an $\mathrm{L}_{n, m}$-frame is strictly condensed if and only if $\left(\leq \circ R_{i} \circ \leq\right)=R_{i}$ and $\left(\geq \circ S_{j} \circ \geq\right)=S_{j}$ for all $i$ and $j$.

These models can be used to interpret modal extensions of Int, Int ${ }^{\partial}$ and Bi -int with $n$ boxes and $m$ diamonds. The logical connectives from $L$ are interpreted
in the underlying intuitionistic Kripke model $(X, \leq, V)$ as usual and, as stated, the interpretations of $\square_{i}$ and $\diamond_{j}$ are defined as in classical modal logic, via the relations $R_{i}$ and $S_{j}$. That is,

$$
\begin{array}{lll}
\mathfrak{M}, x \Vdash \square_{i} \phi & \text { iff } & \text { for all } y \in X, x R_{i} y \text { implies } \mathfrak{M}, y \Vdash \phi \\
\mathfrak{M}, x \Vdash \diamond_{j} \phi & \text { iff } & \mathfrak{M}, y \Vdash \phi \text { for some } y \text { with } x S_{j} y .
\end{array}
$$

We write $x{ }^{\longleftrightarrow} \mathrm{L}_{n, m} x^{\prime}$ if two states satisfy precisely the same $\mathrm{L}_{n, m}$-formulae.
We shall sometimes write $\left(X, \leq,\left(R_{i}\right),\left(S_{j}\right), V\right)$ for a modal $\mathrm{L}_{n, m}$-model. Besides, we remark that (strictly condensed) $\mathrm{L}_{\square}$-models are precisely (strictly condensed) $H \square$-models from [19], and (strictly condensed) $L_{\diamond}$-models can be found in op. cit. under the name of (strictly condensed) $H \diamond$-models. We have the following notion of bisimulation for these models:

Definition 5.2. Let $\mathfrak{M}=\left(X, \leq,\left(R_{i}\right),\left(S_{j}\right), V\right)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime},\left(R_{i}^{\prime}\right),\left(S_{j}^{\prime}\right), V^{\prime}\right)$ be two modal $\mathrm{L}_{n, m}$-models and $B \subseteq X \times X^{\prime}$ a relation. We call $B$ a $\square_{i}$-zigzag if for all $\left(x, x^{\prime}\right) \in B$ the following conditions hold:
( $\square_{i}$-zig) If $x R_{i} y$ then there exists $y^{\prime} \in X^{\prime}$ such that $x^{\prime} R_{i}^{\prime} y^{\prime}$ and $y B y^{\prime}$;
( $\square_{i}$-zag) If $x^{\prime} R_{i}^{\prime} y^{\prime}$ then there exists $y \in X$ such that $x R_{i} y$ and $y B y^{\prime}$;
We call $B$ a $\diamond_{j}$-zigzag if the same conditions hold for $S_{j}$ instead of $R_{i}$.
An $\mathrm{L}_{n, m}$-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ is a relation $B \subseteq X \times X^{\prime}$ which is an L-bisimulation between the underlying intuitionistic Kripke models and which is a $\square_{i}$-zigzag and $\diamond_{j}$-zigzag for all $i \in\{1, \ldots, n\}$ and $j \in\{1, \ldots, m\}$.

We remark that one can quotient with bisimilarity:
Remark 5.3. Let $\mathfrak{M}=\left(X, \leq,\left(R_{i}\right),\left(S_{j}\right), V\right)$ be an $\mathrm{L}_{n, m}$-model. It is easy to see that the collection of $\mathrm{L}_{n, m}$-bisimulation on a model is closed under all unions. Therefore, the relation $B$ of bisimilarity on $\mathfrak{M}$ is again a bisimulation. Moreover, $B$ is an equivalence relation: it is reflexive because the identity on $X$ is a bisimulation, symmetric because the inverse of a bisimulation on $X$ is again a bisimulation, and transitive because bisimulations are closed under composition.

Let $X_{B}$ denote the quotient of $X$ with the equivalence relation $X$ and write $\bar{x} \in X_{B}$ for the equivalence class of $x \in X$. For each of the relations $Z$ on $X$, define a relation $Z_{B}$ on $X / B$ via $\bar{x} Z_{B} \bar{y}$ if there are $x^{\prime} \in \bar{x}$ and $y^{\prime} \in \bar{y}$ such that $x^{\prime} Z y^{\prime}$. Finally, for $p \in \operatorname{Prop}$ let $V_{B}(p)=\{\bar{x} \mid x \in V(p)\}$. Then it follows from a straightforward verification that the tuple

$$
\mathfrak{M}_{B}=\left(X_{B}, \leq_{B},\left(\left(R_{B}\right)_{i}\right),\left(\left(S_{B}\right)_{i}\right), V\right)
$$

is an $\mathrm{L}_{n, m}$-model and the graph of the quotient map $q: X \rightarrow X_{B}$ is a bisimulation between $\mathfrak{M}$ and $\mathfrak{M}_{B}$. Consequently, if $\mathfrak{M}$ is in a Hennesy-Milner class, then we can that the quotient with respect to logical equivalence.

Remark 5.4. When equipped with a suitable notion of (bounded) morphism, the collection of $\mathrm{L}_{n, m}$-frames forms a category. This category is isomorphic to a category of dialgebras [32], and the language $\mathrm{L}_{n, m}$ arises as a dialgebraic logic. Interestingly, on the level of frames, the bisimulations defined in Definition 5.2 correspond precisely to dialgebra bisimulations (or cospans) in the category of $\mathrm{L}_{n, m}$-frames.

A straightforward inductive proof yields:
Proposition 5.5. Let $x$ and $x^{\prime}$ be two states in $\mathrm{L}_{n, m}$-models $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$. Then $x \rightleftharpoons \mathrm{~L}_{n, m} x^{\prime}$ implies $x{ }^{\rightsquigarrow} \mathrm{L}_{n, m} x^{\prime}$.

In order to get a suitable notion of (pre-)image-compactness for the relations $R_{i}, S_{j}$ we extend the notion of a general frame to this modal setting.

Definition 5.6. A general $\mathrm{L}_{n, m}$-frame consists of a modal $\mathrm{L}_{n, m}$-frame $(X, \leq$ $\left., R_{1}, \ldots, R_{n}, S_{1}, \ldots, S_{m}\right)$ and a collection $A \subseteq \mathcal{U} p(X, \leq)$ such that $(X, \leq, A)$ is a general L-frame and $A$ is closed under:

$$
\begin{aligned}
\oplus_{i}: \mathcal{U p}(X, \leq) & \rightarrow \mathcal{U} p(X, \leq): a \mapsto\left\{x \in X \mid R_{i}[x] \subseteq a\right\} \\
\varpi_{j}: \mathcal{U} p(X, \leq) & \rightarrow \mathcal{U} p(X, \leq): a \mapsto\left\{x \in X \mid x S_{j} y \text { for some } y \in a\right\}
\end{aligned}
$$

for all $i \in\{1, \ldots, n\}$ and $j \in\{1, \ldots, m\}$. The corresponding notion of a general $\mathrm{L}_{n, m}$-model arises from adjoining such a frame with an admissible valuation, i.e., a map $V: \operatorname{Prop} \rightarrow A$.

A relation $R_{i}$ in an $\mathrm{L}_{n, m}$-model $\left(X, \leq,\left(R_{i}\right),\left(S_{j}\right), V\right)$ is called (pre-)imagecompact if there exists $A \subseteq \mathcal{U} p(X, \leq)$ such that $\left(X, \leq,\left(R_{i}\right),\left(S_{j}\right), A, V\right)$ is a general $\mathrm{L}_{n, m}$-model and $R_{i}[x]=\left\{y \in X \mid x R_{i} y\right\}$ (resp. $R_{i}^{-1}[x]=\{y \in X \mid$ $\left.y R_{i} x\right\}$ ) is compact in $\tau_{A}$ for every $x \in X$. We similarly define (pre-)imagecompactness for $S_{j}$.

Remark 5.7. The definition of (pre-)image-compactness crucially depends on the underlying base logic. In particular, we never speak about an image compact relation in an intuitionistic Kripke frame: we speak about an image compact relation in a Int-, Int ${ }^{\partial}$ - or Bi -int-frame. For a relation to qualify as image compact, we need to exhibit a system $A$ of admissible subsets that is closed under the operations of the base logic. That is, a choice of admissibles may exhibit a relation as image compact in an Int-frame, but there may be no choice of admissibles $A^{\prime}$ that exhibits the same relation as image-compact in an Bi-int-frame, for example, if $A$ is not closed under $\preceq$. This subtlety is caused by the fact that we treat three base logics simultaneously.

For our Hennessy-Milner type results, we need to restrict to the strictly condensed models. Although this may seem like a harsh restriction, in fact every $L_{n, m}$-model can be turned into a strictly condensed one without changing the interpretation of formulae, by merely readjusting the relations $R_{i}$ and $S_{j}$. We explicitly give this construction for $\mathrm{L}_{\square}$-models and leave the general case to the reader.

Proposition 5.8. Let $\mathfrak{M}=(X, \leq, R, V)$ be an $\mathrm{L}_{\square}$-model and set $R^{+}:=(R \circ \leq)$. Then $\mathfrak{M}^{+}=\left(X, \leq, R^{+}, V\right)$ is strictly condensed, and for all $x \in X$ and $\phi \in \mathrm{L}_{\square}$ we have $\mathfrak{M}, x \Vdash \phi$ iff $\mathfrak{M}^{+}, x \Vdash \phi$.

Proof. To see that $\mathfrak{M}^{+}$is strictly condensed, observe that reflexivity and transitivity of $\leq$ imply $\left(\leq \circ R^{+} \circ \leq\right)=(\leq \circ R \circ \leq) \subseteq(R \circ \leq)=R^{+}$. The preservation of truth can be proved by induction on the structure of the formula $\phi$. All cases are trivial except the modal case. For this, we have

$$
\begin{array}{lll}
\mathfrak{M}, x \Vdash \square \phi & \text { iff } & \text { for all } y \in X, x R y \text { implies } \mathfrak{M}, y \Vdash \phi \\
& \text { iff } & \text { for all } y \in X, x(R \circ \leq) y \text { implies } \mathfrak{M}, y \Vdash \phi \\
& \text { iff } & \text { for all } y \in X, x R^{+} y \text { implies } \mathfrak{M}^{+}, y \Vdash \phi \\
& \text { iff } & \mathfrak{M}^{+}, x \Vdash \square \phi .
\end{array}
$$

The second "iff" holds by the fact that truth-sets of formulae are up-closed in $(X, \leq)$, the third one by induction.

An example of this procedure is depicted in Figure 3 below. It is not in general true that either the identity or the relation of logical equivalence between $\mathfrak{M}$ and $\mathfrak{M}^{+}$is a $L_{\square}$-bisimulation, as is witnessed by the following example.

Example 5.9. Let $X=\{x, y, z\}$ be ordered by the pre-order generated by $y \leq z$ and let $R=\{(x, y)\} \subseteq X \times X$. Then $(X, \leq, R)$ is an $\mathrm{L}_{\square}$-frame. Equip this with the valuation $V:\{p, q\} \rightarrow \mathcal{U} p(X, \leq)$ given by $V(p)=\{y, z\}$ and $V(q)=\{z\}$. Then $\mathfrak{M}=(X, \leq, R, V)$ is the $\mathrm{L}_{\square}$-model depicted in Figure 3, The strictly condensed $\mathrm{L}_{\square}$-model $\mathfrak{M}^{+}$is obtained by changing $R$ to $R^{+}=(R \circ \leq)=$ $\{(x, y),(x, z)\}$.

The the relation of logical equivalence between $\mathfrak{M}$ and $\mathfrak{M}^{+}$is simply the identity relation on $X$. It is easy to see that this is not an $\mathrm{L}_{\square}$-bisimulation: in $\mathfrak{M}^{+}$there is an $R$-transition from $x$ to $z$. The only state in $\mathfrak{M}$ that is logically equivalent to $z$ is $z$. But there is no $R_{\square}$-transition from $x$ to $z$ in $\mathfrak{M}$. So there can be no $\mathrm{L}_{\square}$-bisimulation linking $x$ and $x^{\prime}$.


Figure 3: An $\mathrm{L}_{\square}$-model and its condensed version.

### 5.2. Hennessy-Milner Property for Some Modal Intuitionistic Logics

We now restrict our attention to $\mathrm{Int}_{\square}$ and extend the Hennessy-Milner result from Theorem 4.3 to the setting of $\operatorname{Int}_{\square}$ interpreted in strictly condensed $\operatorname{Int}_{\square}$ models.

Theorem 5.10. Let $\mathfrak{M}=(X, \leq, R, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, R^{\prime}, V^{\prime}\right)$ be two strictly condensed $\operatorname{lnt}_{\square}$-models such that $\leq, \leq^{\prime}, R$ and $R^{\prime}$ are image-compact. Then for all $x \in X$ and $x^{\prime} \in X^{\prime}$ we have

$$
x \rightleftharpoons_{\operatorname{lnt}_{\square}} x^{\prime} \quad \text { iff } \quad x \longleftrightarrow \operatorname{lnt}_{\square} x^{\prime} .
$$

Proof. The direction from left to right follows from Proposition 5.5, For the converse, we let $B$ be logical equivalence and we show that it is a Int ${ }_{\square}$-bisimulation. It follows from Lemma 3.2 and Theorem 4.3 that $B$ is an Int-bisimulation, so it remains to show that ( $\square$-zig) and ( $\square$-zag) hold.

Let $x B x^{\prime}$ and $x R y$ and suppose towards a contradiction that there is no $R^{\prime}$-successor $y^{\prime}$ of $x^{\prime}$ which is logically equivalent to $y$. Then for each such $y^{\prime}$ we can find a separating formula. As in Theorem 4.3, using compactness, we get two finite sets $\Phi^{\prime}$ and $\Psi^{\prime}$ such that $y$ satisfies every formula in $\Phi^{\prime}$ and none in $\Psi^{\prime}$, and such that for every $y^{\prime}$ with $x^{\prime} R^{\prime} y^{\prime}$ there either exists $\phi \in \Phi^{\prime}$ such that $\mathfrak{M}^{\prime}, y^{\prime} \Vdash \phi$, or $\psi \in \Psi^{\prime}$ such that $\mathfrak{M}^{\prime}, y^{\prime} \Vdash \psi$.

Let $y^{\prime}$ be an $R^{\prime}$-successor of $x^{\prime}$, then $y^{\prime} \leq^{\prime} z^{\prime}$ implies $x^{\prime} R^{\prime} z^{\prime}$, because $\mathfrak{M}^{\prime}$ is assumed to be strictly condensed. As a consequence $\mathfrak{M}^{\prime}, y^{\prime} \Vdash \bigwedge \Phi^{\prime} \rightarrow \bigvee \Psi^{\prime}$. Since this holds for any $y^{\prime}$ with $x^{\prime} R^{\prime} y^{\prime}$, we have

$$
\mathfrak{M}^{\prime}, x^{\prime} \Vdash \square\left(\bigwedge \Phi^{\prime} \rightarrow \bigvee \Psi^{\prime}\right)
$$

Furthermore, by construction $\mathfrak{M}, y \Vdash \Lambda \Phi^{\prime} \rightarrow \bigvee \Psi^{\prime}$, so

$$
\mathfrak{M}, x \Vdash \square\left(\bigwedge \Phi^{\prime} \rightarrow \bigvee \Psi^{\prime}\right)
$$

This contradicts the assumption that $x$ and $x^{\prime}$ be logically equivalent. Therefore we conclude that there must exist a $y^{\prime} \in X^{\prime}$ which is logically equivalent to $y$ and satisfies $x^{\prime} R^{\prime} y^{\prime}$. Thus ( $\square$-zig) is satisfied. A symmetric argument shows that ( $\square$-zag) is satisfied as well.

The next example shows that a simple adaptation of "porcupine models" exhibits that logical equivalence does not in general imply $\mathrm{Int}_{\square}$-bisimilarity. Note also that in this example both $\leq$ and $\leq^{\prime}$ are image-finite and pre-image-finite.

Example 5.11. Consider the two structures as in Figure 4 where the lines indicate the relations $R$ and $R^{\prime}$. Equip both models with the trivial order, that is, $x \leq y$ iff $x=y$. Then $\mathfrak{B}$ and $\mathfrak{B}^{\prime}$ are two strictly condensed Int $_{\square}$-frames.

Since the orders are taken to be trivial, the interpretation of intuitionistic logic is classical, i.e., every subset of states is an interpretant and the interpretation of $\neg \phi$ is given by taking complements. Moreover, the notion of an Int $_{\square}$-bisimulation reduces to a Kripke bisimulation in the usual sense for normal modal logic, see e.g. [6, Definition 2.16]. Therefore, the argument in Example 2.23 of op. cit. proves that the roots of the two models are logically equivalent but not bisimilar.


Figure 4: Two structures.

### 5.3. Hennessy-Milner Property for Modal Dual- and Bi-Intuitionistic Logic

We now dualise the result of Theorem 5.10 in a similar way as in the proof of Theorem4.9 in order to obtain a Hennessy-Milner theorem for Int ${ }_{\diamond}{ }_{\diamond}$ interpreted in Int ${ }_{\diamond}^{\partial}$-models. This then leads to the general objective of a general HennessyMilner theorem for bi-intuitionistic modal logic with $n$ boxes and $m$ diamonds.

We commence by extending Definition 2.3. Lemma 2.4 and the translation $(\cdot)^{t}$ to the context of modal bi-intuitionistic logic. Extend the involution $(\cdot)^{t}$ on $\mathrm{Bi}-$ int to an involution on $\mathrm{Bi}-\mathrm{int}_{\square \triangleleft}$ by adding to the recursive definition:

$$
(\square \phi)^{t}=\diamond \phi^{t}, \quad(\diamond \phi)^{t}=\square \phi^{t}
$$

This is easily seen to restrict to bijections $(\cdot)^{t}: \operatorname{lnt}_{\square} \rightarrow \operatorname{Int}_{\diamond}^{\partial}$ and $(\cdot)^{t}: \mathrm{Bi}_{\mathrm{B}} \mathrm{int}_{\square} \rightarrow$ $\mathrm{Bi}^{\text {-int }}{ }_{\diamond}$. Furthermore, for $Z \in\{R, S\}$ we define the dual of a $Z$-model $\mathfrak{M}=$ $(X, \leq, Z, V)$ to be $\mathfrak{M}^{\partial}=\left(X, \geq, Z, V^{\partial}\right)$, where $V^{\partial}(p)=X \backslash V(p)$, for $p \in$ Prop. Then $\mathfrak{M}^{\partial \partial}=\mathfrak{M}$, and moreover we have:

Lemma 5.12. The tuple $\mathfrak{M}=(X, \leq, Z, V)$ is a (strictly condensed) $\mathrm{L}_{\square}$-model if and only if $\mathfrak{M}^{\partial}$ is a (strictly condensed) $\mathrm{L}_{\diamond}$-model.

Models and their duals are related in the following manner. This extends Lemma 2.4

Lemma 5.13. Let $\mathfrak{M}=(X, \leq, R, V)$ be a strictly condensed $\mathrm{L}_{\square}$-model and $\phi \in \mathrm{Bi}_{\mathrm{int}}^{\square}$ a formula. Then we have:

$$
\mathfrak{M}, x \Vdash \phi \quad \text { iff } \quad \mathfrak{M}^{\partial}, x \Vdash \phi^{t} .
$$

We have now set ourselves up for the proof of the Hennessy-Milner theorem of dual-intuitionistic logic with an extra diamond-modality.

Theorem 5.14. Let $\mathfrak{M}=(X, \leq, S, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, S^{\prime}, V^{\prime}\right)$ be two strictly condensed $\operatorname{lnt}_{\diamond}^{\partial}$-models such that $\leq$ and $\leq^{\prime}$ are pre-image-compact and $S$ and $S^{\prime}$ are image-compact. Then for all $x \in X$ and $x^{\prime} \in X^{\prime}$ we have

$$
x \rightleftharpoons_{\operatorname{lnt}_{\diamond}^{\partial}} x^{\prime} \quad \text { iff } \quad x{ }^{\sim} \operatorname{lnt}_{\diamond}^{\partial} x^{\prime} .
$$

Proof. Let $B \subseteq X \times X^{\prime}$ be the relation of logical equivalence. Then $B$ is also logical equivalence of $\operatorname{Int}_{\square}$-formulae between $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$. By assumption all relations in these dual models are image-compact, so it follows form Theorem 5.10 that $B$ is an $\operatorname{Int}_{\square}$-bisimulation between $\mathfrak{M}^{\partial}$ and $\left(\mathfrak{M}^{\prime}\right)^{\partial}$. An easy verification then shows that $B$ is an $\operatorname{Int}_{\diamond}^{\partial}$-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$.

Finally, we attain a Hennessy-Milner theorem for the modal bi-intuitionistic logic $\mathrm{Bi}^{- \text {int }_{n, m}}$ interpreted in $\mathrm{L}_{n, m}$-models. This follows from Theorems 5.10 and 5.14 using Lemma 3.2 in a similar way as in the proof of Theorem4.10.
Theorem 5.15. Let $\mathfrak{M}=\left(X, \leq,\left(R_{i}\right),\left(S_{j}\right), V\right)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime},\left(R_{i}^{\prime}\right),\left(S_{j}^{\prime}\right), V^{\prime}\right)$ be two strictly condensed $\mathrm{L}_{n, m}$-models. Furthermore assume that all relations (including $\leq$ and $\leq^{\prime}$ ) are image-compact and additionally that $\leq$ and $\leq^{\prime}$ are pre-image-compact. Then for all $x \in X$ and $x^{\prime} \in X^{\prime}$ we have

$$
x \rightleftharpoons_{\mathrm{Bi}^{-\mathrm{int}_{n, m}}} x^{\prime} \quad \text { iff } \quad x{ }^{m i-\mathrm{int}_{n, m}} x^{\prime} .
$$

A counterexample for the failure of the converse is readily constructed from the frames $\mathfrak{B}$ and $\mathfrak{B}^{\prime}$ in Figure 4 , equipped with trivial orders $\leq$ and $\leq^{\prime}$, and where $n=m=1$ and $R=S$ is given by the edges, that, in general, logical equivalence between modal models does not imply bisimilarity.

## 6. Applications

We investigate several (bi-)intuitionistic modal logics found in the literature, and equip them with a notion of bisimulation accompanied by a Hennessy-Milner theorem.

We consider (descriptive) $\square$-models for the language $\mathrm{Int}_{\square}$ introduced in [23] in Section 6.1, and in Section 6.2 we look at various ways of interpreting Int $_{\square}$ 。 with a single relation for $\square$ and $\diamond$ (in contrast to the approach taken in Section 5. where each modality is interpreted via its own relation). In particular, this includes the well-known semantics for modal intuitionistic logic given by Fischer Servi 21], and Plotkin and Stirling [22].

In Subsection 6.3 we apply our results to intuitionistic epistemic logic 63]. The knowledge operators in this logic behave like $\square$-modalities. Additionally, the logic has a unary "common knowledge" operator $\mathscr{C}$, which behaves differently.

The second half of this section is devoted to tense bi-intuitionistic logic. In Subsections 6.4, 6.5 and 6.6 we investigate three different ways of defining its semantics. The corresponding notion of bisimulation requires the relations interpreting the modalities to look both forward and backwards. In each of these cases, we give a Hennessy-Milner class.

### 6.1. Wolter/Zakarhyashev Models

In [23], the authors introduce $\square$-models as a semantics for Int $_{\square}$. These coincide with general strictly condensed $L_{\square}$ in the sense of Definition 5.1 with the additional property that the underlying order is a partial order (rather than a pre-order). That is:

Definition 6.1. A $\square$-frame is a tuple $(X, \leq, R, A)$ such that

- $(X, \leq)$ is a partially ordered set;
- $R \subseteq X \times X$ is a relation satisfying $(\leq \circ R \circ \leq)=R$;
- $A \subseteq \mathcal{U} p(X, \leq)$ is a collection of upsets containing $\emptyset$ and $X$ which is closed under $\cap, \cup, \exists$ and $\square$ (cf. Definition 5.6).
A $\square$-frame is called descriptive if $(X, \leq, A)$ is a descriptive intuitionistic Kripke frame [28, Section 8.4] and

$$
x R y \quad \text { iff } \quad \forall a \in A(x \in \amalg a \text { implies } y \in a)
$$

A $\square$-model is a $\square$-frame together with an admissible valuation $V$ : Prop $\rightarrow A$ of the proposition letters.

Formulae in $\mathrm{Int}_{\square}$ are interpreted as usual. Since $\square$-models are simply special cases of strictly condensed $\mathrm{Int}_{\square}$-models, we already have a truth-preserving notion of bisimulation. Moreover, Theorem 5.10 gives rise to a Hennessy-Milner theorem for $\square$-models, where image-compactness is now taken with respect to the general frame structure encompassed in the definition of a $\square$-model.

Corollary 6.2. Let $x$ and $x^{\prime}$ be two states in two $\square$-models all of whose relations are image-compact. Then $x \rightleftharpoons_{\operatorname{lnt}_{\square}} x^{\prime}$ if and only if $x \leadsto \operatorname{lnt}_{\square} x^{\prime}$.

In particular, this holds for all descriptive $\square$-models.
Proposition 6.3. Let $\mathfrak{M}=(X, \leq, R, A)$ be a descriptive $\square$-frame. Then $R$ is image-compact.

Proof. The descriptive intuitionistic Kripke frame $(X, \leq, A)$ underlying $\mathfrak{M}$ can be viewed as an Esakia space $\left(X, \leq, \tau_{A}\right)$, where $\tau_{A}$ is the patch topology defined in Definition 2.5 29]. In particular this means that $\left(X, \tau_{A}\right)$ is a compact topological space. By definition, for any $x \in X$ the set $\{y \in X \mid x \leq y\}$ is closed in $\tau_{A}$, so $\leq$ is image-compact. Furthermore, by definition of a descriptive $\square$-frame we have $R[x]=\bigcap\{a \in A \mid x \in \boxplus a\}$ and since this is the intersection of clopen sets, it is closed in $\tau_{A}$, hence compact.

### 6.2. Božić/Došen Models

In [19], the authors define a $\square \diamond$-model to be a strictly condensed Int $_{\square}$-model $(X, \leq, R)$ which is simultaneously an $\mathrm{Int}_{\diamond}$-model. That is, $(X, \leq)$ is a pre-order and $R$ is a relation on $X$ that satisfies $(\leq \circ R \circ \leq)=R$ and $(\geq \circ R) \subseteq(R \circ \geq)$. These are used to interpret Int $_{\square \diamond}$-formulae in the usual way.

It is straightforward to see that an $\operatorname{lnt}_{\square}$-bisimulation between $\square \diamond$-models preserves all formulae in $\mathrm{Int}_{\square \diamond}$, in particular also those involving $\diamond$. Thus, if $x$ and $x^{\prime}$ are two states in two $\square \diamond$-models with all image-compact relations, then we have a chain of implications:

$$
x \rightleftharpoons_{\operatorname{Int}_{\square}} x^{\prime} \quad \Rightarrow \quad x \longleftrightarrow \operatorname{lnt}_{\square \Delta} x^{\prime} \quad \Rightarrow \quad x \not \operatorname{lnt}_{\square} x^{\prime} \quad \Rightarrow \quad x \rightleftharpoons_{\operatorname{Int}_{\square}} x^{\prime}
$$

This implies:

Corollary 6.4. Let $x$ and $x^{\prime}$ be two states in two $\square \diamond$-models with all imagecompact relations. Then $x \longleftrightarrow \operatorname{lnt}_{\square \Delta} x^{\prime}$ if and only if $x \rightleftharpoons_{\operatorname{lnt}_{\square}} x^{\prime}$.
We note that $\square \diamond$-models are special cases of the models used by e.g. Fischer Servi and Plotkin and Sterling to interpret Int $_{\square \diamond}$, see [22, Section 1] and 21, Section 2]. We refer to these models as FS-models, introduced formally next.

Definition 6.5. An FS-model is a tuple $\mathfrak{M}=(X, \leq, R, V)$ consisting of an intuitionistic Kripke model $(X, \leq, V)$ and a relation $R \subseteq X \times X$ that satisfies $(R \circ \leq) \subseteq(\leq \circ R)$ and $(\geq \circ R) \subseteq(R \circ \geq)$.
In such a model, the interpretation of intuitionistic connectives and $\diamond$ is as usual. However, if we interpret $\square \phi$ as in Definition 5.1 we are no longer guaranteed an upset in $(X, \leq)$. This is remedied by putting

$$
\mathfrak{M}, x \Vdash \square \phi \quad \text { iff } \quad \text { for all } y \in X, x(\leq \circ R) y \text { implies } \mathfrak{M}, y \Vdash \phi .
$$

In the special case where

$$
\begin{equation*}
(\leq \circ R) \subseteq R \tag{3}
\end{equation*}
$$

the interpretation of $\square$ coincides with the one given in Definition 5.1, i.e., without the additional quantification over $\leq$ in between. Moreover, if this is the case then $(X, \leq, R, V)$ is a strictly condensed $\square \diamond$-model. Therefore, we call an FS-model satisfying (3) strictly condensed. Then we have:
Corollary 6.6. Let $x$ and $x^{\prime}$ be two states in two strictly condensed FS-models with all image-compact relations. Then $x \leadsto \operatorname{lnt}_{\square} x^{\prime}$ if and only if $x \rightleftharpoons_{\operatorname{lnt}_{\square}} x^{\prime}$.

### 6.3. Intuitionistic Epistemic Logic

Intuitionistic epistemic logic describes a system of the knowledge of $n$ agents 33]. The logical language used for this is EK, and is constructed from propositional variables, intuitionistic connectives, and additional unary modal operators $\mathscr{K}_{i}$ for every $i \in\{1, \ldots, n\}$ and $\mathscr{C}$. The intuitive meaning of $\mathscr{K}_{i} \phi$ is "agent $i$ knows that $\phi "$ and $\mathscr{C} \phi$ means that $\phi$ is common knowledge. This language can be interpreted in EK-models [33, Definitions 2 and 3]. We give the definition of these models in a slightly reformulated way, so that the connection with $\mathrm{Int}_{\square}$ models is easier to see.
Definition 6.7. An EK-model is a tuple $\left(X, \leq, R_{1}, \ldots, R_{n}, V\right)$ consisting of an intuitionistic Kripke model $(X, \leq, V)$ and relations $R_{i} \subseteq X \times X$ satisfying $\left(\leq \circ R_{i}\right) \subseteq R_{i}$.

The interpretation of intuitionistic connectives is as usual, and the interpretation of $\mathscr{K}_{i}$ is as for boxes:

$$
\mathfrak{M}, x \Vdash \mathscr{K}_{i} \phi \quad \text { iff } \quad \text { for all } y \in X, x R_{i} y \text { implies } \mathfrak{M}, y \Vdash \phi .
$$

The interpretation $\mathscr{C}$ is best described via a new relation $R^{*}$. Let $R=R_{1} \cup$ $\cdots \cup R_{n}$ and let $R^{*}$ be the collection of all pairs $(x, y)$ such that $y$ is reachable from $x$ via a finite number of $R$-transitions. Then

$$
\mathfrak{M}, x \Vdash \mathscr{C} \phi \quad \text { iff } \quad \text { for all } y \in X, x R^{*} y \text { implies } \mathfrak{M}, y \Vdash \phi .
$$

Of course, EK-models are special cases of $\operatorname{Int}_{n, 0}$-models and the interpretation of the $\mathscr{K}_{i}$ corresponds to the $n$ boxes in such a model. A straightforward verification shows that $\operatorname{Int}_{n, 0}$-bisimulations also preserve the operator $\mathscr{C}$, so that we have:

Lemma 6.8. Let $x$ and $x^{\prime}$ be two states in two EK-models $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ which are linked by an $\operatorname{Int}_{n, 0}$-bisimulation. Then $x{ }^{m} \mathrm{EK}^{\prime} x^{\prime}$, that is, $x$ and $x^{\prime}$ satisfy precisely the same EK-formulae.

Conversely, if two states $x$ and $x^{\prime}$ in two EK-models are logically equivalent, then in particular they satisfy the same $\operatorname{Int}_{n, 0}$-formulae, i.e., we have $x{ }^{m} \operatorname{lnt}_{n, 0} x^{\prime}$. If $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ (viewed as $\operatorname{Int}_{n, 0}$-models) are strictly condensed and all their relations are image-compact, then it follows from Theorem 5.10 and Lemma 3.2 that $x$ and $x^{\prime}$ are linked by an $\operatorname{Int}_{n, 0}$-bisimulation. By the previous lemma this in turn implies $x{ }^{4} \overbrace{\text { EK }} x^{\prime}$. Thus we have:

Corollary 6.9. Let $x$ and $x^{\prime}$ be two states in two strictly condensed EK-models all of whose relations are image-compact. Then

$$
x \not \overbrace{\text { EK }} x^{\prime} \quad \text { iff } \quad x \rightleftharpoons_{\operatorname{lnt}_{n, 0}} x^{\prime} .
$$

Therefore $\operatorname{Int}_{n, 0}$-bisimulations provide a suitable notion of bisimulation between EK-models.

### 6.4. Tense Bi-Intuitionistic Logic in Tense Models

Tense bi-intuitionistic logic is obtained from the modal bi-intuitionistic logic Bi-int ${ }_{\square} \diamond$ by extending it with tense operators $\bullet$ corresponding to $\square$ and $\diamond$, respectively. We call this language Tense $=\mathrm{Bi}-\mathrm{int}_{2,2}$. Classically, is interpreted using the converse relation of $\square$. Since we assume no connection between $\square$ and $\diamond$, we get an additional tense operator $\square$ which is interpreted using the converse relation of $\diamond$.

In this subsection we adapt $\mathrm{Bi}_{\mathrm{int}}^{\square \triangleleft}{ }^{-}$models (Definition 5.1) to allow interpretation of Tense-formulae, i.e., we make sure that the truth-set of every formula is still up-closed. In the next two subsections we investigate two more ways to define semantics for tense bi-intuitionistic logic. If $R$ is a relation on $X$, we write $\breve{R}=\{(x, y) \mid y R x\}$ for the converse relation.

Let $(X, \leq, R, S, V)$ be a $\mathrm{Bi}^{- \text {int }_{\square \diamond}}$-model for Bi -int ${ }_{\square \diamond}$. As stated, we want to use the converse relations $\breve{S}$ and $\breve{R}$ to interpret $■$ and , respectively. Therefore, a possible semantics for Tense is given by Bi-int ${ }_{2,2}$-models $\left(X, \leq, R_{1}, R_{2}, S_{1}, S_{2}, V\right)$ that satisfy $R_{2}=\breve{S}_{1}$ and $S_{2}=\breve{R}_{1}$. This identification leads to the additional coherence conditions $\left(\geq \circ \breve{R}_{1}\right) \subseteq\left(\breve{R}_{1} \circ \geq\right)$, and similarly for $S_{1}$. Thus, we can also view such a model as a Bi-int ${ }_{1,1}$-model with additional coherence conditions. This is reflected in the following definition of a tense model.

Definition 6.10. A tense model is a tuple $(X, \leq, R, S, V)$ consisting of an intuitionistic Kripke model $(X, \leq, V)$ and two relations $R, S \subseteq X \times X$ satisfying

$$
(\leq \circ R)=(R \circ \leq) \quad \text { and } \quad(\geq \circ S)=(S \circ \geq)
$$

The interpretation of the tense operators in a tense model $\mathfrak{M}=(X, \leq, R, S, V)$ is given by

$$
\begin{array}{rll}
\mathfrak{M}, x \Vdash \phi & \text { iff } & \mathfrak{M}, y \Vdash \phi \text { for some } y \text { with } y R x \\
\mathfrak{M}, x \Vdash ゅ \phi & \text { iff } & \text { for all } y \in X, y S x \text { implies } \mathfrak{M}, y \Vdash \phi .
\end{array}
$$

Note that this corresponds precisely to the usual interpretation of box and diamond in the Bi-int ${ }_{1,1}$-model $(X, \leq, \breve{S}, \breve{R}, V)$. As a consequence, persistence still holds, i.e., the truth-set of every formula is up-closed in $(X, \leq)$.

To define a tense bisimulation between two tense models $(X, \leq, R, S, V)$ and ( $X^{\prime}, \leq^{\prime}, R^{\prime}, S^{\prime}, V^{\prime}$ ) we simply use the notion of a Bi -int ${ }_{2,2}$-bisimulation between $(X, \leq, R, \breve{S}, S, \breve{R}, V)$ and $\left(X^{\prime}, \leq^{\prime}, R^{\prime}, \breve{S}^{\prime}, S^{\prime}, \breve{R}^{\prime}, V^{\prime}\right)$ from Definition 5.2, Explicitly, this can be defined as follows:

Definition 6.11. By a tense bisimulation between two tense models $\mathfrak{M}=(X, \leq$ $, R, S, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, R^{\prime}, S^{\prime}, V^{\prime}\right)$ we mean a Bi-int-bisimulation $B \subseteq X \times X$ between the underlying intuitionistic Kripke models such that for all $\left(x, x^{\prime}\right) \in B$ and $Z \in\{R, \breve{S}, S, \breve{R}\}$ we have:

- If $x Z y$ then there exists $y^{\prime} \in X^{\prime}$ such that $x^{\prime} Z^{\prime} y^{\prime}$ and $y B y^{\prime}$;
- If $x^{\prime} Z^{\prime} y^{\prime}$ then there exists $y \in X$ such that $x Z y$ and $y B y^{\prime}$.

The notion of tense bisimilarity is defined as usual, and denoted $\rightleftharpoons$ Tense.
It follows from Proposition5.5 that Tense-bisimilar states satisfy the same Tenseformulae.

We call a tense model $(X, \leq, R, S, V)$ strictly condensed if $(\leq \circ R \circ \leq) \subseteq R$ and $(\geq \circ S \circ \geq) \subseteq S$. A straightforward verification shows that this is the case if and only if $(\leq \circ \breve{S} \circ \leq) \subseteq \breve{S}$ and $(\geq \circ \breve{R} \circ \geq) \subseteq \breve{R}$, so a tense model is strictly condensed if and only if the Bi-int ${ }_{2,2}$-model $(\bar{X}, \leq, R, \breve{S}, S, \breve{R}, V)$ is strictly condensed in the sense of Definition5.1. We define (pre-)image-compactness of relations in a tense model $(X, \leq, R, S, V)$ as if it were a Bi -int ${ }_{\square \diamond}$-model. As a corollary of Theorem 5.15 we then obtain:

Corollary 6.12. Let $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ be strictly condensed tense models all of whose relations are both image-compact and pre-image-compact. Suppose $x \in \mathfrak{M}$ and $x^{\prime} \in \mathfrak{M}^{\prime}$. Then

$$
x \not{ }_{\text {Tense }} x^{\prime} \quad \text { iff } \quad x \rightleftharpoons_{\text {Tense }} x^{\prime} \text {. }
$$

We leave the construction of counterexamples showing that we cannot drop the conditions of (pre-)image-compactness of the relations in Corollary 6.12 to the reader.

### 6.5. Tense Bi-Intuitionistic Logic by Goré, Postniece and Tiu

An alternative semantics for Tense is introduced in 24, Section 6]. The authors define a model, which we shall refer to as a GPT-model, to be a tuple $(X, \leq$
$, R, S, V)$ such that $(X, \leq, V)$ is an intuitionistic Kripke model and $R, S$ are relations on $X$ satisfying

$$
\begin{equation*}
(R \circ \leq) \subseteq(\leq \circ R) \quad \text { and } \quad(\geq \circ S) \subseteq(S \circ \geq) \tag{4}
\end{equation*}
$$

The interpretation of the modalities is then given by

$$
\begin{array}{rll}
\mathfrak{M}, x \Vdash \square \phi & \text { iff } & \text { for all } y \in X, x(\leq \circ R) y \text { implies } \mathfrak{M}, y \Vdash \phi \\
\mathfrak{M}, x \Vdash \diamond \phi & \text { iff } & \text { there exists } y \in X \text { such that } x S y \text { and } \mathfrak{M}, y \Vdash \phi \\
\mathfrak{M}, x \Vdash ■ \phi & \text { iff } & \text { for all } y \in X, x(\leq \circ \breve{S}) y \text { implies } \mathfrak{M}, y \Vdash \phi \\
\mathfrak{M}, x \Vdash \phi & \text { iff } & \text { there exists } y \in X \text { such that } x \breve{R} y \text { and } \mathfrak{M}, y \Vdash \phi
\end{array}
$$

We can define a bisimulation between such models in the same way as in Definition 6.11 above. They are easily seen to preserve truth, despite the changed interpretation of the $\square$-modalities. If a GPT-model $\mathfrak{M}=(X, \leq, R, S, V)$ satisfies

$$
\begin{equation*}
(\leq \circ R) \subseteq R, \quad(\leq \circ \breve{S}) \subseteq \breve{S} \tag{5}
\end{equation*}
$$

then the interpretation of $\square$ and $\square$ is the same as in Subsection 6.4 i.e., a state satisfies $\square \phi$ (resp. ■ $\phi$ ) if all $R$-successors (resp. $\breve{S}$-successors) satisfy $\phi$. A GPT-model that satisfies (5) will be called strictly condensed. Indeed, these are strictly condensed frames in the sense of Subsection 6.4 above, because

$$
\begin{array}{rlrl}
(\leq \circ R \circ \leq) & \subseteq(\leq \circ \leq \circ R) & (\text { By (4) })  \tag{4}\\
& \subseteq(\leq \circ R) \\
& \subseteq R & (\leq \text { is transitive }) \\
& (\text { By (15) })
\end{array}
$$

and similarly $(\geq \circ S \circ \geq) \subseteq S$. Since furthermore the interpretation of formulae is the same as for tense models, Corollary 6.12 now carries over to:

Corollary 6.13. Let $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ be strictly condensed GPT-models all of whose relations are both image-compact and pre-image-compact. Suppose $x \in \mathfrak{M}$ and $x^{\prime} \in \mathfrak{M}^{\prime}$. Then logical equivalence implies tense bisimilarity.

As is the case for $\mathrm{L}_{n, m}$-models (see Proposition 5.8), we can turn every GPTmodel into a strictly condensed one by only modifying the relations $R$ and $S$.

Proposition 6.14. For every GPT-model $\mathfrak{M}=(X, \leq, R, S, V)$ we can find a strictly condensed GPT-model $\mathfrak{M}^{+}=\left(X, \leq, R^{+}, S^{+}, V\right)$ whose underlying intuitionistic Kripke model remains unchanged and which satisfies for all $x \in X$ and $\phi \in$ Tense:

$$
\mathfrak{M}, x \Vdash \phi \quad \text { iff } \quad \mathfrak{M}^{+}, x \Vdash \phi .
$$

Proof. Define $R^{+}=(\leq \circ R)$ and $S^{+}=(S \circ \geq)$. Then reflexivity and transitivity of $\leq$ prove $\left(\leq \circ R^{+}\right)=R^{+}$and $\left(R^{+} \circ \leq\right) \subseteq\left(\leq \circ R^{+}\right)$. Besides, $\left(\geq \circ S^{+}\right) \subseteq\left(S^{+} \circ \geq\right)$, and clearly $S^{+}=(S \circ \geq)$ implies $\left(\leq \circ S^{+}\right) \subseteq S^{+}$. So $\mathfrak{M}^{+}$is indeed a strictly condensed GPT-model.

We now prove that the theory of the individual states is unchanged, by induction on the structure of $\phi$. The only non-trivial cases are the ones involving the modalities. We show the cases $\square \phi$ and $\diamond \phi$. Their tense counterparts are similar. We have:

$$
\begin{array}{lll}
\mathfrak{M}, x \Vdash \square \phi & \text { iff } & x(\leq \circ R) y \text { implies } \mathfrak{M}, y \Vdash \phi \\
& \text { iff } & x(\leq \circ \leq \circ R) y \text { implies } \mathfrak{M}, y \Vdash \phi \\
& \text { iff } & x\left(\leq \circ R^{+} y\right) \text { implies } \mathfrak{M}^{+}, y \Vdash \phi \\
& \text { iff } & \mathfrak{M}^{+}, x \Vdash \square \phi
\end{array}
$$

For the diamonds:

$$
\begin{aligned}
\mathfrak{M}, x \Vdash \diamond \phi & \text { iff there exists } y \in X \text { such that } x S y \text { and } \mathfrak{M}, y \Vdash \phi \\
& \text { iff there exists } y \in X \text { such that } x(S \circ \geq) y \text { and } \mathfrak{M}, y \Vdash \phi \\
& \text { iff there exists } y \in X \text { such that } x S^{+} y \text { and } \mathfrak{M}^{+}, y \Vdash \phi \\
& \text { iff } \mathfrak{M}^{+}, x \Vdash \diamond \phi
\end{aligned}
$$

The second "iff" holds by persistence: the direction from left to right is immediate, conversely, if $x S z \geq y$ and $\mathfrak{M}, y \Vdash \phi$, then persistence implies $\mathfrak{M}, z \Vdash \phi$.

### 6.6. Tense Bi-Intuitionistic Logic in $H$-Models

Lastly, we review another approach, taken in [25, 26], where the authors assume additional axioms relating $\square$ and $\diamond$. In particular, in their semantics $\diamond \phi$ is equivalent to $\lrcorner \square \neg \phi$, where $\neg \phi=\phi \rightarrow \perp$ and $\lrcorner \phi=\top \prec \phi$. The interpreting structures they use are $H$-frames [25, Definition 10]. These are precisely strictly condensed $\square$-frames from [19], called strictly condensed $\mathrm{L}_{\square}$-frames in our notation (cf. Definition 5.1). We view an $H$-frame ( $H$-model) as a strictly condensed Bi -int ${ }_{\square}$-frame ( $\mathrm{Bi}^{\left.- \text {int }_{\square}-\text { model }\right) \text {. }}$

Let $\mathfrak{M}=(X, \leq, R, V)$ be an $H$-model. While $\square$ and are interpreted in the same way as in Subsection 6.4, the interpretation of $\square$ and $\diamond$ is given via the so-called left converse of $R$, defined as $\geq \circ R \circ \geq$. Writing (suggestively) $S=(\geq \circ R \circ \geq)$ these modalities are again interpreted as usual, i.e., via

$$
\begin{array}{rll}
\mathfrak{M}, x \Vdash \diamond \phi & \text { iff } & \text { there exists } y \in X \text { such that } x S y \text { and } \mathfrak{M}, y \Vdash \phi \\
\mathfrak{M}, x \Vdash ■ \phi & \text { iff } & y S x \text { implies } \mathfrak{M}, y \Vdash \phi
\end{array}
$$

Therefore, setting $\overline{\mathfrak{M}}=(X, \leq, R, S, V)$ yields a (strictly condensed) tense model $\overline{\mathfrak{M}}$ in the sense of Definition 6.10 which satisfies $\mathfrak{M}, x \Vdash \phi$ iff $\overline{\mathfrak{M}}, x \Vdash \phi$. To see

[^1]that $\overline{\mathfrak{M}}$ is strictly condensed, note that we have $\leq \circ R \circ \leq=R$ by definition, and it follows from reflexivity and transitivity of $\leq$ that
$$
(\geq \circ S \circ \geq)=(\geq \circ \geq \circ R \circ \geq \circ \geq)=(\geq \circ R \circ \geq)=S
$$

The obvious notion of bisimulation between $H$-models is:
Definition 6.15. An $H$-bisimulation between two $H$-models $(X, \leq, R, V)$ and $\left(X^{\prime}, \leq^{\prime}, R^{\prime}, V^{\prime}\right)$ is a Bi-int-bisimulation $B$ between the underlying intuitionistic Kripke models that additional is a $\square$-zigzag and a -zigzag. (That is, both $R$ and $\breve{R}$ satisfy the zigzag conditions.) $H$-bisimilarity is denoted by $\rightleftharpoons_{H}$.

In other words, $B$ is an $H$-bisimulation if and only if it is a $\mathrm{Bi}^{- \text {int }_{\square} \text {-bisimulation }}$ between the Bi -int ${ }_{1,1}$-models $(X, \leq, R, \breve{R}, V)$ and $\left(X^{\prime}, \leq^{\prime}, R^{\prime}, \breve{R}^{\prime}, V^{\prime}\right)$. Besides, a straightforward verification shows that such an $H$-bisimulation between $\mathfrak{M}$ and $\mathfrak{M}^{\prime}$ is also a tense bisimulation between $\overline{\mathfrak{M}}$ and $\overline{\mathfrak{M}}^{\prime}$. Therefore, it preserves truth of all Tense-formulae.

For the converse, suppose $\mathfrak{M}=(X, \leq, R, V)$ and $\mathfrak{M}^{\prime}=\left(X^{\prime}, \leq^{\prime}, R^{\prime}, V^{\prime}\right)$ are two $H$-models all of whose relations are image-compact and pre-image-compact. Then $(X, \leq, R, \breve{R}, V)$ and ( $\left.X^{\prime}, \leq^{\prime}, R^{\prime}, \breve{R}^{\prime}, V^{\prime}\right)$ are strictly condensed Bi -int $\mathrm{t}_{1,1^{-}}$ models in the sense of Definition 5.1. Moreover, they satisfy all preconditions of Theorem 5.15. If $x$ and $x^{\prime}$ are two states in $\mathfrak{M}$ an $\mathfrak{M}^{\prime}$ that satisfy the same Tense-formulae, then in particular $x \leadsto \mathrm{Bi}_{\mathrm{int}}{ }^{\prime} x^{\prime}$, so by Theorem 5.15 there is a Bi -int -bisimulation $B$ linking them. But by definition $B$ is precisely an $H$-bisimulation. Summarising:

$$
x \rightleftharpoons_{H} x^{\prime} \Rightarrow x{ }^{\mu} \overbrace{\text { Tense }} x^{\prime} \Rightarrow x \overbrace{\text { Bi-int }_{\square}} x^{\prime} \Rightarrow x \rightleftharpoons_{H} x^{\prime}
$$

Thus we have proved:
Corollary 6.16. Between any two $H$-models whose relations are image-compact and pre-image-compact, we have $x{ }^{\mu}$ Tense $^{\prime} x^{\prime}$ if and only if $x \rightleftharpoons_{H} x^{\prime}$.

Remark 6.17. One might wonder why we did not employ the results from Subsection 6.4 in order to obtain a Hennessy-Milner result for $H$-models. This would require stipulating $S=(\geq \circ R \circ \geq)$ to be image-compact and pre-image-compact, on top of the preconditions of Corollary 6.16. Indeed, it does necessarily follow from $\leq$ and $R$ being (pre-)image-compact. The current approach circumvents this.

## 7. Image-Compactness Versus Saturation

We detail the relation between image-compactness and notions saturation for normal modal logic over a classical base, and for intuitionistic logic.

### 7.1. Modal Saturation in Classical Modal Logic

We can interpret classical modal logic, that is, the language Int ${ }_{\square}$, in Int $_{\square}$-models where $\leq$ is equality, and recover the classical semantics. In particular, this implies that every subset is up-closed and intuitionistic negation is the same as classical negation. Indeed, such an $\operatorname{Int}_{\square}$-model is simply a Kripke model in the usual sense. We write ML for the language of classical normal modal logic.

If the orders $\leq$ are trivial, then the definition of an $\mathrm{Int}_{\square}$-bisimulation reduces to a relation that preserves truth of proposition letters and satisfies ( $\square-\mathrm{zig}$ ) and ( $\square$-zag). In other words, it is a Kripke bisimulation for classical modal logic in the usual sense, see e.g. [6, Definition 2.16]. In this setting there is a wellknown Hennessy-Milner result for the class of so-called m-saturated models [6, Proposition 2.54]. We recall the definition of $m$-saturation.
Definition 7.1. Let $\mathfrak{M}=(X, R, V)$ be a Kripke model and $a \subseteq X$. Then a set $\Sigma$ of formulae is called satisfiable in $a$ if there exists a world $x \in a$ which satisfies each $\phi \in \Sigma$. A set $\Sigma$ is called finitely satisfiable in $a$ if every finite subset of $\Sigma$ is satisfiable in $a$. The model $\mathfrak{M}$ is called m-saturated if for all $x \in X$ and $\Sigma \subseteq \mathrm{ML}$ it satisfies:

If $\Sigma$ is finitely satisfiable in the set of successors of $x$, then $\Sigma$ is satisfiable in the set of successors of $x$.

Our results subsume the Hennessy-Milner result for m-saturated models in the following sense: a Kripke model $(X, R, V)$ is image-compact if and only if it is m -saturated. This result, together with the notion of image-compact relations for Kripke frames, also appears in [12].

Proposition 7.2. Let $\mathfrak{M}=(X, R, V)$ be a Kripke model. Then $\mathfrak{M}$ is imagecompact if and only if it is m-saturated.

Proof. Let $x \in X$ and let $\Sigma$ be a set of formulae that is finitely satisfiable in the set $R[x]$ of $R$-successors of $x$. Suppose towards a contradiction that $\Sigma$ is not satisfiable in $R[x]$. Then for each $y \in R[x]$ there is a $\phi \in \Sigma$ such that $\mathfrak{M}, y \nVdash \phi$, hence $\left\{\llbracket \neg \phi \rrbracket^{\mathfrak{M}} \mid \phi \in \Sigma\right\}$ is an open cover of $R[x]$. Note that the truth set of every formula is clopen in $\tau_{A}$. By compactness of $R[x]$ we then find a finite subset $\Sigma^{\prime} \subseteq \Sigma$ such that $R[x] \subseteq \bigcup_{\phi \in \Sigma^{\prime}} \llbracket \neg \phi \rrbracket^{\mathfrak{M}}$. But that implies that the finite set $\Sigma^{\prime}$ is not satisfiable, a contradiction with the assumption that $\Sigma$ is finitely satisfiable.

Conversely, suppose $\mathfrak{M}$ is m-saturated. Let $A=\left\{\llbracket \phi \rrbracket^{\mathfrak{M}} \mid \phi \in \mathrm{ML}\right\}$. Then clearly $(X, R, A, V)$ is a general Kripke model. We prove that $R[x]$ is compact for every $x$. By the Alexander subbase theorem it suffices to prove that every open cover consisting of subbase elements has a finite subcover, and since $-A=A$ (because $X \backslash \llbracket \phi \rrbracket^{\mathfrak{M}}=\llbracket \neg \phi \rrbracket^{\mathfrak{M}}$ by classicality) this subbase consists exclusively of truth-sets of formulae. So suppose $R[x] \subseteq \bigcup_{\phi \in \Sigma} \llbracket \phi \rrbracket^{\mathfrak{M}}$, for some set $\Sigma$ of formulae. Then clearly the set $\{\neg \phi \mid \phi \in \Sigma\}$ is not satisfiable, hence (since $\mathfrak{M}$ is m-saturated) there must be a finite $\Sigma^{\prime} \subseteq \Sigma$ such that $\left\{\neg \phi \mid \phi \in \Sigma^{\prime}\right\}$ is not satisfiable in $R[x]$. But that implies $R[x] \subseteq \bigcup_{\phi \in \Sigma^{\prime}} \llbracket \phi \rrbracket^{\mathfrak{M}}$, which gives the desired finite subcover.

In [34] the collection of descriptive Kripke models was identified as a HennessyMilner class. If $(X, R, A, V)$ is a descriptive Kripke model, then for all $\left(X, \tau_{A}\right)$ is a Stone space. Moreover $R[x]$ is closed in $\left(X, \tau_{A}\right)$ for all $x \in X$, hence compact. Therefore, the Hennessy-Milner property for the collection of descriptive Kripke models also follows from our results.

In [10], Hennessy-Milner type results are formulated for so-called weak-strict languages. Such languages are interpreted in Kripke structures. One condition for obtaining such a result, is that the models be SW-saturated (Definition 3.5.1 and Lemma 3.5.8 in op. cit.), which the prove to be equivalent to the customary notion of modal saturation in Proposition 3.5.2.

### 7.2. Saturation for Intuitionistic Logic

In 7] several Hennessy-Milner properties for Int-bisimulations on intuitionistic Kripke models are given. The strongest of these uses the notion of local saturation, an adaptation of m -saturation from Definition 7.1

Definition 7.3. An intuitionistic Kripke model $\mathfrak{M}=(X, \leq, V)$ is locally saturated if for all $x \in X$ and disjoint sets of Int-formulae $\Theta_{s}, \Theta_{r}$ the following holds: If for all finite subsets $\theta_{s} \subseteq \Theta_{s}$ and $\theta_{r} \subseteq \Theta_{r}$ there are worlds $y, y^{\prime} \in \uparrow \leq x$ such that $\mathfrak{M}, y \Vdash \bigwedge \theta_{s}$ and $y^{\prime} \Vdash \bigvee \theta_{r}$, then there is a world $z \in \uparrow \leq x$ which satisfies every formula in $\Theta_{s}$ and refutes every formula in $\Theta_{r}$.

It is shown in [7, Theorem 21] that logical equivalence on a locally saturated intuitionistic Kripke model implies Int-bisimilairty. We shall now show that an intuitionistic Kripke model is locally saturated if and only if it is image-compact. Therefore, Theorem 4.3 is equivalent to loc. cit.

Proposition 7.4. An intuitionistic Kripke model $\mathfrak{M}=(X, \leq, V)$ is locally saturated if and only if it is image-compact.

Proof. Suppose $\mathfrak{M}$ is locally saturated and let $x \in X$. Define $A=\{\llbracket \phi \rrbracket \mid \phi \in$ $\operatorname{lnt}\}$. Then clearly $(X, \leq, A)$ is a general frame. We will show that every finite subcover of $\uparrow \leq x=\{y \in X \mid x \leq y\}$ consisting of subbasic opens in $\tau_{A}$ has a finite subcover. By the Alexander subbase theorem this then proves that $\uparrow \leq x$ is compact in the topological space $\left(X, \tau_{A}\right)$. Let

$$
\begin{equation*}
\bigcup_{i \in I} \llbracket \phi_{i} \rrbracket^{\mathfrak{M}} \cup \bigcup_{j \in J}\left(X \backslash \llbracket \psi_{j} \rrbracket^{\mathfrak{M}}\right) \tag{6}
\end{equation*}
$$

be an open cover of $\uparrow \leq x$ and suppose towards a contradiction that it does not have a finite subcover. Then for every finite $I^{\prime} \subseteq I$ and $J^{\prime} \subseteq J$ there exists $y \in \uparrow \leq x$ such that $y \notin \bigcup_{i \in I^{\prime}} \llbracket \phi_{i} \rrbracket^{\mathfrak{M}} \cup \bigcup_{j \in J^{\prime}}\left(X \backslash \llbracket \psi_{j} \rrbracket^{\mathfrak{M}}\right)$, i.e., $\mathfrak{M}, y \Vdash \bigwedge_{j \in J^{\prime}} \psi_{j}$ and $\mathfrak{M}, y \Vdash \bigvee_{i \in I^{\prime}} \phi_{i}$. Thus, setting $\Theta_{s}=\left\{\psi_{j} \mid j \in J\right\}$ and $\Theta_{s}=\left\{\phi_{i} \mid i \in I\right\}$, the precondition of weak saturatedness for $\uparrow_{\leq} x$ are is satisfied. However, there is no single $y \in \uparrow \leq x$ which satisfies every $\psi_{j} \in \Theta_{s}$ and refutes every $\phi_{i} \in \Theta_{r}$, because then $y$ would not be in the open cover in (6). This contradicts the fact that $(X, \leq, V)$ is locally saturated. So the assumption that (6) has no finite subcover must be wrong, and we conclude that $\mathfrak{M}$ is image-compact.

Conversely, suppose $\mathfrak{M}$ is not locally saturated. Then there exists $x \in X$ and collections of formulae $\Theta_{s}, \Theta_{r}$ such that for all finite subsets $\theta_{s} \subseteq \Theta_{s}$ and $\theta_{r} \subseteq \Theta_{r}$ we can find $y, y^{\prime} \in \uparrow_{\leq} x$ such that $\mathfrak{M}, y \Vdash \bigwedge \theta_{s}$ and $\mathfrak{M}, y^{\prime} \Vdash \bigvee \theta_{r}$ while there is no $x$-successor which satisfies all of $\Theta_{s}$ and refutes all formulae in $\Theta_{r}$. This means that

$$
\bigcup\left\{\llbracket \phi \rrbracket^{\mathfrak{M}} \mid \phi \in \Theta_{r}\right\} \cup \bigcup\left\{\llbracket \psi \rrbracket^{\mathfrak{M}} \mid \psi \in \Theta_{s}\right\}
$$

covers $\uparrow \leq x$ but has no finite subcover. Therefore $\mathfrak{M}$ is not image-compact.

## 8. Conclusion and Further Research

We have investigated the notion of image-compactness and pre-image-compactness for relational models that can be used to interpret classical, intuitionistic, dual-intuitionistic and bi-intuitionistic (modal) logic. This notion allowed an efficient formulation of Hennessy-Milner theorems for Kripke-style bisimulations between such models. In classical modal logic and intuitionistic (nonmodal) logic, our results match well-known Hennessy-Milner results [6, Proposition 2.54], [7, Theorem 21], [34, Corollary 3.9], while for modal (dual- and bi-)intuitionistic logic we have described previously unknown Hennessy-Milner classes. In particular, the current approach generalises the results for (modal) bi-intuitionistic logic that were subject of the predecessor paper of the current paper 27].

There are many interesting directions for further research. Firstly, we have not addressed intuitionistic logic enriched with a diamond-modality, i.e., Int ${ }_{\diamond}$, interpreted in Int ${ }_{\diamond}$-models. Inspection of the proof of Theorem5.10 shows that this no longer holds for diamonds. It would be interesting to investigate conditions for which $\stackrel{\sim}{>}$ Int $_{\diamond}$ implies $\rightleftharpoons$ Int $_{\diamond}$. Dually, this then gives rise to a Hennessy-Milner theorem for dual-intuitionistic logic with a box-modality.

Second, there is the question on how to generalise this to $n$-ary box- and diamond-like operators (see e.g. [6, Definition 1.23]). These are interpreted via $(n+1)$-ary relations, i.e., $x \Vdash \diamond\left(\phi_{1}, \ldots, \phi_{n}\right)$ if there exist $y_{1}, \ldots, y_{n}$ such that $\left(x, y_{1}, \ldots, y_{n}\right) \in S$ and $y_{i} \Vdash \phi_{i}$ for all $i \in\{1, \ldots, n\}$. We expect that similar techniques as the ones presented in this paper will give rise to Hennessy-Milner properties for this generalisation of normal modal logic.

Furthermore, in [9] intuitionistic logic is interpreted in topological spaces. These are then equipped with an additional relation that is used to interpret modalities $\square$ and $\diamond$ and their tense counterparts. In case the underlying topological space is an Alexandrov space, and hence corresponds to a pre-order, the intuitionistic connectives are interpreted as usual, and the modalities like in [21]. It would be interesting to see whether notions of (pre-)image-compactness can be extended to this setting, and how they correspond to the notion saturation given in [9].

Finally, we wonder whether the notion of image-compactness can be used or adapted to obtain Hennessy-Milner results for non-normal modal extensions of classical or (dual- or bi-)intuitionistic logic. In case of monotone modal
logic over a classical base [35, 36] this has been done in 37]. It would be interesting to see how this generalises to monotone modal intuitionistic logic. Other interesting candidates for similar investigations are conditional logic [38, 39, 40] and instantial neighbourhood logic [41, 42].

Acknowledgements. We would like to thank the anonymous referees for for the comprehensive comments and suggestions. Specifically, the references to related work helped embed our paper more closely into the body of existing research.

## 9. Bibliography

## References

[1] M. Hennessy, R. Milner, Algebraic laws for nondeterminism and concurrency, Journal of the Association for Computing Machinery 32 (1) (1985) 137-161.
[2] J. v. Benthem, Modal correspondence theory, Ph.D. thesis, Mathematisch Instituut \& Instituut voor Grondslagenonderzoek, University of Amsterdam (1976).
[3] R. Milner, A Calculus of Communicating Systems, Lecture Notes in Computer Science, Springer-Verlag, Berlin, Heidelberg, 1980.
[4] D. Park, Concurrency and automata on infinite sequences, in: P. Deussen (Ed.), Theoretical Computer Science, Springer Berlin Heidelberg, Berlin, Heidelberg, 1981, pp. 167-183.
[5] P. Aczel, Non-well-founded Sets, no. 14 in CSLI Lecture Notes, CSLI Publications, Stanford, 1988.
[6] P. Blackburn, M. d. Rijke, Y. Venema, Modal Logic, Cambridge Tracts in Theoretical Computer Science, Cambridge University Press, 2001.
[7] A. Patterson, Bisimulation and propositional intuitionistic logic, in: A. Mazurkiewicz, J. Winkowski (Eds.), CONCUR '97: Concurrency Theory, Springer Berlin Heidelberg, Berlin, Heidelberg, 1997, pp. 347-360.
[8] S. Kripke, Semantical analysis of intuitionistic logic I, in: J. Crossley, M. Dummett (Eds.), Formal Systems and Recursive Functions, Vol. 40 of Studies in Logic and the Foundations of Mathematics, Elsevier, 1965, pp. 92-130.
[9] J. M. Davoren, On intuitionistic modal and tense logics and their classical companion logics: Topological semantics and bisimulations, Annals of Pure and Applied Logic 161 (2009) 349-367.
[10] F. B. Moliner, Strict-weak languages: An analysis of strict implication, Ph.D. thesis, Universitat de Barcelona (2004).
[11] A. Přenosil, A duality for distributive unimodal logic, in: R. Goré, B. Kooi, A. Kurucz (Eds.), Proc. AiML 2014, College Publications, 2014, pp. 423438.
[12] M. M. Bonsangue, M. Z. Kwiatkowska, Re-interpreting the modal $\mu$ calculus, in: Modal Logic and Process Algebra, CLSI Publications, 1995, pp. 65-83.
[13] T. Crolard, A formulae-as-types interpretation of subtractive logic, Journal of Logic and Computation 14 (4) (2004) 529-570.
[14] L. Tranchini, Natural deduction for bi-intuitionistic logic, Journal of Applied Logic 25 (2017) 72-96.
[15] G. Restall, Extending intuitionistic logic with subtraction, available at http://consequently.org/writing/ (1997).
[16] C. Rauszer, A formalization of the propositional calculus of $\mathrm{H}-\mathrm{B}$ logic, Studia Logica 33 (1) (1974) 23-34.
[17] C. Rauszer, An Algebraic and Kripke-Style Approach to a Certain Extension of Intuitionistic Logic, Dissertationes Mathematicae, Polish Scientific Publishers, 1980.
[18] R. Goré, I. Shillito, Bi-intuitionistic logics: A new instance of an old problem, in: N. Olivetti, R. Verbrugge, S. Negri, G. Sandu (Eds.), Proc. AIML 2020, 2020, pp. 269-288.
[19] M. Božić, K. Došen, Models for normal intuitionistic modal logics, Studia Logica 43 (1984) 217-245.
[20] H. Ono, On some intuitionistic modal logics, Publications of the Research Institute for Mathematical Sciences 13 (1977) 687-722.
[21] G. Fischer Servi, Semantics for a class of intuitionistic modal calculi, in: M. dalla Chiara (Ed.), Italian Studies in the Philosophy of Science, D. Reidel Publishing Company, 1981, pp. 59-72.
[22] G. Plotkin, C. Stirling, A framework for intuitionistic modal logics: extended abstract, in: Proceedings of the 1986 Conference on Theoretical Aspects of Reasoning about Knowledge, TARK '86, Morgan Kaufmann Publishers Inc., San Francisco, CA, USA, 1986, p. 399-406.
[23] F. Wolter, M. Zakharyaschev, Intuitionistic modal logic, in: A. Cantini, E. Casari, P. Minari (Eds.), Logic and Foundations of Mathematics: Selected Contributed Papers of the Tenth International Congress of Logic, Methodology and Philosophy of Science, Florence, August 1995, Springer Netherlands, Dordrecht, 1999, pp. 227-238.
[24] R. Gore, L. Postniece, A. Tiu, Cut-elimination and proof search for biintuitionistic tense logic, Advances in Modal Logic 20108 (2010) 156-177.
[25] J. Stell, R. Schmidt, D. Rydeheard, A bi-intuitionistic modal logic: Foundations and automation, Journal of Logical and Algebraic Methods in Programming 85 (2016) 500-519.
[26] K. Sano, J. Stell, Strong completeness and the finite model property for bi-intuitionistic stable tense logics, Electronic Proceedings in Theoretical Computer Science 243 (2017) 105-121.
[27] J. de Groot, D. Pattinson, Hennessy-milner properties for (modal) biintuitionistic logic, in: R. Iemhoff, M. Moortgat, R. de Queiroz (Eds.), Logic, Language, Information, and Computation, Springer Berlin Heidelberg, Berlin, Heidelberg, 2019, pp. 161-176.
[28] A. Chagrov, M. Zakharyaschev, Modal Logic, Oxford University Press, 1997.
[29] L. Esakia, Topological Kripke models, Soviet Mathematics Doklady 15 (1974) 147-151.
[30] J. Goguen, R. Burstall, Institutions: Abstract Model Theory for Specification and Programming, Journal of the Association for Computing Machinery 39 (1) (1992) 95-146.
[31] G. Badia, Bi-simulating in bi-intuitionistic logic, Studia Logica 104 (2016) 1037-1050.
[32] J. d. Groot, D. Pattinson, Modal intuitionistic logics as dialgebraic logics, in: Proc. LICS'20, Association for Computing Machinery, New York, 2020, pp. 355-369.
[33] G. Jäger, M. Marti, Intuitionistic common knowledge or belief, Journal of Applied Logic 18 (2016) 150-163.
[34] N. Bezhanishvili, G. Fontaine, Y. Venema, Vietoris bisimulations, Journal of Logic and Computation 20 (5) (2010) 1017-1040.
[35] H. Hansen, Monotonic modal logics, master's thesis, Institute for Logic, Language and Computation, University of Amsterdam (2003).
[36] H. Hansen, C. Kupke, A coalgebraic perspective on monotone modal logic, Electronic Notes in Theoretical Computer Science 106 (2004) 121-143.
[37] S. Celani, Saturated neighbourhood models of monotonic modal logics, Revista de la Unión Matemática Argentina 1 (49) (2008) 111-121.
[38] A. Baltag, G. Cinà, Bisimulation for conditional modalities, Studia Logica 106 (1) (2018) 1-33.
[39] Y. Weiss, Basic intuitionistic conditional logic, Journal of Philosophical Logic 48 (3) (2019) 447-469.
[40] I. Ciardelli, X. Liu, Intuitionistic conditional logics, Journal of Philosophical Logic 49 (2020) 807-832.
[41] J. van Benthem, N. Bezhanishvili, S. Enqvist, J. Yu, Instantial neighbourhood logic, Review of Symbolic Logic 10 (1) (2017) 116-144.
[42] N. Bezhanishvili, S. Enqvist, J. d. Groot, Duality for instantial neighbourhood logic via coalgebra, in: D. Petrişan, J. Rot (Eds.), Proc. CMCS 2020, Springer International Publishing, Cham, 2020, pp. 32-54.


[^0]:    Email addresses: jim.degroot@anu.edu.au (Jim de Groot ( $\boxtimes$ )) dirk.pattinson@anu.edu.au (Dirk Pattinson)

[^1]:    ${ }^{1}$ This is the converse of $\cup R$ in [25], which may seem odd. But verifying $\llbracket \diamond \phi \rrbracket=\llbracket \phi \rrbracket \oplus \cup R=$ $\{x \in X \mid \exists y: y(\cup R) x$ and $y \in \llbracket \phi \rrbracket\}=\{x \in X \mid \exists y: y(\leq \circ \breve{R} \circ \leq) x$ and $y \in \llbracket \phi \rrbracket\}=\{x \in$ $X \mid \exists y: x(\geq \circ R \circ \geq) y$ and $y \in \llbracket \phi \rrbracket\}$ shows that this is indeed how we interpret $\diamond$. A similar verification shows that we get the correct interpretation for

