Probabilistic squares and hexagons of opposition under coherence $\stackrel{\Leftrightarrow}{\curvearrowright}$

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Abstract

Various semantics for studying the square of opposition and the hexagon of opposition have been proposed recently. We interpret *sentences* by imprecise (set-valued) probability assessments on a finite sequence of conditional events. We introduce the *acceptability* of a sentence within coherence-based probability theory. We analyze the relations of the square and of the hexagon in terms of acceptability. Then, we show how to construct probabilistic versions of the square and of the hexagon of opposition by forming suitable tripartitions of the set of all coherent assessments. Finally, as an application, we present new versions of the square and of the hexagon involving generalized quantifiers.

Keywords: coherence, conditional events, hexagon of opposition, imprecise probability, square of opposition, quantified sentences, tripartition

1. Introduction

There is a long history of investigations on the square of opposition spanning over two millenia [4, 38]. A square of opposition represents logical key

 $^{^{\}text{tr}}$ This is a substantially extended version of a paper ([45]) presented at SMPS 2016 (Soft Methods in Probability and Statistics 2016) conference held in Rome in September 12–14, 2016.

relations among basic sentence types in a diagrammatic way. The basic sentence types, traditionally denoted by A (universal affirmative: "Every S is P"), E (universal negative: "No S is P"), I (particular affirmative: "Some S are P"), and O (particular negative: "Some S are not P"), constitute the corners of the square. The diagonals and the sides of the square of opposition are formed by the following logical relations among the basic sentence types: A and E are *contraries* (i.e., they cannot both be true), I and O are subcontraries (i.e., they cannot both be false), A and O as well as E and I are *contradictories* (i.e., they cannot both be true and they cannot both be false), I is a subaltern of A and O is a subaltern of E (i.e., A entails I and E entails O; for a visual representation see, e.g., Figure 3 below, and cover the probabilities for seeing the traditional square of opposition). In the early 1950ies, the square of opposition was expanded to the hexagon of opposition, by adding a sentence at the top and another one at the bottom of the square (see, e.g., Figure 5). Recently, the square of opposition as well as the hexagon of opposition and its extensions have been investigated from various semantic points of view (see, e.g., [3, 4, 11, 20, 21, 22, 28, 34, 35, 36]). In this paper we present a probabilistic analysis of the square of opposition under coherence, introduce the hexagon of opposition under coherence, and study the semantics of basic key relations among quantified statements.

After preliminary notions (Section 2), we introduce, based on g-coherence, a (probabilistic) notion of sentences and their acceptability and show how to construct squares of opposition under coherence from suitable tripartitions (Section 3). Then, we present an application of our square to the study of generalized quantifiers (Section 4). In Section 5 we introduce the *hexagon of opposition* under coherence. Section 6 concludes the paper by some remarks on future work.

2. Preliminary Notions

The coherence-based approach to probability and to other uncertain measures has been adopted by many authors (see, e.g., [5, 6, 9, 10, 14, 15, 16, 17, 18, 24, 26, 30, 31, 41, 42, 44]); we therefore recall only selected key features of coherence and its generalizations in this section.

An event E is a two-valued logical entity which can be either true or false. The indicator of E is a two-valued numerical quantity which is 1, or 0, according to whether the event E is true, or false, respectively. We use the same symbols for events and their indicators. We denote by \top the sure event (i.e., tautology or logical truth) and by \perp the impossible event (i.e., contradiction or logical falsehood). Moreover, given two events E and H, we denote by $E \wedge H$ (resp., $E \vee H$) conjunction (resp., disjunction). To simplify notation, we will use the product EH to denote the conjunction $E \wedge H$, which also denotes the indicator of $E \wedge H$. We denote by \overline{E} the negation of E.

Given two events E and H, with $H \neq \perp$, the conditional event E|H is defined as a three-valued logical entity which is true if EH (i.e., $E \wedge H$) is true, false if EH is true, and indetermined (void) if H is false ([19, p. 307]). In terms of the betting metaphor, if you assess p(E|H) = p, then you are willing to pay (resp., to receive) an amount p and to receive (resp., to pay) 1, or 0, or p, according to whether EH is true, or EH is true, or H is true (bet called off), respectively. For defining coherence, consider a real function $p: \mathcal{F} \to \mathcal{R}$, where \mathcal{F} is an arbitrary family of conditional events. Consider a finite sub-family $\mathcal{F}_n = (E_1|H_1, \ldots, E_n|H_n) \subseteq \mathcal{F}$, and the vector $\mathcal{P}_n = (p_1, \ldots, p_n)$, where $p_i = p(E_i | H_i)$, $i = 1, \ldots, n$. We denote by \mathcal{H}_n the disjunction $H_1 \vee \cdots \vee H_n$. With the pair $(\mathcal{F}_n, \mathcal{P}_n)$ we associate the random gain $\mathcal{G} = \sum_{i=1}^{n} s_i H_i(E_i - p_i)$, where s_1, \ldots, s_n are *n* arbitrary real numbers. \mathcal{G} represents the net gain of n transactions, where for each transaction its meaning is specified by the sign of s_i (*plus* for buying or *minus* for selling) and its scaling is specified by the magnitude of s_i . Denoting by $G_{\mathcal{H}_n}$ the set of values of \mathcal{G} restricted to \mathcal{H}_n , we recall

Definition 1. The function p defined on \mathcal{F} is called *coherent* if and only if, for every integer n, for every finite sub-family $\mathcal{F}_n \subseteq \mathcal{F}$ and for every s_1, \ldots, s_n , it holds that: $\min G_{\mathcal{H}_n} \leq 0 \leq \max G_{\mathcal{H}_n}$.

We say that p is *incoherent* if and only if p is not coherent.

As shown by Definition 1, a probability assessment is coherent if and only if, in any finite combination of n bets, it does not happen that the values in the set $G_{\mathcal{H}_n}$ are all positive, or all negative (no Dutch Book). Moreover, coherence of p(E|H) requires that $p(E|H) \in [0, 1]$ for every $E|H \in \mathcal{F}$. If p on \mathcal{F} is coherent, we call it a conditional probability on \mathcal{F} (see, e.g., [1, 17, 47]). Notice that, if p is coherent, then p also satisfies all the well known properties of finitely additive conditional probability (while the converse does not hold; see, e.g., [17, Example 13] or [23, Example 8]).

In what follows \mathcal{F} will denote finite sequence of conditional events. Let $\mathcal{F} = (E_1|H_1, \ldots, E_n|H_n)$. We denote by \mathcal{P} a (precise) probability assessment $\mathcal{P} = (p_1, \ldots, p_n)$ on \mathcal{F} , where $p_j = p(E_j|H_j) \in [0, 1], j = 1, \ldots, n$. Moreover, we denote by Π the set of all coherent precise assessments on \mathcal{F} . We recall

that when there are no logical relations among the events $E_1, H_1, \ldots, E_n, H_n$ involved in \mathcal{F} , that is $E_1, H_1, \ldots, E_n, H_n$ are logically independent, then the set Π associated with \mathcal{F} is the whole unit hypercube $[0,1]^n$. If there are logical relations, then the set Π could be a strict subset of $[0,1]^n$. As it is well known $\Pi \neq \emptyset$; therefore, $\emptyset \neq \Pi \subseteq [0,1]^n$. If not stated otherwise, we do not make any assumptions concerning logical independence.

Definition 2. An imprecise, or set-valued, assessment \mathcal{I} on a family of conditional events \mathcal{F} is a (possibly empty) set of precise assessments \mathcal{P} on \mathcal{F} .

Definition 2 states that an *imprecise (probability) assessment* \mathcal{I} on a sequence of n conditional events \mathcal{F} is just a (possibly empty) subset of $[0, 1]^n$ ([25, 27, 28]). For instance, think about an agent (like Pythagoras) who considers only rational numbers to evaluate the probability of an event E|H. Pythagoras' evaluation can be represented by the imprecise assessment $\mathcal{I} = [0, 1] \cap \mathbb{Q}$ on E|H. Moreover, a constraint like p(E|H) > 0 can be represented by the imprecise assessment $\mathcal{I} = [0, 1] \cap \mathbb{Q}$ on the imprecise assessment $\mathcal{I} = [0, 1] \cap \mathbb{Q}$ on the imprecise assessment $\mathcal{I} = [0, 1] \cap \mathbb{R}$.

Given an imprecise assessment \mathcal{I} we denote by $\overline{\mathcal{I}}$ the complementary imprecise assessment of \mathcal{I} , i.e. $\overline{\mathcal{I}} = [0,1]^n \backslash \mathcal{I}$. We now recall the notions of g-coherence and total coherence in the general case of imprecise (in the sense of set-valued) probability assessments [28].

Definition 3 (g-coherence). Given a sequence of n conditional events \mathcal{F} . An imprecise assessment $\mathcal{I} \subseteq [0,1]^n$ on \mathcal{F} is g-coherent iff there exists a coherent precise assessment \mathcal{P} on \mathcal{F} such that $\mathcal{P} \in \mathcal{I}$.

Definition 4 (t-coherence). An imprecise assessment \mathcal{I} on \mathcal{F} is totally coherent (t-coherent) iff the following two conditions are satisfied: (i) \mathcal{I} is non-empty; (ii) if $\mathcal{P} \in \mathcal{I}$, then \mathcal{P} is a coherent precise assessment on \mathcal{F} .

Definition 5 (t-coherent part). Given a sequence of n conditional events \mathcal{F} . Let Π be the set of all coherent assessments on \mathcal{F} . We denote by π : $\wp([0,1]^n) \to \wp(\Pi)$ the function defined by $\pi(\mathcal{I}) = \Pi \cap \mathcal{I}$, for any imprecise assessment $\mathcal{I} \in \wp([0,1]^n)$. Moreover, for each subset $\mathcal{I} \in \wp([0,1]^n)$ we call $\pi(\mathcal{I})$ the t-coherent part of \mathcal{I} .

Of course, if $\pi(\mathcal{I}) \neq \emptyset$, then \mathcal{I} is g-coherent and $\pi(\mathcal{I})$ is t-coherent.

3. From Imprecise Assessments to the Square of Opposition

In this section we consider imprecise assessments on a given sequence \mathcal{F} of *n* conditional events. In our approach, a sentence *s* is a pair $(\mathcal{F}, \mathcal{I})$, where $\mathcal{I} \subseteq [0, 1]^n$ is an imprecise assessment on \mathcal{F} . We introduce the following equivalence relation under t-coherence:

Definition 6. Given two sentences $s_1 : (\mathcal{F}, \mathcal{I}_1)$ and $s_2 : (\mathcal{F}, \mathcal{I}_2)$, s_1 and s_2 are equivalent (under t-coherence), denoted by $s_1 \equiv s_2$, iff $\pi(\mathcal{I}_1) = \pi(\mathcal{I}_2)$.

Definition 7. Given three sentences $s : (\mathcal{F}, \mathcal{I}), s_1 : (\mathcal{F}, \mathcal{I}_1), and s_2 : (\mathcal{F}, \mathcal{I}_2).$ We define: $s_1 \wedge s_2 : (\mathcal{F}, \mathcal{I}_1 \cap \mathcal{I}_2)$ (conjunction); $s_1 \vee s_2 : (\mathcal{F}, \mathcal{I}_1 \cup \mathcal{I}_2)$ (disjunction); $\bar{s} : (\mathcal{F}, \bar{\mathcal{I}}), where \bar{\mathcal{I}} = [0, 1]^n \backslash \mathcal{I}$ (negation).

Remark 1. As the basic operations among sentences are defined by settheoretical operations, they inherit the corresponding properties (including associativity, commutativity, De Morgan's law, etc.). Moreover, as $\pi(\mathcal{I}_1 \cap \mathcal{I}_2) = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2)$, by setting $s_1^* = (\mathcal{F}, \pi(\mathcal{I}_1))$, $s_2^* = (\mathcal{F}, \pi(\mathcal{I}_2))$ and $(s_1 \wedge s_2)^* : (\mathcal{F}, \pi(\mathcal{I}_1 \cap \mathcal{I}_2))$, it follows that $(s_1 \wedge s_2) \equiv (s_1 \wedge s_2)^* \equiv s_1^* \wedge s_2^*$. Likewise, $s_1 \vee s_2 \equiv (s_1 \vee s_2)^* \equiv s_1^* \vee s_2^*$.

As we interpret the basic sentence types involved in the square of opposition by imprecise probability assessments on sequences of conditional events, we will introduce the following notion of acceptability, which serves as a semantic bridge between basic sentence types and imprecise assessments:

Definition 8. A sentence $s : (\mathcal{F}, \mathcal{I})$ is (resp., is not) acceptable iff the assessment \mathcal{I} on \mathcal{F} is (resp., is not) g-coherent, i.e. $\pi(\mathcal{I})$ is not (resp., is) empty.

Remark 2. If $s_1 \wedge s_2$ is acceptable, then s_1 is acceptable and s_2 is acceptable. However, the converse does not hold, indeed $s_1 : (E|H, \{1\})$ is acceptable and $s_2 : (E|H), \{0\})$ is acceptable, but $s_1 \wedge s_2 : (E|H, \emptyset)$ is not acceptable (as $\pi(\emptyset) = \emptyset$).

Definition 9. Given two sentences $s_1 : (\mathcal{F}, \mathcal{I}_1)$ and $s_2 : (\mathcal{F}, \mathcal{I}_2)$, we say, under coherence: s_1 and s_2 are contraries iff the sentence $s_1 \wedge s_2$ is not acceptable;¹ s_1 and s_2 are subcontraries iff $\overline{s}_1 \wedge \overline{s}_2$ is not acceptable; s_1 and

¹ Some definitions of contrariety additionally require that " s_1 and s_2 can both be acceptable". For reasons stated in [28], we omit this additional requirement. Similarly, *mutatis mutandis*, in our definition of subcontrariety.

 s_2 are contradictories iff s_1 and s_2 are both, contraries and subcontraries; s_2 is a subaltern of s_1 iff the sentence $s_1 \wedge \overline{s}_2$ is not acceptable.

Remark 3. By Remark 1, we observe that two sentences s_1 and s_2 are contraries if and only if $\pi(\mathcal{I}_1 \cap \mathcal{I}_2) = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \emptyset$. Moreover, two sentences s_1 and s_2 are subcontraries if and only if $\pi(\overline{\mathcal{I}}_1 \cap \overline{\mathcal{I}}_2) = \pi(\overline{\mathcal{I}}_1) \cap \pi(\overline{\mathcal{I}}_2) = \emptyset$, that is (by De Morgan's law) if and only if $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2) = \Pi$. Then, two sentences s_1 and s_2 are contradictories if and only if $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2) = \Pi$, that is if and only if $s_2 = \overline{s}_1$ (and, of course, $s_1 = \overline{s}_2$). Given two sentences s_1, s_2 we also observe that s_2 is a subaltern of s_1 if and only if $\Pi \cap (\mathcal{I}_1 \cap \overline{\mathcal{I}}_2) = \emptyset$, which also amounts to say that $\Pi \cap \mathcal{I}_1 \subseteq \Pi \cap \mathcal{I}_2$, that is if and only if $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_2)$. For instance, $s_1 \vee s_2$ is a subaltern of s_1 and also of s_2 ; similarly, s_1 is a subaltern of $s_1 \wedge s_2$, and s_2 is a subaltern of $s_1 \wedge s_2$. Furthermore, if s_1 is not acceptable, that is $\pi(\mathcal{I}_1) = \emptyset$, then any sentence s_2 is a subaltern of s_1 . For example, the sentence $s_1 : (E|\overline{E}, \{1\})$ is not acceptable because $\Pi = \{0\}$ and then any sentence $s_2 : (E|\overline{E}, \mathcal{I})$, where $\mathcal{I} \subseteq [0, 1]$, is a subaltern of s_1 .

Based on the relations given in Definition 9 we define a square of opposition as follows.

Definition 10. Let $s_k : (\mathcal{F}, \mathcal{I}_k), k = 1, 2, 3, 4$, be four sentences. We call the ordered quadruple (s_1, s_2, s_3, s_4) a square of opposition (under coherence), iff the following relations among the four sentences hold:

- (a) s_1 and s_2 are contraries, i.e., $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \emptyset$;
- (b) s_3 and s_4 are subcontraries, i.e., $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) = \Pi$;
- (c) s_1 and s_4 are contradictories, i.e., $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_4) = \Pi;$ s_2 and s_3 are contradictories, i.e., $\pi(\mathcal{I}_2) \cap \pi(\mathcal{I}_3) = \emptyset$ and $\pi(\mathcal{I}_2) \cup \pi(\mathcal{I}_3) = \Pi;$
- (d) s_3 is a subaltern of s_1 , i.e., $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_3)$; s_4 is a subaltern of s_2 , i.e., $\pi(\mathcal{I}_2) \subseteq \pi(\mathcal{I}_4)$.

Figure 1 shows the square of opposition based on Definition 10.

Remark 4. Based on Definition 10, we observe that in order to verify if a quadruple of sentences (s_1, s_2, s_3, s_4) , where $s_k : (\mathcal{F}, \mathcal{I}_k)$ and k = 1, 2, 3, 4, is a

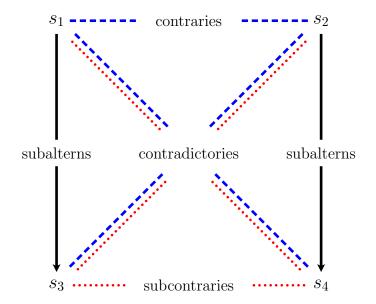


Figure 1: Probabilistic square of opposition defined by the quadruple (s_1, s_2, s_3, s_4) . The arrows indicate subalternation, dashed lines indicate contraries, and dotted lines indicate sub-contraries. Contradictories are indicated by combined dotted and dashed lines.

square of opposition, it is necessary and sufficient to check that the quadruple (s'_1, s'_2, s'_3, s'_4) , where $s'_k = (\mathcal{F}, \mathcal{I}'_k)$, $\mathcal{I}'_k = \pi(\mathcal{I}_k)$, is a square of opposition. Then, we say that two squares (s_1, s_2, s_3, s_4) and (s'_1, s'_2, s'_3, s'_4) coincide iff $\pi(\mathcal{I}_k) = \pi(\mathcal{I}'_k)$ for each k. Moreover, based on Definition 10, we observe that (s_1, s_2, s_3, s_4) is a square of opposition iff (s_2, s_1, s_4, s_3) is a square of opposition.

Definition 11. An (ordered) tripartition of a set \mathfrak{S} is a triple $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, where $\mathcal{D}_1, \mathcal{D}_2$, and \mathcal{D}_3 are subsets of \mathfrak{S} , such that the following conditions are satisfied: (i) $\mathcal{D}_i \cap \mathcal{D}_j = \emptyset$, $i \neq j$ for all i, j = 1, 2, 3; (ii); $\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3 = \mathfrak{S}$.

Theorem 1. Given any sequence of n conditional events \mathcal{F} and a quadruple (s_1, s_2, s_3, s_4) of sentences, with $s_k : (\mathcal{F}, \mathcal{I}_k)$, k = 1, 2, 3, 4. Define $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$, and $\mathcal{D}_3 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$. Then, the quadruple (s_1, s_2, s_3, s_4) is a square of opposition if and only if $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is a tripartition of (the non-empty set) Π such that: $\pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$, $\pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$.

Proof. (\Rightarrow). We assume that $\mathcal{D}_1 = \pi(\mathcal{I}_1), \mathcal{D}_2 = \pi(\mathcal{I}_2), \text{ and } \mathcal{D}_3 = \pi(\mathcal{I}_3) \cap$ $\pi(\mathcal{I}_4)$. Of course, $\mathcal{D}_i \subseteq \Pi$, i = 1, 2, 3. We now prove that: (i) $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$; (*ii*) $\mathcal{D}_3 = \prod (\mathcal{D}_1 \cup \mathcal{D}_2)$. (*i*) From condition (a) in Definition 10, as s_1 and s_2 are contraries, it follows that $\mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$. (ii) We first prove that $\mathcal{D}_3 \subseteq \Pi \setminus (\mathcal{D}_1 \cup \mathcal{D}_2)$. This trivially follows when $\mathcal{D}_3 = \emptyset$. If $\mathcal{D}_3 \neq \emptyset$, then let $x \in \mathcal{D}_3 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$. As $x \in \pi(\mathcal{I}_3)$, from condition (c) in Definition 10, we obtain $x \notin \pi(\mathcal{I}_2)$. Likewise, as $x \in \pi(\mathcal{I}_4)$, from condition (c) in Definition 10, we obtain $x \notin \pi(\mathcal{I}_1)$. Then, $x \in \Pi$ and $x \notin (\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2))$, that is $x \in \Pi \setminus (\mathcal{D}_1 \cup \mathcal{D}_1)$ \mathcal{D}_2). We now prove that $\Pi \setminus (\mathcal{D}_1 \cup \mathcal{D}_2) \subseteq \mathcal{D}_3$. This trivially follows when $\Pi \setminus (\mathcal{D}_1 \cup \mathcal{D}_2) = \emptyset$. If $\Pi \setminus (\mathcal{D}_1 \cup \mathcal{D}_2) \neq \emptyset$, let $x \in \Pi \setminus (\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_2))$. As $x \in \mathbb{N}$ $\Pi \setminus \pi(\mathcal{I}_1)$, from condition (c) in Definition 10, we obtain $x \in \pi(\mathcal{I}_4)$. Likewise, as $x \in \Pi \setminus \pi(\mathcal{I}_2)$ from condition (c) in Definition 10, we obtain $x \in \pi(\mathcal{I}_3)$. Then, $x \in (\pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)) = \mathcal{D}_3$. Therefore $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ is a tripartition of Π . By our assumption, $\pi(\mathcal{I}_1) = \mathcal{D}_1$ and $\pi(\mathcal{I}_2) = \mathcal{D}_2$. We observe that $\pi(\mathcal{I}_3) \cap \mathcal{D}_3 = \mathcal{D}_3$; moreover, from conditions (c) and (d), we obtain $\pi(\mathcal{I}_3) \cap$ $\mathcal{D}_2 = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_2) = \emptyset$ and $\pi(\mathcal{I}_3) \cap \mathcal{D}_1 = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_3) = \pi(\mathcal{I}_1) = \mathcal{D}_1;$ then $\pi(\mathcal{I}_3) = \pi(\mathcal{I}_3) \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) = \mathcal{D}_1 \cup \mathcal{D}_3$. Likewise, we observe that $\pi(\mathcal{I}_4) \cap \mathcal{D}_3 = \mathcal{D}_3$; moreover, from conditions (c),(d) in Definition 10, we obtain $\mathcal{D}_1 \cap \pi(\mathcal{I}_4) = \pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \emptyset \text{ and } \mathcal{D}_2 \cap \pi(\mathcal{I}_4) = \pi(\mathcal{I}_2) \cap \pi(\mathcal{I}_4) = \pi(\mathcal{I}_2) = \mathcal{D}_2;$ then $\pi(\mathcal{I}_4) = \pi(\mathcal{I}_4) \cap (\mathcal{D}_1 \cup \mathcal{D}_2 \cup \mathcal{D}_3) = \mathcal{D}_2 \cup \mathcal{D}_3.$ (\Leftarrow) Assume that $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, where $\mathcal{D}_1 = \pi(\mathcal{I}_1), \mathcal{D}_2 = \pi(\mathcal{I}_2), \mathcal{D}_3 = \pi(\mathcal{I}_3) \cap$ $\pi(\mathcal{I}_4)$, is a tripartition of Π such that $\mathcal{D}_1 \cup \mathcal{D}_3 = \pi(\mathcal{I}_3)$ and $\mathcal{D}_2 \cup \mathcal{D}_3 = \pi(\mathcal{I}_4)$, we prove that the quadruple (s_1, s_2, s_3, s_4) satisfies conditions (a), (b), (c), and (d) in Definition 10. We observe that $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_2) = \mathcal{D}_1 \cap \mathcal{D}_2 = \emptyset$, which coincides with (a). Condition (b) is satisfied because $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) =$ $\mathcal{D}_1 \cup \mathcal{D}_3 \cup \mathcal{D}_2 \cup \mathcal{D}_3 = \Pi$. Moreover, $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \mathcal{D}_1 \cap (\mathcal{D}_2 \cup \mathcal{D}_3) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_4) = \mathcal{D}_1 \cup (\mathcal{D}_2 \cup \mathcal{D}_3) = \Pi$; likewise, $\pi(\mathcal{I}_2) \cap \pi(\mathcal{I}_3) = \mathcal{D}_2 \cap$ $(\mathcal{D}_1 \cup \mathcal{D}_3) = \emptyset$ and $\pi(\mathcal{I}_2) \cup \pi(\mathcal{I}_3) = \mathcal{D}_2 \cup (\mathcal{D}_1 \cup \mathcal{D}_3) = \Pi$. Thus, the conditions in (c) are satisfied. Finally, $\pi(\mathcal{I}_1) = \mathcal{D}_1 \subseteq \mathcal{D}_1 \cup \mathcal{D}_3 = \pi(\mathcal{I}_3)$ and $\pi(\mathcal{I}_2) = \mathcal{D}_2 \subseteq \mathcal{D}_2 \cup \mathcal{D}_3 = \pi(\mathcal{I}_4)$ which satisfy conditions in (d).

A method to construct a square of opposition by starting from a tripartition of Π is given in the following result (see also [20]).

Corollary 1. Given any sequence of n conditional events \mathcal{F} and a tripartition $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ of Π , then the quadruple (s_1, s_2, s_3, s_4) , with $s_k : (\mathcal{F}, \mathcal{I}_k)$, k = 1, 2, 3, 4 and $\pi(\mathcal{I}_1) = \mathcal{D}_1, \pi(\mathcal{I}_2) = \mathcal{D}_2, \pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3, \pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$ is a square of opposition. *Proof.* The proof immediately follows by observing $\pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4) = \mathcal{D}_3$ and by the (\Leftarrow) side proof of Theorem 1.

The following result allows to construct a square of opposition by starting from a tripartition of the whole set $[0, 1]^n$:

Corollary 2. Given a tripartition $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of $[0, 1]^n$, let $\mathcal{I}_1 = \mathcal{B}_1, \mathcal{I}_2 = \mathcal{B}_2, \mathcal{I}_3 = \mathcal{B}_1 \cup \mathcal{B}_3$, and $\mathcal{I}_4 = \mathcal{B}_2 \cup \mathcal{B}_3$. For any sequence of n conditional events \mathcal{F} , the quadruple (s_1, s_2, s_3, s_4) , where $s_k : (\mathcal{F}, \mathcal{I}_k), k = 1, 2, 3, 4$, is a square of opposition.

Proof. Let \mathcal{F} be any sequence of n conditional events and Π be the associated set of all coherent precise assessments. We set $\mathcal{D}_i = \pi(\mathcal{B}_i)$, i = 1, 2, 3. Of course, $(\pi(\mathcal{B}_1), \pi(\mathcal{B}_2), \pi(\mathcal{B}_3))$ is a tripartition of Π . Moreover, $\pi(\mathcal{I}_1) = \mathcal{D}_1$, $\pi(\mathcal{I}_2) = \mathcal{D}_2, \pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3, \pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$. Then, by applying Corollary 1 we obtain that (s_1, s_2, s_3, s_4) is a square of opposition.

Traditionally the square of opposition can be constructed based on the fragmented square of opposition which requires only the contrariety and contradiction relations (which goes back to Aristotle's *De Interpretatione* 6–7, 17b.17–26, see [38, Section 2]). This result also holds in our framework:

Theorem 2. The quadruple (s_1, s_2, s_3, s_4) of sentences, with $s_k : (\mathcal{F}, \mathcal{I}_k)$, k = 1, 2, 3, 4, is a square of opposition iff relations (a) and (c) in Definition 10 are satisfied.

Proof. (\Rightarrow) It follows directly from Definition 10. (\Leftarrow) We prove that (d) and (b) in Definition 10 follow from (a) and (c). If $\pi(\mathcal{I}_1) = \emptyset$, then of course $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_3)$. If $\pi(\mathcal{I}_1) \neq \emptyset$, let $x \in \pi(\mathcal{I}_1) \subseteq \Pi$, from (a) it follows that $x \notin \pi(\mathcal{I}_2)$, and since (c) requires $\pi(\mathcal{I}_2) \cup \pi(\mathcal{I}_3) = \Pi$, we obtain $x \in \pi(\mathcal{I}_3)$. Thus, $\pi(\mathcal{I}_1) \subseteq \pi(\mathcal{I}_3)$; likewise, $\pi(\mathcal{I}_2) \subseteq \pi(\mathcal{I}_4)$. Therefore, (d) is satisfied. Now we prove that (b) is satisfied, i.e. $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) = \Pi$. Of course, $\pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4) \subseteq \Pi$. Let $x \in \Pi$. If $x \notin \pi(\mathcal{I}_3)$, then, $x \in \pi(\mathcal{I}_2)$ from (c). Moreover, from (d), $x \in \pi(\mathcal{I}_4)$. Then, $\Pi \subseteq \pi(\mathcal{I}_3) \cup \pi(\mathcal{I}_4)$. Therefore, (b) is satisfied.

Corollary 3. The quadruple (s_1, s_2, s_3, s_4) of sentences, with $s_k : (\mathcal{F}, \mathcal{I}_k)$, k = 1, 2, 3, 4, is a square of opposition if and only if $(s_1, s_2, s_3, s_4) = (s_1, s_2, \overline{s_2}, \overline{s_1})$ with s_1 and s_2 being contraries.

$$\begin{array}{c|cccc} \mathcal{B}_2(x) & \mathcal{B}_3(x) & \mathcal{B}_1(x) \\ \hline 0 & 1-x & x & 1 \end{array}$$

Figure 2: Example of a tripartition $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$ of [0, 1], with $x \in]\frac{1}{2}, 1]$.

Proof. Of course, if (s_1, s_2, s_3, s_4) is a square of opposition, then s_1 and s_2 are contraries. Moreover, s_1 and s_4 are contradictories, that is: $\pi(\mathcal{I}_1) \cap \pi(\mathcal{I}_4) = \emptyset$ and $\pi(\mathcal{I}_1) \cup \pi(\mathcal{I}_4) = \Pi$. Therefore, $\Pi \setminus \pi(\mathcal{I}_4) = \pi(\mathcal{I}_1)$, which amounts to $s_4 = \bar{s}_1$. Similarly, as s_2 and s_3 are contradictories, it holds that $s_3 = \bar{s}_2$. Conversely, assume that s_1 and s_2 are contraries. By instantiating Theorem 2 with $s_3 = \bar{s}_2$ and with $s_4 = \bar{s}_1$, it follows that the quadruple $(s_1, s_2, \bar{s}_2, \bar{s}_1)$ is a square of opposition.

4. Square of Opposition and Generalized Quantifiers

Let \mathcal{F} be a conditional event P|S (where $S \neq \bot$) and $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$ be a tripartition of [0, 1], where $\mathcal{B}_1(x) = [x, 1], \mathcal{B}_2(x) = [0, 1 - x], \mathcal{B}_3(x) = [1 - x, x[\text{ and } x \in]\frac{1}{2}, 1]$ (see Figure 2).

Consider the quadruple of sentences (A(x), E(x), I(x), O(x)), with A(x): $(P|S, \mathcal{I}_{A(x)}), E(x) : (P|S, \mathcal{I}_{E(x)}), I(x) : (P|S, \mathcal{I}_{I(x)}), O(x) : (P|S, \mathcal{I}_{O(x)}),$ where $\mathcal{I}_{A(x)} = \mathcal{B}_1(x) = [x, 1], \ \mathcal{I}_{E(x)} = \mathcal{B}_2(x) = [0, 1 - x], \ \mathcal{I}_{I(x)} =$ $\mathcal{B}_1(x) \cup \mathcal{B}_3(x) = [1 - x, 1], \text{ and } \mathcal{I}_{O(x)} = \mathcal{B}_2(x) \cup \mathcal{B}_3(x) = [0, x[.$ By applying Corollary 2 with $(s_1, s_2, s_3, s_4) = (A(x), E(x), I(x), O(x))$, it follows that (A(x), E(x), I(x), O(x)) is a square of opposition for any $x \in [\frac{1}{2}, 1]$ (see Figure 3). We recall that in presence of some logical relations between Pand S the set Π could be a strict subset of [0,1]. In particular, we have the following three cases (see, [29, 30]): (i) if $P \wedge S \neq \bot$ and $P \wedge S \neq S$, then $\Pi = [0, 1];$ (ii) if $P \wedge S = S$, then $\Pi = \{1\};$ (iii) if $P \wedge S = \bot$, then $\Pi = \{0\}$. The quadruple (A(x), E(x), I(x), O(x)), with the threshold $\frac{1}{2} < x \leq 1$, is a square of opposition in each of the three cases. In particular we obtain: case (i) $\pi(\mathcal{I}_{A(x)}) = \mathcal{I}_{A(x)}, \ \pi(\mathcal{I}_{E(x)}) = \mathcal{I}_{E(x)}, \\ \pi(\mathcal{I}_{I(x)}) = \mathcal{I}_{I(x)}, \ \text{and} \ \pi(\mathcal{I}_{O(x)}) = \mathcal{I}_{O(x)};$ case (ii): $\pi(\mathcal{I}_{A(x)}) = \{1\}, \ \pi(\mathcal{I}_{E(x)}) = \emptyset, \ \pi(\mathcal{I}_{I(x)}) = \{1\}, \ \text{and} \ \pi(\mathcal{I}_{O(x)}) = \emptyset;$ case (iii): $\pi(\mathcal{I}_{A(x)}) = \emptyset, \ \pi(\mathcal{I}_{E(x)}) = \{1\}, \ \pi(\mathcal{I}_{I(x)}) = \emptyset, \ \text{and} \ \pi(\mathcal{I}_{O(x)}) = \{1\}.$ We note that in cases (ii) and (iii) we obtain degenerated squares each, where—apart from the contradictory relations—all relations are strengthened. Specifically, both contrary and the subcontrary become contradictory

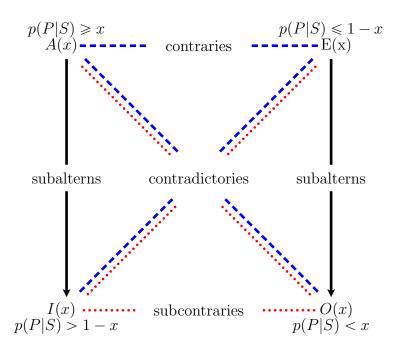


Figure 3: Probabilistic square of opposition $\mathbf{S}(x)$ defined on the four sentence types (A(x), E(x), I(x), O(x)) with the threshold $x \in]\frac{1}{2}, 1]$ (see also Table 1). It provides a new interpretation of the traditional square of opposition (see, e.g., [38]), where the corners are labeled by "Every S is P" (A), "No S is P" (E), "Some S is P" (I), and "Some S is not P" (O).

relations. Moreover, both subalternation relations become symmetric. As by coherence $p(P|S) + p(\bar{P}|S) = 1$, a sentence $s : (P|S, \mathcal{I})$ is equivalent to the sentence $s' : (\bar{P}|S, \bar{\mathcal{I}})$, where $\bar{\mathcal{I}} = [0, 1] \setminus \mathcal{I}$. Table 1 presents generalization of basic sentence types A(x), E(x), I(x), and O(x) involving generalized quantifiers Q. The generalized quantifiers are defined on a threshold $x > \frac{1}{2}$. The value of the threshold may be context dependent and provides lots of flexibility for modeling various instances of generalized quantifiers (like "most", "almost all").

Given two thresholds x_1 and x_2 , with $\frac{1}{2} < x_2 < x_1 \leq 1$, we analyze the relations among the same sentence types in the two squares of opposition $\mathbf{S}(x_1)$ and $\mathbf{S}(x_2)$, with $\mathbf{S}(x_i) = (A(x_i), E(x_i), I(x_i), O(x_i))$, i = 1, 2. It can be easily proved that: $A(x_2)$ is a subaltern of $A(x_1)$, $E(x_2)$ is a subaltern of

| Sentence | | Probability constraints | Assessment on $P S$ |
|----------|-------------------------------------|----------------------------|--|
| A(x): | $(Q_{\geq x} S \text{ are } P)$ | $p(P S) \ge x$ | $\mathcal{I}_{A(x)} = [x, 1]$ |
| E(x): | $(Q_{\geq x} S \text{ are not } P)$ | $p(\bar{P} S) \geqslant x$ | $\mathcal{I}_{E(x)} = [0, 1 - x]$ |
| I(x): | $(Q_{>1-x} S \text{ are } P)$ | p(P S) > 1 - x | $\mathcal{I}_{I(x)} =]1 - x, 1]$ |
| O(x): | $(Q_{>1-x} S \text{ are not } P)$ | $p(\bar{P} S) > 1 - x$ | $\mathcal{I}_{O(x)} = \left[0, x\right[$ |
| A(1): | (Every S is P) | p(P S) = 1 | $\mathcal{I}_A = \{1\}$ |
| E(1): | (No S is P) | $p(\bar{P} S) = 1$ | $\mathcal{I}_E = \{0\}$ |
| I(1): | (Some S is P) | p(P S) > 0 | $\mathcal{I}_I =]0, 1]$ |
| O(1): | (Some S is not P) | $p(\bar{P} S) > 0$ | $\mathcal{I}_O = [0, 1[$ |

Table 1: Probabilistic interpretation of the sentence types A, E, I, and O involving generalized quantifiers Q defined by a threshold x (with $x \in]\frac{1}{2}, 1]$) on the subject S and predicate P and the respective imprecise probabilistic assessments $\mathcal{I}_{A(x)}, \mathcal{I}_{E(x)}, \mathcal{I}_{I(x)}, and$ $\mathcal{I}_{O(x)}$ on the conditional event P|S (above). When x = 1, we obtain our probabilistic interpretation of the traditional sentence types A, E, I, and O (below).

 $E(x_1)$, $I(x_1)$ is a subaltern of $I(x_2)$, and $O(x_1)$ is a subaltern of $O(x_2)$. In the extreme case x = 1 we obtain the probabilistic interpretation under coherence of the basic sentence types involved in the traditional square of opposition (A, E, I, O) (see [27, 28] for the *default square of opposition*, involving defaults and negated defaults).

In agreement with De Morgan (as pointed out by [20]) by the quadruple (a, e, i, o) we denotes the square of opposition obtained from (A, E, I, O) when the events P and S are replaced by \overline{P} and \overline{S} , respectively. Specifically, $a: (\overline{P}|\overline{S}, \{1\}), e: (\overline{P}|\overline{S}, \{0\}), i: (\overline{P}|\overline{S}, [0, 1]), \text{ and } o: (\overline{P}|\overline{S}, [0, 1[).$

In the general case when P and S are logically independent it can be proved that the set of all coherent assessments on $(P|S, \overline{P}|\overline{S})$ is the square $[0,1]^2$ (see e.g. [17]; see also [14, Proposition 1] [15, Theorem 4]). Thus, in the general case there are no relations between any two sentences s_1 and s_2 , where $s_1 \in \{A, E, I, O\}$ and $s_2 \in \{a, e, i, o\}$. Therefore, the two squares (A, E, I, O) and (a, e, i, o) do not form a cube of opposition (with these two squares as opposite facing sides).

5. Hexagon of Opposition

Compared to the millennia long history of investigations on the square of opposition, the hexagon of opposition was discovered fairly recently, namely in the 1950ies. The hexagon generalizes the square by adding the disjunction of the top vertices of the square to build a new vertex at the top and by adding the conjunction of the bottom vertices of the square to build a new vertex at the bottom. According to Béziau ([2]), the hexagon of opposition was introduced by the French priest and logician Augustin Sesmat ([48]) and by the philosopher Robert Blanché ([7]), who worked out the full structure of the hexagon of opposition (for his main work on the hexagon of opposition see [8]). Jaspers and Seuren ([33]) trace the history of the hexagon back also to the American philosopher Paul Jacoby ([32], see also [20]). In this section we will use the tools developed in Section 3, to construct a *hexagon of opposition* by starting from a square of opposition. More precisely, given a traditional square of opposition (A, E, I, O), by setting $U = A \lor E, Y = I \land O$, the tuple (A, E, I, O, U, Y) defines a hexagon of opposition. Accordingly, we define the (probabilistic) hexagon of opposition in our approach as follows:

Definition 12 (Hexagon of opposition). Let $s_k : (\mathcal{F}, \mathcal{I}_k), k = 1, 2, 3, 4, 5, 6$, be six sentences. We call the ordered tuple $(s_1, s_2, s_3, s_4, s_5, s_6)$ a hexagon of opposition (under coherence), if and only if the following relations among the six sentences hold:

- (i) (s_1, s_2, s_3, s_4) is a square of opposition;
- (*ii*) $s_5 = s_1 \lor s_2$;
- (*iii*) $s_6 = s_3 \wedge s_4$.

Figure 4 shows the probabilistic hexagon of opposition as given by Definition 12.

Theorem 3. Let $s_k : (\mathcal{F}, \mathcal{I}_k)$, k = 1, 2, 3, 4, 5, 6, be six sentences. The tuple $(s_1, s_2, s_3, s_4, s_5, s_6)$ is a hexagon of opposition, if and only if $(s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_2, \overline{s}_2, \overline{s}_1, s_1 \lor s_2, \overline{s}_1 \land \overline{s}_2)$, with s_1 and s_2 being contraries.

Proof. (\Rightarrow). Let $(s_1, s_2, s_3, s_4, s_5, s_6)$ be a hexagon of opposition. Then, as (s_1, s_2, s_3, s_4) is a square of opposition, s_1 and s_2 are contraries. Moreover, by Corollary 3, it follows that $(s_1, s_2, s_3, s_4) = (s_1, s_2, \overline{s_2}, \overline{s_1})$. Then, by Definition 12, $s_5 = s_1 \lor s_2$ and $s_6 = s_3 \land s_4 = \overline{s_1} \land \overline{s_2}$. Therefore, $(s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_2, \overline{s_2}, \overline{s_1}, s_1 \lor s_2, \overline{s_1} \land \overline{s_2})$.

 (\Leftarrow) . Let $(s_1, s_2, s_3, s_4, s_5, s_6) = (s_1, s_2, \overline{s}_2, \overline{s}_1, s_1 \lor s_2, \overline{s}_1 \land \overline{s}_2)$, with s_1 and s_2 being contraries. From Corollary 3, it follows that (s_1, s_2, s_3, s_4) is a square

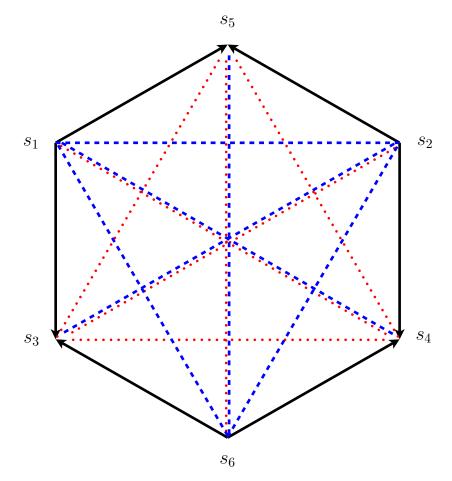


Figure 4: Probabilistic hexagon of opposition defined on the six sentence types $(s_1, s_2, s_3, s_4, s_5, s_6)$, where (s_1, s_2, s_3, s_4) is a square of opposition, $s_5 = s_1 \lor s_2$, and $s_6 = s_3 \land s_4$ (see Definition 12). For the meaning of the lines see Figure 1.

of opposition. Then, by relations (ii) and (iii) in Definition 12, it follows that $(s_1, s_2, s_3, s_4, s_5, s_6)$ is a hexagon of opposition.

Remark 5. Assume that s_1 and s_2 are contraries. Then, by Corollary 3, the quadruple $(s_1, s_2, \overline{s}_2, \overline{s}_1)$ is a square of opposition, and by Definition 12, the tuple $(s_1, s_2, \overline{s}_2, \overline{s}_1, s_1 \lor s_2, \overline{s}_1 \land \overline{s}_2)$ is a hexagon of opposition.

We now consider relations among a tripartition of the set of all coherent assessments Π and a hexagon of opposition.

Remark 6. Given a hexagon of opposition $(s_1, s_2, s_3, s_4, s_5, s_6)$, we observe that the sentence $s_6 = s_3 \wedge s_4$ represents the pair $(\mathcal{F}, \mathcal{I}_6)$, where $\mathcal{I}_6 = \mathcal{I}_3 \cap \mathcal{I}_4$. Moreover, by Remark 1, $\pi(\mathcal{I}_6) = \pi(\mathcal{I}_3 \cap \mathcal{I}_4) = \pi(\mathcal{I}_3) \cap \pi(\mathcal{I}_4)$. Therefore, based on Theorem 1, the triple $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$, where $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$, and $\mathcal{D}_3 = \pi(\mathcal{I}_6)$, is a tripartition of Π . Conversely, based on Corollary 1, given a tripartition $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ of Π , the sequence $(s_1, s_2, s_3, s_4, s_5, s_6)$ where $s_k : (\mathcal{F}, \mathcal{I}_k), \ k = 1, \ldots, 6$, with $\pi(\mathcal{I}_1) = \mathcal{D}_1, \ \pi(\mathcal{I}_2) = \mathcal{D}_2, \ \pi(\mathcal{I}_3) = \mathcal{D}_1 \cup \mathcal{D}_3, \ \pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3, \ \pi(\mathcal{I}_5) = \mathcal{D}_1 \cup \mathcal{D}_2, \ and \ \pi(\mathcal{I}_6) = \mathcal{D}_3$, is a hexagon of opposition (see also [11, 20, 21]).

Next, we consider relations among a tripartition of $[0, 1]^n$ and a hexagon of opposition.

Remark 7. Based on Corollary 2, we can also construct a hexagon of opposition by starting from a tripartition of the whole set $[0,1]^n$. Specifically, given a tripartition $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$ of $[0,1]^n$, let $\mathcal{I}_1 = \mathcal{B}_1, \mathcal{I}_2 = \mathcal{B}_2, \mathcal{I}_3 = \mathcal{B}_1 \cup \mathcal{B}_3$, $\mathcal{I}_4 = \mathcal{B}_2 \cup \mathcal{B}_3, \mathcal{I}_5 = \mathcal{B}_1 \cup \mathcal{B}_2$, and $\mathcal{I}_6 = \mathcal{B}_3$. For any sequence of n conditional events \mathcal{F} , the tuple $(s_1, s_2, s_3, s_4, s_5, s_6)$, where $s_k : (\mathcal{F}, \mathcal{I}_k), k = 1, \ldots, 6$, is a hexagon of opposition.

Theorem 4. Given a hexagon of opposition $(s_1, s_2, s_3, s_4, s_5, s_6)$, by Definition 12 all relations among the basic sentence types in the square (s_1, s_2, s_3, s_4) hold. Moreover, by Theorem 3 (and also by Remark 3), the following relations hold:

- (i) s_1 and s_6 are contraries (since $s_6 = \bar{s}_2 \wedge \bar{s}_1$ and $\pi(\mathcal{I}_1 \cap \bar{\mathcal{I}}_2 \cap \bar{\mathcal{I}}_1) = \emptyset$);
- (ii) s_2 and s_6 are contraries (since $s_6 = \bar{s}_2 \wedge \bar{s}_1$ and $\pi(\mathcal{I}_2 \cap \bar{\mathcal{I}}_2 \cap \bar{\mathcal{I}}_1) = \emptyset$);
- (iii) s_3 is a subaltern of s_6 (since $s_6 = s_3 \wedge s_4$);

- (iv) s_4 is a subaltern of s_6 (since $s_6 = s_3 \wedge s_4$);
- (v) s_5 is a subaltern of s_1 (since $s_5 = s_1 \lor s_2$);
- (vi) s_5 is a subaltern of s_2 (since $s_5 = s_1 \lor s_2$);
- (vii) s_5 and s_3 are subcontraries (as $s_5 = s_1 \lor s_2$ and $s_3 = \overline{s}_2$, hence $\pi(\mathcal{I}_1 \cup \mathcal{I}_2) \cup \overline{\mathcal{I}}_2) = \Pi$);
- (viii) s_5 and s_4 are subcontraries (as $s_5 = s_1 \lor s_2$ and $s_4 = \bar{s}_1$, hence $\pi(\mathcal{I}_1 \cup \mathcal{I}_2) \cup \bar{\mathcal{I}}_1) = \Pi$);
 - (ix) s_5 and s_6 are contradictories (as $s_5 = s_1 \lor s_2$, $s_6 = s_3 \land s_4 = \bar{s}_2 \land \bar{s}_1$, hence $\pi((\mathcal{I}_1 \cup \mathcal{I}_2) \cap (\bar{\mathcal{I}}_1 \cap \bar{\mathcal{I}}_2)) = \emptyset$) and $\pi((\mathcal{I}_1 \cup \mathcal{I}_2) \cup (\bar{\mathcal{I}}_1 \cap \bar{\mathcal{I}}_2)) = \Pi)$.

Figure 4 illustrates all the relations in the hexagon of opposition described in Theorem 4. This figure also shows the two triangles $T_1 : (s_1, s_2, s_6)$ and $T_2 : (s_3, s_4, s_5)$. We note that the sides of T_1 consist of contrary relations, whereas the sides of T_2 consist of subcontrary relations. Moreover, the coherent part of the imprecise assessments defined by sentences in T_1 (i.e., $\mathcal{D}_1 = \pi(\mathcal{I}_1)$, $\mathcal{D}_2 = \pi(\mathcal{I}_2)$ and $\mathcal{D}_3 = \pi(\mathcal{I}_6)$) forms a tripartition $(\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ of Π . Whereas, the imprecise assessments defined by sentences in T_2 are such that $\pi(\mathcal{I}_3) =$ $\mathcal{D}_1 \cup \mathcal{D}_3, \pi(\mathcal{I}_4) = \mathcal{D}_2 \cup \mathcal{D}_3$, and $\pi(\mathcal{I}_5) = \mathcal{D}_1 \cup \mathcal{D}_2$.

By basing the hexagon of opposition on the square of opposition (A(x), E(x), I(x), O(x)) (as introduced in Section 4) we obtain the following hexagon of opposition: (A(x), E(x), I(x), O(x), U(x), Y(x)) with $x \in]1/2, 1]$, where U(x) denotes $A(x) \lor E(x)$ and Y(x) denotes $I(x) \land O(x)$ (see Table 2). Figure 5 illustrates the hexagon (A(x), E(x), I(x), O(x), U(x), Y(x)) with $x \in]1/2, 1]$.

We now consider a generalization of the hexagon of opposition (A(x), E(x), I(x), O(x), U(x), Y(x)) by considering *n* conditional events. In particular, let $\mathcal{F} = (P_1|S_1, \ldots, P_n|S_n)$ be a sequence of *n* conditional events. Exploiting Remark 7, we construct a hexagon of opposition by considering the following tripartition of $[0, 1]^n$: $(\mathcal{B}_1(x), \mathcal{B}_2(x), \mathcal{B}_3(x))$, with $x \in [1/2, 1]$, where

$$\mathcal{B}_1(x) = \{ (p_1, \dots, p_n) \in [0, 1]^n : \sum_{i=1}^n \frac{p_i}{n} \ge x \}, \\ \mathcal{B}_2(x) = \{ (p_1, \dots, p_n) \in [0, 1]^n : \sum_{i=1}^n \frac{p_i}{n} \le 1 - x \}, \\ \mathcal{B}_3(x) = \{ (p_1, \dots, p_n) \in [0, 1]^n : 1 - x < \sum_{i=1}^n \frac{p_i}{n} < x \}.$$

We obtain the following (generalized) hexagon of opposition (A(x), E(x), I(x), O(x), U(x), Y(x)), with the quantified statements

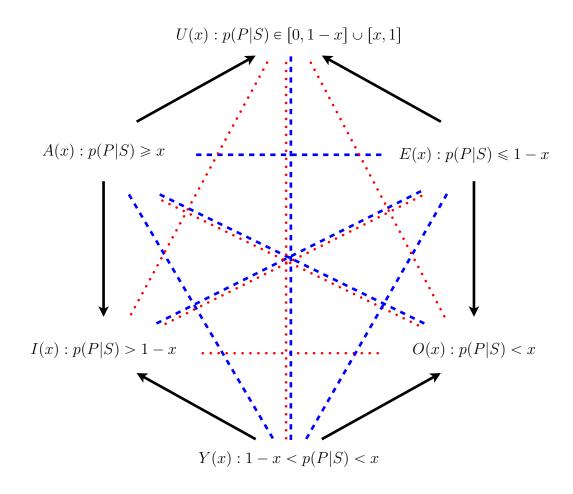


Figure 5: Probabilistic hexagon of opposition defined on the six sentence types with the threshold $x \in]\frac{1}{2}, 1]$ (see also Table 1). It provides a new interpretation of the hexagon of opposition, which we compose of the probabilistic square of opposition and the two additional vertices U(x) (i.e., $A(x) \vee E(x)$; top) and Y(x) (i.e., $I(x) \wedge O(x)$; bottom). For the meaning of the lines see Figure 1.

| Sentence | | Probability constr. | Assessment on $P S$ |
|----------|--|---|---|
| U(x): | $A(x) \lor E(x)$ | $p(P S) \ge x$ or $p(\bar{P} S) \ge x$ | $\mathcal{I}_{U(x)} = [0, 1 - x] \cup [x, 1]$ |
| Y(x): | $I(x) \wedge O(x)$ | $\begin{cases} p(P S) > 1 - x\\ p(\bar{P} S) > 1 - x \end{cases}$ | $\mathcal{I}_{Y(x)} =]1 - x, x[$ |
| U(1): | Every S is P or No S is P | p(P S) = 1 or $p(\overline{P} S) = 1$ | $\mathcal{I}_U = \{0\} \cup \{1\}$ |
| Y(1): | Some S is P and Some S is \overline{P} | $ \left\{ \begin{array}{l} p(P S) > 0 \\ p(\bar{P} S) > 0 \end{array} \right. $ | $\mathcal{I}_Y =]0,1[$ |

Table 2: Probabilistic interpretation of the sentence types at the top (U) and at the bottom (Y) of the hexagon of opposition involving generalized quantifiers Q defined by a threshold x (with $x \in]\frac{1}{2}, 1$) on the subject S and predicate P and the respective imprecise probabilistic assessments $\mathcal{I}_{U(x)}$, and $\mathcal{I}_{Y(x)}$ on the conditional event P|S (above). When x = 1, we obtain our probabilistic interpretation of the traditional sentence types U, Y.

 $A(x) : (\mathcal{F}, \mathcal{I}_{A(x)}), E(x) : (\mathcal{F}, \mathcal{I}_{E(x)}), I(x) : (\mathcal{F}, \mathcal{I}_{I(x)}), O(x) : (\mathcal{F}, \mathcal{I}_{O(x)}), U(x) : (\mathcal{F}, \mathcal{I}_{U(x)}), Y(x) : (\mathcal{F}, \mathcal{I}_{Y(x)}),$ where

$$\begin{aligned} \mathcal{I}_{A(x)} &= \mathcal{B}_{1}(x), \ \mathcal{I}_{E(x)} = \mathcal{B}_{2}(x), \ \mathcal{I}_{Y(x)} = \mathcal{B}_{3}(x), \\ \mathcal{I}_{I(x)} &= \mathcal{B}_{1}(x) \cup \mathcal{B}_{3}(x) = \{(p_{1}, \dots, p_{n}) \in [0, 1]^{n} : \sum_{i=1}^{n} \frac{p_{i}}{n} > 1 - x\}, \\ \mathcal{I}_{O(x)} &= \mathcal{B}_{2}(x) \cup \mathcal{B}_{3}(x) = \{(p_{1}, \dots, p_{n}) \in [0, 1]^{n} : \sum_{i=1}^{n} \frac{p_{i}}{n} < x\}, \\ \mathcal{I}_{U(x)} &= \mathcal{B}_{1}(x) \cup \mathcal{B}_{2}(x) = \{(p_{1}, \dots, p_{n}) \in [0, 1]^{n} : \sum_{i=1}^{n} \frac{p_{i}}{n} \geqslant x \text{ or } \sum_{i=1}^{n} \frac{p_{i}}{n} \leqslant 1 - x\}. \end{aligned}$$

6. Concluding Remarks

Finally, we note that conditional probability interpretations of quantified statements were also proposed in psychology (see, e.g., [12, 13, 37, 39, 41, 43, 46]), since generalized quantifiers are psychologically much more plausible compared to the traditional logical quantifiers, as the latter are either too strict (\forall does not allow for exceptions) or too weak (\exists quantifies over at least one object) for formalizing everyday life sentences. Recent experimental data suggests that people negate conditionals and quantified statements mainly by building contraries (in the sense of inferring $p(\neg C|A) = 1 - x$ from the negated p(C|A) = x) but hardly ever by building contradictories (in the sense of inferring p(C|A) < x from the negated p(C|A) = x; see [40, 46]). However, this empirical result calls for further experiments. The square presented in Section 4 and the hexagon presented in Section 5 can serve as a new rationality framework for formal-normative and psychological investigations of basic relations among quantified statements.

Acknowledgement

We thank Deutsche Forschungsgemeinschaft (DFG), Fondation Maison des Sciences de l'Homme (FMSH), and Villa Vigoni for supporting joint meetings at Villa Vigoni where parts of this work originated (Project: "Human Rationality: Probabilistic Points of View"). Niki Pfeifer is supported by his DFG project PF 740/2-2 (within the SPP1516 "New Frameworks of Rationality"). Giuseppe Sanfilippo is supported by the INdAM–GNAMPA Project (2016 Grant U 2016/000391).

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