Direct type-specific conic fitting and eigenvalue bias correction

Matthew Harker [a,*], Paul O'Leary [a], Paul Zsombor-Murray [b]

[a] *Institute for Automation, University of Leoben, Austria*
[b] *Centre for Intelligent Machines, McGill University, Canada*

A new method to fit specific types of conics to scattered data points is introduced. Direct, specific fitting of ellipses and hyperbolae is achieved by imposing a quadratic constraint on the conic coefficients, whereby an improved partitioning of the design matrix is devised so as to improve computational efficiency and numerical stability by eliminating redundant aspects of the fitting procedure. Fitting of parabolas is achieved by determining an orthogonal basis vector set in the Grassmannian space of the quadratic terms' coefficients. The linear combination of the basis vectors that fulfills the parabolic condition and has a minimum residual norm is determined using Lagrange multipliers. This is the first known direct solution for parabola specific fitting. Furthermore, the inherent bias of a linear conic fit is addressed. We propose a linear method of correcting this bias, producing better geometric fits which are still constrained to specific conic type.

Keywords: Curve fitting; Conics; Constrained least squares

1. Introduction

This paper addresses the problem of fitting a specific type of conic to scattered data, e.g. finding the best hyperbolic approximation to a set of data points. Solutions are provided for all three types of conic, i.e. hyperbolae, ellipses and parabolas, together with their degenerate forms. This is especially useful when *a-priori* knowledge of the problem indicates the type of conic to be fit.

This problem was addressed by Nievergelt [13], however the quadratic constraint used by him leads to a general fit. The result of the fit is tested for its type; if it is not of the sought type then he proceeds to solve a *geodetic* equation leading to the nearest conic of the desired type. Quadratic constrained least squares was first successfully applied by Fitzgibbon et al. [3,16] to the problem of ellipse specific fitting – a task which was considered to be fundamentally non-linear up to that time. The work of Fitzgibbon et al. was extended by O'Leary et al. [14] to solve ellipse and hyperbola specific fitting. However, a parabola specific fit cannot be solved using standard quadratic constrained least squares since it requires a zero constraint, which leads to the trivial solution using the method of Lagrange multipliers.

Fitzgibbon et al. [3] noted that the linear specific algorithm for ellipses is biased, by which the ellipses tend away from parabolic solutions. This is a bias with respect to the semi-axes of the conic sections, which is present for the ellipses as well as hyperbolae.

In some cases this produces undesirable results, an issue which as of yet has not been addressed. The most important contributions of this paper are

1. A new linear parabola specific fitting method.
2. An improved matrix partitioning, extending the work of Halíř and Flusser [8]. An incremental orthogonal residualization of the partitioned scatter matrix is performed which corresponds to a generalization of the Eckart-Young-Mirsky matrix approximation theorem [6].

* Corresponding author. Tel.: +43 3842 402 5308; fax: +43 3842 402 5302.
E-mail address: matthew.harker@stud.unileoben.ac.at (M. Harker).

3. A linear method for correcting the eccentricity bias of linear specific fitting based on a pencil of conics, and the so-called approximate mean square (AMS) distance proposed by Taubin [18].

The theoretical background to the proposed methods is presented and verified by comprehensive numerical testing.

2. Geometric background

The notation for the quadratic forms in the projective plane, i.e. the conic sections, used in standard literature on geometry [11] is,

$$\mathbf{p}^{\mathrm{T}}\mathtt{K}\mathbf{p} = [x \quad y \quad w] \begin{bmatrix} a & \frac{b}{2} & \frac{d}{2} \\ \frac{b}{2} & c & \frac{e}{2} \\ \frac{d}{2} & \frac{e}{2} & f \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix} = 0, \tag{1}$$

where $\mathtt{K}$ is the conic matrix, and $\mathbf{p}$ is a homogeneous point with w the homogeneous coordinate. Expanding Eq. (1) delivers the scalar point equation of the conic,

$$ax^2 + bxy + cy^2 + dxw + eyw + fw^2 = 0. \tag{2}$$

The type of conic is identified by the roots of Eq. (2) evaluated at infinity, i.e. with $w = 0$,

$$ax^2 + bxy + cy^2 = 0. \tag{3}$$

Explicitly,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} y. \tag{4}$$

By means of the discriminant, the conics can be identified as follows:

$$b^2 - 4ac \begin{cases} > 0 & \text{real asymptotes} \Rightarrow \text{hyperbola}, \\ = 0 & \text{real parallel asymptotes} \Rightarrow \text{parabola}, \\ < 0 & \text{complex asymptotes} \Rightarrow \text{ellipse}. \end{cases} \tag{5}$$

Hence, to constrain a conic to its type, one must constrain the discriminant to its sign. Fig. 1 shows the space of the quadratic portion of all conic sections. The equation $b^2 - 4ac = 0$ is a second order surface in this space, namely, it is an elliptical cone with the origin as its vertex. This surface represents all parabolas. All ellipses are contained within the cone, whereas all hyperbolae are external to the cone. Since the conic equation is homogeneous, the quadratic portion can be scaled to have unit norm, which is shown by the unit sphere in the figure.

3. Data preparation

Chojnacki et al. [2] showed that ensuring the data is mean-free and scaled to have a root-mean-square distance of $\sqrt{2}$ to the origin improves the numerical performance and statistical behaviour of a fitting algorithm. In a planar fit this involves subtracting centroid coordinates $(\bar{x}, \bar{y})$ from raw data (x_i, y_i) to give so-called *mean-free* coordinates $(\hat{x}_i, \hat{y}_i) = (x_i - \bar{x}, y_i - \bar{y})$. With n data points, the appropriate scaling factor m imposes the metric, with

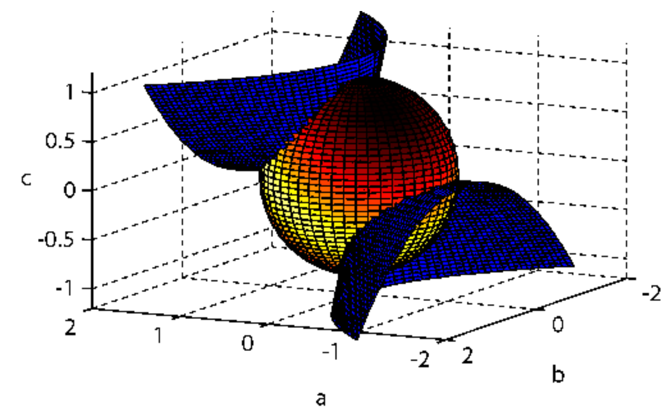

Fig. 1. The space of quadratic forms defined by the coefficients a, b, and c. The unit sphere representing all solutions shows the error density of a specific set of data on its surface by means of a colour gradient. All conics can be represented by the points on the unit sphere, i.e. a two-dimensional manifold in the three-dimensional subspace. The parabolas are constrained by the condition $b^2 - 4ac = 0$; this is the equation of the origin centred elliptical cone that separates the ellipses from the hyperbolae.

$$m = \sqrt{\frac{2n}{\sum_{i=1}^{n}(\hat{x}_i^2 + \hat{y}_i^2)}}. \tag{6}$$

The data set therefore becomes $(x_i, y_i) \triangleq (m\hat{x}_i, m\hat{y}_i)$. The algorithms presented in this paper assume that the data is mean-free and scaled as described here.

4. Reduction of the scatter matrix

The conic equation, i.e. Eq. (2), can be written as a product of vectors, i.e.

$$\mathbf{dz} = [x^2 \quad xy \quad y^2 \quad x \quad y \quad 1][a \quad b \quad c \quad d \quad e \quad f]^{\mathrm{T}} = 0 \tag{7}$$

with $w = 1$. The "design" $\mathbf{d}$ and coefficient $\mathbf{z}$ vectors are, respectively, termed the dual-Grassmannian and Grassmannian coordinates of the conics. Given n points, the vector $\mathbf{d}$ becomes the n-row design matrix $\mathtt{D}$. This results in a vector $\mathbf{r}$ which is the residual vector of the n points in the conic equation whose norm is to be minimized, and corresponds to the algebraic distances of the points to the conic. The partitioning of the design matrix $\mathtt{D}$ and coefficient vector $\mathbf{z}$ is proposed as follows:

$$\mathtt{D}\mathbf{z} = \mathbf{r} = [\mathtt{D}_2 \quad \mathtt{D}_1 \quad \mathtt{D}_0] \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{z}_1 \\ z_0 \end{bmatrix} = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}. \tag{8}$$

In this case, the matrices are partitioned into groupings of their quadratic, linear, and constant terms, i.e.

$$\mathtt{D}_2 = \begin{bmatrix} x_1^2 & x_1y_1 & y_1^2 \\ \vdots & \vdots & \vdots \\ x_n^2 & x_ny_n & y_n^2 \end{bmatrix}, \quad \mathtt{D}_1 = \begin{bmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}, \quad \text{and } \mathtt{D} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{9}$$

and therefore,

$$\mathbf{z}_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \; \mathbf{z}_1 = \begin{bmatrix} d \\ e \end{bmatrix} \text{ and } z_0 = f. \tag{10}$$

The motivation behind this partitioning is the fact that the column of ones is statistically invariant; similarly, the statistical nature of the quadratic data is different to that of the linear data. Moreover, we wish to impose a constraint on the quadratic coefficients only. With the partitioning, the sum of the squared residuals is,

$$\begin{bmatrix} \mathbf{z}_2^{\mathrm{T}} & \mathbf{z}_1^{\mathrm{T}} & z_0 \end{bmatrix} \begin{bmatrix} \mathtt{S}_{22} & \mathtt{S}_{21} & \mathtt{S}_{20} \\ \mathtt{S}_{12} & \mathtt{S}_{11} & \mathtt{S}_{10} \\ \mathtt{S}_{02} & \mathtt{S}_{01} & \mathtt{S}_{00} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{z}_1 \\ z_0 \end{bmatrix} = \mathbf{r}^{\mathrm{T}}\mathbf{r}, \tag{11}$$

where the scatter matrices $\mathtt{S}_{ij}$, are defined as, $\mathtt{S}_{ij} \triangleq \mathtt{D}_i^{\mathrm{T}}\mathtt{D}_j$, noting of course that $\mathtt{S}_{ij} = \mathtt{S}_{ji}^{\mathrm{T}}$ and $\mathtt{S}_{00} = n$. The unique minimum of the least squares problem occurs when the set of partial derivatives of Eq. (11) are equal to zero, i.e. when

$$\mathtt{S}_{22}\mathbf{z}_2 + \mathtt{S}_{21}\mathbf{z}_1 + \mathtt{S}_{20}z_0 = \mathbf{0} \tag{12}$$

$$\mathtt{S}_{21}^{\mathrm{T}}\mathbf{z}_2 + \mathtt{S}_{11}\mathbf{z}_1 + \mathtt{S}_{10}z_0 = \mathbf{0} \tag{13}$$

$$\mathtt{S}_{20}^{\mathrm{T}}\mathbf{z}_2 + \mathtt{S}_{10}^{\mathrm{T}}\mathbf{z}_1 + nz_0 = 0. \tag{14}$$

Noting that the solution to this set of equations is actually the trivial solution, we must in fact put a constraint on the coefficients. The partial derivative with respect to z_0, Eq. (14), implies that,

$$z_0 = -\frac{1}{n}\begin{bmatrix} \mathtt{S}_{20}^{\mathrm{T}} & \mathtt{S}_{10}^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{z}_1 \end{bmatrix} = -\begin{bmatrix} \overline{x^2} & \overline{xy} & \overline{y^2} & \overline{x} & \overline{y} \end{bmatrix} \begin{bmatrix} \mathbf{z}_2 \\ \mathbf{z}_1 \end{bmatrix}. \tag{15}$$

This directly states that the linear fit, which in this space is a hyperplane, must pass through the centroid of the points in the corresponding space. Nievergelt [12] applies this to the fitting of hyperplanes and hyperspheres. Therefore, with mean-free planar data, we should apply a transformation in the hyperspace such that,

$$\hat{\mathtt{D}}_2 \triangleq \begin{bmatrix} x_1^2 - \overline{x^2} & x_1y_1 - \overline{xy} & y_1^2 - \overline{y^2} \\ \vdots & \vdots & \vdots \\ x_n^2 - \overline{x^2} & x_ny_n - \overline{xy} & y_n^2 - \overline{y^2} \end{bmatrix}, \tag{16}$$

and redefine the quadratic design matrix and associated scatter matrices accordingly as "mean-free", i.e.

$$\hat{\mathtt{S}}_{22} = \hat{\mathtt{D}}_2^{\mathrm{T}}\hat{\mathtt{D}}_2 \quad \text{and} \quad \hat{\mathtt{S}}_{21} = \hat{\mathtt{D}}_2^{\mathrm{T}}\mathtt{D}_1. \tag{17}$$

Since we have assumed the data is mean-free and scaled according to Section 3, the scatter matrix $\hat{\mathtt{S}}_{11}$ and $\mathtt{S}_{11}$ are in fact equivalent. With this transformation, Eq. (15) is satisfied by $z_0 = 0$. This effectively forces the hyperplane through the centroid of the data, satisfying the partial derivative with respect to the coordinate z_0. This transformation not only ensures the Euclidean invariance of the fit, but also reduces the dimensionality of the problem. The column of ones, $\mathtt{D}_0$, is redundant to the problem at hand. This is a reduction of dimensionality that has been overlooked in past literature. The problem is reduced to determining the orientation of the hyperplane to be fit, as the relative shift is now known. The reduced system of partial derivatives is now,

$$\hat{\mathtt{S}}_{22}\mathbf{z}_2 + \hat{\mathtt{S}}_{21}\mathbf{z}_1 = \mathbf{0} \tag{18}$$

$$\hat{\mathtt{S}}_{21}^{\mathrm{T}}\mathbf{z}_2 + \hat{\mathtt{S}}_{11}\mathbf{z}_1 = \mathbf{0}. \tag{19}$$

Solving the partial derivative with respect to $\mathbf{z}_1$, Eq. (19), for the linear terms' coefficient vector yields $\mathbf{z}_1$ when $\mathbf{z}_2$ is held constant, i.e.

$$\mathbf{z}_1 = -\hat{\mathtt{S}}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}}\mathbf{z}_2. \tag{20}$$

Substitution of this relation and $z_0 = 0$ into the least squares problem in Eq. (11) results in a function in the quadratic coefficients only, and free of the redundant column of ones $\mathtt{D}_0$, i.e.

$$\mathbf{z}_2^{\mathrm{T}}\left(\hat{\mathtt{S}}_{22} - \hat{\mathtt{S}}_{21}\hat{\mathtt{S}}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}}\right)\mathbf{z}_2 = \mathbf{r}^{\mathrm{T}}\mathbf{r}. \tag{21}$$

The matrix,

$$\mathtt{M} \triangleq \hat{\mathtt{S}}_{22} - \hat{\mathtt{S}}_{21}\hat{\mathtt{S}}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}} \tag{22}$$

$$= \hat{\mathtt{D}}_2^{\mathrm{T}}(\mathtt{I}_n - \hat{\mathtt{D}}_1\hat{\mathtt{D}}_1^{+})\hat{\mathtt{D}}_2, \tag{23}$$

is the reduced scatter matrix sought, and is the *Schur Complement* [15] of $\hat{\mathtt{S}}_{11}$ in the scatter matrix. The matrix $\hat{\mathtt{D}}_1^{+}$ is the pseudo-inverse matrix of $\hat{\mathtt{D}}_1$, i.e. a least squares mapping based on the planar data. The matrix product $\hat{\mathtt{D}}_1\hat{\mathtt{D}}_1^{+}$ is the set of orthogonal projections on to the range space of $\hat{\mathtt{D}}_1$, and is – to a scaling factor – the covariance of the residuals of the linear portion of the data. Specifically, if σ_1 and σ_2 are the singular values of $\hat{\mathtt{D}}_1$ which correspond to the residual vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, then,

$$\hat{\mathtt{D}}_1\hat{\mathtt{D}}_1^{+} = \frac{1}{\sigma_1^2}\mathbf{r}_1\mathbf{r}_1^{\mathrm{T}} + \frac{1}{\sigma_2^2}\mathbf{r}_2\mathbf{r}_2^{\mathrm{T}}. \tag{24}$$

This implies that matrix $\mathtt{M}$ is the result of subtracting the quadratic residual elements predicted by the linear portion from the residuals of the quadratic portion. This allows an optimization to be performed in the space of the coefficients $\mathbf{z}_2$, a subspace for which corresponding $\mathbf{z}_1$ vectors have a residual of minimal norm. In other words, the mapping in Eq. (20) corresponds to $\mathbf{z}_1 = -\hat{\mathtt{D}}_1^{+}\hat{\mathtt{D}}_2\mathbf{z}_2$, and is thus the least squares mapping of $\mathbf{z}_1$ from the residual vector of $\mathbf{z}_2$, and essentially refits the linear portion given the specific quadratic coefficients. The Schur Complement mentioned above that leads to the dimensional reduction of the problem in fact corresponds to the generalization of the Eckart-Young-Mirsky matrix approximation theorem proposed by Golub et al. [6]. Essentially, the scatter matrix is approximated by a matrix of lower rank, i.e. the reduced scatter matrix $\mathtt{M}$.

5. Fitting conics with a quadratic constraint

The problem of the linear fitting of a conic with a quadratic constraint on the roots at infinity can now be stated as,

$$\mathbf{z}_2^{\mathrm{T}}\mathtt{M}\mathbf{z}_2 = \min_{\mathbf{z}_2 \neq \mathbf{0}} \quad \text{subject to} \quad \mathbf{z}_2^{\mathrm{T}}\mathtt{C}\mathbf{z}_2 = \alpha, \tag{25}$$

where the matrix $\mathtt{C}$ describes a quadratic constraint on the coefficients a, b, and c. As Bookstein [1] showed, the minimization problem can be stated as a Lagrange multiplier problem, and solved as a generalized eigenvector problem. Combining the function to be minimized and the constraint with a Lagrange multiplier, results in the system

$$H(\mathbf{z}_2, \lambda) = \mathbf{z}_2^{\mathrm{T}}\mathtt{M}\mathbf{z}_2 + \lambda(\mathbf{z}_2^{\mathrm{T}}\mathtt{C}\mathbf{z}_2 - \alpha), \tag{26}$$

which is solved for its partial derivatives with respect to $\mathbf{z}_2$ and λ, i.e.

$$\mathtt{M}\mathbf{z}_2 + \lambda\mathtt{C}\mathbf{z}_2 = \mathbf{0} \tag{27}$$

$$\mathbf{z}_2^{\mathrm{T}}\mathtt{C}\mathbf{z}_2 = \alpha. \tag{28}$$

Solving Eq. (27) as a generalized eigenvector problem yields eigenvectors which minimize $\mathbf{z}_2^{\mathrm{T}}\mathtt{M}\mathbf{z}_2$. The *inertia* of an eigensystem is, in short, the set of signs of its eigenvalues. By the *Sylvester Law of Inertia* [7] it can be shown that the inertia of the generalized eigenvalue problem has the same inertia as that of the eigenvalues of the matrix $\mathtt{C}^{-1}$ i.e.

$$\operatorname{sign}(\lambda(\mathtt{M}, \mathtt{C})) = \operatorname{sign}(\lambda(\mathtt{C}^{-1})), \tag{29}$$

since the matrix $\mathtt{M}$ is positive (semi)definite [3]. Moreover, if $(\lambda_i, \mathbf{e}_i)$ is a solution to the generalized eigenvector problem then,

$$\operatorname{sign}(\lambda_i) = \operatorname{sign}(\mathbf{e}_i^{\mathrm{T}}\mathtt{C}\mathbf{e}_i). \tag{30}$$

The combination of Eqs. (29) and (30) states that the sign of the constraint, $\mathbf{e}_i^{\mathrm{T}}\mathtt{C}\mathbf{e}_i$, takes on the sign of the eigenvalues of the matrix $\mathtt{C}^{-1}$ for each generalized eigenvector $\mathbf{e}_i$. This fact is essential to specific fitting since the sign of the constraint defines the conic type. Further simplification occurs if the matrix $\mathtt{C}$ is non-singular, in which case the generalized eigenvector problem can be solved as the eigenvector problem,

$$\mathtt{C}^{-1}\mathtt{M}\mathbf{z}_2 = \lambda\mathbf{z}_2. \tag{31}$$

With an approach proposed by O'Leary and Zsombor-Murray [14] the constraint,

$$b^2 - 4ac = \mathbf{z}_2^{\mathrm{T}} \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ -2 & 0 & 0 \end{bmatrix} \mathbf{z}_2 = \alpha \tag{32}$$

is applied. In this case the matrix $\mathtt{C}^{-1}$ has the eigenvalues $-\frac{1}{2}$, $\frac{1}{2}$, and 1, that is, there is always one elliptical solution, and two hyperbolic solutions. It was shown that two of the resulting eigenvectors correspond to the best elliptical and best hyperbolic solutions. The solutions are extracted by evaluating the condition $b^2 - 4ac$ in terms of the resulting eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, i.e.

$$\kappa_i = \mathbf{e}_i^{\mathrm{T}}\mathtt{C}\mathbf{e}_i. \tag{33}$$

The ellipse is found as,

$${}_e\mathbf{z}_2 = \mathbf{e}_u \quad \text{where} \quad u = \min_i(\kappa_i), \tag{34}$$

and the hyperbola as,

$${}_h\mathbf{z}_2 = \mathbf{e}_v \quad \text{where} \quad v = \min_{i \neq u, \kappa_i > 0} |\lambda_i|, \tag{35}$$

and λ_i is the generalized eigenvalue corresponding to the ith eigenvector. In other words, the ellipse is the eigenvector whose corresponding condition value is the lone negative value. The two positive values correspond to two hyperbolic solutions, whereby the solution with the eigenvalue of minimum magnitude is selected. This is due to the fact that the eigenvectors are local extrema of the Rayleigh quotient [7],

$$\lambda = \frac{\mathbf{z}_2^{\mathrm{T}}\mathtt{M}\mathbf{z}_2}{\mathbf{z}_2^{\mathrm{T}}\mathtt{C}\mathbf{z}_2}. \tag{36}$$

The resulting best fitting conics are hence constrained minima of the algebraic distance cost function.

6. The parabola

If one wishes to fit a parabola to scattered data, the eigenvector problem in Eq. (31) cannot be applied, as the explicit solution to the Lagrange multiplier problem results in the trivial solution when applying the null constraint $b^2 - 4ac = 0$. Also, the *secular equation* proposed by Gander [5], does not apply when $\alpha = 0$. If the constraint matrix $\mathtt{C}$ is the identity matrix, then the system is solved with the constraint $a^2 + b^2 + c^2 = 1$, which is implicit in the evaluation of eigenvector and singular value problems. The resulting eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, with the corresponding eigenvalues λ_1, λ_2, and λ_3, form an orthonormal basis vector set for the space of the coefficients of the quadratic terms of all conics. The eigenvalues and corresponding vectors should be ordered such that,

$$|\lambda_1| \geqslant |\lambda_2| \geqslant |\lambda_3| \geqslant 0, \tag{37}$$

since usually, the eigenvector corresponding to the eigenvalue of smallest magnitude is the best fit solution to the linear conic fit. The above constraint ensures that all solutions lie on the unit sphere centred at the origin. The required condition for a parabola, i.e. $b^2 - 4ac = 0$, is the equation of a quadric; specifically, it is an elliptical cone (see Fig. 1) that forms the boundary between the ellipses and hyperbolae in this space. The elliptical solutions lay within the cone, whereas the hyperbolic solutions are external to the cone. The curve of intersection of the two quadrics, i.e. the cone and the sphere, is a fourth order curve which represents the parabolic solutions of unit norm. To fit a parabola, we must find a minimizing point on this cone. We therefore take the quadratic coefficients of the parabola to be a linear combination of the eigenvectors of the matrix $\mathtt{M}$, i.e.

$$ {}_p\mathbf{z}_2 = \mathbf{e}_3 + s\mathbf{e}_2 + t\mathbf{e}_1. \tag{38} $$

Since the singular values of the matrix $\mathtt{M}$, i.e. the square-roots of the eigenvalues of $\mathtt{M}^{\mathrm{T}}\mathtt{M}$, are the 2-norm distances of the respective vectors to the null-space of $\mathtt{M}$, then the eigenvector associated with the smallest singular value can be considered the minimizing solution. Thus, we assume $\mathbf{e}_3$ is the best fit, and use combinations of the other two eigenvectors to find optimal parabolic solutions. Since the equations are homogeneous, only two parameters are needed to fully describe the space. The error associated with taking this linear combination is the magnitude of the resulting residual vector, that is,

$$ \begin{aligned} \|\mathtt{M}({}_p\mathbf{z}_2)\| \triangleq \Delta(s,t) &= ({}_p\mathbf{z}_2)^{\mathrm{T}}\mathtt{M}^{\mathrm{T}}\mathtt{M}({}_p\mathbf{z}_2) \\ &= (\mathbf{e}_3 + s\mathbf{e}_2 + t\mathbf{e}_1)^{\mathrm{T}}\mathtt{M}^{\mathrm{T}}\mathtt{M}(\mathbf{e}_3 + s\mathbf{e}_2 + t\mathbf{e}_1) \\ &= \mathbf{e}_3^{\mathrm{T}}\mathtt{M}^{\mathrm{T}}\mathtt{M}\mathbf{e}_3 + \mathbf{e}_2^{\mathrm{T}}\mathtt{M}^{\mathrm{T}}\mathtt{M}\mathbf{e}_2 s^2 + \mathbf{e}_1^{\mathrm{T}}\mathtt{M}\mathbf{e}_1 t^2 \\ &= \lambda_3^2 + \lambda_2^2 s^2 + \lambda_1^2 t^2. \end{aligned} \tag{39} $$

In the quadratic coefficient space, this error function is essentially an ellipsoid-shaped error density with semi-axes proportional to the singular values of the reduced scatter matrix $\mathtt{M}$. Referring again to Fig. 1, this error density is shown on the unit sphere as a colour gradient, whereby the darker the colour, the higher algebraic error is for the corresponding conic. In the space of the parameters s and t, it is an ellipse-shaped error density with semi-axes proportional to the first and second largest singular values. The constraint to ensure that ${}_p\mathbf{z}_2$ is indeed parabolic is found by expanding the constraint $b^2 - 4ac = 0$ in terms of the assumed solution with parametric coefficients a, b and c, i.e.

$$ (e_{32} + e_{22}s + e_{12}t)^2 - 4(e_{31} + e_{21}s + e_{11}t)(e_{33} + e_{23}s + e_{13}t) = 0, \tag{40} $$

where e_{ij} is the jth element of the ith eigenvector. Expanding this expression yields an expression in the form,

$$ C(s,t) = \gamma_1 s^2 + \gamma_2 st + \gamma_3 t^2 + \gamma_4 s + \gamma_5 t + \gamma_6 = 0, \tag{41} $$

which is a conic in the parameter space. The problem is thus to minimize the error function of Eq. (39), i.e. $\Delta(s,t)$, upon the points of the constraint conic. This can be formulated as the Lagrange multiplier[1] problem,

$$ H(s,t,\mu) = \Delta(s,t) + \mu C(s,t). \tag{42} $$

Upon solving the partial derivatives of $H(s,t,\mu)$, a fourth order polynomial in μ is obtained. Defining the coefficients,

$$ \begin{aligned} &\alpha_1 = \lambda_1^2, \quad \alpha_2 = \lambda_2^2, \quad \alpha_3 = \alpha_1\alpha_2 \\ &k_1 = 4\gamma_3\gamma_6 - \gamma_5^2, \quad k_2 = \gamma_2\gamma_6 - \frac{1}{2}\gamma_4\gamma_5, \quad k_3 = \frac{1}{2}\gamma_2\gamma_5 - \gamma_3\gamma_4 \\ &k_4 = 4\gamma_6\gamma_1 - \gamma_4^2, \quad k_5 = 4\gamma_1\gamma_3 - \gamma_2^2, \quad k_6 = \gamma_2\gamma_4 - 2\gamma_1\gamma_5 \\ &k_7 = -4(\gamma_1\alpha_1 + \alpha_2\gamma_3), \quad k_8 = \gamma_1 k_1 - \gamma_2 k_2 + \gamma_4 k_3, \end{aligned} \tag{43} $$

the polynomial coefficients are given as,

$$ \begin{aligned} &K_4 = k_5 k_8 \quad K_3 = 2k_7 k_8 \\ &K_2 = 4[(2\gamma_2 k_2 + 4k_8)\alpha_3 + \gamma_1 k_4 \alpha_1^2 + \gamma_3 K_1 \alpha_2^2] \\ &K_1 = -8\alpha_3(k_1\alpha_2 + k_4\alpha_1) \quad K_0 = 16\gamma_6\alpha_3^2. \end{aligned} \tag{44} $$

Thus, solving

$$ K_4\mu^4 + K_3\mu^3 + K_2\mu^2 + K_1\mu + K_0 = 0, \tag{45} $$

yields four solutions for μ. The best fitting parabola can be extracted as it corresponds usually, but not always, to the real Lagrange multiplier with the smallest magnitude, i.e.

$$ \mu_* = \min_i |\mu_i|, \quad \mu_i \in \mathbb{R}. \tag{46} $$

Backsubstitution for the corresponding s_* and t_* is in the form,

$$ s_* = \frac{2\mu_*}{u_*}(k_3\mu_* + \alpha_1\gamma_4) \quad \text{and} \quad t_* = \frac{\mu_*}{u_*}(k_6\mu_* + 2\alpha_2\gamma_5), \tag{47} $$

where

$$ u_* = k_5\mu_*^2 + k_7\mu_* + 4\alpha_3. \tag{48} $$

The quadratic coefficients of the parabola are found by backsubstitution of the s_* and t_* into the linear combination of the eigenvectors, i.e. ${}_p\mathbf{z}_2 = \mathbf{e}_3 + s_*\mathbf{e}_2 + t_*\mathbf{e}_1$.

7. Backsubstitution

Given the quadratic solution vectors of the conics, $\mathbf{z}_2$, be it the ellipse ${}_e\mathbf{z}_2$, hyperbola ${}_h\mathbf{z}_2$, or parabola ${}_p\mathbf{z}_2$, backsubstitution is the same. The quadratic coefficients are known, and thus the directions of the asymptotes are also known. The backsubstitution then determines the shift of the conic centre as well as its scaling factor as to how far it is from the mere product of the asymptotes, i.e. a degenerate conic. The backsubstitution can be accomplished in concise matrix form, that is,

$$ \mathbf{z} = \begin{bmatrix} & \mathtt{I}_3 & \\ & -\mathtt{S}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}} & \\ -\overline{x^2} & -\overline{xy} & -\overline{y^2} \end{bmatrix} \mathbf{z}_2 \triangleq \mathtt{B}\mathbf{z}_2, \tag{49} $$

where $\mathtt{I}_3$ is the 3×3 identity matrix. Thus, $\mathtt{B}$ is a 6×3 matrix, and the resulting vector $\mathbf{z}$ is the set of corresponding conic coefficients, i.e. the Grassmannian coefficients. As noted above, the mapping of the linear portion is a least squares mapping from the quadratic residual vector, and the constant term corresponds to pushing the hyperplane back to fit through the actual centroid of the data in the hyperspace.

[1] Eigenvalue and Lagrange multiplier problems are analogous, and both use λ. To avoid confusion, μ has been chosen to denote the Lagrange multiplier here.

In the plane, the transformation which will place the conic back onto the original data is the same transformation that was applied to prepare data, but in the form of the similarity transformation $\mathtt{K}_* = \mathtt{T}^{\mathrm{T}}\mathtt{K}\mathtt{T}$, where

$$\mathtt{T} = \begin{bmatrix} m & 0 & -m\bar{x} \\ 0 & m & -m\bar{y} \\ 0 & 0 & 1 \end{bmatrix}. \tag{50}$$

8. Bias correction and fitting with a pencil of conics

8.1. Correction of low ellipse eccentricity bias

As discussed in Fitzgibbon et al. [3] the ellipse specific fitting algorithm is biased to fitting ellipses of low eccentricity. The ellipse and hyperbola specific fits provide a means of correcting this bias while maintaining the specific conic type. This is based on a pencil of conics defined by the ellipse and hyperbola, and the so-called approximate mean square distance (AMS) proposed by Taubin [17,18]. Upon the backsubstitution step for the ellipse and hyperbola, let

$$\mathbf{z}_e = \begin{bmatrix} \mathtt{I}_3 \\ -\hat{\mathtt{S}}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}} \end{bmatrix} {}_e\mathbf{z}_2 \quad \text{and} \quad \mathbf{z}_h = \begin{bmatrix} \mathtt{I}_3 \\ -\hat{\mathtt{S}}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}} \end{bmatrix} {}_h\mathbf{z}_2 \tag{51}$$

due to the dimensionality required for the proposed solution. The remaining coordinate, z_0, will again be determined by forcing the hyperplane through the centroid. A pencil of conics is described by the two conic sections and a single parameter, μ, in the form,

$$\mathbf{z}(\mu) = (1-\mu)\mathbf{z}_e + \mu\mathbf{z}_h, \quad \mu \in \mathbb{R}. \tag{52}$$

More commonly the pencil is described by the conic matrices; however, it is equivalently described by the Grassmannian coefficients. In this case, the pencil is defined such that $\mathbf{z}(0) = \mathbf{z}_e$, and $\mathbf{z}(1) = \mathbf{z}_h$. The pencil of conics describes all conics which pass through the four intersection points of the ellipse and hyperbola. We combine this notion with the so-called approximate mean square distance, given by

$$\varepsilon^2(\mathbf{z}) = \frac{\|\hat{\mathtt{D}}\mathbf{z}\|^2}{\|\hat{\mathtt{D}}_x\mathbf{z}\|^2 + \|\hat{\mathtt{D}}_y\mathbf{z}\|^2}, \tag{53}$$

which is the sum-of-squared distances from the points to the first order approximations of the conic about each point. The matrices $\hat{\mathtt{D}}_x$ and $\hat{\mathtt{D}}_y$ are, respectively, the partial derivatives of the design matrix with respect to x and y, i.e.

$$\hat{\mathtt{D}}_x = \begin{bmatrix} 2x_1 & y_1 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2x_n & y_n & 0 & 1 & 0 \end{bmatrix} \quad \text{and} \quad \hat{\mathtt{D}}_y = \begin{bmatrix} 0 & x_1 & 2y_1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & x_n & 2y_n & 0 & 1 \end{bmatrix}. \tag{54}$$

Substituting the conic pencil, $\mathbf{z} = \mathbf{z}(\mu)$, this can be alternatively expressed as,

$$\varepsilon^2(\mu) = \frac{\mathbf{z}(\mu)^{\mathrm{T}}\hat{\mathtt{S}}\mathbf{z}(\mu)}{\mathbf{z}(\mu)^{\mathrm{T}}\hat{\mathtt{S}}_{xy}\mathbf{z}(\mu)}, \tag{55}$$

where

$$\hat{\mathtt{S}}_{xy} = \hat{\mathtt{D}}_x^{\mathrm{T}}\hat{\mathtt{D}}_x + \hat{\mathtt{D}}_y^{\mathrm{T}}\hat{\mathtt{D}}_y. \tag{56}$$

and the scatter matrix

$$\hat{\mathtt{S}} = \begin{bmatrix} \hat{\mathtt{S}}_{22} & \hat{\mathtt{S}}_{21} \\ \hat{\mathtt{S}}_{21}^{\mathrm{T}} & \hat{\mathtt{S}}_{11} \end{bmatrix}, \tag{57}$$

applies. The matrix $\hat{\mathtt{S}}_{xy}$ can in fact be composed of elements of the matrix $\hat{\mathtt{S}}$ [4], and hence does not require the matrix multiplication. Defining the constants,

$$\begin{aligned} \sigma_1 &= \mathbf{z}_e^{\mathrm{T}}\hat{\mathtt{S}}\mathbf{z}_e \quad & \sigma_4 &= \mathbf{z}_e^{\mathrm{T}}\hat{\mathtt{S}}_{xy}\mathbf{z}_e \\ \sigma_2 &= \mathbf{z}_e^{\mathrm{T}}\hat{\mathtt{S}}\mathbf{z}_h \quad & \sigma_5 &= \mathbf{z}_e^{\mathrm{T}}\hat{\mathtt{S}}_{xy}\mathbf{z}_h \\ \sigma_3 &= \mathbf{z}_h^{\mathrm{T}}\hat{\mathtt{S}}\mathbf{z}_h \quad & \sigma_6 &= \mathbf{z}_h^{\mathrm{T}}\hat{\mathtt{S}}_{xy}\mathbf{z}_h, \end{aligned} \tag{58}$$

the sum of the squared error is expressed as

$$\varepsilon^2(\mu) = \frac{(1-\mu)^2\sigma_1 + 2(1-\mu)\mu\sigma_2 + \mu^2\sigma_3}{(1-\mu)^2\sigma_4 + 2(1-\mu)\mu\sigma_5 + \mu^2\sigma_6}. \tag{59}$$

We consequently have a single parameter problem, with extrema that can be computed directly from a first derivative,

$$\frac{\mathrm{d}(\varepsilon^2)}{\mathrm{d}\mu} = \frac{2(Q_2\mu^2 + Q_1\mu + Q_0)}{(R_2\mu^2 + R_2\mu + R_0)^2} = 0, \tag{60}$$

where

$$\begin{aligned} Q_2 &= \sigma_2\sigma_4 - \sigma_2\sigma_6 - \sigma_3\sigma_4 + \sigma_3\sigma_5 - \sigma_1\sigma_5 + \sigma_1\sigma_6 \\ Q_1 &= \sigma_3\sigma_4 + 2\sigma_1\sigma_5 - \sigma_1\sigma_6 - 2\sigma_2\sigma_4 \\ Q_0 &= \sigma_2\sigma_4 - \sigma_1\sigma_5. \end{aligned} \tag{61}$$

The roots of Eq. (60) are therefore given by the quadratic formula,

$$\mu_{1,2} = \frac{-Q \pm \sqrt{Q_1^2 - 4Q_2Q_0}}{2Q_2}. \tag{62}$$

The minimizing solution can be identified by evaluating the sign of the second derivative of the error squared function for these values of μ. The coefficients in the denominator are required for the second derivative evaluation and are given by,

$$\begin{aligned} R_2 &= \sigma_4 - 2\sigma_5 + \sigma_6 \\ R_1 &= 2\sigma_5 - 2\sigma_4 \\ R_0 &= \sigma_4. \end{aligned} \tag{63}$$

The pencil of conics essentially describes a line in the five-dimensional Grassmannian space. As the pencil of conics is defined by an ellipse and hyperbola, i.e. two points on this line, the pencil necessarily contains conics of both types. There are two parabolas which separate the ellipses from the hyperbolas in the pencil, which can be identified by solving an equation quadratic in μ, i.e.

$$(\kappa_e + \kappa_h)\mu^2 - 2\kappa_e\mu + \kappa_e = 0, \tag{64}$$

with κ_i defined as in Eq. (33). Since $\kappa_e < 0$ and $\kappa_h > 0$, the two roots representing parabolas are always real. If the value of μ which minimizes the approximate mean square error lies outside the range of μ which define ellipses, then we can simply limit the value of μ to a value which still defines an ellipse. Such a limit could therefore be the limiting parabola itself. Another possibility is to describe the conic eccentricity as a function of μ and solve with a limiting value of eccentricity, which would prevent a solution that is too close to being a parabola. Other limits are possible, which could for example be application dependent.

Once the optimal value, $\mu = \mu_*$, is found, then backsubstitution entails evaluating the conic pencil at this point, i.e. $\mathbf{z}_* = \mathbf{z}(\mu_*)$, and then substituting for the remaining coefficient, i.e.

$$z_0 = -\begin{bmatrix} \overline{x^2} & \overline{xy} & \overline{y^2} & 0 & 0 \end{bmatrix} \mathbf{z}_*. \tag{65}$$

This conic must finally be transformed back to the original data by the similarity transformation defined by $\mathtt{T}$.

8.2. Correcting eccentricity for hyperbolae

Similar to the case of ellipse specific fitting, the hyperbola specific fit too has an eccentricity bias. As the solution tends away from a parabola, the eccentricity tends to be higher. In some cases a pair of asymptotes may describe the data well, but the wrong pair of branches result, i.e. those with higher eccentricity. This case is shown in Fig. 4(b). The method of bias correction proposed above functions identically for hyperbolae as for ellipses. This is shown to improve the solution from a geometrical point of view, while still maintaining the hyperbola specific solution. This is an important result, as the hyperbolae tend to be neglected in the literature, although they still play an important role in metric vision applications.

8.3. Fitting conics with the conic pencil

The methodology of subspace optimization can also be used for fitting conics such as parabolas and degenerate conics [9]. The parabolas, for example, are found by the roots of Eq. (64). The empirical data presented in Section 10.2 shows adequate evidence that the subspace defined by the ellipse and hyperbola specific solutions is ideal for one-dimensional sub-optimization.

9. Summary of algorithm

The algorithm resulting from the above analysis can be summarized as follows:

1. Generate a scaled, mean-free set of data points,

$$_m x_i = m(x_i - \bar{x}) \quad \text{and} \quad _m y_i = m(y_i - \bar{y}). \tag{66}$$

2. Perform a linear regression on the mean-free $_m x_i$ and $_m y_i$. If the residual is too small, stop the algorithm, since the data is best described by a line.
3. Generate the quadratic design matrix, and remove the mean values from the columns. Compute the scatter matrix with the linear prediction removed from the mean-free quadratic terms, i.e.

$$\mathtt{M} = \hat{\mathtt{S}}_{22} - \hat{\mathtt{S}}_{21}\hat{\mathtt{S}}_{11}^{-1}\hat{\mathtt{S}}_{21}^{\mathrm{T}}. \tag{67}$$

4. For ellipses and hyperbolae, determine the eigenvectors of $\mathtt{C}^{-1}\mathtt{M}$, where $\mathtt{C}$ defines the constraint $b^2 - 4ac = \alpha$. Select the quadratic portion of the elliptical and hyperbolic solutions by means of the eigenvalues, and values of the constraint evaluated for each eigenvector.
5. For parabolas, solve the standard eigenvector problem. Determine and solve the fourth order polynomial,

$$K_4\mu^4 + K_3\mu^3 + K_2\mu^2 + K_1\mu + K_0 = 0 \tag{68}$$

and backsubstitute the real μ with the smallest magnitude to obtain the quadratic parabola coefficients.

6. Backsubstitute the quadratic coefficients of the desired conic into $\mathbf{z} = \mathtt{B}\mathbf{z}_2$. Find the conic matrix from the Grassmannian coefficients, and apply the similarity transformation $\mathtt{K}_* = \mathtt{T}^{\mathrm{T}}\mathtt{K}\mathtt{T}$.
7. If the goal is a better geometric fit, determine and solve the quadratic equation,

$$Q_2\mu^2 + Q_1\mu + Q_0 = 0 \tag{69}$$

and backsubstitute the minimizing μ into the conic pencil. Perform backsubstitution for z_0 and apply the similarity transformation $\mathtt{K}_* = \mathtt{T}^{\mathrm{T}}\mathtt{K}\mathtt{T}$.

10. Numerical tests

10.1. Specific fitting algorithms

The conic forms which are most commonly encountered in metric vision were used to test the algorithm. The five test cases – i.e. elliptical, hyperbolic, parabolic, degenerate hyperbolic, and elliptical arc data – are shown row-wise in Fig. 2. All three conic types were fitted; the ellipse, hyperbola and parabola solutions are shown column-wise. The results show that the algorithm always produces the specific types of conics, regardless of the nature of the data. The tests were performed with random noise with standard deviations of 3% of the amplitude of the respective x and y data.

10.2. Bias correction algorithm

The following tests were used to test the bias correction algorithm:

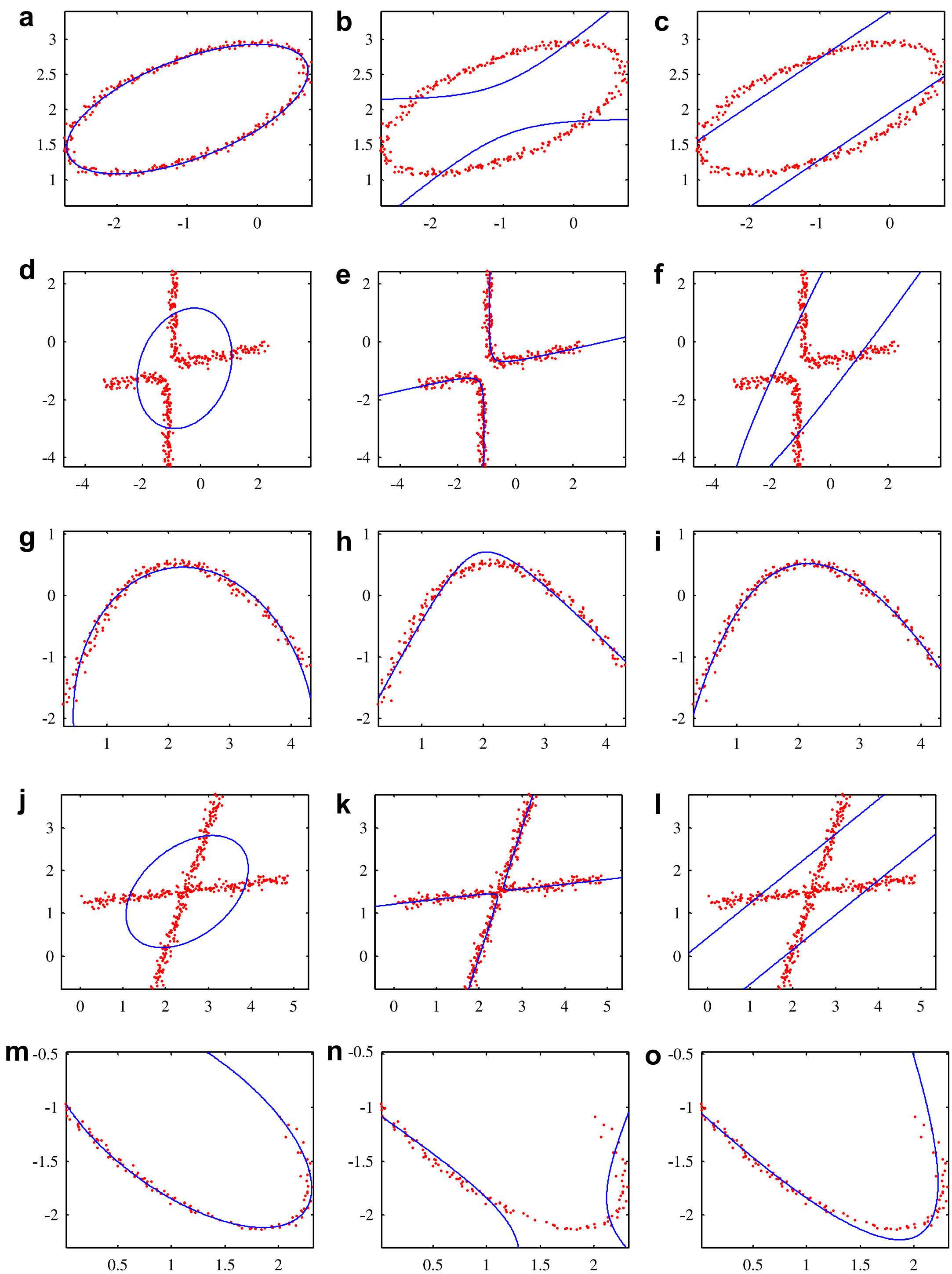

Fig. 2. An ellipse (a), hyperbola (b), and parabola (c) fit to noisy elliptical data. An ellipse (d), hyperbola (e), and parabola (f) fit to noisy hyperbolic data. An ellipse (g), hyperbola (h), and parabola (i) fit to noisy parabolic data. An ellipse (j), hyperbola (k), and parabola (l) fit to noisy degenerate data. An ellipse (m), hyperbola (n), and parabola (o) fit to noisy elliptical arc data.

1. A set of data which described an arc of a parabola was corrupted with progressively more noise to compare the bias correction algorithm proposed here to the standard AMS algorithm.
2. An elliptical arc was repeatedly corrupted with the same amount of noise to test the improvement factor of the bias correction to the standard ellipse specific algorithm. A similar test appears in Fitzgibbon et al. [3].

Fig. 3 shows the results of the first test. The random noise is measured in percent standard deviation (σ_{Noise}) as described above. Fig. 3(a) shows the data at the maximum noise level of 10%, a level much higher than normally encountered in image processing. The two fits, i.e. the method of Taubin and the method proposed here, are qualitatively identical. Fig. 3(b) shows the geometric error of both methods as a function of increasing noise level. What

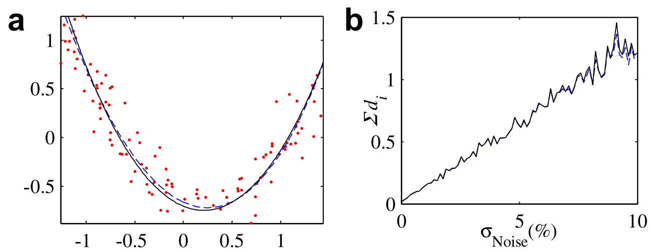

Fig. 3. Comparison of the AMS fit () and the method presented here (−). (a) The two fitted conics at the maximum noise level of 10%. (b) The sum of squared geometric error for each conic, i.e. the performance of the algorithms for increasingly noisy data. The results show that the two fits are virtually indistinguishable, whereby the algorithm presented here retains its type-specific fitting property.

can be concluded from this test is that the method presented here will provide virtually identical results at low noise levels. This indicates that the specific fitting algorithm has identified an optimal one-dimensional subspace, to which the best AMS fitting conic belongs. That is, this is an excellent subspace within which to perform sub-optimization procedures. Also of note from this test is the high level of linearity in the degradation of geometric error with respect to the noise level.

The second test compares the three algorithms; the ellipse specific, the bias correction, and the AMS method of Taubin. Fig. 4(a) shows a random iteration of the half-ellipse test, with the noise level at a 3% standard deviation. Table 1 shows the measures of error used; the mean centre point given in a percentage of the respective semi-axis of the original ellipse; and the relative geometric error with respect to the AMS algorithm; both measures are averaged over 100 iterations. The low-eccentricity bias of the ellipse specific method consistently predicts a low centre point, i.e. a slightly flattened ellipse. The relative error shows that the bias correction of the ellipse actually improves the fit in terms of geometric error. Although the AMS method performs well in any case, it is not ellipse specific, and in less controlled cases may return hyperbolae.

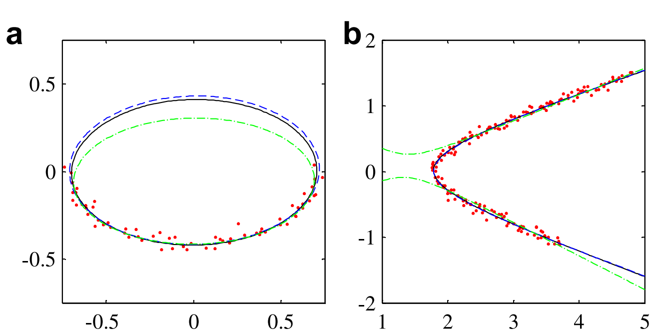

Fig. 4. Comparison of ellipse specific (), AMS (), and the method presented here (−). (a) A random result from the half-ellipse test. The low-eccentricity bias of the ellipse specific method predicts a flattened ellipse, missing points near the edges of the data set. (b) A similar test for the hyperbola. Incorrect branches appear due to the eccentricity bias. Note that the bias-correction solution is virtually indistinguishable from the AMS solution, the difference being that the bias-correction solution is type-specific.

Table 1
Statistical performance of the algorithms for 100 iterations of the half-ellipse fit

Method	Mean centre point (% of semi-axis)	Avg. relative geometric error (%)
Ellipse specific	(0.10, −13.0)	6.3
Taubin AMS	(0.18, 1.3)	0.0
Bias correction	(0.16, −0.7)	1.3
Original ellipse	(0.00, 0.00)	–

The results of a similar test for the bias correction of hyperbola are shown in Fig. 4(b). The improvement of the bias correction is much more pronounced than for the ellipse, in that a different set of hyperbola branches result. Also of note in the comparison of algorithms is the fact that the AMS algorithm requires a sixth order eigen-decomposition, which is a numerical procedure, whereas the method proposed here requires roots of a cubic and a quadratic, both of which have closed form solutions.

11. Conclusions

The above proposed algorithm provides a new and efficient method for the linear fitting of conics of specific types. The column of ones, common to previous methods is now implicitly in the problem, rather than explicitly. The efficiency arises from this decrease in dimensionality of the problem. The three solutions delivered by the algorithm are also guaranteed to be each the best ellipse, hyperbola, and parabola. An algorithm for correcting the eccentricity bias of the elliptical and hyperbolic solutions was presented, and was shown to improve the geometric error of the fitting while maintaining the specific conic type. An additional bias correction to which the pencil of conics could be applied is the statistical bias of conic fitting proposed by Kanatani [10]. The linear and specific fitting has applications in automatic inspection or prejudicial perception, where fast and accurate fitting is required for real time inspection of shape manufacturing.

References

[1] Fred Bookstein, Fitting conic sections to scattered data, Computer Graphics and Image Processing 9 (1979) 56–71.
[2] Wojciech Chojnacki, Michael J. Brooks, Anton van den Hengel, Darren Gawley, Revisiting Hartley's normalized eight-point algorithm, IEEE Transactions on Pattern Analysis and Machine Intelligence 25 (9) (2003) 1172–1177.
[3] Andrew Fitzgibbon, Maurizio Pilu, Robert Fischer, Direct least square fitting of ellipses, IEEE Transactions on Pattern Analysis and Machine Intelligence 21 (5) (1999) 476–480.
[4] Andrew W. Fitzgibbon, Robert B, Fisher, A buyer's guide to conic fitting, in: British Machine Vision Conference, 1995, pp. 513–522.
[5] W. Gander, Least squares with a quadratic constraint, Numerische Mathematik 36 (1981) 291–307.
[6] G.H. Golub, A. Hoffman, G.W. Stewart, A generalization of the Eckart-Young-Mirsky matrix approximation theorem, Linear Algebra and its Applications 88/89 (1987) 317–327.
[7] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd ed., John Hopkins University Press, Baltimore, 1996.

[8] Radim Halíř, Jan Flusser, Numerically stable direct least squares fitting of ellipses. Technical Report citeseer.nj.nec.com/350661.html, Department of Software Engineering, Charles University, Czech Republic, 2000.
[9] Matthew Harker, Paul O'Leary, Paul Zsombor-Murray, Fitting conics of specific types to scattered data, Technical Report, Institute for Automation, University of Leoben, 2004.
[10] Kenichi Kanatani, Statistical bias of conic fitting and renormalization, IEEE Transactions on Pattern Analysis and Machine Intelligence 16 (3) (1994) 320–326.
[11] Felix Klein, Elementary Mathematics from an Advanced Standpoint: Geometry, Dover Publications, New York, 1940.
[12] Yves Nievergelt, Hyperspheres and hyperplanes fitted seamlessly by algebraic constrained total least-squares, Linear Algebra and its Applications 331 (2001) 43–59.
[13] Yves Nievergelt, Fitting conics of specific types to data, Linear Algebra and its Applications 378 (2004) 1–30.
[14] Paul O'Leary, Paul Zsombor-Murray, Direct and specific least-square fitting of hyperbolae and ellipses, Journal of Electronic Imaging 13 (3) (2004) 492–503.
[15] D.V. Ouellette, Schur complements and statistics, Linear Algebra and its Applications 36 (1981) 187–295.
[16] Maurizio Pilu, Andrew Fitzgibbon, Robert Fisher, Ellipse-specific direct least-square fitting, in: IEEE ICIP, Lausanne, Switzerland, 1996.
[17] Paul L. Rosin, Assessing error of fit functions for ellipses, Graphical Models and Image Processing 58 (5) (1996) 494–502.
[18] Gabriel Taubin, Estimation of planar curves, surfaces and nonplanar space curves defined by implicit equations with applications to edge and range image segmentation, IEEE Transactions on Pattern Analysis and Machine Intelligence 13 (11) (1991) 1115–1138.