Fuzzy *h*-ideals of hemirings

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Abstract

A characterization of an *h*-hemiregular hemiring in terms of a fuzzy *h*-ideal is provided. Some properties of prime fuzzy *h*-ideals of *h*-hemiregular hemirings are investigated. It is proved that a fuzzy subset ζ of a hemiring *S* is a prime fuzzy left (right) *h*-ideal of *S* if and only if ζ is two-valued, $\zeta(0) = 1$, and the set of all *x* in *S* such that $\zeta(x) = 1$ is a prime (left) right *h*-ideal of *S*. Finally, the similar properties for maximal fuzzy left (right) *h*-ideals of hemirings are considered.

Keywords: Hemiring; *H*-hemiregular hemiring; Fuzzy *h*-ideal; Prime fuzzy *h*-ideal; Normal fuzzy *h*-ideal

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1. Introduction

There are many concepts of universal algebras generalizing an associative ring $(R, +, \cdot)$. Some of them – in particular, nearrings and several kinds of semirings – have been proven very useful. Nearrings arise from rings by cancelling either the axioms of left or those of right distributivity. The second type of those algebras $(S, +, \cdot)$, called semirings (and sometimes halfrings), share the same properties as a ring except that (S, +) is assumed to be a semigroup rather than a commutative group. Semirings, ordered semirings and hemirings appear in a natural manner in some applications to the theory of automata and formal languages (see [1]). It is a well known result that regular languages form so-called star semirings. According to the well known theorem of Kleene, the languages, or sets of words, recognized by finite-state automata are precisely

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those that are obtained from letters of input alphabets by the application of the operations sum (union), product, and star (Kleene closure). If a language is represented as a formal series with the coefficients in a Boolean semiring, then the Kleene theorem can be well described by the Schützenberger Representation theorem. Moreover, if the coefficient semiring is a field, then the corresponding syntactic algebra of the series (see [21] for details) has a finite rank if and only if the series are rational.

Many-valued logic has been proposed to model phenomena in which uncertainty and vagueness are involved. One of the most general classes of the manyvalued logic is the BL-logic defined as the logic of continuous *t*-norms. But in fact, BL-logic is a commutative lattice-ordered semiring. So, Lukasiewicz logic, Gödel logic and Product logic, as special cases of BL-logic, are special cases of semirings.

The class of K-fuzzy semirings $(K \cup \{+\infty\}, \min, \max)$, where K denotes a subset of the power set of R which is closed under the operations min, +, or max, has many interesting applications. Min-max-plus computations (and suitable semirings) are used in several areas, e.g., in differential equations. Continuous timed Petri nets can be modelled by using generalized polynomial recurrent equations in the "(min,+) semiring" (see [4]). It is interesting to observe that the fuzzy calculus, which is used for artificial intelligence purposes, indeed involves essentially "(min,max) semirings" (see [5] for more details and references). Moreover, the same hemirings can be used to study fundamental concepts of the automata theory such as nondeterminism (cf. [19]). Many other applications with references can be found in a guide to the literature on semirings and their applications [7].

Ideals of semirings play a central role in the structure theory and are useful for many purposes. However, they do not in general coincide with the usual ring ideals if S is a ring and, for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semirings. Indeed, many results in rings apparently have no analogues in semirings using only ideals. M.Henriksen defined in [10] a more restricted class of ideals in semirings, which is called the class of k-ideals, with the property that if the semiring S is a ring then a complex in S is a k-ideal if and only if it is a ring ideal. Another more restricted class of ideals has been given in hemirings by K.Iizuka [11]. However, in an additively commutative semiring S, ideals of a semiring coincide with "ideals" of a ring, provided that the semiring is a hemiring. We now call this ideal an h-ideal of the hemiring S. The properties of h-ideals and also k-ideals of hemirings were thoroughly investigated by D.R. La Torre in [20] and by using the hideals and k-ideals, D.R. La Torre established some analogous ring theorems for hemirings. Other important results were obtained in [3], [6], [14-17] and [22]. Recently, Y.B.Jun [12] considered the fuzzy setting of *h*-ideals of hemirings.

In this paper, we introduce the concept of h-hemiregularity as a generalization of the regularity in rings. Next we describe prime fuzzy h-ideals of hemirings and characterize prime fuzzy h-ideals of h-hemiregular hemirings by fuzzy h-ideals. Finally, we investigate the properties of normal and maximal fuzzy left h-ideals of hemirings.

2. Preliminaries

Recall that a *semiring* is an algebraic system $(S, +, \cdot)$ consisting of a nonempty set S together with two binary operations on S called addition and multiplication (denoted in the usual manner) such that (S, +) and (S, \cdot) are semigroups and the following distributive laws

$$a(b+c) = ab + ac$$
 and $(a+b)c = ac + bc$

are satisfied for all $a, b, c \in S$.

By zero of a semiring $(S, +, \cdot)$ we mean an element $0 \in S$ such that $0 \cdot x = x \cdot 0 = 0$ and 0 + x = x + 0 = x for all $x \in S$. A semiring with zero and a commutative semigroup (S, +) is called a *hemiring*.

Example 2.1. (i) A simple example of an infinite hemiring is the set of all non-negative integers with usual addition and multiplication.

(ii) The set $S = \cdot$	$\{0, 1, 2, 3\}$	with the	following	Cavley	tables:
(11) 110 000 0	$ 0, \pm, \pm, 0 $	1 11 1011 0110	10mo mmg	00,10,	000100.

+	0	1	2	3			0	1	2	3
0	0	1	2	3	-	0	0	0	0	0
1	1	1	2	3			0			
2	2	2	2	3		2	0	1	1	1
3	3	$ \begin{array}{c} 1 \\ 1 \\ 2 \\ 3 \end{array} $	3	2		3	0	1	1	1

is a finite hemiring.

A *left ideal* of a semiring is a subset A of S closed with respect to the addition and such that $SA \subseteq A$. A left ideal A of S is called a *left k-ideal* if for any $y, z \in A$ and $x \in S$ from x + y = z it follows $x \in A$. A *right k-ideal* is defined analogously.

A left ideal A of a hemiring S is called a *left h-ideal* if for any $x, z \in S$ and $a, b \in A$ from x + a + z = b + z it follows $x \in A$. A *right h-ideal* is defined analogously. Every left (respectively, right) *h*-ideal is a left (respectively, right) *k*-ideal. The converse is not true (cf. [20]).

A fuzzy set μ of a semiring S is called a $\mathit{fuzzy}\ \mathit{left}\ \mathit{ideal}\$ if for all $x,y\in S$ we have

- $(F_1) \quad \mu(x+y) \ge \min\{\mu(x), \mu(y)\},\$
- $(F_2) \quad \mu(xy) \ge \mu(y).$

Note that a fuzzy left ideal μ of a semiring S with zero 0 satisfies also the inequality $\mu(0) \ge \mu(x)$ for all $x \in S$.

Definition 2.2. A fuzzy left ideal μ of S is called a *fuzzy left k-ideal* if for all $x, y, z \in S$

 $x + y = z \longrightarrow \mu(x) \ge \min\{\mu(y), \mu(z)\}.$

A fuzzy right k-ideal is defined analogously. The basic properties of fuzzy k-ideals in semirings are described by Baik and Kim in [2].

Definition 2.3. A fuzzy left ideal μ of a hemiring S is called a *fuzzy left* h-ideal if for all $a, b, x, z \in S$

 $x + a + z = b + z \longrightarrow \mu(x) \ge \min\{\mu(a), \mu(b)\}.$

A fuzzy right h-ideal is defined similarly. Of course, every fuzzy left (respectively, right) h-ideal is a fuzzy left (respectively, right) k-ideal. The converse is not true (cf. [12]).

From the Transfer Principle in fuzzy set theory, cf. [13], it follows that a fuzzy set μ defined on X can be characterized by level subsets, i.e. by sets of the form

$$U(\mu; t) = \{ x \in X \, | \, \mu(x) \ge t \},\$$

where $t \in [0, 1]$. Namely, as it is proved in [13], for any algebraic system $\mathfrak{A} = (X, \mathbb{F})$, where \mathbb{F} is a family of operations (also partial) defined on X, the Transfer Principle can be formulated in the following way:

Lemma 2.4. A fuzzy set μ defined on \mathfrak{A} has the property \mathcal{P} if and only if all non-empty level subsets $U(\mu; t)$ have the property \mathcal{P} .

For example, a fuzzy set μ of a semiring S is a fuzzy left ideal if and only if all non-empty level subsets $U(\mu; t)$ are left ideals of S. Similarly, a fuzzy set μ in a hemiring S is a fuzzy left h-ideal of S if and only if each non-empty level subset $U(\mu; t)$ is a left h-ideal of S.

As a simple consequence of the above property, we obtain the following proposition, which was first proved in [12].

Proposition 2.5. Let A be a non-empty subset of a hemiring S. Then a fuzzy set μ_A defined by

$$\mu_A(x) = \begin{cases} t & \text{if } x \in A \\ s & \text{otherwise} \end{cases}$$

where $0 \le s < t \le 1$, is a fuzzy left h-ideal of S if and only if A is a left h-ideal of S.

Definition 2.6. Let μ and ν be fuzzy sets in a hemiring S. Then the h-product of μ and ν is defined by

$$(\mu \circ_h \nu)(x) = \sup_{x+a_1b_1+z=a_2b_2+z} (\min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\})$$

and $(\mu \circ_h \nu)(x) = 0$ if x cannot be expressed as $x + a_1b_1 + z = a_2b_2 + z$.

Lemma 2.7. If μ and ν are fuzzy left h-ideals in a hemiring S, then so is $\mu \cap \nu$, where $\mu \cap \nu$ is defined by

$$(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$$

for all $x \in S$. Moreover, if μ and ν are a fuzzy right h-ideal and a fuzzy left h-ideal, respectively, then $\mu \circ_h \nu \subseteq \mu \cap \nu$.

3. *H*-hemiregularity

Definition 3.1. A hemiring S is said to be *h*-hemiregular if for each $a \in S$, there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$.

It is not difficult to observe that in the case of rings the h-hemiregularity coincides with the classical regularity of rings.

Example 3.2. Let S be the set of all non-negative integers N_0 with an element ∞ such that $\infty \ge x$ for all $x \in N_0$. Consider two operations: $a + b = \max\{a, b\}$ and $a \cdot b = \min\{a, b\}$. It is easy to check that $(S, +, \cdot)$ is an *h*-hemiregular hemiring.

The *h*-closure \overline{A} of A in a hemiring S is defined as

$$\overline{A} = \{ x \in S \mid x + a_1 + z = a_2 + z \text{ for some } a_1, a_2 \in A, z \in S \}.$$

It is clear that if A is a left ideal of S, then \overline{A} is the smallest left h-ideal of S containing A. We also have $\overline{\overline{A}} = \overline{A}$ for each $A \subseteq S$. Moreover, $A \subseteq B \subseteq S$ implies $\overline{\overline{A}} \subseteq \overline{B}$.

Lemma 3.3. Let S be a hemiring and $A, B \subseteq S$, then $\overline{AB} = \overline{\overline{A} \ \overline{B}}$.

Proof. Because $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$, then $AB \subseteq \overline{A} \ \overline{B}$, and in consequence, $\overline{AB} \subseteq \overline{\overline{A} \ \overline{B}}$.

To prove the converse inclusion, let
$$x \in A$$
 and $y \in B$. Then there exist $a_i \in A$, $b_i \in B$ and $z_1, z_2 \in S$ such that $x + a_1 + z_1 = a_2 + z_1$ and $y + b_1 + z_2 = b_2 + z_2$.
Putting $z = xz_2 + 2a_1z_2 + z_1y + 2z_1b_1 + z_1z_2$, we see that $z \in S$ and
 $a_2b_2 + a_1b_1 + z$
 $= a_2b_2 + a_1b_1 + xz_2 + 2a_1z_2 + z_1y + 2z_1b_1 + z_1z_2$
 $= a_2b_2 + xz_2 + a_1z_2 + z_1(y + b_1 + z_2) + a_1b_1 + z_1b_1 + a_1z_2$
 $= a_2b_2 + xz_2 + a_1z_2 + z_1(y + b_1 + z_2) + a_1b_1 + z_1b_1 + a_1z_2$
 $= a_2b_2 + xz_2 + a_1z_2 + z_1(z_2 + b_2) + a_1b_1 + z_1b_1 + a_1z_2$
 $= a_2b_2 + xz_2 + a_1z_2 + z_1z_2 + z_1b_2 + a_1b_1 + z_1b_1 + a_1z_2$
 $= a_2b_2 + (x + a_1 + z_1)z_2 + z_1b_2 + a_1b_1 + z_1b_1 + a_1z_2$
 $= a_2b_2 + (a_2 + z_1)z_2 + z_1b_2 + a_1b_1 + z_1b_1 + a_1z_2$
 $= a_2(b_2 + z_2) + z_1(b_2 + z_2) + a_1b_1 + z_1b_1 + a_1z_2$
 $= (a_2 + z_1)(b_2 + z_2) + a_1b_1 + z_1b_1 + a_1z_2$
 $= (x + a_1 + z_1)(y + b_1 + z_2) + a_1b_1 + z_1b_1 + a_1z_2$
 $= xy + a_1b_1 + xb_1 + a_1y + xz_2 + a_1z_2 + z_1y + z_1b_1 + z_1z_2 + a_1b_1 + z_1b_1 + a_1z_2$
 $= xy + (a_1b_1 + xb_1 + a_1y + xz_2 + a_1z_2 + z_1y + z_1b_1 + z_1z_2 + a_1b_1 + z_1z_2)$
 $= xy + (a_1 + x + z_1)b_1 + a_1(y + b_1 + z_2) + (xz_2 + a_1z_2 + z_1y + z_1b_1 + z_1z_2)$
 $= xy + (a_2 + z_1)b_1 + a_1(b_2 + z_2) + (xz_2 + a_1z_2 + z_1y + z_1b_1 + z_1z_2)$
 $= xy + (a_2 + z_1)b_1 + a_1(b_2 + z_2) + (xz_2 + a_1z_2 + z_1y + z_1b_1 + z_1z_2)$
 $= xy + (a_2 + z_1)b_1 + a_1(b_2 + z_2) + (xz_2 + a_1z_2 + z_1y + z_1b_1 + z_1z_2)$

 $= xy + a_2b_1 + a_1b_2 + z.$

So, $a_2b_2 + a_1b_1 + z = xy + a_2b_1 + a_1b_2 + z$, whence we can deduce $xy \in \overline{AB}$ because $a_ib_j \in AB$ and $z \in S$. This means that $xy \in \overline{AB}$ for $x \in \overline{A}, y \in \overline{B}$.

Now let $z' \in \overline{A} \ \overline{B}$ be arbitrary. Then $z' = \sum_{i=1}^{n} x_i y_i$ for some $x_i \in \overline{A}$ and $\underline{y_i \in \overline{B}}$. Thus $z' \in \overline{AB}$, i.e. $\overline{A} \ \overline{B} \subseteq \overline{AB}$, whence $\overline{\overline{A} \ \overline{B}} \subseteq \overline{\overline{AB}} = \overline{AB}$. Therefore $\overline{\overline{A} \ \overline{B}} = \overline{AB}$. \Box

Lemma 3.4. If A and B are, respectively, right and left h-ideals of a hemiring S, then $\overline{AB} \subseteq A \cap B$.

Proof. Let $x \in \overline{AB}$, then $x + \sum_{i=1}^{m} a_i b_i + z = \sum_{j=1}^{n} a'_j b'_j + z$ for $a_i, a'_j \in A$, $b_i, b'_j \in B$ and $z \in S$. Since A is a right h-ideal of S and (S, +) is a commutative semigroup, elements $\sum_{i=1}^{m} a_i b_i$, $\sum_{j=1}^{n} a'_j b'_j$ are in A, and in consequence, $x \in A$. Similarly, we can prove that $x \in B$. So, $x \in A \cap B$, i.e. $\overline{AB} \subseteq A \cap B$. \Box

Lemma 3.5. A hemiring S is h-hemiregular if and only if for any right h-ideal A and any left h-ideal B we have $\overline{AB} = A \cap B$.

Proof. Assume that S is h-hemiregular and $a \in A \cap B$. Then there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$. Since A is a right h-ideal of S, we have $ax_i \in A$ and $ax_ia \in AB$ for i = 1, 2. Thus $a \in \overline{AB}$, which implies $A \cap B \subseteq \overline{AB}$. This, by Lemma 3.4, gives $\overline{AB} = A \cap B$.

Conversely, let $a \in S$. Then, as it is not difficult to verify, $aS + N_0 a$, where $N_0 = \{0, 1, 2, \ldots\}$, is the principal right ideal of S generated by a. Consequently, $(aS + N_0 a)$ is a right h-ideal of S. Therefore

$$\overline{(aS+N_0a)} = \overline{(aS+N_0a)} \cap S = \overline{(aS+N_0a)}S = \overline{(aS+N_0a)}S = \overline{aS}$$

because S is trivially an h-ideal of itself. Thus

$$a = a \cdot 0 + 1 \cdot a \in aS + N_0 a \subseteq \overline{(aS + N_0 a)} = \overline{aS}$$

Similarly, $a \in \overline{Sa}$. Hence

$$a \in \overline{aS} \cap \overline{Sa} = \overline{\overline{aS} \cdot \overline{Sa}} = \overline{aSSa} \subseteq \overline{aSa}$$

since \overline{aS} and \overline{Sa} are, respectively, right and left *h*-ideals of *S*. This shows that there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$. So, *S* is a *h*-hemiregular hemiring. \Box

Now, we characterize h-hemiregular hemirings by fuzzy h-ideals.

Theorem 3.6. A hemiring S is h-hemiregular if and only if for any fuzzy right h-ideal μ and fuzzy left h-ideal ν we have $\mu \circ_h \nu = \mu \cap \nu$.

Proof. Let S be an h-hemiregular hemiring. Then $\mu \circ_h \nu \subseteq \mu \cap \nu$ by Lemma 2.7. For any $a \in S$ there exist $x_1, x_2, z \in S$ such that $a + ax_1a + z = ax_2a + z$. Thus

$$(\mu \circ_h \nu)(a) = \sup_{\substack{a+ax_1a+z=ax_2a+z\\ \geq \min\{\mu(ax_1), \mu(ax_2), \nu(a)\}} (\min\{\mu(a), \nu(a)\})$$

$$\geq \min\{\mu(a), \nu(a)\}$$

$$= (\mu \cap \nu)(a),$$

i.e. $\mu \cap \nu \subseteq \mu \circ_h \nu$, whence $\mu \circ_h \nu = \mu \cap \nu$.

Conversely, let C and D be, respectively right and left h-ideals of S. Then, as it is not difficult to check (cf. [12]), their characteristic functions χ_C and χ_D are, respectively, fuzzy right h-ideal and fuzzy left h-ideal. Moreover, by Lemma 3.4, $\overline{CD} \subseteq C \cap D$. Let $a \in C \cap D$. Then $\chi_C(a) = 1 = \chi_D(a)$. Thus

$$(\chi_C \circ_h \chi_D)(a) = (\chi_C \cap \chi_D)(a) = \min\{\chi_C(a), \chi_D(a)\} = 1.$$

So, $\min\{\chi_C(a_1), \chi_D(b_1), \chi_C(a_2), \chi_D(b_2)\} = 1$ for some a_1, a_2, b_1, b_2 satisfying the equality $a + a_1b_1 + z = a_2b_2 + z$. But then $\chi_C(a_i) = 1 = \chi_D(b_i)$ for i = 1, 2, which implies $a_i \in C$ and $b_i \in D$. Therefore $a \in \overline{CD}$. Hence $\overline{CD} = C \cap D$. Lemma 3.5 completes the proof. \Box

4. Prime fuzzy left *h*-ideals

A prime left (right) *h*-ideals of hemirings are defined in the same way as prime ideals in rings, i.e. we say that left (right) *h*-ideal *P* of a hemiring *S* is prime if $P \neq S$ and for any two left (right) *h*-ideals *A* and *B* of *S* from $AB \subseteq P$ it follows either $A \subseteq P$ or $B \subseteq P$.

Definition 4.1. A fuzzy left (right) *h*-ideal ζ of a hemiring *S* is said to be *prime* if ζ is a non-constant function and for any two fuzzy left (right) *h*-ideals μ and ν of *S*, $\mu \circ_h \nu \subseteq \zeta$ implies $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$.

Example 4.2. The fuzzy subset

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0.2 & \text{otherwise} \end{cases}$$

defined on the set N_0 of all non-negative integers is a prime fuzzy left *h*-ideal of a hemiring $(N_0, +, \cdot)$.

Using the Transfer Principle (cf. Lemma 2.4) we can prove the following:

Proposition 4.3. A fuzzy set χ_P of a hemiring S is a prime fuzzy left (right) h-ideal of S if and only if P is a prime left (right) h-ideal of S.

Theorem 4.4. A fuzzy subset ζ of a hemiring S is a prime fuzzy left (right) h-ideal of S if and only if

- (i) $\zeta^0 = \{x \in S \mid \zeta(x) = \zeta(0)\}$ is a prime left (right) h-ideal of S,
- (ii) Im $\zeta = \{\zeta(x) \mid x \in S\}$ contains exactly two elements,
- (*iii*) $\zeta(0) = 1$.

Proof. We prove this theorem only for left *h*-ideals. For right *h*-ideals the proof is very similar.

Let ζ be a prime fuzzy left *h*-ideal. Then it is easy to check that ζ^0 is a prime left *h*-ideal. Suppose that Im ζ has more than two values. Then there exist two elements $a, b \in S \setminus \zeta^0$ such that $\zeta(a) \neq \zeta(b)$. Without loss of generality we can assume that $\zeta(a) < \zeta(b)$. Since ζ is a fuzzy left *h*-ideal and $b \notin \zeta^0$, it follows that $\zeta(a) < \zeta(b) < \zeta(0)$. So, there exist $r, t \in [0, 1]$ such that

$$\zeta(a) < r < \zeta(b) < t < \zeta(0). \tag{*}$$

Let ν and ω be fuzzy left *h*-ideals defined by $\nu = r\chi_{\langle a \rangle}$ and $\omega = t\chi_{\langle b \rangle}$, where $\chi_{\langle a \rangle}$, $\chi_{\langle b \rangle}$ are characteristic functions of ideals generated by *a* and *b*, respectively. Then, for any $x \in S$, which cannot be expressed in the form $x + a_1b_1 + z = a_2b_2 + z$, where $z \in S$, $a_1, a_2 \in \langle a \rangle$ and $b_1, b_2 \in \langle b \rangle$, we have $(\nu \circ_h \omega)(x) = 0$. Otherwise,

$$(\nu \circ_h \omega)(x) = \sup_{x+a_1b_1+z=a_2b_2+z} (\min\{\nu(a_1), \nu(a_2), \omega(b_1), \omega(b_2)\}) = \min\{r, t\} = r.$$

Since ζ is a fuzzy left *h*-ideal, from $x + a_1b_1 + z = a_2b_2 + z$ it follows that

$$\zeta(x) \ge \min\{\zeta(a_1b_1), \zeta(a_2b_2)\} \ge \min\{\zeta(b_1), \zeta(b_2)\} \ge r.$$

So, $(\nu \circ_h \omega)(x) \leq \zeta(x)$, whence $\nu \circ_h \omega \subseteq \zeta$, which implies $\nu \subseteq \zeta$ or $\omega \subseteq \zeta$ because ζ is a fuzzy prime left *h*-ideal. Therefore $\nu(a) = r \leq \zeta(a)$ or $\omega(b) = t \leq \zeta(b)$ which contradicts to (*). Consequently, Im μ contains exactly two elements.

To prove (*iii*) suppose that ζ is a prime fuzzy left *h*-ideal and $\zeta(0) \neq 1$. Then, according to (*ii*), Im $\zeta = \{\alpha_1, \alpha_2\}$, where $0 \leq \alpha_1 < \alpha_2 < 1$. Since $\zeta(0) = \zeta(0 \cdot x) \geq \zeta(x)$ for all $x \in S$, we have $\zeta(0) = \alpha_2$. Thus

$$\zeta(x) = \begin{cases} \alpha_2 & \text{if } x \in \zeta^0\\ \alpha_1 & \text{otherwise.} \end{cases}$$

Consider, for fixed $a \in \zeta^0$ and $b \in S \setminus \zeta^0$, two fuzzy subsets

$$\mu(x) = \begin{cases} t & \text{if } x \in \langle a \rangle \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \nu(x) = \begin{cases} r & \text{if } x \in \langle b \rangle \\ 0 & \text{otherwise,} \end{cases}$$

where $0 \le \alpha_1 < r < \alpha_2 < t \le 1$.

It is clear that μ and ν are fuzzy left *h*-ideals of *S*.

If x does not satisfy the equality $x + a_1b_1 + z = a_2b_2 + z$, where $a_1, a_2 \in \langle a \rangle$, $b_1, b_2 \in \langle b \rangle$ and $z \in S$, then $(\mu \circ_h \nu)(x) = 0$. Otherwise,

$$(\mu \circ_h \nu)(x) = \sup_{x+a_1b_1+z=a_2b_2+z} (\min\{\mu(a_1), \mu(a_2), \nu(b_1), \nu(b_2)\}) = \min\{t, r\} = r.$$

By (i), ζ^0 is a prime left *h*-ideal. If $a_1, a_2 \in \langle a \rangle$, then $a_1, a_2 \in \zeta^0$ because $a \in \zeta^0$ and $\langle a \rangle \subseteq \zeta^0$. This implies $x \in \zeta^0$. Thus $\zeta(x) = \alpha_2 > r = (\mu \circ_h \nu)(x)$. Therefore, $\mu \circ_h \nu \subseteq \zeta$. But $\mu(a) = t > \alpha_2 = \zeta(a)$ and $\nu(b) = r > \alpha_1 = \zeta(b)$, which gives $\mu \not\subseteq \zeta$ and $\nu \not\subseteq \zeta$. This contradicts to the assumption that ζ is a prime fuzzy left *h*-ideal of *S*. Hence $\zeta(0) = 1$.

Conversely, assume that the above conditions are satisfied. Then $\zeta(0) = 1$ and $\operatorname{Im}\zeta = \{\alpha, 1\}$ for some $0 \leq \alpha < 1$. Moreover, $\zeta(x + y) \geq \min\{\zeta(x), \zeta(y)\}$ for $x, y \in S$ because $\zeta(x + y) < \min\{\zeta(x), \zeta(y)\}$ implies $\zeta(x) = \zeta(y) = 1$, i.e. $x, y \in \zeta^0$, whence $x + y \in \zeta^0$, and consequently $\zeta(x+y) = 1$, which is impossible. Similarly $\zeta(xy) \geq \zeta(y)$ since $\zeta(y) = 1$ implies $xy \in \zeta^0$, whence $\zeta(xy) = 1$. This means that ζ is a fuzzy left ideal of S. In fact, ζ is a fuzzy left h-ideal. It is prime. Indeed, if there exist two fuzzy left h-ideals $\mu \not\subseteq \zeta$ and $\nu \not\subseteq \zeta$ such that $\mu \circ_h \nu \subseteq \zeta$, then $\mu(x_0) > \zeta(x_0)$ and $\nu(y_0) > \zeta(y_0)$ for some $x_0, y_0 \in S$. It is possible only in the case when $\zeta(x_0) = \zeta(y_0) = \alpha$, i.e. when $x_0, y_0 \notin \zeta^0$. Since ζ^0 is prime, then there exists $r \in S$ such that $x_0ry_0 \notin \zeta^0$. Otherwise, $x_0Sy_0 \subseteq \zeta^0$, whence $(Sx_0)(Sy_0) \subseteq \zeta^0$. So, $(Sx_0)(Sy_0) \subseteq \overline{\zeta^0} = \zeta^0$, because ζ^0 is a left h-ideal of S. Moreover, $\overline{Sx_0} \ \overline{Sy_0} \subseteq \overline{Sx_0} \ \overline{Sy_0} \subseteq \overline{\zeta^0}$. In the first case $\langle x_0 \rangle \langle x_0 \rangle \subseteq \overline{Sx_0} \subseteq \zeta^0$, whence $\langle x_0 \rangle \subseteq \zeta^0$. So, $x_0 \in \langle x_0 \rangle \subseteq \overline{\langle x_0} \subseteq \zeta^0$. This is a contradiction. Also the second case yields a contradiction.

Let $a = x_0 r y_0$. Then $\zeta(a) = \alpha$. Consequently, by the assumption

$$(\mu \circ_h \nu)(a) \le \zeta(a) = \alpha. \qquad (**)$$

Obviously $a + x_0 r y_0 = 2x_0 r y_0$. Thus $a + x_0 (r y_0) + z = (2x_0)(r y_0) + z$ for any $z \in S$. Therefore for $a = x_0 r y_0$ we have

$$(\mu \circ_h \nu)(a) = \sup_{\substack{a+x_1y_1+z=x_2y_2+z\\ \geq \min\{\mu(x_0), \mu(2x_0), \nu(ry_0)\} \geq \min\{\mu(x_0), \nu(ry_0)\}} (\min\{\mu(x_0), \nu(ry_0)\} \geq \min\{\mu(x_0), \nu(y_0)\} > \alpha,$$

since $\mu(x_0) > \alpha$ and $\nu(y_0) > \alpha$.

This contradicts (**). Hence for any fuzzy left *h*-ideals μ and ν of S, $\mu \circ_h \nu \subseteq \zeta$ implies $\mu \subseteq \zeta$ or $\nu \subseteq \zeta$. This completes the proof. \Box

Corollary 4.5. A fuzzy subset ζ of a ring R is a prime fuzzy ideal of R if and only if

- (i) ζ^0 is a prime ideal of R,
- (ii) $Im\zeta$ contains exactly two elements,
- (*iii*) $\zeta(0) = 1$.

Proof. Since rings are special case of hemirings and in the case of rings h-ideals are ideals, the above result is a simple consequence of the above theorem. \Box

Now, we give an example which shows that in Theorem 4.4 the condition (iii) cannot be omitted.

Example 4.6. The set N_0 of all non-negative integers is a hemiring with respect to usual addition and multiplication. Consider the following fuzzy subsets of N_0 :

$$\zeta(n) = \begin{cases} 0.5 & \text{if } n \text{ is even,} \\ 0.2 & \text{otherwise} \end{cases}$$
$$\mu(n) = \begin{cases} 0.7 & \text{if } n \text{ is even,} \\ 0 & \text{otherwise} \end{cases}$$
$$\nu(n) = \begin{cases} 0.3 & \text{if } n = 3k \text{ for some } k \in N_0 \\ 0 & \text{otherwise} \end{cases}$$

Then ζ , μ and ν are fuzzy left *h*-ideals such that $\mu \circ_h \nu \subseteq \zeta$. Moreover, $\zeta^0 = \langle 2 \rangle = 2N_0$ is a prime left *h*-ideal, but $\mu(2) = 0.7 > 0.5 = \zeta(2)$ and $\nu(3) = 0.3 > 0.2 = \zeta(3)$. Thus, ζ is not a prime fuzzy left *h*-ideal of N_0 . \Box

5. Normal fuzzy left *h*-ideals

Definition 5.1. A fuzzy left *h*-ideal μ of a hemiring *S* is said to be *normal* if there exists $x \in S$ such that $\mu(x) = 1$.

Note that if μ is a normal fuzzy left *h*-ideal of *S*, then $\mu(0) = 1$, and hence μ is normal if and only if $\mu(0) = 1$. Obviously, any fuzzy left *h*-ideal containing some normal left *h*-ideal is normal. Indeed, if $\nu \subseteq \mu$ and ν is normal, then $1 = \nu(0) \leq \mu(0)$, and so $\mu(0) = 1$.

Directly from Theorem 4.4 we obtain the following:

Proposition 5.2. Every prime fuzzy h-ideal of a hemiring is normal.

The converse is not true, see the following:

Example 5.3. The fuzzy subset

$$\mu(x) = \begin{cases} 1 & \text{if } x \in \langle 4 \rangle, \\ 0.5 & \text{if } x \in \langle 2 \rangle - \langle 4 \rangle, \\ 0 & \text{otherwise} \end{cases}$$

defined on a hemiring $(N_0, +, \cdot)$, where N_0 is the set of all non-negative integers, is a normal fuzzy left *h*-ideal of N_0 , which is not prime.

Proposition 5.4. Given a fuzzy left h-ideal μ of a hemiring S, let μ^+ be a fuzzy set in S defined by $\mu^+(x) = \mu(x) + 1 - \mu(0)$ for all $x \in S$. Then μ^+ is a normal fuzzy left h-ideal of S which contains μ .

Proof. For all $x, y \in S$ we have $\mu^+(0) = \mu(0) + 1 - \mu(0) = 1$ and

$$\mu^{+}(x+y) = \mu(x+y) + 1 - \mu(0) \ge \min\{\mu(x), \mu(y)\} + 1 - \mu(0)$$

= min{\mu(x) + 1 - \mu(0), \mu(y) + 1 - \mu(0)}
= min{\mu(x), \mu^{+}(y)},

which proves (F_1) . Similarly,

$$\mu^+(xy) = \mu(xy) + 1 - \mu(0) \ge \mu(y) + 1 - \mu(0) = \mu^+(y),$$

verifies (F_2) . Hence μ^+ is a fuzzy left ideal of S.

Now, let $a, b, x, z \in S$ be such that x + a + z = b + z. Then

$$\mu^{+}(x) = \mu(x) + 1 - \mu(0) \ge \min\{\mu(a), \mu(b)\} + 1 - \mu(0)$$

= min{\mu(a) + 1 - \mu(0), \mu(b) + 1 - \mu(0)}
= min{\mu(a), \mu^{+}(b)}.

Therefore, μ^+ is a normal fuzzy left *h*-ideal of *S*, and obviously $\mu \subseteq \mu^+$. \Box

Corollary 5.5. Let μ and μ^+ be as in Proposition 5.4. If there exists $x \in S$ such that $\mu^+(x) = 0$, then $\mu(x) = 0$.

Of course, for any left *h*-ideal A of S, the characteristic function χ_A of A is a normal fuzzy left *h*-ideal of S. It is clear that μ is a normal fuzzy left *h*-ideal of S if and only if $\mu^+ = \mu$.

Proposition 5.6. If μ is a fuzzy left h-ideal of S, then $(\mu^+)^+ = \mu^+$. Moreover, if μ is normal, then $(\mu^+)^+ = \mu$.

Proof. Straightforward. \Box

Theorem 5.7. Let μ be a fuzzy left h-ideal of a hemiring S and let f: $[0,\mu(0)] \rightarrow [0,1]$ be an increasing function. Then a fuzzy set $\mu_f : S \rightarrow [0,1]$ defined by $\mu_f(x) = f(\mu(x))$ is a fuzzy left h-ideal of S. In particular, if $f(\mu(0)) = 1$, then μ_f is normal; if $f(t) \geq t$ for all $t \in [0,\mu(0)]$, then $\mu \subseteq \mu_f$.

Proof. Indeed, for all $x, y \in S$ we have

$$\mu_f(x+y) = f(\mu(x+y)) \ge f(\min\{\mu(x), \mu(y)\})$$

= min{f(\mu(x)), f(\mu(y))}
= min{\mu_f(x), \mu_f(y)},

which proves (F_1) . Similarly,

$$\mu_f(xy) = f(\mu(xy)) \ge f(\mu(y)) = \mu_f(y),$$

which proves (F_2) . Hence μ_f is a fuzzy left ideal of S. If $a, b, x, z \in S$ are such that x + a + z = b + z, then

$$\mu_f(x) = f(\mu(x)) \ge f(\min\{\mu(a), \mu(b)\})$$

= min{f(\mu(a)), f(\mu(b))}
= min{\mu_f(a), \mu_f(b)}.

Therefore μ_f is a fuzzy left *h*-ideal of *S*. If $f(\mu(0)) = 1$, then μ is normal. Assume that $f(t) = f(\mu(x)) \ge \mu(x)$ for any $x \in S$, which proves $\mu \subseteq \mu_f$. \Box

Let $\mathcal{N}(S)$ denote the set of all normal fuzzy left *h*-ideals of *S*. Note that $\mathcal{N}(S)$ is a poset under the set inclusion.

Theorem 5.8. Let $\mu \in \mathcal{N}(S)$ be non-constant such that it is a maximal element of $(\mathcal{N}(S), \subseteq)$. Then μ takes only two values 0 and 1.

Proof. Since μ is normal, we have $\mu(0) = 1$. Let $\mu(x) \neq 1$ for some $x \in S$. We claim that $\mu(x) = 0$. If not, then there exists $x_0 \in S$ such that $0 < \mu(x_0) < 1$. Define on S a fuzzy set ν putting $\nu(x) = (\mu(x) + \mu(x_0))/2$ for all $x \in S$. Then, clearly ν is well-defined and for all $x, y \in S$ we have

$$\begin{split} \nu(x+y) &= \frac{1}{2}(\mu(x+y) + \mu(x_0)) \geq \frac{1}{2}(\min\{\mu(x), \mu(y)\} + \mu(x_0)) \\ &= \min\{\frac{1}{2}(\mu(x) + \mu(x_0)), \frac{1}{2}(\mu(y) + \mu(x_0))\} \\ &= \min\{\nu(x), \nu(y)\}, \end{split}$$

which proves (F_1) . Similarly,

$$\nu(xy) = \frac{1}{2}(\mu(xy) + \mu(x_0)) \ge \frac{1}{2}(\mu(y) + \mu(x_0)) = \nu(y),$$

which proves (F_2) . Hence ν is a fuzzy left ideal of S.

Moreover, for $a, b, x, z \in S$ be such that x + a + z = b + z we have

$$\nu(x) = \frac{1}{2}(\mu(x) + \mu(x_0))$$

$$\geq \frac{1}{2}(\min\{\mu(a), \mu(b)\} + \mu(x_0))$$

$$= \min\{\frac{1}{2}(\mu(a) + \mu(x_0)), \frac{1}{2}(\mu(b) + \mu(x_0))\}$$

$$= \min\{\nu(a), \nu(b)\}.$$

Therefore, ν is a fuzzy left *h*-ideal of *S*. By Proposition 5.4 ν^+ is a maximal fuzzy left *h*-ideal of *S*. Note that

$$\nu^{+}(x_{0}) = \nu(x_{0}) + 1 - \nu(0)$$

= $\frac{1}{2}(\mu(x_{0}) + \mu(x_{0})) + 1 - \frac{1}{2}(\mu(0) + \mu(x_{0}))$
= $\frac{1}{2}(\mu(x_{0}) + 1) = \nu(x_{0})$

and $\nu^+(x_0) < 1 = \nu^+(0)$. Hence ν^+ is non-constant, and μ is not a maximal element of $\mathcal{N}(S)$. This is a contradiction. \Box

Definition 5.9. A non-constant fuzzy left *h*-ideal μ of *S* is called *maximal* if μ^+ is a maximal element of $\mathcal{N}(S)$.

Theorem 5.10. If a fuzzy left h-ideal μ of S is maximal, then

- (i) μ is normal,
- (ii) μ takes only the values 0 and 1,
- (iii) $\chi_{\mu^0} = \mu$, (iv) μ^0 is a maximal left h-ideal of S.

Proof. Let μ be a maximal fuzzy left *h*-ideal of *S*. Then μ^+ is a non-constant maximal element of the poset $(\mathcal{N}(S), \subseteq)$. It follows from Theorem 5.8 that μ^+ takes only the values 0 and 1. Note that $\mu^+(x) = 1$ if and only if $\mu(x) = \mu(0)$, and $\mu^+(x) = 0$ if and only if $\mu(x) = \mu(0) - 1$. By Corollary 5.5, we have $\mu(x) = 0$, and so $\mu(0) = 1$. Hence μ is normal and $\mu^+ = \mu$. This proves (i) and (ii).

(iii) Obvious.

(iv) It is clear that we can prove $\mu^0 = \{x \in S \mid \mu(0) = 1\}$ is a left *h*-ideal. Obviously $\mu^0 \neq S$ because μ takes two values. Let A be a left h-ideal containing μ^0 . Then $\mu_{\mu^0} \subseteq \mu_A$, and in consequence, $\mu = \mu_{\mu^0} \subseteq \mu_A$. Since μ is normal, μ_A also is normal and takes only two values: 0 and 1. But, by the assumption, μ is maximal, so $\mu = \mu_A$ or $\mu = \omega$, where $\omega(x) = 1$ for all $x \in S$. In the last case $\mu^0 = S$, which is impossible. So, $\mu = \mu_A$, i.e. $\mu_A = \chi_A$. Therefore $\mu^0 = A$. \Box

Definition 5.11. A normal fuzzy left *h*-ideal μ of a hemiring *S* is said to be completely normal if there exists $x \in S$ such that $\mu(x) = 0$.

Denote by $\mathcal{C}(S)$ the set of all completely normal fuzzy left *h*-ideals of *S*. We note that $\mathcal{C}(S) \subseteq \mathcal{N}(S)$ and the restriction of the partial ordering \subseteq of $\mathcal{N}(S)$ gives a partial ordering of $\mathcal{C}(S)$.

Proposition 5.12. Any non-constant maximal element of $(\mathcal{N}(S), \subseteq)$ is also a maximal element of $(\mathcal{C}(S), \subseteq)$.

Proof. Let μ be a non-constant maximal element of $(\mathcal{N}(S), \subseteq)$. By Theorem 5.10, μ takes only the values 0 and 1, and so $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in S$. Hence $\mu \in \mathcal{C}(S)$. Assume that there exists $\nu \in \mathcal{C}(S)$ such that $\mu \subseteq \nu$. It follows that $\mu \subseteq \nu$ in $\mathcal{N}(S)$. Since μ is maximal in $(\mathcal{N}(S), \subseteq)$ and ν is nonconstant, therefore $\mu = \nu$. Thus μ is maximal element of $(\mathcal{C}(S), \subseteq)$, which ends the proof. \Box

Theorem 5.13. Every maximal fuzzy left h-ideal of a hemiring S is completely normal.

Proof. Let μ be a maximal fuzzy left *h*-ideal of *S*. Then by Theorem 5.10, μ is normal and $\mu = \mu^+$ takes only the values 0 and 1. Since μ is non-constant, it follows that $\mu(0) = 1$ and $\mu(x) = 0$ for some $x \in S$. Hence μ is completely normal, which ends the proof. \Box

6. Conclusions

In the present paper, we showed that the basic results of fuzzy sets in hemirings are similar, but not identical, with the corresponding results for semirings. So, it is important for us to study different types of fuzzy sets among hemirings, semirings and rings. In our opinion the future study of fuzzy sets in hemirings and semirings can be connected with (1) investigating semiprime fuzzy h-ideals; (2) establishing a fuzzy spectrum of a hemiring; (3) finding intuitionistic and/or interval-valued fuzzy sets and triangular norms. The obtained results can be used to solve some social networks problems and to decide whether the corresponding graph is balanced or clusterable.

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