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Embedding Hamiltonian cycles in alternating group graphs under conditional fault model

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A B S T R A C T

In this paper, assuming that each node is incident with two or more fault-free links, we show that an n-dimensional alternating group graph can tolerate up to $4n-13$ link faults, where $n \geqslant 4$, while retaining a fault-free Hamiltonian cycle. The proof is computer-assisted. The result is optimal with respect to the number of link faults tolerated. Previously, without the assumption, at most $2n-6$ link faults can be tolerated for the same problem and the same graph.

1. Introduction

The performance of interconnection networks (networks for short) heavily relies on their topologies. In recent decades, a lot of network topologies have been proposed in the literature [1,9,16,22,27] for the purpose of connecting hundreds or thousands of processors. Among them, the alternating group graph [22] owns many favorable properties such as sublogarithmic degree and diameter, vertex and edge symmetry, recursive structure, maximal fault tolerance, and strong resilience. These properties are all desired when we are building an interconnection topology for a parallel and distributed system. The alternating group graph is an instance of Cayley graphs [1].

Rings are one of the most fundamental networks for parallel and distributed computation, and many simple and efficient ring algorithms for solving various algebra or graph problems can be found in [2]. They can be also used as control/data flow structures for distributed computation in networks (see [29]). In [22], cycles of lengths ranging from 3 to $n!/2$ were embedded in an n-dimensional alternating group graph of $n!/2$ nodes, in order to use the advantages of rings. In [31], the alternating group graph was shown to be panpositionable Hamiltonian. Besides, the alternating group graph can embed grids [22], trees [22], and paths of all possible lengths between every two nodes [6].

Since node or link faults may occur to networks, it is significant to consider faulty networks. Previously, many fundamental problems such as diameter [11,23,30], routing [4], gossiping [12], and embedding [3,5–7,15,18–21,28,32,33] were studied on various faulty networks. Among them, two fault models were considered; one was the random fault model [4,6,7,11,12,18–20,25,28], and the other was the conditional fault model [3,5,15,21,23,30,32,33]. The *random fault*

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model assumed that the faults might occur everywhere without any restriction, whereas the *conditional fault model* assumed that the distribution of faults was subject to some constraint, e.g., two or more fault-free links incident to each node.

In this paper, under the conditional fault model and with the assumption of at least two fault-free links incident to each node, we show that an n-dimensional alternating group graph can tolerate up to $4n-13$ link faults, where $n \geqslant 4$, while retaining a fault-free Hamiltonian cycle. The result is optimal with respect to the number of link faults tolerated. For the same problem, at most $2n-6$ link faults can be tolerated if the random fault model is adopted (see [20]). With our result, all parallel algorithms developed on rings of lengths up to $n!/2$ can be executed as well on an n-dimensional alternating group graph provided every node is incident with at least two fault-free links, even if it has $4n-13$ link faults.

Previous results with the same fault model and assumption are described as follows, which are all embeddings of fault-free paths or cycles in faulty networks. An n-dimensional hypercube (n-cube for short) with at most $2n-5$ link faults is strongly Hamiltonian laceable [32]. An n-cube with at most 2^{n-1} link faults is Hamiltonian if these link faults form a matching and there are two fault-free links in each dimension whose distance is odd [13]. Given any fault-free link, an n-cube with at most $2n-5$ link faults contains fault-free cycles of even lengths ranging from 6 to 2^n that each contain the link [33]. An n-dimensional star graph with at most $2n-7$ link faults is Hamiltonian [15] and strongly Hamiltonian laceable [34]. An n-dimensional crossed cube with at most $2n-5$ link faults is Hamiltonian [21]. An m-ary n-cube with at most $4n-5$ link faults is Hamiltonian [3], where $m \geqslant 3$.

On the other hand, under the same fault model, but with a different assumption of each node having at least k fault-free neighbors, the minimum number of node faults whose removal may disconnect an n-cube can increase to $(n-k)2^k$, where $1 \leqslant k \leqslant \lfloor n/2 \rfloor$ [24]. Such a minimum number was called the *R_k-node-connectivity*. There is a lower bound of $m^d((n-d-1)(m-1)(s+1)+(m-s-1))$ on the R_k-node-connectivity of an m-ary n-cube [35], where $d = \lfloor k/(m-1) \rfloor$ and $s = k \bmod (m-1)$. In particular, when $k = 1$, the R_1-node-connectivity of an n-cube (m-ary n-cube) can increase to $2n-2$ ($4n-2$ if $m \geqslant 4$, and $4n-3$ if $m = 3$) [14,10], and the R_1-node connectivities of cube-connected cycles, undirected binary de Bruijn networks and Kautz graphs are all greater by one than their normal node connectivities [26]. Besides, the diameters of an n-cube with $2n-3$ node faults [23] and an n-dimensional star graph with $2n-5$ node faults [30] are greater by two in the worst case than their normal diameters.

In the next section, the structure of the alternating group graph is first reviewed. Some necessary definitions, notations and fundamental properties of the alternating group graph are then introduced. In Section 3, it is shown that there exists a fault-free Hamiltonian cycle in an n-dimensional alternating group graph with up to $4n-13$ link faults under the conditional fault model and with our assumption. Finally, in Section 4, this paper concludes with some remarks.

2. Preliminaries

It is convenient to represent a network with a graph G, where each vertex (edge) of G uniquely represents a node (link) of the network. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. Throughout this paper, we use node and vertex, link and edge, and network and graph, interchangeably.

Let $u = a_1a_2\cdots a_n$ be a permutation of $1,2,\ldots,n$. A pair of symbols a_i and a_j in u are said to be an *inversion* if $a_i < a_j$ and $i > j$, and u is an *even permutation* if it has an even number of inversions. There are $n!/2$ even permutations of $1,2,\ldots,n$. The following is a formal definition of the alternating group graph.

Definition 1 [22]. An *n-dimensional alternating group graph*, denoted by AG_n, has the node set $V(AG_n) = \{a_1a_2\cdots a_n | a_1a_2\cdots a_n$ is an even permutation of $1,2,\ldots,n\}$ and the link set $E(AG_n) = \{(a_1a_2\cdots a_n, a_2a_ia_3a_4\cdots a_{i-2}a_{i-1}a_1a_{i+1}a_{i+2}\cdots a_n), (a_1a_2\cdots a_n, a_ia_1a_3a_4\cdots a_{i-2}a_{i-1}a_2a_{i+1}a_{i+2}\cdots a_n) | a_1a_2\cdots a_n \in V(AG_n)$ and $3 \leqslant i \leqslant n\}$.

The two links $(a_1a_2\cdots a_n, a_2a_ia_3a_4\cdots a_{i-2}a_{i-1}a_1a_{i+1}a_{i+2}\cdots a_n)$ and $(a_1a_2\cdots a_n, a_ia_1a_3a_4\cdots a_{i-2}a_{i-1}a_2a_{i+1}a_{i+2}\cdots a_n)$ are referred to as *i-dimensional links* of $a_1a_2\cdots a_n$, where $3 \leqslant i \leqslant n$. Intuitively, $a_2a_ia_3a_4\cdots a_{i-2}a_{i-1}a_1a_{i+1}a_{i+2}\cdots a_n$ ($a_ia_1a_3a_4\cdots a_{i-2}a_{i-1}a_2a_{i+1}a_{i+2}\cdots a_n$) is obtained from $a_1a_2\cdots a_n$ by shifting the three elements a_1, a_2, a_i left (right) cyclically, while retaining the other $n-3$ elements $a_3,a_4,\ldots,a_{i-1},a_{i+1},a_{i+2},\ldots,a_n$ stationary. Fig. 1 illustrates the topologies of AG_3 and AG_4. It is not difficult to see that AG_n has $n!/2$ nodes and $(n-2)n!/2$ links. Besides, AG_n is regular of degree $2(n-2)$. Throughout this paper, we use $E^{(i)}(AG_n)$ to denote the set of all i-dimensional links in AG_n.

It was shown in [22] that AG_n is both node symmetric and link symmetric. Besides, AG_n is recursive, as explained below. It can be observed from Fig. 1 that AG_4 consists of four embedded AG_3's, denoted by $AG_4^{(1)}$, $AG_4^{(2)}$, $AG_4^{(3)}$, and $AG_4^{(4)}$. In general, AG_n comprises n embedded AG_{n-1}'s: $AG_n^{(i)}$ for $1 \leqslant i \leqslant n$, where each node in $AG_n^{(i)}$ has the rightmost digit i. For $I \subseteq \{1,2,\ldots,n\}$, we let AG_n^I denote the subgraph of AG_n induced by $\bigcup_{k \in I} V(AG_n^{(k)})$, and for $p \neq q$, we let $\widetilde{E}_{p,q}(AG_n)$ denote the set of n-dimensional links in AG_n that connect $AG_n^{(p)}$ and $AG_n^{(q)}$.

For each $u \in V(G)$, let $\deg(u)$ denote the *degree* of u, which is the number of links incident to u, and let $\delta(G) = \min\{\deg(u) | u \in V(G)\}$ be the minimal node degree of G. By $P_{v_0,v_t} = \langle v_0, v_1, v_2, \ldots, v_{t-1}, v_t \rangle$ we denote a path from node v_0 to node v_t, where $v_1, v_2, \ldots, v_{t-1}$ are intermediate nodes. When $v_0 = v_t, P_{v_0,v_t}$ forms a cycle. A path may contain another path as its subpath. For example, $P_{v_0,v_t} = \langle v_0, v_1, \ldots, v_i, P_{v_i,v_j}, v_j, \ldots, v_t \rangle$ contains a subpath, i.e., P_{v_i,v_j}, as its subpath.

A path (cycle) in G is called a *Hamiltonian path* (*Hamiltonian cycle*) if it contains every node of G exactly once. G is called *Hamiltonian* if it has a Hamiltonian cycle, and *Hamiltonian-connected* if it has a Hamiltonian path between every two nodes.

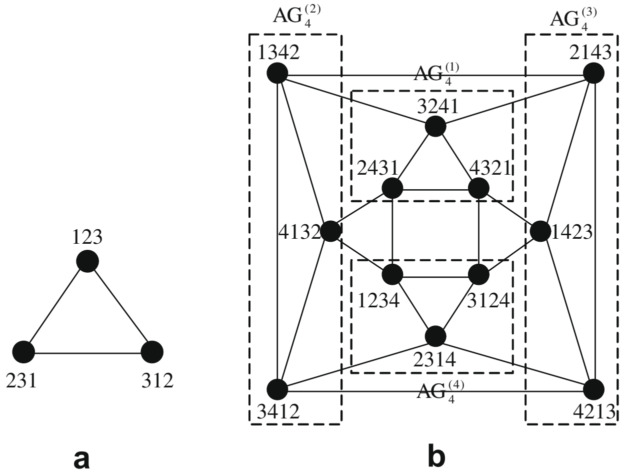

Fig. 1. The topologies of (a) AG_3 and (b) AG_4.

Since AG_n is isomorphic to the $(n, n-2)$-arrangement graph (see [8]), the following lemma can be obtained, as an immediate consequence of Hsu [20].

Lemma 1 [20]. *Suppose that $F_v \subset V(\mathrm{AG}_n)$ and $F_e \subset E(\mathrm{AG}_n)$. Then, $\mathrm{AG}_n - F_v - F_e$ is Hamiltonian if $|F_v| + |F_e| \leqslant 2n - 6$, and Hamiltonian-connected if $|F_v| + |F_e| \leqslant 2n - 7$, where $n \geqslant 4$ and $\mathrm{AG}_n - F_v - F_e$ denotes the subgraph of AG_n that is obtained by removing all nodes in F_v and all links in F_e from AG_n (the links incident to the nodes of F_v are removed automatically, but they are not counted in $|F_e|$).*

The following lemma is an immediate consequence of Definition 1.

Lemma 2. *$|\widetilde{E}_{p,q}(\mathrm{AG}_n)| = (n-2)!$ for all $p, q \in \{1, 2, \ldots, n\}$ and $p \neq q$, where $n \geqslant 4$.*

Lemma 3. *Suppose that $F_e \subset E(\mathrm{AG}_n)$ and $I = \{k_1, k_2, \ldots, k_{|I|}\} \subseteq \{1, 2, \ldots, n\}$, where $n \geqslant 5$ and $|I| \geqslant 2$. Then, for any node $s \in \mathrm{AG}_n^{(k_1)}$ and any node $t \in \mathrm{AG}_n^{(k_{|I|})}$, $\mathrm{AG}_n^I - F_e$ contains a Hamiltonian path $P_{s,t}$ that passes through $\mathrm{AG}_n^{(k_1)}, \mathrm{AG}_n^{(k_2)}, \ldots, \mathrm{AG}_n^{(k_{|I|})}$ in this sequence, provided the following two conditions hold:*

(1) *$|\widetilde{E}_{k_j,k_{j+1}}(\mathrm{AG}_n) \setminus F_e| \geqslant 3$ for all $1 \leqslant j < |I|$, where $\setminus$ denotes the set difference;*
(2) *$\mathrm{AG}_n^{(k_j)} - F_e$ is Hamiltonian-connected for all $1 \leqslant j \leqslant |I|$.*

Proof. A desired Hamiltonian path, as depicted in Fig. 2, can result as a consequence of (1) and (2). With (1), nodes $v_1, u_2, v_2, u_3, \ldots, v_{|I|-1}, u_{|I|}$ can be determined, sequentially, so that $u_{|I|} \neq v_{|I|}$, $v_j \neq u_j$, and $(v_j, u_{j+1}) \in E^{(n)}(\mathrm{AG}_n) \setminus F_e$ for all $1 \leqslant j < |I|$. With (2), a Hamiltonian path P_{u_j,v_j} can be found in $\mathrm{AG}_n^{(k_j)} - F_e$ for all $1 \leqslant j \leqslant |I|$. $\square$

3. A fault-free Hamiltonian cycle

In this section, assuming that each node is incident with two or more fault-free links, we show that AG_n can tolerate up to $4n - 13$ link faults, while retaining a fault-free Hamiltonian cycle, where $n \geqslant 4$.

Theorem 1. *Suppose that $F_e \subset E(AG_n)$, where $n \geqslant 4$. If $|F_e| \leqslant 4n - 13$ and $\delta(AG_n - F_e) \geqslant 2$, then $AG_n - F_e$ is Hamiltonian.*

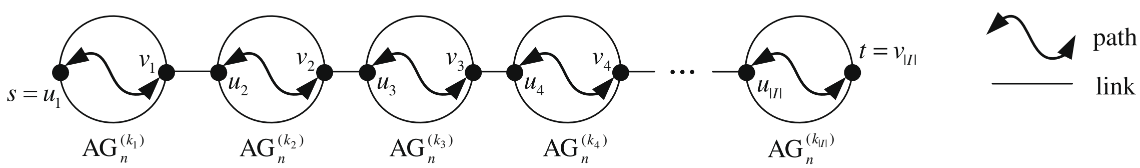

Fig. 2. A Hamiltonian path in $\mathrm{AG}_n^I - F_e$.

Proof. We show the theorem by induction on n. When $n = 4$, the correctness of the theorem can be verified by a computer program (refer to [36]). Suppose that the theorem holds for AG_{n-1}, where $n-1 \geqslant 4$. In the rest of the proof, we show that the theorem also holds for AG_n.

Since AG_n is link symmetric, we assume $|E^{(n)}(AG_n) \cap F_e| \geqslant |E^{(n-1)}(AG_n) \cap F_e| \geqslant \cdots \geqslant |E^{(3)}(AG_n) \cap F_e|$, without loss of generality. When $n \geqslant 8$, we have $|E^{(n)}(AG_n) \cap F_e| \geqslant \lceil (4n-13)/(n-2) \rceil \geqslant 4$. Hence, $|E(AG_n^{(r)}) \cap F_e| \leqslant |F_e \setminus E^{(n)}(AG_n)| \leqslant 4n-17$ for all $1 \leqslant r \leqslant n$. Similarly, when $5 \leqslant n \leqslant 7$, we have $|E^{(n)}(AG_n) \cap F_e| \geqslant 3$ and $|E(AG_n^{(r)}) \cap F_e| \leqslant |F_e \setminus E^{(n)}(AG_n)| \leqslant 4n-16$ for all $1 \leqslant r \leqslant n$.

By Lemma 2, we have $|\widetilde{E}_{p,q}(AG_n)| = (n-2)!$ for all p, $q \in \{1,2,\ldots,n\}$ and $p \neq q$. If $n \geqslant 6$, then $|\widetilde{E}_{p,q}(AG_n)| \geqslant 4n > 4n-10 \geqslant |F_e| + 3$, which implies $|\widetilde{E}_{p,q}(AG_n) \setminus F_e| > 3$. If $n = 5$, then $0 \leqslant |\widetilde{E}_{p,q}(AG_5) \setminus F_e| \leqslant 6$, because $|\widetilde{E}_{p,q}(AG_5)| = 6$ and $|F_e| \leqslant 7$. Suppose that $|E(AG_n^{(\alpha)}) \cap F_e| \geqslant |E(AG_n^{(r)}) \cap F_e|$ for all $r \in \{1,2,\ldots,n\} \setminus \{\alpha\}$, where $1 \leqslant \alpha \leqslant n$. Three cases: $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 2n-9$, $2n-8 \leqslant |E(AG_n^{(\alpha)}) \cap F_e| \leqslant 4n-17$, and $|E(AG_n^{(\alpha)}) \cap F_e| = 4n-16$, are discussed below. Notice that we have $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 4n-17$ when $n \geqslant 8$, and $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 4n-16$ when $5 \leqslant n \leqslant 7$.

Case 1. $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 2n-9$. Since each $AG_n^{(r)}$ is regular of degree $2n-6$, each node in $AG_n^{(r)} - F_e$ has degree at least three, where $1 \leqslant r \leqslant n$. A desired Hamiltonian cycle in $AG_n - F_e$ can be obtained as shown in Fig. 3, where $i_1 = \alpha, \{i_2, i_3, \ldots, i_n\} = \{1,2,\ldots,n\} \setminus \{\alpha\}, u_s, v_s \in V(AG_n^{(i_s)})$ and $u_s \neq v_s$ for $s \in \{1,2,\ldots,n\}$, and (u_1, v_n), $(v_j, u_{j+1}) \in E^{(n)}(AG_n) \setminus F_e$ for $j \in \{1,2,\ldots,n-1\}$. When $n \geqslant 6, (v_1, u_2) \in \widetilde{E}_{i_1,i_2}(AG_n) \setminus F_e$ and $(u_1, v_n) \in \widetilde{E}_{i_1,i_n}(AG_n) \setminus F_e$ can be selected (refer to the third paragraph of the proof). When $n = 5$, they can be also selected if we arrange $i_2, i_3, \ldots, i_n$ so that $|\widetilde{E}_{i_1,i_2}(AG_n) \setminus F_e| \geqslant 1$ and $|\widetilde{E}_{i_1,i_n}(AG_n) \setminus F_e| \geqslant 1$. By Lemma 1, there is a Hamiltonian path P_{u_1,v_1} in $AG_n^{(i_1)} - F_e$.

Next, the existence of a Hamiltonian path P_{u_2,v_n} in $AG_n^I - F_e$ is explained, where $I = \{i_2, i_3, \ldots, i_n\}$. When $n \geqslant 6$, the existence of P_{u_2,v_n} in $AG_n^I - F_e$ can be assured by Lemma 3. The condition (1), i.e., $|\widetilde{E}_{i_j,i_{j+1}}(AG_n) \setminus F_e| \geqslant 3$ for all $2 \leqslant j < n$, of Lemma 3 holds (refer to the third paragraph of the proof). The condition (2), i.e., $AG_n^{(i_j)} - F_e$ is Hamiltonian-connected for all $2 \leqslant j \leqslant n$, of Lemma 3 holds as a consequence of Lemma 1.

When $n = 5$, we have $0 \leqslant |\widetilde{E}_{p,q}(AG_5) \setminus F_e| \leqslant 6$ for all p, $q \in I$ and $p \neq q$ (refer to the third paragraph of the proof). If $|\widetilde{E}_{p,q}(AG_5) \setminus F_e| \geqslant 3$ for all $p, q \in I$ and $p \neq q$, the existence of a Hamiltonian path P_{u_2,v_5} in $AG_5^I - F_e$ can be assured by Lemma 3, similar to the situation of $n > 5$. If there exist $p', q' \in I$ and $p' \neq q'$ satisfying $|\widetilde{E}_{p',q'}(AG_5) \setminus F_e| < 3$, then the pair of p' and q' is unique, for otherwise $|F_e| \geqslant 2|\widetilde{E}_{p',q'}(AG_5) \cap F_e| \geqslant 8$, a contradiction. After arranging i_2, i_3, i_4, i_5 so that $\{p', q'\} \notin \{\{i_2, i_3\}, \{i_3, i_4\}, \{i_4, i_5\}\}$, the existence of a Hamiltonian path P_{u_2,v_5} in $AG_5^I - F_e$ can be assured by Lemma 3 similarly.

Case 2. $2n-8 \leqslant |E(AG_n^{(\alpha)}) \cap F_e| \leqslant 4n-17$. Recall that $|F_e \setminus E^{(n)}(AG_n)| \leqslant 4n-16$ and $|E(AG_n^{(\alpha)}) \cap F_e| \geqslant |E(AG_n^{(r)}) \cap F_e|$ for all $r \in \{1,2,\ldots,n\} \setminus \{\alpha\}$, where $1 \leqslant \alpha \leqslant n$ (refer to the second and third paragraphs of the proof). It suffices to consider two situations: (1) $|E(AG_n^{(\alpha)}) \cap F_e| = 2n-8$, $|E(AG_n^{(\beta)}) \cap F_e| = 2n-8$ for some $\beta \in \{1,2,\ldots,n\} \setminus \{\alpha\}$, and $|E(AG_n^{(r)}) \cap F_e| = 0$ for all $r \in \{1,2,\ldots,n\} \setminus \{\alpha, \beta\}$ and (2) $|E(AG_n^{(r)}) \cap F_e| \leqslant 2n-9$ for all $r \in \{1,2,\ldots,n\} \setminus \{\alpha\}$.

First we consider the situation (1). Since $|F_e| \leqslant 4n-13$ and $|E(AG_n^{(\alpha)}) \cap F_e| + |E(AG_n^{(\beta)}) \cap F_e| = 4n-16$, we have $|E^{(n)}(AG_n) \cap F_e| \leqslant 3$. However, $|E^{(n)}(AG_n) \cap F_e| \geqslant 3$ if $5 \leqslant n \leqslant 7$ and $|E^{(n)}(AG_n) \cap F_e| \geqslant 4$ if $n \geqslant 8$ (refer to the second paragraph of the proof). Therefore, we have $|E^{(n)}(AG_n) \cap F_e| = 3$, and the situation (1) occurs only when $5 \leqslant n \leqslant 7$. Moreover, since each $AG_n^{(r)}$ is regular of degree $2n-6$, each node in $AG_n^{(r)} - F_e$ has degree at least two, where $1 \leqslant r \leqslant n$. A desired Hamiltonian cycle in $AG_n - F_e$ can be obtained as shown in Fig. 4, where $i_1 = \alpha$, $i_2 = \beta$, and $\{i_3, i_4, \ldots, i_n\} = \{1,2,\ldots,n\} \setminus \{\alpha, \beta\}$.

Refer to Fig. 4 again. The induction hypothesis assures a Hamiltonian cycle C_1 in $AG_n^{(i_1)} - F_e$ and a Hamiltonian cycle C_2 in $AG_n^{(i_2)} - F_e$. Notice that $|E^{(n)}(AG_n) \cap F_e| = 3$, $|\widetilde{E}_{p,q}(AG_n)| = (n-2)! \geqslant 6$ for all p, $q \in \{1,2,\ldots,n\}$ and $p \neq q$, and each node is incident with two links in $E^{(n)}(AG_n)$. It is easy to understand that $(v_1, u_2) \in \widetilde{E}_{i_1,i_2}(AG_n) \setminus F_e$ and a neighboring node u_1 (v_2) of v_1 (u_2) in C_1 (C_2) having $(u_1, v_n) \in \widetilde{E}_{i_1,i_n}(AG_n) \setminus F_e((v_2, u_3) \in \widetilde{E}_{i_2,i_3}(AG_n) \setminus F_e)$ can be found. By Lemma 3, there exists a Hamiltonian path P_{u_3,v_n} in $AG_n^I - F_e$, where $I = \{i_3, i_4, \ldots, i_n\}$.

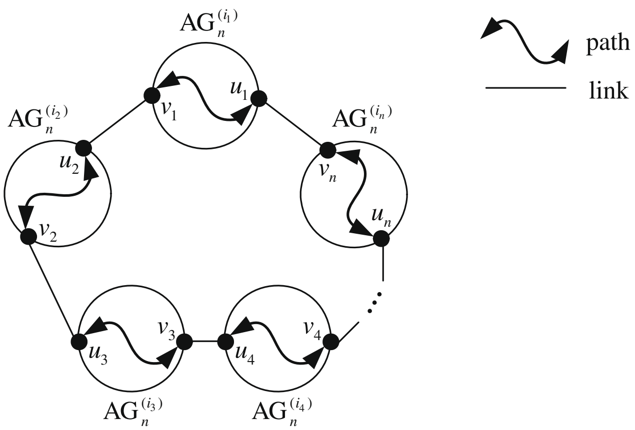

Fig. 3. A Hamiltonian cycle in $AG_n - F_e$ for $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 2n-9$.

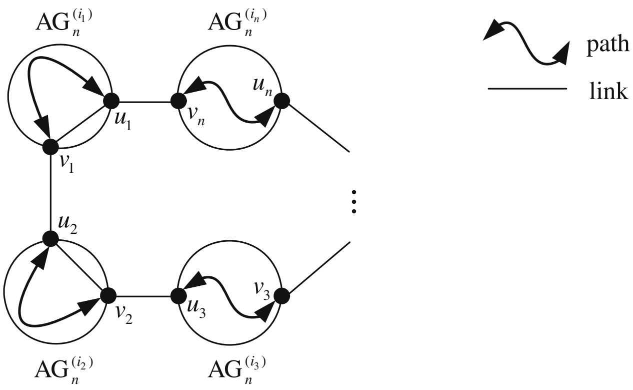

Fig. 4. A Hamiltonian cycle in $AG_n - F_e$ for $|E(AG_n^{(\alpha)}) \cap F_e| = |E(AG_n^{(\beta)}) \cap F_e| = 2n - 8$.

Next we consider the situation (2). Two cases: $\delta(AG_n^{(\alpha)} - F_e) \geqslant 1$ and $\delta(AG_n^{(\alpha)} - F_e) = 0$, are discussed below.

Case 2.1. $\delta(AG_n^{(\alpha)} - F_e) \geqslant 1$. A desired Hamiltonian cycle in $AG_n - F_e$ can be obtained as shown in Fig. 5a, where $i_1 = \alpha$ and $\{i_2, i_3, \ldots, i_n\} = \{1, 2, \ldots, n\} \backslash \{\alpha\}$. We first assume $\delta(AG_n^{(\alpha)} - F_e) \geqslant 2$. The induction hypothesis assures a Hamiltonian cycle C_1 in $AG_n^{(i_1)} - F_e$. Since $|F_e| \leqslant 4n - 13$ and $|E(AG_n^{(\alpha)}) \cap F_e| \geqslant 2n - 8$, a link, i.e., (u_1, v_1), can be selected from C_1 so that there exist (v_1, u_2), $(u_1, v_n) \in E^{(n)}(AG_n) \backslash F_e$, for otherwise $|E^{(n)}(AG_n) \cap F_e| \geqslant ((n-1)!/2)/2 > 2n - 5$, a contradiction. Then a Hamiltonian path P_{u_2, v_n} in $AG_n^I - F_e$ can be assured by Lemma 3 (refer to Case 1), where $I = \{i_2, i_3, \ldots, i_n\}$.

Then we assume $\delta(AG_n^{(\alpha)} - F_e) = 1$. Refer to Fig. 5a again. Since $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 4n - 17$, there exists a unique node of degree one in $AG_n^{(\alpha)} - F_e$. We let v_1 be this node. Since $\delta(AG_n - F_e) \geqslant 2$, there exists an n-dimensional link of v_1 that is not contained in F_e. We let $(v_1, u_2) \in E^{(n)}(AG_n) \backslash F_e$ be this link, where $u_2 \in V(AG_n^{(i_2)})$ is assumed. Since $|F_e| \leqslant 4n - 13$ and $|E(AG_n^{(\alpha)}) \cap F_e| \geqslant 2n - 8$, a link, i.e., (v_1, u_1), can be selected from $E(AG_n^{(i_1)}) \cap F_e$ so that there exists $(u_1, v_n) \in E^{(n)}(AG_n) \backslash F_e$, where $v_n \in V(AG_n^{(i_n)})$, for otherwise $|E^{(n)}(AG_n) \cap F_e| \geqslant 2(2n - 7) > 2n - 5$, a contradiction. The induction hypothesis assures a Hamiltonian cycle C_1 in $AG_n^{(i_1)} - (F_e \setminus \{(v_1, u_1)\})$. Since $(v_1, u_1) \in C_1$, there exists a Hamiltonian path $P_{u_1, v_1} (= C_1 - (v_1, u_1))$ in $AG_n^{(i_1)} - F_e$. A Hamiltonian path P_{u_2, v_n} in $AG_n^I - F_e$ can be assured by Lemma 3 (refer to Case 1), where $I = \{i_2, i_3, \ldots, i_n\}$.

Case 2.2. $\delta(AG_n^{(\alpha)} - F_e) = 0$. A desired Hamiltonian cycle in $AG_n - F_e$ can be obtained as shown in Fig. 5b, where $i_1 = \alpha$ and $\{i_2, i_3, \ldots, i_n\} = \{1, 2, \ldots, n\} \backslash \{\alpha\}$. This case occurs only when $n \geqslant 6$, because $|E(AG_n^{(\alpha)}) \cap F_e| \leqslant 4n - 17$ and $AG_n^{(\alpha)}$ is regular of degree $2n - 6$. There exists a unique node of degree zero in $AG_n^{(\alpha)} - F_e$. We let w be this node. Since $\delta(AG_n - F_e) \geqslant 2$, the two n-dimensional links of w are not contained in F_e. We let (w, v_3), $(w, u_4) \in E^{(n)}(AG_n) \backslash F_e$ be these two links, where $v_3 \in V(AG_n^{(i_3)})$ and $u_4 \in V(AG_n^{(i_4)})$ are assumed. Refer to Fig. 5b again.

The two links (v_1, u_2), (u_1, v_n) can be found, because $|\widetilde{E}_{i_1, r}(AG_n)| = (n-2)!$ for all $r \in \{i_2, i_3, \ldots, i_n\}$ and $|E^{(n)}(AG_n) \cap F_e| \leqslant (4n - 13) - (2n - 8) = 2n - 5$. Let $F_v = \{w\}$ and $F'_e = (E(AG_n^{(i_1)}) \cap F_e) \setminus \{(w, z) | (w, z) \in E(AG_n^{(i_1)})\}$, where $|F_v| + |F'_e| \leqslant 2n - 10$. By Lemma 1, there exists a Hamiltonian path P_{u_1, v_1} in $AG_n^{(i_1)} - F_v - F'_e$. By Lemma 3, there exist a Hamiltonian path P_{u_2, v_3} in $AG_n^{I_1} - F_e$ and a Hamiltonian path P_{u_4, v_n} in $AG_n^{I_2} - F_e$, where $I_1 = \{i_2, i_3\}$ and $I_2 = \{i_4, i_5, \ldots, i_n\}$.

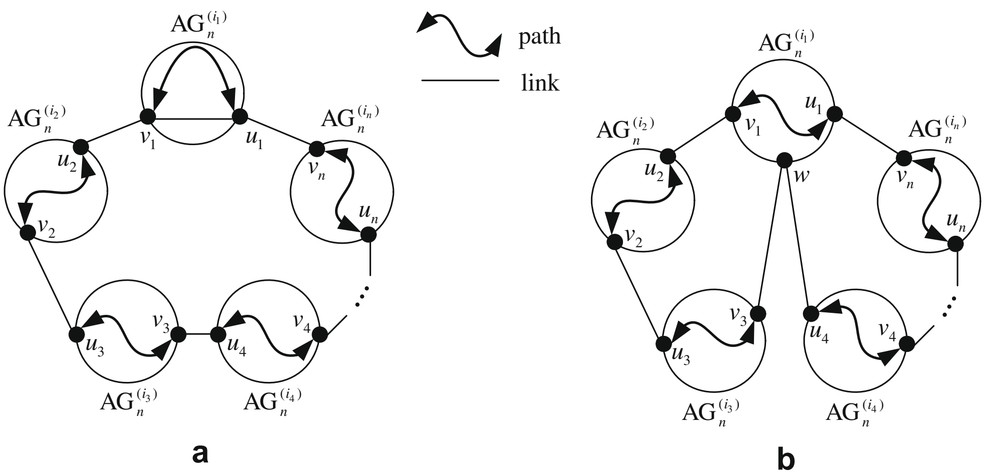

Fig. 5. Hamiltonian cycles in $AG_n - F_e$ for $|E(AG_n^{(\alpha)}) \cap F_e| \geqslant 2n - 8$ and $|E(AG_n^{(r)}) \cap F_e| \leqslant 2n - 9$ for all $r \in \{1, 2, \ldots, n\} \backslash \{\alpha\}$. (a) $\delta(AG_n^{(\alpha)} - F_e) \geqslant 1$. (b) $\delta(AG_n^{(\alpha)} - F_e) = 0$.

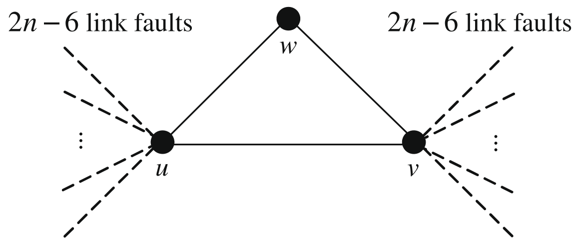

Fig. 6. A situation of no fault-free Hamiltonian cycle.

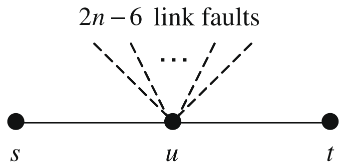

Fig. 7. A situation of no fault-free Hamiltonian path from s to t.

Case 3. $|E(\mathrm{AG}_n^{(\alpha)}) \cap F_e| = 4n - 16$. Recall that when $n \geqslant 8$, we have $|E(\mathrm{AG}_n^{(r)}) \cap F_e| \leqslant 4n - 17$ for all $1 \leqslant r \leqslant n$, and when $5 \leqslant n \leqslant 7$, we have $|E^{(n)}(\mathrm{AG}_n) \cap F_e| \geqslant 3$ (refer to the second paragraph of the proof). Hence, this case occurs only when $5 \leqslant n \leqslant 7$, and has $|E^{(n)}(\mathrm{AG}_n) \cap F_e| = 3$ and $|E(\mathrm{AG}_n^{(r)}) \cap F_e| = 0$ for all $r \in \{1,2,\ldots,n\} \backslash \{\alpha\}$, as a consequence of $|F_e| \leqslant 4n - 13$. Further, since $|\widetilde{E}_{p,q}(\mathrm{AG}_n)| = (n-2)! \geqslant 6$, we have $|\widetilde{E}_{p,q}(\mathrm{AG}_n) \setminus F_e| \geqslant 3$ for all p, $q \in \{1,2,\ldots,n\}$ and $p \neq q$. Three cases: $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) \geqslant 2$, $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) = 1$, and $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) = 0$, are discussed below.

Case 3.1. $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) \geqslant 2$. A desired Hamiltonian cycle in $\mathrm{AG}_n - F_e$ can be obtained as shown in Fig. 5a, where $i_1 = \alpha$ and $\{i_2, i_3, \ldots, i_n\} = \{1,2,\ldots,n\} \backslash \{\alpha\}$. Since $|F_e| \leqslant 4n - 13$, the link (u_1, v_1) can be selected from $E(\mathrm{AG}_n^{(i_1)}) \cap F_e$ so that there exist (v_1, u_2), $(u_1, v_n) \in E^{(n)}(\mathrm{AG}_n) \backslash F_e$. The induction hypothesis assures a Hamiltonian cycle C_1 in $\mathrm{AG}_n^{(i_1)} - (F_e \setminus \{(u_1, v_1)\})$. If C_1 contains (u_1, v_1), then a Hamiltonian path P_{u_2, v_n} in $\mathrm{AG}_n^I - F_e$ can be assured by Lemma 3, where $I = \{i_2, i_3, \ldots, i_n\}$. Otherwise, a desired Hamiltonian cycle in $\mathrm{AG}_n - F_e$ can be obtained all the same as Case 2.1 (when $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) \geqslant 2$).

Case 3.2. $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) = 1$. A desired Hamiltonian cycle in $\mathrm{AG}_n - F_e$ can be obtained as shown in Fig. 5a, where $i_1 = \alpha$ and $\{i_2, i_3, \ldots, i_n\} = \{1,2,\ldots,n\} \backslash \{\alpha\}$. Since $|E(\mathrm{AG}_n^{(\alpha)}) \cap F_e| = 4n - 16$, there exists a unique node of degree one in $\mathrm{AG}_n^{(\alpha)} - F_e$. We let v_1 be this node. Since $\delta(\mathrm{AG}_n - F_e) \geqslant 2$, there exists an n-dimensional link of v_1 that is not contained in F_e. We let $(v_1, u_2) \in E^{(n)}(\mathrm{AG}_n) \backslash F_e$ be this link, where $u_2 \in V(\mathrm{AG}_n^{(i_2)})$ is assumed. The link (v_1, u_1) can be selected from $E(\mathrm{AG}_n^{(i_1)}) \cap F_e$ so that there exists $(u_1, v_n) \in E^{(n)}(\mathrm{AG}_n) \backslash F_e$, where $v_n \in V(\mathrm{AG}_n^{(i_n)})$, for otherwise $|E^{(n)}(\mathrm{AG}_n) \cap F_e| \geqslant 2(2n-7) > 3$, a contradiction. The induction hypothesis assures a Hamiltonian cycle C_1 in $\mathrm{AG}_n^{(i_1)} - (F_e \setminus \{(v_1, u_1)\})$. Since $(v_1, u_1) \in C_1$, there exists a Hamiltonian path $P_{u_1, v_1} (= C_1 - (v_1, u_1))$ in $\mathrm{AG}_n^{(i_1)} - F_e$. A Hamiltonian path P_{u_2, v_n} in $\mathrm{AG}_n^I - F_e$ can be assured by Lemma 3, where $I = \{i_2, i_3, \ldots, i_n\}$.

Case 3.3. $\delta(\mathrm{AG}_n^{(\alpha)} - F_e) = 0$. A desired Hamiltonian cycle in $\mathrm{AG}_n - F_e$ can be obtained as shown in Fig. 5b, where $i_1 = \alpha$ and $\{i_2, i_3, \ldots, i_n\} = \{1,2,\ldots,n\} \backslash \{\alpha\}$. There exists a unique node of degree zero in $\mathrm{AG}_n^{(\alpha)} - F_e$. We let w be this node. Since $\delta(\mathrm{AG}_n - F_e) \geqslant 2$, the two n-dimensional links of w are not contained in F_e. We let (w, v_3), $(w, u_4) \in E^{(n)}(\mathrm{AG}_n) \backslash F_e$ be these two links, where $v_3 \in V(\mathrm{AG}_n^{(i_3)})$ and $u_4 \in V(\mathrm{AG}_n^{(i_4)})$ are assumed. Since $|\widetilde{E}_{i_1, r}(\mathrm{AG}_n)| = (n-2)! \geqslant 6$ for all $r \in \{i_2, i_3, \ldots, i_n\}$ and $|E^{(n)}(\mathrm{AG}_n) \cap F_e| = 3$, the two links (v_1, u_2) and (u_1, v_n) can be selected from $\widetilde{E}_{i_1, i_2}(\mathrm{AG}_n) \setminus F_e$ and $\widetilde{E}_{i_1, i_n}(\mathrm{AG}_n) \setminus F_e$, respectively. Let $F_v = \{w\}$ and $F'_e = (E(\mathrm{AG}_n^{(i_1)}) \cap F_e) \setminus \{(w,z) | (w,z) \in E(\mathrm{AG}_n^{(i_1)})\}$, where $|F_v| + |F'_e| \leqslant 2n - 9$. By Lemma 1, there exists a Hamiltonian path P_{u_1, v_1} in $\mathrm{AG}_n^{(i_1)} - F_v - F'_e$. By Lemma 3, there exist a Hamiltonian path P_{u_2, v_3} in $\mathrm{AG}_n^{I_1} - F_e$ and a Hamiltonian path P_{u_4, v_n} in $\mathrm{AG}_n^{I_2} - F_e$, where $I_1 = \{i_2, i_3\}$ and $I_2 = \{i_4, i_5, \ldots, i_n\}$. $\square$

4. Concluding remarks

It is both practically significant and theoretically interesting to investigate the fault-tolerant capability of a multiprocessor system. Most of previous work adopted the random fault model, which assumed that the faults might occur anywhere without any restriction. Under the random fault model, an n-dimensional alternating group graph can tolerate up to $2n - 6$ link faults or node faults, while retaining a fault-free Hamiltonian cycle (see Lemma 1). There was another fault model, i.e., the conditional fault model, which assumed that the fault distribution must satisfy some properties.

In this paper, adopting the conditional fault model and assuming that there were two or more fault-free links incident to each node, we showed that an n-dimensional alternating group graph contained a fault-free Hamiltonian cycle, even if there were up to $4n - 13$ link faults, where $n \geqslant 4$. Our result is optimal with respect to the number of link faults tolerated. Refer to Fig. 6, where a distribution of $4n - 12$ link faults over an n-dimensional alternating group graph is shown. It is easy to see that there is no fault-free Hamiltonian cycle for this situation.

On the other hand, consider the fault-tolerant Hamiltonian-connectedness problem, i.e., finding a fault-free Hamiltonian path between every pair of distinct nodes. If the problem is solved on an n-dimensional alternating group graph under the random fault model, the maximal number of tolerable link faults is $2n-7$ (see Lemma 1). For the same problem and same graph, it is not possible to increase the maximal number of tolerable link faults, if the conditional fault model is adopted instead. Refer to Fig. 7, where a distribution of $2n-6$ link faults over an n-dimensional alternating group graph is shown. It is easy to see that there is no fault-free Hamiltonian path from s to t for this situation.

Since an n-dimensional alternating group graph is isomorphic to an $(n, n-2)$-arrangement graph, the embedding methods and results proposed in this paper are useful to those people who are interesting in the fault-tolerant hamiltonicity of the arrangement graph under the conditional fault model. Besides, interested readers may try to solve the pancycle problem [17] on the alternating group graph under the conditional fault model.

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