# Quasi Conjunction, Quasi Disjunction, T-norms and T-conorms: Probabilistic Aspects 

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#### Abstract

We make a probabilistic analysis related to some inference rules which play an important role in nonmonotonic reasoning. In a coherence-based setting, we study the extensions of a probability assessment defined on $n$ conditional events to their quasi conjunction, and by exploiting duality, to their quasi disjunction. The lower and upper bounds coincide with some well known t-norms and t-conorms: minimum, product, Lukasiewicz, and Hamacher t-norms and their dual t-conorms. On this basis we obtain Quasi And and Quasi Or rules. These are rules for which any finite family of conditional events p-entails the associated quasi conjunction and quasi disjunction. We examine some cases of logical dependencies, and we study the relations among coherence, inclusion for conditional events, and pentailment. We also consider the Or rule, where quasi conjunction and quasi disjunction of premises coincide with the conclusion. We analyze further aspects of quasi conjunction and quasi disjunction, by computing probabilistic bounds on premises from bounds on conclusions. Finally, we consider biconditional events, and we introduce the notion of an $n$-conditional event. Then we give a probabilistic interpretation for a generalized Loop rule. In an appendix we provide explicit expressions for the Hamacher t-norm and t-conorm in the unitary hypercube.


Keywords: coherence, lower/upper probability bounds, quasi conjunction/disjunction, t-norms/conorms, Goodman-Nguyen inclusion relation, generalized Loop rule.

## 1. Introduction

In classical (monotonic) logic, if a conclusion $C$ follows from some premises, then $C$ also follows when the set of premises is enlarged; that is, adding premises never invalidates any conclusions. In contrast, in (nonmonotonic) commonsense reasoning we are typically in a situation of partial knowledge, and a conclusion reached from a set of premises may be retracted when some premises are added. Nonmonotonic reasoning is a relevant topic in the field of artificial intelligence, and has been studied in literature by many symbolic and numerical formalisms (see, e.g. [6, 8, 9, 22, 54]). A remarkable theory related to nonmonotonic reasoning has been proposed by Adams in his probabilistic logic of conditionals ([1]). We recall that the approach of Adams can be developed with full generality by exploiting coherence-based probabilistic reasoning ([26]). In the setting of coherence conditional probabilities can be directly assigned, and zero probabilities for conditioning events can be properly managed (see, e.g. [7, 10, 11, 12, $18,35,40,44,62]$ ). The coherence-based approach is applied in many fields: statistical analysis, decision theory, probabilistic default reasoning and fuzzy theory. It allows one to manage incomplete probabilistic assignments in a situation of vague or partial knowledge (see, e.g. [13, 14, 15, 17, 19, 37, 55, 56, 64]). A basic notion in the work of Adams is the quasi conjunction of conditionals. This logical operation also plays a relevant role in [22] (see also [6]), where a suitable Quasi And rule is introduced to characterize entailment from a knowledge base. In the present article, besides quasi conjunction, we study by duality the quasi disjunction of conditional events and the

[^0]associated Quasi Or rule.
Theoretical tools which play a relevant role in artificial intelligence and fuzzy logic are $t$-norms and $t$-conorms. These allow one to extend the Boolean operations of conjunction and disjunction to the setting of multi-valued logics. Tnorms (first proposed in [58]) and $t$-conorms were introduced in [63] and are a subclass of aggregation functions ([46, 47, 48]). They play a basic role in decision theory, information and data fusion, probability theory and risk management.
In this paper we give many insights about probabilistic default reasoning in the setting of coherence, by making a probabilistic analysis of the Quasi And, Quasi Or and Loop inference rules. Some results were already given without proof in [39]. To begin, we recall some basic notions and results regarding coherence, probabilistic default reasoning, and the Hamacher t-norm/t-conorm (Section 2). Then, we show that some well known t-norms and t-conorms appear as lower and upper bounds when we propagate probability assessments on a finite family of conditional events to the associated quasi conjunction. By these bounds we obtain the Quasi And rule. We also consider special cases of logical dependencies associated with the Goodman-Nguyen inclusion relation ([45]) and with the compound probability theorem. Then, we give two results which identify the strict relationship holding among coherence, the Goodman-Nguyen inclusion relation, and p-entailment (Section 3). We deepen a further aspect of the Quasi And rule by determining the probability bounds on the premises from given bounds on the conclusion of the rule (Section 4). By exploiting duality, we give analogous results for the quasi disjunction of conditional events, and we obtain the Quasi Or rule. We also examine the Or rule, and we show that quasi conjunction and quasi disjunction of the premises of this rule both coincide with its conclusion (Section 5). In a similar way, we then enrich the Quasi Or rule by determining the probability bounds on the premises from given bounds on the conclusion of the rule (Section 6). We consider biconditional events, and we introduce the notion of an $n$-conditional event, by means of which we give a probabilistic semantics to a generalized Loop rule (Section 7). Finally, we give some conclusions and perspectives on future work (Section 8). We illustrate notions and results with a table and some figures.
The results given in this work may be useful for the treatment of uncertainty in many applications of statistics and artificial intelligence, in particular for the probabilistic approach to inference rules in nonmonotonic reasoning, for the psychology of uncertain reasoning, and for probabilistic reasoning in the semantic web (see, e.g., [38, 51, 57, 60, 61]).

## 2. Some Preliminary Notions

In this section we first discuss some basic notions regarding coherence. Then, we recall the notions of pconsistency and p-entailment of Adams ([]]) within the setting of coherence.

### 2.1. Basic notions on coherence

As in the approach of de Finetti, events represent uncertain facts described by (non ambiguous) logical propositions. An event $A$ is a two-valued logical entity which can be true $(T)$, or false $(F)$. The indicator of $A$ is a two-valued numerical quantity which is 1 , or 0 , according to whether $A$ is true, or false. We denote by $\Omega$ the sure event and by $\emptyset$ the impossible one. We use the same symbols for events and their indicators. Moreover, we denote by $A \wedge B$ (resp., $A \vee B$ ) the logical intersection, or conjunction (resp., logical union, or disjunction). To simplify notations, in many cases we denote the conjunction between $A$ and $B$ as the product $A B$. We denote by $A^{c}$ the negation of $A$. Of course, the truth values for conjunctions, disjunctions and negations are obtained by applying the propositional calculus. Given any events $A$ and $B$, we simply write $A \subseteq B$ to denote that $A$ logically implies $B$, that is $A B^{c}=\emptyset$, which means that $A$ and $B^{c}$ cannot be both true. Given $n$ events $A_{1}, \ldots, A_{n}$, as $A_{i} \vee A_{i}^{c}=\Omega, \quad i=1, \ldots, n$, by expanding the expression $\bigwedge_{i=1}^{n}\left(A_{i} \vee A_{i}^{c}\right)$, we can represent $\Omega$ as the disjunction of $2^{n}$ logical conjunctions, some of which may be impossible. The remaining ones are the atoms, or constituents, generated by $A_{1}, \ldots, A_{n}$. We recall that $A_{1}, \ldots, A_{n}$ are logically independent when the number of atoms generated by them is $2^{n}$. Of course, in case of some logical dependencies among $A_{1}, \ldots, A_{n}$ the number of atoms is less than $2^{n}$. For instance, given two logically incompatible events $A, B$, as $A B=\emptyset$ the atoms are: $A B^{c}, A^{c} B, A^{c} B^{c}$. We remark that, to introduce the basic notions, an equivalent approach is that of considering a Boolean algebra $\mathcal{B}$ whose elements are interpreted as events. In this way events would be combined by means of the Boolean operations; then to say that $A_{1}, \ldots, A_{n}$ are logically independent would mean that the subalgebra generated by them has $2^{n}$ atoms. Concerning conditional events, given two events $A, B$, with $A \neq \emptyset$, in our approach the conditional event $B \mid A$ is defined as a three-valued logical entity which is true ( T ), or false ( F ), or void
(V), according to whether $A B$ is true, or $A B^{c}$ is true, or $A^{c}$ is true, respectively. We recall that, agreeing to the betting metaphor, if you assess $P(B \mid A)=p$, then you are willing to pay an amount $p$ and to receive 1 , or 0 , or $p$, according to whether $A B$ is true, or $A B^{c}$ is true, or $A^{c}$ is true (bet called off), respectively. Given a real function $P: \mathcal{F} \rightarrow \mathcal{R}$, where $\mathcal{F}$ is an arbitrary family of conditional events, let us consider a subfamily $\mathcal{F}_{n}=\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\} \subseteq \mathcal{F}$, and the vector $\mathcal{P}_{n}=\left(p_{1}, \ldots, p_{n}\right)$, where $p_{i}=P\left(E_{i} \mid H_{i}\right), i=1, \ldots, n$. We denote by $\mathcal{H}_{n}$ the disjunction $H_{1} \vee \cdots \vee H_{n}$. As $E_{i} H_{i} \vee E_{i}^{c} H_{i} \vee H_{i}^{c}=\Omega, \quad i=1, \ldots, n$, by expanding the expression $\bigwedge_{i=1}^{n}\left(E_{i} H_{i} \vee E_{i}^{c} H_{i} \vee H_{i}^{c}\right)$, we can represent $\Omega$ as the disjunction of $3^{n}$ logical conjunctions, some of which may be impossible. The remaining ones are the atoms, or constituents, generated by the family $\mathcal{F}_{n}$ and, of course, are a partition of $\Omega$. We denote by $C_{1}, \ldots, C_{m}$ the constituents contained in $\mathcal{H}_{n}$ and (if $\mathcal{H}_{n} \neq \Omega$ ) by $C_{0}$ the remaining constituent $\mathcal{H}_{n}^{c}=H_{1}^{c} \cdots H_{n}^{c}$, so that

$$
\mathcal{H}_{n}=C_{1} \vee \cdots \vee C_{m}, \quad \Omega=\mathcal{H}_{n}^{c} \vee \mathcal{H}_{n}=C_{0} \vee C_{1} \vee \cdots \vee C_{m}, \quad m+1 \leq 3^{n} .
$$

Interpretation with the betting scheme. With the pair $\left(\mathcal{F}_{n}, \mathcal{P}_{n}\right)$ we associate the random gain $\mathcal{G}=\sum_{i=1}^{n} s_{i} H_{i}\left(E_{i}-p_{i}\right)$, where $s_{1}, \ldots, s_{n}$ are $n$ arbitrary real numbers. We observe that $\mathcal{G}$ is the difference between the amount that you receive, $\sum_{i=1}^{n} s_{i}\left(E_{i} H_{i}+p_{i} H_{i}^{c}\right)$, and the amount that you pay, $\sum_{i=1}^{n} s_{i} p_{i}$, and represents the net gain from engaging each transaction $H_{i}\left(E_{i}-p_{i}\right)$, the scaling and meaning (buy or sell) of the transaction being specified by the magnitude and the sign of $s_{i}$ respectively. Let $g_{h}$ be the value of $\mathcal{G}$ when $C_{h}$ is true; of course, $g_{0}=0$. Denoting by $G_{\mathcal{H}_{n}}=\left\{g_{1}, \ldots, g_{m}\right\}$ the set of values of $\mathcal{G}$ restricted to $\mathcal{H}_{n}$, we have
Definition 1. The function $P$ defined on $\mathcal{F}$ is said to be coherent if and only if, for every integer $n$, for every finite sub-family $\mathcal{F}_{n} \subseteq \mathcal{F}$ and for every $s_{1}, \ldots, s_{n}$, one has: $\min G_{\mathcal{H}_{n}} \leq 0 \leq \max G_{\mathcal{H}_{n}}$.

Notice that the condition $\min G_{\mathcal{H}_{n}} \leq 0 \leq \max G_{\mathcal{H}_{n}}$ can be written in two equivalent ways: $\min G_{\mathcal{H}_{n}} \leq 0$, or $\max G_{\mathcal{H}_{n}} \geq 0$. As shown by Definition 1, a probability assessment is coherent if and only if, in any finite combination of $n$ bets, it does not happen that the values $g_{1}, \ldots, g_{m}$ are all positive, or all negative (no Dutch Book).

Coherence with penalty criterion. An equivalent notion of coherence for unconditional events and random quantities was introduced by de Finetti ([24, 25, 26]) using the penalty criterion associated with the quadratic scoring rule. Such a penalty criterion has been extended to the case of conditional events in [30]. With the pair $\left(\mathcal{F}_{n}, \mathcal{P}_{n}\right)$ we associate the loss $\mathcal{L}=\sum_{i=1}^{n} H_{i}\left(E_{i}-p_{i}\right)^{2}$; we denote by $L_{h}$ the value of $\mathcal{L}$ if $C_{h}$ is true. If you specify the assessment $\mathcal{P}_{n}$ on $\mathcal{F}_{n}$ as representing your belief's degrees, you are required to pay a penalty $L_{h}$ when $C_{h}$ is true. Then, we have
Definition 2. The function $P$ defined on $\mathcal{F}$ is said to be coherent if and only if there does not exist an integer $n$, a finite sub-family $\mathcal{F}_{n} \subseteq \mathcal{F}$, and an assessment $\mathcal{P}_{n}{ }^{*}=\left(p_{1}^{*}, \ldots, p_{n}^{*}\right)$ on $\mathcal{F}_{n}$ such that, for the loss $\mathcal{L}^{*}=\sum_{i=1}^{n} H_{i}\left(E_{i}-p_{i}^{*}\right)^{2}$, associated with $\left(\mathcal{F}_{n}, \mathcal{P}_{n}^{*}\right)$, it results $\mathcal{L}^{*} \leq \mathcal{L}$ and $\mathcal{L}^{*} \neq \mathcal{L}$; that is $L_{h}^{*} \leq L_{h}, h=1, \ldots, m$, with $L_{h}^{*}<L_{h}$ in at least one case.

We can develop a geometrical approach to coherence by associating, with each constituent $C_{h}$ contained in $\mathcal{H}_{n}$, a point $Q_{h}=\left(q_{h 1}, \ldots, q_{h n}\right)$, where $q_{h j}=1$, or 0 , or $p_{j}$, according to whether $C_{h} \subseteq E_{j} H_{j}$, or $C_{h} \subseteq E_{j}^{c} H_{j}$, or $C_{h} \subseteq H_{j}^{c}$. Then, denoting by $I$ the convex hull of $Q_{1}, \ldots, Q_{m}$, the following characterization of coherence w.r.t. penalty criterion can be given ([30, Theorem 4.4], see also [12, 31])

Theorem 1. The function $P$ defined on $\mathcal{F}$ is coherent if and only if, for every finite sub-family $\mathcal{F}_{n} \subseteq \mathcal{F}$, one has $\mathcal{P}_{n} \in I$.

Equivalence between betting scheme and penalty criterion. The betting scheme and the penalty criterion are equivalent, as can be proved by the following steps ([34]):

1. The condition $\mathcal{P}_{n} \in \mathcal{I}$ amounts to solvability of the following system $(\Sigma)$ in the unknowns $\lambda_{1}, \ldots, \lambda_{m}$

$$
\left\{\begin{array}{l}
\sum_{h=1}^{m} q_{h j} \lambda_{h}=p_{j}, \quad j=1, \ldots, n ; \\
\sum_{h=1}^{m} \lambda_{h}=1, \quad \lambda_{h} \geq 0, h=1, \ldots, m .
\end{array}\right.
$$

We say that system $(\Sigma)$ is associated with the pair $\left(\mathcal{F}_{n}, \mathcal{P}_{n}\right)$.
2. Let $\mathbf{x}=\left(x_{1}, \ldots, x_{m}\right), \mathbf{y}=\left(y_{1}, \ldots, y_{n}\right)^{t}$ and $A=\left(a_{i j}\right)$ be, respectively, a row $m$-vector, a column $n$-vector and a $m \times n$-matrix. The vector $\mathbf{x}$ is said semipositive if $x_{i} \geq 0, \forall i$, and $x_{1}+\cdots+x_{m}>0$. Then, we have (cf. [28, Theorem 2.9])

Theorem 2. Exactly one of the following alternatives holds.
(i) the equation $\mathbf{x} A=0$ has a semipositive solution;
(ii) the inequality $A \mathbf{y}>0$ has a solution.
3. By choosing $a_{i j}=q_{i j}-p_{j}, \forall i, j$, the solvability of $\mathbf{x} A=0$ means that $\mathcal{P}_{n} \in \mathcal{I}$, while the solvability of $A \mathbf{y}>0$ means that, choosing $s_{i}=y_{i}, \forall i$, one has $\min G_{\mathcal{H}_{n}}>0$. Hence, by applying Theorem 2 with $A=\left(q_{i j}-p_{j}\right)$, we obtain $\max G_{\mathcal{H}_{n}} \geq 0$ if and only if $(\Sigma)$ is solvable. In other words, $\max G_{\mathcal{H}_{n}} \geq 0$ if and only if $\mathcal{P}_{n} \in \mathcal{I}$. Therefore, Definition 1 and Definition 2 are equivalent.

### 2.2. Coherence Checking

Given the assessment $\mathcal{P}_{n}$ on $\mathcal{F}_{n}$, let $S$ be the set of solutions $\Lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ of the system $(\Sigma)$. Then, assuming $S \neq \emptyset$, define

$$
\begin{aligned}
& \Phi_{j}(\Lambda)=\Phi_{j}\left(\lambda_{1}, \ldots, \lambda_{m}\right)=\sum_{r: C_{r} \subseteq H_{j}} \lambda_{r}, \quad j=1, \ldots, n ; \Lambda \in S ; \\
& M_{j}=\max _{\Lambda \in S} \Phi_{j}(\Lambda), \quad j=1, \ldots, n ; \quad I_{0}=\left\{j: M_{j}=0\right\}
\end{aligned}
$$

We observe that, assuming $\mathcal{P}_{n}$ coherent, each solution $\Lambda=\left(\lambda_{1}, \ldots, \lambda_{m}\right)$ of system $(\Sigma)$ is a coherent extension of the assessment $\mathcal{P}_{n}$ on $\mathcal{F}_{n}$ to the family $\left\{C_{1}\left|\mathcal{H}_{n}, \ldots, C_{m}\right| \mathcal{H}_{n}\right\}$. Then, by the additive property, the quantity $\Phi_{j}(\Lambda)$ is the conditional probability $P\left(H_{j} \mid \mathcal{H}_{n}\right)$ and the quantity $M_{j}$ is the upper probability $P^{*}\left(H_{j} \mid \mathcal{H}_{n}\right)$ over all the solutions $\Lambda$ of system $(\Sigma)$. Of course, $j \in I_{0}$ if and only if $P^{*}\left(H_{j} \mid \mathcal{H}_{n}\right)=0$. Notice that $I_{0} \subset\{1, \ldots, n\}$. We denote by $\left(\mathcal{F}_{0}, \mathcal{P}_{0}\right)$ the pair associated with $I_{0}$. Given the pair $\left(\mathcal{F}_{n}, \mathcal{P}_{n}\right)$ and a subset $J \subset J_{n}=\{1, \ldots, n\}$, we denote by $\left(\mathcal{F}_{J}, \mathcal{P}_{J}\right)$ the pair associated with $J$ and by $\Sigma_{J}$ the corresponding system. We observe that $\left(\Sigma_{J}\right)$ is solvable if and only if $\mathcal{P}_{J} \in \mathcal{I}_{J}$, where $I_{J}$ is the convex hull associated with the pair $\left(\mathcal{F}_{J}, \mathcal{P}_{J}\right)$. Then, we have ([32, Theorem 3.2]; see also [10, 33])
Theorem 3. Given a probability assessment $\mathcal{P}_{n}$ on the family $\mathcal{F}_{n}$, if the system $(\Sigma)$ associated with $\left(\mathcal{F}_{n}, \mathcal{P}_{n}\right)$ is solvable, then for every $J \subset\{1, \ldots, n\}$, such that $J \backslash I_{0} \neq \emptyset$, the system $\left(\Sigma_{J}\right)$ associated with $\left(\mathcal{F}_{J}, \mathcal{P}_{J}\right)$ is solvable too.

The previous result says that the condition $\mathcal{P}_{n} \in \mathcal{I}$ implies $\mathcal{P}_{J} \in I_{J}$ when $J \backslash I_{0} \neq \emptyset$. We observe that, if $\mathcal{P}_{n} \in \mathcal{I}$, then for every nonempty subset $J$ of $J_{n} \backslash I_{0}$ it holds that $J \backslash I_{0}=J \neq \emptyset$; hence, by Theorem 1 , the subassessment $\mathcal{P}_{J_{n} \backslash I_{0}}$ on the subfamily $\mathcal{F}_{J_{n} \backslash I_{0}}$ is coherent. In particular, when $I_{0}$ is empty, coherence of $\mathcal{P}_{n}$ amounts to solvability of system $(\Sigma)$, that is to condition $\mathcal{P}_{n} \in I$. When $I_{0}$ is not empty, coherence of $\mathcal{P}_{n}$ amounts to the validity of both conditions $\mathcal{P}_{n} \in \mathcal{I}$ and $\mathcal{P}_{0}$ coherent, as shown by the result below ([32, Theorem 3.3])
Theorem 4. The assessment $\mathcal{P}_{n}$ on $\mathcal{F}_{n}$ is coherent if and only if the following conditions are satisfied: (i) $\mathcal{P}_{n} \in \mathcal{I}$; (ii) if $I_{0} \neq \emptyset$, then $\mathcal{P}_{0}$ is coherent.

### 2.3. Basic notions on probabilistic default reasoning

Given a conditional knowledge base $\mathcal{K}_{n}=\left\{H_{i} \mid \sim E_{i}, i=1,2, \ldots, n\right\}$, we denote by $\mathcal{F}_{n}=\left\{E_{i} \mid H_{i}, i=1,2, \ldots, n\right\}$ the associated family of conditional events. We give below, in the setting of coherence, synthetic definitions of the notions of p-consistency and p-entailment of Adams, which are related with [8, Theorem 4.5, Theorem 4.9], [41, Theorem 5], [42, Theorem 6].

Definition 3. The knowledge base $\mathcal{K}_{n}=\left\{H_{i} \quad \mid \sim E_{i}, \quad i=1,2, \ldots, n\right\}$ is $p$-consistent if and only if the assessment $\left(p_{1}, p_{2}, \ldots, p_{n}\right)=(1,1, \ldots, 1)$ on $\mathcal{F}_{n}$ is coherent.

Definition 4. A p-consistent knowledge base $\mathcal{K}_{n}=\left\{H_{i} \mid \sim E_{i}, i=1, \ldots, n\right\}$ p-entails the conditional $A \mid \sim B$, denoted $\mathcal{K}_{n} \Rightarrow_{p} A \mid \sim B$, if and only if, for every coherent assessment ( $p_{1}, p_{2}, \ldots, p_{n}, z$ ) on $\mathcal{F}_{n} \cup\{B \mid A\}$ such that $\left(p_{1}, p_{2}, \ldots, p_{n}\right)=(1,1, \ldots, 1)$, it holds that $z=1$.

The previous definitions of p-consistency and p-entailment are equivalent (see [35, Theorem 8], [41, Theorem 5], [42, Theorem 6]) to that ones given in [35].

Remark 1. We say that a family of conditional events $\mathcal{F}_{n}$ p-entails a conditional event $B \mid A$ when the associated knowledge base $\mathcal{K}_{n}$ p-entails the conditional $A \nvdash B$.

Definition 4 can be generalized to $p$-entailment of a family (of conditional events) $\Gamma$ from another family $\mathcal{F}$ in the following way.

Definition 5. Given two p-consistent finite families of conditional events $\mathcal{F}$ and $\mathcal{S}$, we say that $\mathcal{F}$ p-entails $\mathcal{S}$ if $\mathcal{F}$ p-entails $E \mid H$ for every $E \mid H \in \mathcal{S}$.

We remark that, from Definition4, we trivially have that $\mathcal{F}$ p-entails $E \mid H$, for every $E \mid H \in \mathcal{F}$; then, by Definition 5. it immediately follows

$$
\begin{equation*}
\mathcal{F} \Rightarrow_{p} \mathcal{S}, \quad \forall \mathcal{S} \subseteq \mathcal{F} . \tag{1}
\end{equation*}
$$

Probabilistic default reasoning has been studied by many authors (see, e.g.,[6, 8, 9, 22, 54]); methods and results based on the maximum entropy principle have been given in [50, 65].

### 2.4. Hamacher t-norm and t-conorm

We recall that the Hamacher t-norm, with parameter $\lambda=0$, or Hamacher product, $T_{0}^{H}$ is defined as ([49])

$$
T_{0}^{H}(x, y)= \begin{cases}0, & (x, y)=(0,0),  \tag{2}\\ \frac{x y}{x+y-x y}, & (x, y) \neq(0,0) .\end{cases}
$$

We also recall that the Hamacher t-conorm, with parameter $\lambda=0, S_{0}^{H}$ is

$$
S_{0}^{H}(x, y)= \begin{cases}1, & (x, y)=(1,1)  \tag{3}\\ \frac{x+y-2 x y}{1-x y}, & (x, y) \neq(1,1) .\end{cases}
$$

As is well known, t-norms overlap with copulas ([3, 59]); indeed, commutative associative copulas are t-norms and $t$-norms which satisfy the 1 -Lipschitz condition are copulas. We also recall that some well-known families of $t$-norms receive a different name in the literature when considered as families of copulas. In particular, the Hamacher product is a copula because it satisfies the following necessary and sufficient condition ([3, Theorem 1.4.5]):

Theorem 5. A t-norm T is a copula if and only if it satisfies the Lipschitz condition: $T\left(x_{2}, y\right)-T\left(x_{1}, y\right) \leq x_{2}-x_{1}$, whenever $x_{2} \leq x_{1}$.

Hamacher product is called Ali-Mikhail-Haq copula with parameter 0 ([2, 46, 52, 59]). Further details on t -norms and t -conorms are given in the Appendices.

## 3. Lower and Upper Bounds for Quasi Conjunction

We recall below the notion of quasi conjunction of conditional events as defined in [1].
Definition 6. Given any events $A, H, B, K$, with $H \neq \emptyset, K \neq \emptyset$, the quasi conjunction of the conditional events $A \mid H$ and $B \mid K$ is the conditional event $C(A|H, B| K)=\left(A H \vee H^{c}\right) \wedge\left(B K \vee K^{c}\right) \mid(H \vee K)$, or equivalently $C(A|H, B| K)=$ $\left(A H B K \vee A H K^{c} \vee H^{c} B K\right) \mid(H \vee K)$.

Table 1 shows the truth-table of the quasi conjunction $C(A|H, B| K)$ and of the quasi disjunction $\mathcal{D}(A|H, B| K)$ (see Section 5 . In general, given a family of $n$ conditional events $\mathcal{F}_{n}=\left\{E_{i} \mid H_{i}, i=1, \ldots, n\right\}$, we have

$$
\mathcal{C}\left(\mathcal{F}_{n}\right)=C\left(E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right)=\bigwedge_{i=1}^{n}\left(E_{i} H_{i} \vee H_{i}^{c}\right) \mid\left(\bigvee_{i=1}^{n} H_{i}\right) .
$$

Quasi conjunction is associative; that is, for every subset $J \subset\{1, \ldots, n\}$, it holds that $\mathcal{C}\left(\mathcal{F}_{n}\right)=\mathcal{C}\left(\mathcal{F}_{J} \cup \mathcal{F}_{\Gamma}\right)=$ $C\left[C\left(\mathcal{F}_{J}\right), C\left(\mathcal{F}_{\Gamma}\right)\right]$, where $\Gamma=\{1, \ldots, n\} \backslash J$. An interesting analysis of many three-valued logics studied in the literature has been given by Ciucci and Dubois in [16]. In such a paper the definition of conjunction satisfies left monotonicity, right monotonicity and conformity with Boolean logic; then the authors show that there are 14 different ways of defining conjunction and only 5 of them (one of which defines quasi conjunction) satisfy commutativity and associativity.
Assuming $A, H, B, K$ logically independent, we have ([36], see also [37]):
(i) the probability assessment $(x, y)$ on $\{A|H, B| K\}$ is coherent for every $(x, y) \in[0,1]^{2}$;
(ii) given a coherent assessment $(x, y)$ on $\{A|H, B| K\}$, the extension $\mathcal{P}=(x, y, z)$ on $\mathcal{F}=\{A|H, B| K, C(A|H, B| K)\}$, with $z=P[C(A|H, B| K)]$, is coherent if and only if $z \in[l, u]$, with

$$
l=T_{L}(x, y)=\max (x+y-1,0), \quad u=S_{0}^{H}(x, y)= \begin{cases}\frac{x+y-2 x y}{1-x y}, & (x, y) \neq(1,1)  \tag{4}\\ 1, & (x, y)=(1,1)\end{cases}
$$

where $T_{L}$ is the Lukasiewicz t-norm (see Appendix A) and $S_{0}^{H}$ is the Hamacher t-conorm ${ }^{1}$ with parameter $\lambda=0$. The lower bound $T_{L}$ for the quasi conjunction is the Fréchet-Hoeffding lower bound; both $l$ and $u$ coincide with the Fréchet-Hoeffding bounds if we consider the conjunction of conditional events in the setting of conditional random quantities, as made in [43]. The lower and upper bounds, $l, u$, of $z=P[C(A|H, B| K)]$ can be obtained by studying the coherence of the assessment $\mathcal{P}=(x, y, z)$, based on the geometrical approach described in Section2. The constituents generated by the family $\{A|H, B| K, C(A|H, B| K)\}$ and the corresponding points $Q_{h}$ 's are given in columns 2 and 6 of Table 1] In [36] (see also [37]) the values $l, u$ are computed by observing that the coherence of $\mathcal{P}=(x, y, z)$ simply amounts to the geometrical condition $\mathcal{P} \in \mathcal{I}$, where $\mathcal{I}$ is the convex hull of the points $Q_{1}, Q_{2}, \ldots, Q_{8}$ (associated with the constituents $C_{1}, C_{2}, \ldots, C_{8}$ contained in $H \vee K$ ). We observe that in this case the convex hull $\mathcal{I}$ does not depend on $z$. Figure 1 shows, for given values $x, y$, the convex hull $I$ and the associated interval $[l, u]$ for $z=P[C(A|H, B| K)]$.

| $h$ | $C_{h}$ | $A \mid H$ | $B \mid K$ | $C(A\|H, B\| K)$ | $Q_{h}$ | $\mathcal{D}(A\|H, B\| K)$ | $Q_{h}$ |
| :---: | :--- | :--- | :---: | :---: | :---: | :---: | :---: |
| 0 | $H^{c} K^{c}$ | Void | Void | Void | $(x, y, z)$ | Void | $(x, y, z)$ |
| 1 | $A H B K$ | True | True | True | $(1,1,1)$ | True | $(1,1,1)$ |
| 2 | $A H K^{c}$ | True | Void | True | $(1, y, 1)$ | True | $(1, y, 1)$ |
| 3 | $A H B^{c} K$ | True | False | False | $(1,0,0)$ | True | $(1,0,1)$ |
| 4 | $H^{c} B K$ | Void | True | True | $(x, 1,1)$ | True | $(x, 1,1)$ |
| 5 | $H^{c} B^{c} K$ | Void | False | False | $(x, 0,0)$ | False | $(x, 0,0)$ |
| 6 | $A^{c} H B K$ | False | True | False | $(0,1,0)$ | True | $(0,1,1)$ |
| 7 | $A^{c} H K^{c}$ | False | Void | False | $(0, y, 0)$ | False | $(0, y, 0)$ |
| 8 | $A^{c} H B^{c} K$ | False | False | False | $(0,0,0)$ | False | $(0,0,0)$ |

Table 1: Truth-Table of the quasi conjunction and of the quasi disjunction with the associated points $Q_{h}$ 's.

Remark 2. Notice that, if the events $A, B, H, K$ were not logically independent, then some constituents $C_{h}$ 's (at least one) would become impossible and the lower bound $l$ could increase, while the upper bound $u$ could decrease. To examine this aspect we will consider some special cases of logical dependencies.

### 3.1. The Case $A|H \subseteq B| K$

The notion of logical inclusion among events has been generalized to conditional events by Goodman and Nguyen in [45]. We recall below this generalized notion.

Definition 7. Given two conditional events $A \mid H$ and $B \mid K$, we say that $A \mid H$ implies $B \mid K$, denoted by $A|H \subseteq B| K$, iff $A H$ true implies $B K$ true and $B^{c} K$ true implies $A^{c} H$ true; i.e., iff $A H \subseteq B K$ and $B^{c} K \subseteq A^{c} H$.

Remark 3. Denoting by $t(\cdot)$ the truth value function and assuming the order False $<$ Void $<$ True, then it can be easily verified that

$$
\begin{aligned}
& A|H \subseteq B| K \Leftrightarrow A H B^{c} K=H^{c} B^{c} K=A H K^{c}=\emptyset, \\
& A|H \subseteq B| K \Leftrightarrow t(A \mid H) \leq t(B \mid K) \Leftrightarrow t\left(B^{c} \mid K\right) \leq t\left(A^{c} \mid H\right) \Leftrightarrow B^{c}\left|K \subseteq A^{c}\right| H .
\end{aligned}
$$

Given any conditional events $A|H, B| K$, we denote by $\Pi_{x}$ the set of coherent probability assessments $x$ on $A \mid H$, by $\Pi_{y}$ the set of coherent probability assessments $y$ on $B \mid K$ and by $\Pi$ the set of coherent probability assessments $(x, y)$ on $\{A|H, B| K\}$; moreover we indicate by $T_{x \leq y}$ the triangle $\left\{(x, y) \in[0,1]^{2}: x \leq y\right\}$. We have

[^1]

Figure 1: The convex hull $\mathcal{I}$ associated with the pair $(\mathcal{F}, \mathcal{P})$ in case of quasi conjunction without logical dependencies. The interval $[l, u]$ for $z=P[C(A|H, B| K)]$ is the range of the third coordinate $z$ of each $\mathcal{P} \in \overline{\mathcal{P}_{l} \mathcal{P}_{u}}=\left\{(x, y, z): z \in\left[T_{L}(x, y), S_{0}^{H}(x, y)\right]\right\}$. The segment $\overline{\mathcal{P}_{l} \mathcal{P}_{u}}$ is the intersection between the segment $\{(x, y, z): z \in[0,1]\}$ and the convex hull $\mathcal{I}$.

Theorem 6. Given two conditional events $A|H, B| K$, we have

$$
\begin{equation*}
\Pi \subseteq T_{x \leq y} \Longleftrightarrow A|H \subseteq B| K, \text { or } A H=\emptyset \text {, or } B^{c} K=\emptyset \tag{5}
\end{equation*}
$$

Proof. $(\Rightarrow)$ We will prove that

$$
\begin{equation*}
A|H \nsubseteq B| K, A H \neq \emptyset, B^{c} K \neq \emptyset \Longrightarrow \Pi \nsubseteq T_{x \leq y} . \tag{6}
\end{equation*}
$$

We observe that $A H=\emptyset$ if and only if $\Pi_{x}=\{0\}$ and that $B^{c} K=\emptyset$ if and only if $\Pi_{y}=\{1\}$. Moreover, by Remark 3 it holds

$$
A|H \nsubseteq B| K \Longleftrightarrow A H B^{c} K \vee H^{c} B^{c} K \vee A H K^{c} \neq \emptyset .
$$

Then, in order to prove formula (6), we distinguish three cases:
(i) $A H B^{c} K \neq \emptyset$; (ii) $H^{c} B^{c} K \neq \emptyset, A H \neq \emptyset$; (iii) $A H K^{c} \neq \emptyset, B^{c} K \neq \emptyset$.

In case (i), the assessment ( 1,0 ) on $\{A|H, B| K\}$ is coherent. In case (ii), as $A H \neq \emptyset$ we have $\{1\} \subseteq \Pi_{x}$; then, the assessment ( 1,0 ) on $\{A|H, B| K\}$ is coherent. In case (iii), as $B^{c} K \neq \emptyset$ we have $\{0\} \subseteq \Pi_{y}$; then, the assessment $(1,0)$ on $\{A|H, B| K\}$ is coherent. Then, in each of the three cases the assessment $(1,0)$ is coherent and hence $\Pi \nsubseteq T_{x \leq y}$.
$(\Leftarrow)$ We distinguish three cases:
(a) $A|H \subseteq B| K$; (b) $A H=\emptyset$; (c) $B^{c} K=\emptyset$.
(a) The constituents generated by $\{A|H, B| K\}$ and contained in $H \vee K$ belong to the family:

$$
\left\{A H B K, H^{c} B K, A^{c} H B K, A^{c} H K^{c}, A^{c} H B^{c} K\right\} .
$$

The corresponding points $Q_{h}$ 's belong to the set $\{(1,1),(x, 1),(0,1),(0, y),(0,0)\}$, which has the triangle $T_{x \leq y}$ as convex hull; hence the convex hull $\Pi$ of the points $Q_{h}$ 's is a subset of $T_{x \leq y}$.
(b) Since $\Pi_{x}=\{0\}$ it follows that $\Pi \subseteq\{(0, y): y \in[0,1]\} \subseteq T_{x \leq y}$.
(c) Since $\Pi_{y}=\{1\}$ it follows that $\Pi \subseteq\{(x, 1): x \in[0,1]\} \subseteq T_{x \leq y}$.

The next result, related with Theorem 6 and with the inclusion relation, characterizes the notion of p-entailment between two conditional events.

Theorem 7. Given two conditional events $A|H, B| K$, with $A H \neq \emptyset$, the following assertions are equivalent: (a) $\left(A\left|H \Rightarrow_{p} B\right| K\right)$; (b) $A|H \subseteq B| K$, or $K \subseteq B$; (c) $\Pi \subseteq T_{x \leq y}$.

Proof. As $A H \neq \emptyset$, from Theorem 6 the assertions (b) and (c) are equivalent; hence, we only need to show the equivalence between (a) and (b).
$((\mathrm{a}) \Rightarrow(\mathrm{b}))$. We will prove that

$$
A|H \nsubseteq B| K, A H \neq \emptyset, B^{c} K \neq \emptyset \Longrightarrow A\left|H \not \nRightarrow_{p} B\right| K
$$

Assume that $A|H \nsubseteq B| K, B^{c} K \neq \emptyset$. Then, as in the proof of Theorem6 we distinguish three cases:
(i) $A H B^{c} K \neq \emptyset$; (ii) $H^{c} B^{c} K \neq \emptyset, A H \neq \emptyset$; (iii) $A H K^{c} \neq \emptyset, B^{c} K \neq \emptyset$.

In all three cases the assessment $(1,0)$ is coherent; thus $A\left|H \not \nRightarrow ~_{p} B\right| K$.
$((\mathrm{b}) \Rightarrow(\mathrm{a}))$. We preliminarily observe that $\{A \mid H\}$ is p-consistent. Now, if $A|H \subseteq B| K$, then p-entailment of $B \mid K$ from $A \mid H$ follows from monotonicity of conditional probability w.r.t. inclusion relation. If $K \subseteq B$, then trivially $A \mid H$ p-entails $B \mid K$.

Example 1. Given any events $A, B$, for the conditional events $A \vee B, B \mid A^{c}$ it holds that $B\left|A^{c}=(A \vee B)\right| A^{c} \subseteq A \vee B$. Then, for the assessment $P(A \vee B)=x, P\left(B \mid A^{c}\right)=y$, the necessary condition of coherence $0 \leq y \leq x \leq 1$ must be satisfied. Of course, $P(A \vee B)$ 'high' does not imply $P\left(B \mid A^{c}\right)$ 'high'; for instance, it is coherent to assign $P\left(B \mid A^{c}\right)=0.2$ and $P(A \vee B)=0.8$. Then, the inference of the conditional event $B \mid A^{c}$ from the disjunction $A \vee B$ may be 'weak'. A probabilistic analysis characterizing the cases in which such an inference is 'strong' has been made in [38].

Remark 4. We observe that, under the hypothesis $A|H \subseteq B| K$, the constituents generated by $\{A|H, B| K\}$ belong to the family

$$
\mathcal{H}=\left\{A H B K, H^{c} B K, A^{c} H B K, A^{c} H K^{c}, A^{c} H B^{c} K, H^{c} K^{c}\right\}
$$

The quasi-conjunction is $C(A|H, B| K)=\left(A H \vee H^{c} B K\right) \mid(H \vee K)$ and, as shown by Table 1, for any constituent in $\mathcal{H}$ it holds that

$$
t(A \mid H) \leq t(C(A|H, B| K)) \leq t(B \mid K)
$$

Then, we have (see Remark 3)

$$
\begin{equation*}
A|H \subseteq B| K \Longrightarrow A|H \subseteq C(A|H, B| K) \subseteq B| K \tag{7}
\end{equation*}
$$

As conditional probability is monotonic w.r.t. inclusion relation among conditional events ([45]), it holds that $P(A \mid H) \leq P[C(A|H, B| K)] \leq P(B \mid K)$. As shown by Theorem 6, in our coherence-based approach the monotonic property is obtained without assuming that $P(H)$ and $P(K)$ are positive. The next result establishes that $P[C(A|H, B| K)]$ can coherently assume all the values in the interval $[P(A \mid H), P(B \mid K)]$. We have

Proposition 1. Let be given any coherent assessment ( $x, y$ ) on $\{A|H, B| K\}$, with $A|H \subseteq B| K$ and with no further logical relations. Then, the extension $z=P[C(A|H, B| K)]$ is coherent if and only if $l \leq z \leq u$, where

$$
l=x=\min (x, y)=T_{M}(x, y), u=y=\max (x, y)=S_{M}(x, y)
$$

Proof. We recall that, apart from $A|H \subseteq B| K$, there are no further logical relations; thus it holds that $\Pi=T_{x \leq y}$ (i.e. $0 \leq x \leq y \leq 1)$. Denoting by $[l, u]$ the interval of coherent extensions of the assessment $(x, y)$ to $C(A|H, B| K)$, by (7) it holds that $[l, u] \subseteq[x, y]$. In order to prove that $[l, u]=[x, y]$ it is enough to verify that both the assessments $\mathcal{P}_{l}=(x, y, x)$ and $\mathcal{P}_{u}=(x, y, y)$ are coherent. Given any assessment $\mathcal{P}=(x, y, z)$, with $x \leq y$, we study the coherence of $\mathcal{P}$ by the geometrical approach described in Section 2 The constituents generated by the family and contained in $H \vee K$ are:

$$
C_{1}=A H B K, C_{2}=H^{c} B K, C_{3}=A^{c} H B K, C_{4}=A^{c} H K^{c}, C_{5}=A^{c} H B^{c} K
$$

The corresponding points $Q_{h}$ 's are

$$
Q_{1}=(1,1,1), Q_{2}=(x, 1,1), Q_{3}=(0,1,0), Q_{4}=(0, y, 0), Q_{5}=(0,0,0),
$$

and, in our case, the coherence of $\mathcal{P}$ simply amounts to the geometrical condition $\mathcal{P} \in \mathcal{I}$, where $I$ is the convex hull of the points $Q_{1}, Q_{2}, \ldots, Q_{5}$.

It can be verified that $\mathcal{P}_{l}=x Q_{1}+(y-x) Q_{3}+(1-y) Q_{5}$, so that $\mathcal{P}_{l} \in \mathcal{I}$; hence $l=x$. Concerning $\mathcal{P}_{u}$, we first observe that when $(x, y)=(1,1)$ we have $\mathcal{P}_{u}=(1,1,1)=Q_{1}$, so that $\mathcal{P}_{u} \in \mathcal{I}$; hence $u=y=1$. Assuming $(x, y) \neq(1,1)$, it can be verified that $\mathcal{P}_{u}=\frac{x-x y}{1-x} Q_{1}+\frac{y-x}{1-x} Q_{2}+(1-y) Q_{5}$, so that $\mathcal{P}_{u} \in \mathcal{I}$; hence $u=y$. Therefore, $[l, u]=[x, y]$.

We remark that the lower/upper bound above, $l, u$, may change if we add further logical relations; for instance, if $H=K$, it is $C(A|H, B| H)=A \mid H$, in which case $l=u=x$. Finally, in agreement with Remark 2 we observe that $T_{L}(x, y) \leq \min (x, y) \leq \max (x, y) \leq S_{0}^{H}(x, y)$. We also recall that $T_{M}(x, y)=\min (x, y)$ is the largest t-norm and $S_{M}(x, y)=\max (x, y)$ is the smallest t -conorm ([53]). Figure 2 shows the convex hull $I$ for given values $x, y$, with the associated interval $[l, u]$ for $z=P[C(A|H, B| K)]$, when $A|H \subseteq B| K$.


Figure 2: The convex hull $\mathcal{I}$ associated with the pair $(\mathcal{F}, \mathcal{P})$ when $A|H \subseteq B| K$. The interval $[l, u]$ for $z=P[C(A|H, B| K)]$ is the range of the third coordinate $z$ of each $\mathcal{P} \in \overline{\mathcal{P}_{l} \mathcal{P}_{u}}=\left\{(x, y, z): z \in\left[T_{M}(x, y), S_{M}(x, y)\right]\right\}$. The segment $\overline{\mathcal{P}_{l} \mathcal{P}_{u}}$ is the intersection between the segment $\{(x, y, z): z \in[0,1]\}$ and the convex hull $I$. This intersection is empty for $x>y$ because of $\Pi \subseteq T_{x \leq y}$,

### 3.2. Compound Probability Theorem

We now examine the quasi conjunction of $A \mid H$ and $B \mid A H$, with $A, B, H$ logically independent events. As it can be easily verified, we have $C(A|H, B| A H)=A B \mid H$; moreover, the probability assessment $(x, y)$ on $\{A|H, B| A H\}$ is coherent, for every $(x, y) \in[0,1]^{2}$. Hence, by the compound probability theorem, if the assessment $\mathcal{P}=(x, y, z)$ on $\mathcal{F}=$ $\{A|H, B| A H, A B \mid H\}$ is coherent, then $z=x y$; i.e., $l=u=x \cdot y=T_{P}(x, y)$. In agreement with Remark 2, we observe that $T_{L}(x, y) \leq x y \leq S_{0}^{H}(x, y)$.
We observe that $A|H=A H| H, B|A H=A B H| A H, A B|H=A B H| H$; as $z=x y,\{A H|H, A B H| A H\}$ p-entails $A B H \mid H$ (transitive property). Moreover $A B|H \subseteq B| H$; hence $\{A|H, B| A H\}$ p-entails $B \mid H$ (cut rule).

### 3.3. Lower and Upper Bounds for the Quasi Conjunction of $n$ Conditional Events

In this subsection we generalize formula (4). Let be given $n$ conditional events $E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}$. By the associative property of quasi conjunction, defining $\mathcal{F}_{k}=\left\{E_{1}\left|H_{1}, \ldots, E_{k}\right| H_{k}\right\}$, for each $k=2, \ldots, n$ it holds that $\mathcal{C}\left(\mathcal{F}_{k}\right)=\mathcal{C}\left(C\left(\mathcal{F}_{k-1}\right), E_{k} \mid H_{k}\right)$. Then, we have

Theorem 8. Given a probability assessment $\mathcal{P}_{n}=\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ on $\mathcal{F}_{n}=\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\}$, let $\left[l_{k}, u_{k}\right]$ be the interval of coherent extensions of the assessment $\mathcal{P}_{k}=\left(p_{1}, p_{2}, \ldots, p_{k}\right)$ on the quasi conjunction $C\left(\mathcal{F}_{k}\right)$, where $\mathcal{F}_{k}=$ $\left\{E_{1}\left|H_{1}, \ldots, E_{k}\right| H_{k}\right\}$. Then, assuming $E_{1}, H_{1}, \ldots, E_{n}, H_{n}$ logically independent, for each $k=2, \ldots, n$, we have

$$
\begin{gather*}
l_{k}=T_{L}\left(p_{1}, p_{2}, \ldots, p_{k}\right)=\max \left(p_{1}+p_{2}+\ldots+p_{k}-(k-1), 0\right),  \tag{8}\\
u_{k}=S_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right)= \begin{cases}1, & p_{i}=1 \text { for at least one } i, \\
\frac{\sum_{i=1}^{k} \frac{p_{i}}{1-p_{i}}}{\sum_{i=1}^{k} \frac{p_{i}}{1-p_{i}}+1}, & p_{i}<1 \text { for } i=1, \ldots, k .\end{cases} \tag{9}
\end{gather*}
$$

Proof. Of course, from 4 it is $l_{2}=T_{L}\left(p_{1}, p_{2}\right), \quad u_{2}=S_{0}^{H}\left(p_{1}, p_{2}\right)$. We recall that both $T_{L}$ and $S_{0}^{H}$ are associative. Moreover, as

$$
C\left(\mathcal{F}_{3}\right)=C\left(C\left(\mathcal{F}_{2}\right), E_{3} \mid H_{3}\right), \quad l_{2} \leq P\left[C\left(\mathcal{F}_{2}\right)\right] \leq u_{2},
$$

defining $P\left[C\left(\mathcal{F}_{2}\right)\right]=x$ and observing that the quantities $T_{L}\left(x, p_{3}\right), S_{0}^{H}\left(x, p_{3}\right)$ are non-decreasing functions of $x$, we have

$$
\begin{gathered}
l_{3}=T_{L}\left(l_{2}, p_{3}\right)=T_{L}\left(T_{L}\left(p_{1}, p_{2}\right), p_{3}\right)=T_{L}\left(p_{1}, p_{2}, p_{3}\right) \\
u_{3}=S_{0}^{H}\left(u_{2}, p_{3}\right)=S_{0}^{H}\left(S_{0}^{H}\left(p_{1}, p_{2}\right), p_{3}\right)=S_{0}^{H}\left(p_{1}, p_{2}, p_{3}\right)
\end{gathered}
$$

Considering any $k>3$, we proceed by induction. Assuming

$$
l_{k-1}=T_{L}\left(p_{1}, p_{2}, \ldots, p_{k-1}\right), \quad u_{k-1}=S_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k-1}\right)
$$

as $C\left(\mathscr{F}_{k}\right)=C\left(C\left(\mathcal{F}_{k-1}\right), E_{k} \mid H_{k}\right)$ and $l_{k-1} \leq P\left[C\left(\mathcal{F}_{k-1}\right)\right] \leq u_{k-1}$, defining $P\left[C\left(\mathcal{F}_{k-1}\right)\right]=x$ and observing that the quantities $T_{L}\left(x, p_{k}\right)$ and $S_{0}^{H}\left(x, p_{k}\right)$ are non-decreasing functions of $x$, we have

$$
\begin{aligned}
& l_{k}=T_{L}\left(l_{k-1}, p_{k}\right)=T_{L}\left(p_{1}, p_{2}, \ldots, p_{k}\right) \\
& u_{k}=S_{0}^{H}\left(u_{k-1}, p_{k}\right)=S_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right)
\end{aligned}
$$

The explicit values of $l_{k}$ and $u_{k}$ in (8) and (9) follow by Appendix B and Appendix C
Notice that $\left(p_{1}, p_{2}, \ldots, p_{n}\right)=(1,1, \ldots, 1)$ implies $T_{L}\left(p_{1}, p_{2}, \ldots, p_{n}\right)=1$. Then, from Theorem 8 we obtain the following Quasi And rule (see also [41, Theorem 4]).

Corollary 1. Given a p-consistent family of conditional events $\mathcal{F}_{n}$, we have

$$
\begin{equation*}
\text { (Quasi And) } \quad \mathcal{F}_{n} \Rightarrow_{p} C\left(\mathcal{F}_{n}\right) \tag{10}
\end{equation*}
$$

We observe that, from (1), we obtain ([42, Theorem 5])

$$
\begin{equation*}
\mathcal{F}_{n} \Rightarrow_{p} \mathcal{C}(\mathcal{S}), \quad \forall \mathcal{S} \subseteq \mathcal{F}_{n} . \tag{11}
\end{equation*}
$$

Of course, (11) still holds when there are logical dependencies because in this case the lower bound for quasi conjunction does not decrease, as observed in Remark 2. In the next example we illustrate the key role of quasi conjunction when we study p-entailment. This example has been already examined in [35], by using the inference rules of System $P$ in the setting of coherence.

Example 2 (Linda's example). We start with a given p-consistent family of conditional events $\mathcal{F}$; then, we use the quasi conjunction to check the p-entailment of some further conditional events from $\mathcal{F}$. The family $\mathcal{F}$ concerns various attributes for a given party (the party is great, the party is noisy, Linda and Steve are present, and so on). We introduce the following events:

$$
\begin{array}{ll}
L=\text { "Linda goes to the party"; } & S=\text { "Steve goes to the party"; } \\
G=\text { "the party is great"; } & N=" t h e ~ p a r t y ~ i s ~ n o i s y ", ~
\end{array}
$$

which are assumed to be logically independent. Then, we consider the family $\mathcal{F}=\left\{G|L, S| L, N^{c}|L S, L| S, G^{c} \mid N^{c}\right\}$ and the family of further conditional events

$$
\mathcal{K}=\left\{N^{c}\left|L, L^{c}\right| \Omega, G N^{c}\left|L S, N^{c}\right| S, N^{c} \mid(L \vee S)\right\}
$$

It can be verified that the assessment $(1,1,1,1,1)$ on $\mathcal{F}$ is coherent, i.e. the family $\mathcal{F}$ is p -consistent. By exploiting quasi conjunction, we can verify that $\mathcal{F}$ p-entails $\mathcal{K}$; that is $\mathcal{F}$ p-entails each conditional event in $\mathcal{K}$. Indeed:
(a) concerning $N^{c} \mid L$, for the subset $\mathcal{S}=\left\{S\left|L, N^{c}\right| L S\right\}$ we have $\mathcal{C}(\mathcal{S})=N^{c} S\left|L \subseteq N^{c}\right| L$; thus: $\mathcal{F} \Rightarrow{ }_{p} \mathcal{C}(\mathcal{S}) \Rightarrow{ }_{p} N^{c} \mid L$;
(b) concerning $L^{c} \mid \Omega$, for the subset $\mathcal{S}=\left\{G|L, S| L, N^{c}\left|L S, G^{c}\right| N^{c}\right\}$ we have $\mathcal{C}(\mathcal{S})=G^{c} L^{c} N^{c}\left|\left(L \vee N^{c}\right) \subseteq L^{c}\right| \Omega$; thus: $\mathcal{F} \Rightarrow{ }_{p} C(\mathcal{S}) \Rightarrow{ }_{p} L^{c} \mid \Omega$;
(c) concerning $G N^{c} \mid L S$, for the subset $\mathcal{S}=\left\{G|L, S| L, N^{c} \mid L S\right\}$ we have $C(\mathcal{S})=G N^{c} S\left|L \subseteq G N^{c}\right| L S$; thus: $\mathcal{F} \Rightarrow_{p}$ $C(S)={ }_{p} G N^{c} \mid L S$;
(d) concerning $N^{c} \mid S$, for the subset $\mathcal{S}=\left\{N^{c}|L S, L| S\right\}$ we have $C(\mathcal{S})=L N^{c}\left|S \subseteq N^{c}\right| S$; thus: $\mathcal{F} \Rightarrow_{p} C(\mathcal{S}) \Rightarrow{ }_{p} N^{c} \mid S$;
(e) concerning $N^{c} \mid(L \vee S)$, for the subset $\mathcal{S}=\left\{S\left|L, N^{c}\right| L S, L \mid S\right\}$ we have $\mathcal{C}(\mathcal{S})=L N^{c} S\left|(L \vee S) \subseteq N^{c}\right|(L \vee S)$; thus: $\mathcal{F} \Rightarrow{ }_{p} C(\mathcal{S}) \Rightarrow_{p} N^{c} \mid(L \vee S)$.
We point out that the p-entailment of $\mathcal{K}$ from $\mathcal{F}$ can be also verified by applying Algorithm 2 in [42]. We also observe that, using the basic events $L, S, G, N$, we can define conditional events which are not p-entailed from $\mathcal{F}$. For instance, concerning $G \mid N$, associated with the conditional "if the party is noisy, then the party is great", it can be proved that $\mathcal{F}$ does not p-entail $G \mid N$. Indeed, there is no subset $\mathcal{S} \subseteq \mathcal{F}$, with $\mathcal{S} \neq \emptyset$, such that $C(\mathcal{S}) \Rightarrow_{p} G \mid N$ (see [42, Theorem 6]).
3.4. The Case $E_{1}\left|H_{1} \subseteq E_{2}\right| H_{2} \subseteq \ldots \subseteq E_{n} \mid H_{n}$

In this subsection we give a result on quasi conjunctions when $E_{i}\left|H_{i} \subseteq E_{i+1}\right| H_{i+1}, i=1, \ldots, n-1$. We have
Theorem 9. Given a family $\mathcal{F}_{n}=\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\}$ of conditional events such that $E_{1}\left|H_{1} \subseteq E_{2}\right| H_{2} \subseteq \ldots \subseteq$ $E_{n} \mid H_{n}$, and a coherent probability assessment $\mathcal{P}_{n}=\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ on $\mathcal{F}_{n}$, let $\mathcal{C}\left(\mathcal{F}_{k}\right)$ be the quasi conjunction of $\mathcal{F}_{k}=\left\{E_{i} \mid H_{i}, i=1, \ldots, k\right\}, k=2, \ldots, n$. Moreover, let $\left[l_{k}, u_{k}\right]$ be the interval of coherent extensions on $C\left(\mathcal{F}_{k}\right)$ of the assessment ( $p_{1}, p_{2}, \ldots, p_{k}$ ) on $\mathcal{F}_{k}$. We have: (i) $E_{1}\left|H_{1} \subseteq \mathcal{C}\left(\mathcal{F}_{2}\right) \subseteq \ldots \subseteq \mathcal{C}\left(\mathcal{F}_{n}\right) \subseteq E_{n}\right| H_{n}$; (ii) by assuming no further logical relations, any probability assessment $\left(z_{2}, \ldots, z_{k}\right)$ on $\left\{C\left(\mathcal{F}_{2}\right), \ldots, C\left(\mathcal{F}_{k}\right)\right\}$ is a coherent extension of the assessment $\left(p_{1}, p_{2}, \ldots, p_{k}\right)$ on $\mathcal{F}_{k}$ if and only if $p_{1} \leq z_{2} \leq \cdots \leq z_{k} \leq p_{k}, k=2, \ldots, n$; moreover

$$
l_{k}=\min \left(p_{1}, \ldots, p_{k}\right)=p_{1}, u_{k}=\max \left(p_{1}, \ldots, p_{k}\right)=p_{k}, k=2, \ldots, n .
$$

Proof. (i) By iteratively applying (7) and by the associative property of quasi conjunction, we have $C\left(\mathcal{F}_{k-1}\right) \subseteq C\left(\mathcal{F}_{k}\right) \subseteq$ $E_{k} \mid H_{k}, k=2, \ldots, n$;
(ii) by exploiting the logical relations in point (i), the assertions immediately follow by applying a reasoning similar to that in Remark 4.

### 3.5. Generalized Compound Probability Theorem

In this subsection we generalize the result obtained in Subsection 3.2. Given the family $\mathcal{F}=\left\{A_{1}\left|H, A_{2}\right| A_{1} H, \ldots, A_{n} \mid A_{1} \cdots A_{n-1} H\right\}$, by iteratively exploiting the associative property, we have

$$
\begin{gathered}
C(\mathcal{F})=C\left(C\left(A_{1}\left|H, A_{2}\right| A_{1} H\right), A_{3}\left|A_{1} A_{2} H, \ldots, A_{n}\right| A_{1} \cdots A_{n-1} H\right)= \\
=C\left(A_{1} A_{2}\left|H, A_{3}\right| A_{1} A_{2} H, \ldots, A_{n} \mid A_{1} \cdots A_{n-1} H\right)=\cdots=A_{1} A_{2} \cdots A_{n} \mid H
\end{gathered}
$$

thus, by the compound probability theorem, if the assessment $\mathcal{P}=\left(p_{1}, \ldots, p_{n}, z\right)$ on $\mathcal{F} \cup\{C(\mathcal{F})\}$ is coherent, then

$$
z=l=u=p_{1} p_{2} \cdots p_{n}=T_{P}\left(p_{1}, p_{2}, \ldots, p_{n}\right)
$$

## 4. Further Aspects on Quasi Conjunction: from Bounds on Conclusions to Bounds on Premises in Quasi And rule

In this section, we study the propagation of probability bounds on the conclusion of the Quasi And rule to its premises. We start with the case of two premises $A \mid H$ and $B \mid K$, by examining probabilistic aspects on the lower and upper bounds, $l$ and $u$, for the probability of the conclusion $C(A|H, B| K)$. More precisely, given any number $\gamma \in[0,1]$, we find:
(i) the set $\mathcal{L}_{\gamma}$ of the coherent assessments $(x, y)$ on $\{A|H, B| K\}$ such that, for each $(x, y) \in \mathcal{L}_{\gamma}$, one has $l \geq \gamma$;
(ii) the set $\mathcal{U}_{\gamma}$ of the coherent assessments $(x, y)$ on $\{A|H, B| K\}$ such that, for each $(x, y) \in \mathcal{U}_{\gamma}$, one has $u \leq \gamma$.

Case (i). Of course, $\mathcal{L}_{0}=[0,1]^{2}$; hence we can assume $\gamma>0$. It must be $l=\max \{x+y-1,0\} \geq \gamma$, i.e., $x+y \geq 1+\gamma$ (as $\gamma>0$ ); hence $\mathcal{L}_{\gamma}$ coincides with the triangle having the vertices $(1,1),(1, \gamma),(\gamma, 1)$; that is

$$
\mathcal{L}_{\gamma}=\{(x, y): \gamma \leq x \leq 1,1+\gamma-x \leq y \leq 1\} .
$$

Notice that $\mathcal{L}_{1}=\{(1,1)\}$; moreover, for $\gamma \in(0,1),(\gamma, \gamma) \notin \mathcal{L}_{\gamma}$.
Case (ii). Of course, $\mathcal{U}_{1}=[0,1]^{2}$; hence we can assume $\gamma<1$. We recall that $u=S_{0}^{H}(x, y)$, then in order the inequality $S_{0}^{H}(x, y) \leq \gamma$ be satisfied, it must be $x<1$ and $y<1$. Thus, $u \leq \gamma$ if and only if $\frac{x+y-2 x y}{1-x y} \leq \gamma$. Given any $x<1, y<1$, we have

$$
\begin{equation*}
u-x=\frac{y(1-x)^{2}}{1-x y} \geq 0, \quad u-y=\frac{x(1-y)^{2}}{1-x y} \geq 0 \tag{12}
\end{equation*}
$$

then, from $u \leq \gamma$ it follows $x \leq \gamma, y \leq \gamma$; hence $\mathcal{U}_{\gamma} \subseteq[0, \gamma]^{2}$. Then, taking into account that $x \leq \gamma$ and hence

$$
1-(2-\gamma) x=1-2 x+\gamma x \geq 1-2 x+x^{2}=(1-x)^{2}>0
$$

we have

$$
\begin{equation*}
\frac{x+y-2 x y}{1-x y} \leq \gamma \Longleftrightarrow y \leq \frac{\gamma-x}{1-(2-\gamma) x} \tag{13}
\end{equation*}
$$

therefore

$$
\mathcal{U}_{\gamma}=\left\{(x, y): 0 \leq x \leq \gamma, \quad y \leq \frac{\gamma-x}{1-(2-\gamma) x}\right\}
$$

Notice that $\mathcal{U}_{0}=\{(0,0)\}$; moreover, for $x=y=\gamma \in(0,1)$, it is $u=\frac{2 \gamma}{1+\gamma}>\gamma$; hence, for $\gamma \in(0,1), \mathcal{U}_{\gamma}$ is a strict subset of $[0, \gamma]^{2}$.
Of course, for every $(x, y) \notin \mathcal{L}_{\gamma} \cup \mathcal{U}_{\gamma}$, it is $l<\gamma<u$. Figure 3 displays the sets $\mathcal{L}_{\gamma}, \mathcal{U}_{\gamma}$ when $\gamma=0.6$.


Figure 3: The sets $\mathcal{L}_{\gamma}, \mathcal{U}_{\gamma}$.

In the next result we determine in general the sets $\mathcal{L}_{\gamma}, \mathcal{U}_{\gamma}$.
Theorem 10. Given a coherent assessment $\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ on the family $\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\}$, where $E_{1}, H_{1}, \ldots, E_{n}, H_{n}$ are logically independent, we have

$$
\begin{align*}
& \mathcal{L}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: p_{1}+\cdots+p_{n} \geq \gamma+n-1\right\}, \gamma>0,  \tag{14}\\
& \mathcal{U}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: 0 \leq p_{1} \leq \gamma, p_{k+1} \leq r_{k}, k=1, \ldots, n-1\right\}, \gamma<1,
\end{align*}
$$

where $r_{k}=\frac{\gamma-u_{k}}{1-(2-\gamma) u_{k}}, u_{k}=S_{0}^{H}\left(p_{1}, \ldots, p_{k}\right)$, with $\mathcal{L}_{0}=\mathcal{U}_{1}=[0,1]^{n}$.
Proof. Of course, $\mathcal{L}_{0}=[0,1]^{n}$, so that we can assume $\gamma>0$. It must be $l_{n}=\max \left(p_{1}+\cdots+p_{n}-(n-1), 0\right) \geq \gamma$, that is, as $\gamma>0, p_{1}+\cdots+p_{n} \geq \gamma+n-1$. Hence: $\mathcal{L}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: p_{1}+\cdots+p_{n} \geq \gamma+n-1\right\}$.
We observe that $\mathcal{L}_{\gamma}$ is a convex polyhedron with vertices the points

$$
\begin{aligned}
& V_{1}=(\gamma, 1, \ldots, 1), V_{2}=(1, \gamma, 1, \ldots, 1), \cdots, \\
& V_{n}=(1, \ldots, 1, \gamma), V_{n+1}=(1,1, \ldots, 1) .
\end{aligned}
$$

Moreover, the convex hull of the vertices $V_{1}, \ldots, V_{n}$ is the subset of the points $\left(p_{1}, \ldots, p_{n}\right)$ of $\mathcal{L}_{\gamma}$ such that $l_{n}=\gamma$, that is such that $p_{1}+\cdots+p_{n}=\gamma+n-1$.
Now, let us determine the set $\mathcal{U}_{\gamma}$. Of course, $\mathcal{U}_{1}=[0,1]^{n}$, so that we can assume $\gamma<1$. We recall that $u_{2}, \ldots, u_{n}$ are the upper bounds on $C\left(\mathcal{F}_{2}\right), \ldots, C\left(\mathcal{F}_{n}\right)$ associated with $\left(p_{1}, \ldots, p_{n}\right)$. Then, from the relations

$$
C\left(\mathscr{F}_{k+1}\right)=C\left(C\left(\mathscr{F}_{k}\right), E_{k+1} \mid H_{k+1}\right), k=2, \ldots, n-1
$$

by applying (12) with $x=u_{k}, y=p_{k+1}$, we have that in order the inequality $u_{k+1} \leq \gamma$ be satisfied, it must be $u_{k} \leq$ $\gamma, p_{k+1} \leq \gamma, k=2, \ldots, n-1$. Therefore

$$
u_{n} \leq \gamma \Longrightarrow p_{1} \leq \gamma, \ldots, p_{n} \leq \gamma, u_{2} \leq \gamma, \ldots, u_{n-1} \leq \gamma
$$

so that $\mathcal{U}_{\gamma} \subseteq[0, \gamma]^{n}$. By iteratively applying $[13$, we obtain

$$
\begin{gathered}
0 \leq p_{1} \leq \gamma, \quad p_{2} \leq \frac{\gamma-p_{1}}{1-(2-\gamma) p_{1}} \quad \Longrightarrow \quad u_{2} \leq \gamma, \\
0 \leq u_{2} \leq \gamma, \quad p_{3} \leq \frac{\gamma-u_{2}}{1-(2-\gamma) u_{2}} \quad \Longrightarrow u_{3} \leq \gamma, \\
\vdots \\
0 \leq u_{n-1} \leq \gamma, \quad p_{n} \leq \frac{\gamma-u_{n-1}}{1-(2-\gamma) u_{n-1}} \quad \Longrightarrow u_{n} \leq \gamma .
\end{gathered}
$$

Therefore, observing that $u_{1}=p_{1}$,

$$
\mathcal{U}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: 0 \leq p_{1} \leq \gamma, \quad p_{k+1} \leq \frac{\gamma-u_{k}}{1-(2-\gamma) u_{k}}, k=1, \ldots, n-1\right\} .
$$

We observe that $\mathcal{U}_{0}=\{(0, \ldots, 0)\}$; moreover, for $p_{1}=\cdots=p_{n}=\gamma \in(0,1)$, we obtain (by induction)

$$
u_{2}=\frac{2 \gamma}{1+\gamma}>\gamma, u_{3}=\frac{3 \gamma}{1+2 \gamma}>\gamma, \cdots, u_{n}=\frac{n \gamma}{1+(n-1) \gamma}>\gamma
$$

hence, for $\gamma \in(0,1), \mathcal{U}_{\gamma}$ is a strict subset of $[0, \gamma]^{n}$.
Of course, for every $\left(p_{1}, \ldots, p_{n}\right) \notin \mathcal{L}_{\gamma} \cup \mathcal{U}_{\gamma}$, it is $l_{n}<\gamma<u_{n}$. As an example, for $p_{1}=\cdots=p_{n}=\gamma \in(0,1)$, one has

$$
l_{n}=\max (n \gamma-(n-1), 0)<\gamma<u_{n}=\frac{n \gamma}{1+(n-1) \gamma} .
$$

## 5. Lower and Upper Bounds for Quasi Disjunction

We recall below the notion of quasi disjunction of conditional events as defined in [1].
Definition 8. Given any events $A, H, B, K$, with $H \neq \emptyset, K \neq \emptyset$, the quasi disjunction of the conditional events $A \mid H$ and $B \mid K$ is the conditional event $\mathcal{D}(A|H, B| K)=(A H \vee B K) \mid(H \vee K)$.

The constituents generated by the family $\{A|H, B| K, \mathcal{D}(A|H, B| K)\}$ and the corresponding points $Q_{h}$ 's are given in columns 2 and 8 of Table 1. In general, given a family of $n$ conditional events $\mathcal{F}_{n}=\left\{E_{i} \mid H_{i}, i=1, \ldots, n\right\}$, it is $\mathcal{D}\left(\mathscr{F}_{n}\right)=\mathcal{D}\left(E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right)=\left(\bigvee_{i=1}^{n} E_{i} H_{i}\right) \mid\left(\bigvee_{i=1}^{n} H_{i}\right)$. Quasi disjunction is associative; that is, for every subset $J \subset\{1, \ldots, n\}$, we have $\mathcal{D}\left(\mathcal{F}_{n}\right)=\mathcal{D}\left(\mathcal{F}_{J} \cup \mathcal{F}_{\Gamma}\right)=\mathcal{D}\left[\mathcal{D}\left(\mathcal{F}_{J}\right), \mathcal{D}\left(\mathcal{F}_{\Gamma}\right)\right]$, where $\Gamma=\{1, \ldots, n\} \backslash J$.
Remark 5. We recall that the quasi conjunction of $A \mid H$ and $B \mid K$ can also be written as $C(A|H, B| K)=\left(A \vee H^{c}\right) \wedge(B \vee$ $\left.K^{c}\right) \mid(H \vee K)$; then, based on the usual negation operation $(E \mid H)^{c}=E^{c} \mid H$, it holds that

$$
\begin{align*}
& {\left[C\left(A^{c}\left|H, B^{c}\right| K\right)\right]^{c}=\left[\left(A^{c} \vee H^{c}\right) \wedge\left(B^{c} \vee K^{c}\right) \mid(H \vee K)\right]^{c}=} \\
& =(A H \vee B K) \mid(H \vee K)=\mathcal{D}(A|H, B| K), \tag{15}
\end{align*}
$$

which represents the De Morgan duality between quasi conjunction and quasi disjunction. We also have $\mathcal{D}(A|H, B| K) \vee$ $C\left(A^{c}\left|H, B^{c}\right| K\right)=\Omega \mid(H \vee K)$ and $\mathcal{D}(A|H, B| K) \wedge \mathcal{C}\left(A^{c}\left|H, B^{c}\right| K\right)=\emptyset \mid(H \vee K)$. From 15) it follows

$$
\begin{equation*}
P[\mathcal{D}(A|H, B| K)]=1-P\left[C\left(A^{c}\left|H, B^{c}\right| K\right)\right] \tag{16}
\end{equation*}
$$

which will be exploited in the next result.
Proposition 2. Assuming $A, H, B, K$ logically independent, we have:
(i) the probability assessment $(x, y)$ on $\{A|H, B| K\}$ is coherent for every $(x, y) \in[0,1]^{2}$;
(ii) given a coherent assessment $(x, y)$ on $\{A|H, B| K\}$, the assessment $\mathcal{P}=(x, y, z)$ on $\mathcal{F}=\{A|H, B| K, \mathcal{D}(A|H, B| K)\}$, with $z=P[\mathcal{D}(A|H, B| K)]$, is a coherent extension of $(x, y)$ if and only if $z \in[l, u]$, where

$$
l=T_{0}^{H}(x, y)=\left\{\begin{array}{ll}
\frac{x y}{x+y-x y}, & (x, y) \neq(0,0), \\
0, & (x, y)=(0,0),
\end{array} \quad u=S_{L}(x, y)=\min (x+y, 1)\right.
$$

Proof. We observe that, by (4], the extension $\gamma=P\left[C\left(A^{c}\left|H, B^{c}\right| K\right)\right]$ of the assessment $P(A \mid H)=x, P(B \mid K)=y$ is coherent if and only if $\gamma^{\prime} \leq \gamma \leq \gamma^{\prime \prime}$, where $\gamma^{\prime}=T_{L}(1-x, 1-y), \gamma^{\prime \prime}=S_{0}^{H}(1-x, 1-y)$. Then, based on 16) and on the results given in Appendix A, it follows that

$$
l=1-S_{0}^{H}(1-x, 1-y)=T_{0}^{H}(x, y), \quad u=1-T_{L}(1-x, 1-y)=S_{L}(x, y) .
$$

In Figure 4 is shown the convex hull $I$ for given values $x, y$, with the associated interval $[l, u]$ of coherent extensions $z=P[\mathcal{D}(A|H, B| K)]$. As for quasi conjunction, the convex hull $I$ does not depend on $z$. In the next subsections we examine some particular cases.

### 5.1. The Dual of Compound Probability Theorem

Given any logically independent events $A, B, H$, with $A^{c} H \neq \emptyset$, the assessment $(x, y)$ on $\left\{A|H, B| A^{c} H\right\}$ is coherent, for every $(x, y) \in[0,1]^{2}$. We have $\mathcal{D}\left(A|H, B| A^{c} H\right)=(A \vee B) \mid H$ and, defining $z=P(A \vee B \mid H)$, by 16 and by the results in Subsection 3.2, we have

$$
\begin{aligned}
& z=P\left(\mathcal{D}\left(A|H, B| A^{c} H\right)\right)=1-P\left(C\left(A^{c}\left|H, B^{c}\right| A^{c} H\right)\right)= \\
& =1-T_{P}(1-x, 1-y)=x+y-x y=S_{P}(x, y) ;
\end{aligned}
$$

that is $z$ is equal to the probabilistic sum of $x, y$.

### 5.2. The case $A|H \subseteq B| K$

From $A|H \subseteq B| K$ we have $\mathcal{D}(A|H, B| K)=(B K) \mid(H \vee K)$. Then, as shown by Table 1 and by Remark 4 , it holds that

$$
t(A \mid H) \leq t(C(A|H, B| K)) \leq t(\mathcal{D}(A|H, B| K)) \leq t(B \mid K)
$$

Then: $A|H \subseteq B| K$ implies $A|H \subseteq C(A|H, B| K) \subseteq \mathcal{D}(A|H, B| K) \subseteq B| K$. We recall that, by Remark $3, A|H \subseteq B| K$ amounts to $B^{c}\left|K \subseteq A^{c}\right| H$; then, given the assessment $P(A \mid H)=x, P(B \mid K)=y$, where $x \leq y$, by applying Proposition 1 to the family $\left\{B^{c}\left|K, A^{c}\right| H\right\}$, the extension $\gamma=P\left[C\left(B^{c}\left|K, A^{c}\right| H\right)\right]$ of $(x, y)$ is coherent if and only if $\gamma^{\prime} \leq \gamma \leq \gamma^{\prime \prime}$, where $\gamma^{\prime}=1-y, \gamma^{\prime \prime}=1-x$. Then, by [16], the extension $z=P[\mathcal{D}(A|H, B| K)]$ of $(x, y)$ is coherent if and only if $l \leq z \leq u$, where $l=x=\min (x, y), u=y=\max (x, y)$.


Figure 4: The convex hull $I$ associated with the pair $(\mathcal{F}, \mathcal{P})$ in case of quasi disjunction without logical relations. The interval $[l, u]$ for $z=$ $P[\mathcal{D}(A|H, B| K)]$ is the range of the third coordinate $z$ of each $\mathcal{P} \in \overline{\mathcal{P}_{l} \mathcal{P}_{u}}=\left\{(x, y, z): z \in\left[T_{0}^{H}(x, y), S_{L}(x, y)\right]\right\}$. The segment $\overline{\mathcal{P}_{l} \mathcal{P}_{u}}$ is the intersection between the segment $\{(x, y, z): z \in[0,1]\}$ and the convex hull $\mathcal{I}$.

### 5.3. Quasi Conjunction, Quasi Disjunction and Or Rule.

We recall that in Or rule with premises $H \mid \sim A$ and $K \nsim A$ the conclusion is $H \vee K \mid \sim A$. Moreover, for the conditional events $A \mid H$ and $A \mid K$ associated with the premises, we have

$$
C(A|H, A| K)=\mathcal{D}(A|H, A| K)=A \mid(H \vee K),
$$

which is the conditional event associated with the conclusion $H \vee K \mid \sim A$ of Or rule. In [35] it has been proved that, under logical independence of $A, H, K$, the assessment $z=P(A \mid(H \vee K)$ is a coherent extension of the assessment $(x, y)$ on $\{A|H, A| K\}$ if and only if $z \in[l, u]$, with

$$
\begin{equation*}
l=T_{0}^{H}(x, y), \quad u=S_{0}^{H}(x, y) . \tag{17}
\end{equation*}
$$

The convex hull $I$ for given values $x, y$ and the associated interval $[l, u]$ for $z=P[\mathcal{D}(A|H, A| K)]$ are shown in Figure (5)

### 5.4. Lower and Upper Bounds for the Quasi Disjunction of $n$ Conditional Events

Given the family $\mathcal{F}_{n}=\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\}$, let us consider the quasi disjunction $\mathcal{D}\left(\mathcal{F}_{n}\right)$ of the conditional events in $\mathcal{F}_{n}$. By the associative property of quasi disjunction, defining $\mathcal{F}_{k}=\left\{E_{1}\left|H_{1}, \ldots, E_{k}\right| H_{k}\right\}$, for each $k=2, \ldots, n$ it holds that $\mathcal{D}\left(\mathcal{F}_{k}\right)=\mathcal{D}\left(\mathcal{D}\left(\mathcal{F}_{k-1}\right), E_{k} \mid H_{k}\right)$. Then, denoting by $T_{0}^{H}$ the Hamacher t-norm with parameter $\lambda=0$ and by $S_{L}$ the Lukasiewicz t-conorm (see Appendix B and Appendix C), we have

Theorem 11. Given a probability assessment $\mathcal{P}_{n}=\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ on $\mathcal{F}_{n}=\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\}$, let $\left[l_{k}, u_{k}\right]$ be the interval of coherent extensions of the assessment $\mathcal{P}_{k}=\left(p_{1}, p_{2}, \ldots, p_{k}\right)$ on the quasi disjunction $\mathcal{D}\left(\mathscr{F}_{k}\right)$, where $\mathcal{F}_{k}=$ $\left\{E_{1}\left|H_{1}, \ldots, E_{k}\right| H_{k}\right\}$. Then, assuming $E_{1}, H_{1}, \ldots, E_{n}, H_{n}$ logically independent, for each $k=2, \ldots, n$, we have

$$
l_{k}=T_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right), \quad u_{k}=S_{L}\left(p_{1}, p_{2}, \ldots, p_{k}\right)
$$

Proof. Of course, from Proposition 2 it is $l_{2}=T_{0}^{H}\left(p_{1}, p_{2}\right)$ and $u_{2}=S_{L}\left(p_{1}, p_{2}\right)$. The rest of the proof is similar to that one in Theorem 8 .


Figure 5: Convex hull $\mathcal{I}$ associated with the pair $(\mathcal{F}, \mathcal{P})$ for the Or rule. The interval $[l, u]$ for $z=P[\mathcal{D}(A|H, A| K)]=P[C(A|H, A| K)]$ is the range of the third coordinate $z$ of each $\mathcal{P} \in \overline{\mathcal{P}_{l} \mathcal{P}_{u}}=\left\{(x, y, z): z \in\left[T_{0}^{H}(x, y), S_{0}^{H}(x, y)\right]\right\}$. The segment $\overline{\mathcal{P}_{l} \mathcal{P}_{u}}$ is the intersection between the segment $\{(x, y, z): z \in[0,1]\}$ and the convex hull $\mathcal{I}$.

Remark 6. Given any conditional events $A \mid H$ and $B \mid K$, as shown in Table 1 it holds that $t(C(A|H, B| K)) \leq t(\mathcal{D}(A|H, B| K))$, which amounts to $C(A|H, B| K)) \subseteq \mathcal{D}(A|H, B| K)$. In general, given a finite family of conditional events $\mathcal{F}_{n}$, we have $t\left(C\left(\mathcal{F}_{n}\right)\right) \leq t\left(\mathcal{D}\left(\mathcal{F}_{n}\right)\right)$, that is $C\left(\mathcal{F}_{n}\right) \subseteq \mathcal{D}\left(\mathcal{F}_{n}\right)$, so that $P\left[C\left(\mathcal{F}_{n}\right)\right] \leq P\left[\mathcal{D}\left(\mathcal{F}_{n}\right)\right]$. Thus, if the family $\mathcal{F}_{n}$ is p-consistent, then $\mathcal{F}_{n} \Rightarrow_{p} \mathcal{C}\left(\mathcal{F}_{n}\right) \Rightarrow_{p} \mathcal{D}\left(\mathcal{F}_{n}\right)$ and we obtain the following Quasi Or rule

$$
\text { (Quasi Or) } \quad \mathcal{F}_{n} \Rightarrow_{p} \mathcal{D}\left(\mathcal{F}_{n}\right) .
$$

We observe that Quasi Or rule also follows directly from Theorem 11

### 5.5. General Or Rule

Let us consider the general Or rule (see [37]), where the premises are the conditional events $E\left|H_{1}, \ldots, E\right| H_{n}$ and the conclusion is the conditional event $E \mid\left(H_{1} \vee H_{2} \vee \ldots, \vee H_{n}\right)$. By the associative property of quasi disjunction, defining $\mathcal{F}_{k}=\left\{E\left|H_{1}, \ldots, E\right| H_{k}\right\}$, for each $k=2, \ldots, n$ it holds that

$$
\mathcal{D}\left(\mathcal{F}_{k}\right)=\mathcal{D}\left(\mathcal{D}\left(\mathcal{F}_{k-1}\right), E_{k} \mid H_{k}\right)=E \mid\left(H_{1} \vee H_{2} \vee \ldots, \vee H_{k}\right) .
$$

We also observe that $\mathcal{D}\left(\mathcal{F}_{k}\right)=C\left(\mathcal{F}_{k}\right)$. Then, by exploiting the notions of t-norm, t-conorm, quasi disjunction and quasi conjunction, Theorem 9 in [37] can be written as

Theorem 12. Given a probability assessment $\mathcal{P}_{n}=\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ on $\mathcal{F}_{n}=\left\{E\left|H_{1}, E\right| H_{2}, \ldots, E \mid H_{n}\right\}$, let $\left[l_{k}, u_{k}\right]$ be the interval of coherent extensions of the assessment $\mathcal{P}_{k}=\left(p_{1}, p_{2}, \ldots, p_{k}\right)$ on the quasi disjunction $\mathcal{D}\left(\mathcal{F}_{k}\right)$, where $\mathcal{F}_{k}=\left\{E\left|H_{1}, \ldots, E\right| H_{k}\right\}$. Then, assuming $E, H_{1}, \ldots, H_{n}$ logically independent, for each $k=2, \ldots, n$, we have

$$
l_{k}=T_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right), \quad u_{k}=S_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right)
$$

Proof. Of course, from 17) it is $l_{2}=T_{0}^{H}\left(p_{1}, p_{2}\right)$ and $u_{2}=S_{0}^{H}\left(p_{1}, p_{2}\right)$. The rest of the proof is similar to that one in Theorem 8

In [37, Theorem 9]), by implicitly assuming $\left(p_{1}, \ldots, p_{k}\right) \in(0,1)^{k}$, it has been proved by a direct probabilistic analysis that

$$
l_{k}=\frac{1}{1+\sum_{i=1}^{k} \frac{1-p_{i}}{p_{i}}}, \quad u_{k}=\frac{\sum_{i=1}^{k} \frac{p_{i}}{1-p_{i}}}{1+\sum_{i=1}^{k} \frac{p_{i}}{1-p_{i}}} .
$$

By adopting the conventions $\frac{1}{\infty}=0, \frac{1}{0}=\infty, \frac{\infty}{\infty}=1$, the previous formulas hold in general for every $\left(p_{1}, \ldots, p_{k}\right) \in$ $[0,1]^{k}$. In Appendix C the previous expressions for the Hamacher t-norm and t-conorm have been derived by using the notion of additive generator.

Example 3. An application of Or rule is obtained by imagining a medical scenario with a disease $E$ and $n$ symptoms $H_{1}, \ldots, H_{n}$, with $P\left(E \mid H_{i}\right)=p_{i}, i=1, \ldots, n$, and $P\left(E \mid\left(H_{1} \vee \cdots \vee H_{n}\right) \in\left[l_{n}, u_{n}\right]\right.$. If, for instance, $p_{1}=\cdots=p_{n}=1-\varepsilon$, from Theorem 12 it follows $l_{n}=T_{0}^{H}(1-\varepsilon, \ldots, 1-\varepsilon)=\frac{1-\varepsilon}{1+(n-1) \varepsilon}$ and $u_{n}=S_{0}^{H}(1-\varepsilon, \ldots, 1-\varepsilon)=\frac{n(1-\varepsilon)}{\varepsilon+n(1-\varepsilon \varepsilon}$. Then: (i) for $\varepsilon \rightarrow 0$, we have $l_{n} \rightarrow 1$ and $u_{n} \rightarrow 1$; (ii) for $n \rightarrow+\infty$ we have $l_{n} \rightarrow 0$ and $u_{n} \rightarrow 1$. As we can see, in the second case the interval $\left[l_{n}, u_{n}\right]$ gets wider and wider as the number of premises increases. An interesting related phenomenon where additional information leads to less informative conclusion is the pseudodiagnosticity task, studied in the psychology of uncertain reasoning ([51, 66]).

## 6. Further Aspects on Quasi Disjunction: from Bounds on Conclusions to Bounds on Premises in Quasi Or rule

In this section, we study the propagation of probability bounds on the conclusion of the Quasi Or rule to its premises. We start with the case of two premises $A \mid H$ and $B \mid K$, by examining probabilistic aspects on the lower and upper bounds, $l$ and $u$, for the probability of the conclusion $\mathcal{D}(A|H, B| K)$. More precisely, given any number $\gamma \in[0,1]$, we find:
(i) the set $\mathbf{L}_{\gamma}$ of the coherent assessments $(x, y)$ on $\{A|H, B| K\}$ such that, for each $(x, y) \in \mathbf{L}_{\gamma}$, one has $l \geq \gamma$;
(ii) the set $\mathbf{U}_{\gamma}$ of the coherent assessments ( $x, y$ ) on $\{A|H, B| K\}$ such that, for each $(x, y) \in \mathbf{U}_{\gamma}$, one has $u \leq \gamma$.

Case (i). Let be given $\gamma \in[0,1]$. We denote by $\mathbf{L}_{\gamma}$ the set of coherent assessments ( $x, y$ ) on $\{A|H, B| K\}$ which imply $z \geq \gamma$. Of course, $\mathbf{L}_{0}=[0,1]^{2}$; hence we can assume $\gamma>0$. We recall that $l=T_{0}^{H}(x, y)$, then in order the inequality $T_{0}^{H}(x, y) \geq \gamma$ be satisfied, it must be $x>0$ and $y>0$. Thus, $l \geq \gamma$ if and only if $\frac{x y}{x+y-x y} \geq \gamma$. We have

$$
\begin{equation*}
x-l=\frac{x^{2}(1-y)}{x+y-x y} \geq 0, \quad y-l=\frac{y^{2}(1-x)}{x+y-x y} \geq 0 ; \tag{19}
\end{equation*}
$$

then, from $l \geq \gamma$ it follows $x \geq \gamma, y \geq \gamma$; thus $\mathbf{L}_{\gamma} \subseteq[\gamma, 1]^{2}$. Then, taking into account that $x \geq \gamma$ and hence $x(1+\gamma)-\gamma>0$, we have

$$
\begin{equation*}
\frac{x y}{x+y-x y} \geq \gamma \Longleftrightarrow y \geq \frac{\gamma x}{x(1+\gamma)-\gamma} \tag{20}
\end{equation*}
$$

therefore

$$
\mathbf{L}_{\gamma}=\left\{(x, y): \gamma \leq x \leq 1, \quad y \geq \frac{\gamma x}{x-\gamma+\gamma x}\right\} .
$$

Notice that $\mathbf{L}_{1}=\{(1,1)\}$; for $x=y=\gamma \in(0,1)$, it is $l=\frac{\gamma}{2-\gamma}<\gamma$; hence, for $\gamma \in(0,1), \mathbf{L}_{\gamma}$ is a strict subset of $[\gamma, 1]^{2}$. Case (ii). Of course, $\mathbf{U}_{1}=[0,1]^{2}$; hence we can assume $\gamma<1$. It must be $u=\min \{x+y, 1\} \leq \gamma$, i.e., $x+y \leq \gamma$ (as $\gamma<1)$; hence $\mathbf{U}_{\gamma}$ coincides with the triangle having the vertices $(0,0),(0, \gamma),(\gamma, 0)$; that is

$$
\mathbf{U}_{\gamma}=\{(x, y): 0 \leq x \leq \gamma, 0 \leq y \leq \gamma-x\}
$$

Notice that $\mathbf{U}_{0}=\{(0,0)\}$; moreover, for $\gamma \in(0,1),(\gamma, \gamma) \notin \mathbf{U}_{\gamma}$.
Of course, for every $(x, y) \notin \mathbf{L}_{\gamma} \cup \mathbf{U}_{\gamma}$, it is $l<\gamma<u$.
Figure 6 displays the sets $\mathbf{L}_{\gamma}, \mathbf{U}_{\gamma}$ when $\gamma=0.4$. In the next result we determine in general the sets $\mathbf{L}_{\gamma}, \mathbf{U}_{\gamma}$.


Figure 6: The sets $\mathbf{L}_{\gamma}, \mathbf{U}_{\gamma}$.

Theorem 13. Let be given the family $\mathcal{F}_{n}=\left\{E_{1}\left|H_{1}, \ldots, E_{n}\right| H_{n}\right\}$, with the events $E_{1}, H_{1}, \ldots, E_{n}, H_{n}$ logically independent. Moreover, for any given $\gamma \in[0,1]$ let $\mathbf{L}_{\gamma}\left(\right.$ resp. $\left.\mathbf{U}_{\gamma}\right)$ be the set of the coherent assessments $\left(p_{1}, p_{2}, \ldots, p_{n}\right)$ on $\mathcal{F}_{n}$ such that, for each $\left(p_{1}, p_{2}, \ldots, p_{n}\right) \in \mathbf{L}_{\gamma}\left(\right.$ resp. $\left.\left(p_{1}, p_{2}, \ldots, p_{n}\right) \in \mathbf{U}_{\gamma}\right)$, one has $l \geq \gamma(r e s p . u \leq \gamma)$, where $l$ is the lower bound (resp. $u$ is the upper bound) of the coherent extensions $z=P\left[\mathcal{D}\left(\mathcal{F}_{n}\right)\right]$. We have

$$
\begin{align*}
& \mathbf{U}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: p_{1}+\cdots+p_{n} \leq \gamma\right\}, \gamma<1, \\
& \mathbf{L}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: \gamma \leq p_{1} \leq 1, r_{k} \leq p_{k+1}, k=1, \ldots, n-1\right\}, \gamma>0, \tag{21}
\end{align*}
$$

where $r_{k}=\frac{\gamma l_{k}}{l_{k}-\gamma+\gamma l_{k}}, l_{k}=T_{0}^{H}\left(p_{1}, \ldots, p_{k}\right)$, with $\mathbf{L}_{0}=\mathbf{U}_{1}=[0,1]^{n}$.
Proof. Of course, $\mathbf{U}_{1}=[0,1]^{n}$, so that we can assume $\gamma<1$. It must be $u_{n}=\min \left(p_{1}+\cdots+p_{n}, 1\right) \leq \gamma$, that is, as $\gamma<1, p_{1}+\cdots+p_{n} \leq \gamma$. Hence: $\mathbf{U}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: p_{1}+\cdots+p_{n} \leq \gamma\right\}$.
We observe that $\mathbf{U}_{\gamma}$ is a convex polyhedron with vertices the points

$$
\begin{aligned}
& V_{1}=(\gamma, 0, \ldots, 0), V_{2}=(0, \gamma, 0, \ldots, 0), \cdots, \\
& V_{n}=(0, \ldots, 0, \gamma), V_{n+1}=(0,0, \ldots, 0)
\end{aligned}
$$

Moreover, the convex hull of the vertices $V_{1}, \ldots, V_{n}$ is the subset of the points $\left(p_{1}, \ldots, p_{n}\right)$ of $\mathbf{U}_{\gamma}$ such that $u_{n}=\gamma$, that is such that $p_{1}+\cdots+p_{n}=\gamma$.

Of course, $\mathbf{L}_{0}=[0,1]^{n}$, so that we can assume $\gamma>0$. We recall that $l_{2}, \ldots, l_{n}$ are the lower bounds on $\mathcal{D}\left(\mathscr{F}_{2}\right), \ldots, \mathcal{D}\left(\mathscr{F}_{n}\right)$ associated with $\left(p_{1}, \ldots, p_{n}\right)$. Then, from the relations

$$
\mathcal{D}\left(\mathcal{F}_{k+1}\right)=\mathcal{D}\left(\mathcal{D}\left(\mathcal{F}_{k}\right), E_{k+1} \mid H_{k+1}\right), k=2, \ldots, n-1,
$$

by applying (19) with $x=l_{k}, y=p_{k+1}$, we have that in order the inequality $l_{k+1} \geq \gamma$ be satisfied, it must be $l_{k} \geq \gamma, p_{k+1} \geq \gamma, k=2, \ldots, n-1$. Therefore

$$
l_{n} \geq \gamma \Longrightarrow p_{1} \geq \gamma, \ldots, p_{n} \geq \gamma, l_{2} \geq \gamma, \ldots, l_{n-1} \geq \gamma
$$

so that $\mathbf{L}_{\gamma} \subseteq[\gamma, 1]^{n}$. By iteratively applying $\sqrt{20}$, we obtain

$$
\begin{gathered}
\gamma \leq p_{1} \leq 1, \quad p_{2} \geq \frac{\gamma p_{1}}{p_{1}(1+\gamma)-\gamma} \quad \Longrightarrow \quad l_{2} \geq \gamma \\
\gamma \leq l_{2} \leq 1, \quad p_{3} \geq \frac{\gamma l_{2}}{l_{2}(1+\gamma)-\gamma} \quad \Longrightarrow \quad l_{3} \geq \gamma \\
\gamma \leq l_{n-1} \leq 1, \quad p_{n} \geq \frac{\gamma l_{n-1}}{l_{n-1}(1+\gamma)-\gamma} \quad \Longrightarrow \quad l_{n} \geq \gamma
\end{gathered}
$$

Therefore, observing that $l_{1}=p_{1}$, we have,

$$
\mathbf{L}_{\gamma}=\left\{\left(p_{1}, \ldots, p_{n}\right) \in[0,1]^{n}: \gamma \leq p_{1} \leq 1, \quad p_{k+1} \geq \frac{\gamma l_{k}}{l_{k}(1+\gamma)-\gamma}, k=1, \ldots, n-1\right\}
$$

We observe that $\mathbf{L}_{1}=\{(1, \ldots, 1)\}$; moreover, for $p_{1}=\cdots=p_{n}=\gamma \in(0,1)$, we obtain (by induction)

$$
l_{2}=\frac{\gamma}{2-\gamma}<\gamma, l_{3}=\frac{\gamma}{3-2 \gamma}<\gamma, l_{4}=\frac{\gamma}{4-3 \gamma} \cdots, l_{n}=\frac{\gamma}{n-(n-1) \gamma}<\gamma
$$

hence, for $\gamma \in(0,1), \mathbf{L}_{\gamma}$ is a strict subset of $[0, \gamma]^{n}$.

## 7. Biconditional Events, $\boldsymbol{n}$-Conditional Events and Loop Rule

We now examine the quasi conjunction of $A \mid B$ and $B \mid A$, with $A, B$ logically independent events. We have

$$
C(A|B, B| A)=\left(A B \vee B^{c}\right) \wedge\left(B A \vee A^{c}\right)|(A \vee B)=A B|(A \vee B) .
$$

We observe that the conditional event $A B \mid(A \vee B)$ captures the notion of biconditional event ${ }^{2} A \nmid \vdash B$ considered by some authors as the "conjunction" between $A \mid B$ and $B \mid A$ and has the same truth table of the "defective biconditional" discussed in [29]; see also [27]. It can be easily verified that, for every pair $(x, y) \in[0,1] \times[0,1]$ the probability assessment $(x, y)$ on $\{A|B, B| A\}$ is coherent. Given any coherent assessment $(x, y)$ on $\{A|B, B| A\}$, the probability assessment $z=P(A \nvdash B)$, is a coherent extension of $(x, y)$ if and only if

$$
z= \begin{cases}0 & (x, y)=(0,0) \\ \frac{x y}{x+y-x y} & (x, y) \neq(0,0)\end{cases}
$$

We can study the coherence of the assessment $\mathcal{P}=(x, y, z)$ on the family

$$
\mathcal{F}=\{A|B, B| A, A \dashv B\}=\{A|B, B| A, A B \mid(A \vee B)\},
$$

by the geometrical approach described in Section 2 . In such a case, as the events of the family are not logically independent, the constituents generated by the family and contained in $A \vee B$ are: $C_{1}=A B, C_{2}=A B^{c}, C_{3}=A^{c} B$. We distinguish two cases: (i) $(x, y) \neq(0,0)$; (ii) $(x, y)=(0,0)$.
(i) If $(x, y) \neq(0,0)$ the corresponding points $Q_{h}$ 's are $Q_{1}=(1,1,1), Q_{2}=(x, 0,0), Q_{3}=(0, y, 0)$, and, in our case, the coherence of $\mathcal{P}$ simply amounts to the geometrical condition $\mathcal{P} \in I$, where $I$ is the triangle with vertices $Q_{1}, Q_{2}, Q_{3}$. Based on the equation of the plane containing $I$, we have that $\mathcal{P}$ is coherent if and only if: $z=\frac{x y}{x+y-x y}$.
(ii) If $(x, y)=(0,0)$, then $Q_{2}=Q_{3}=(0,0,0)$ and the convex hull $I$ is the segment $Q_{1} Q_{2}$. Then, $\mathcal{P}=(0,0, z)$ is

[^2]coherent if and only if $z=0$.
Then, the value $z$ is a coherent extension of $(x, y)$ if and only if
\[

z=T_{0}^{H}(x, y)= $$
\begin{cases}0 & (x, y)=(0,0) \\ \frac{x y}{x+y-x y} & (x, y) \neq(0,0),\end{cases}
$$
\]

where $T_{0}^{H}(x, y)$ is the Hamacher t-norm, with parameter $\lambda=0$, defined by formula 22. In agreement with Remark 2, we observe that

$$
T_{L}(x, y) \leq T_{0}^{H}(x, y) \leq S_{0}^{H}(x, y)
$$

### 7.1. Generalizing Biconditional Events: An Application to Loop rule

As shown before, given any (non impossible) events $A_{1}, A_{2}$, the biconditional event associated with them is given by

$$
A_{1} \dashv\left|A_{2}=C\left(A_{2}\left|A_{1}, A_{1}\right| A_{2}\right)=A_{1} A_{2}\right|\left(A_{1} \vee A_{2}\right) .
$$

The notion of biconditional event can be generalized by defining the $n$-conditional event associated with $n$ (non impossible) events $A_{1}, \ldots, A_{n}$ as

$$
A_{1} \nvdash A_{2} \dashv \vdash \cdots \nvdash A_{n}=C\left(A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right) .
$$

Let $C_{0}, C_{1}, \ldots, C_{m}$ be the constituents generated by the conditional events $A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}$. We set $C_{0}=$ $A_{1}^{c} A_{2}^{c} \cdots A_{n}^{c}$ and $C_{1}=A_{1} A_{2} \cdots A_{n}$; then, for each $h=2, \ldots, m$, it is $C_{h}=A_{i_{1}} \cdots A_{i_{r}} A_{i_{r+1}}^{c} \cdots A_{i_{n}}^{c}$, with $1 \leq r<n$. As it can be easily verified, the truth value of the $n$-conditional associated with $C_{h}$ is true, or false, or void, according to whether $h=1$, or $h>1$, or $h=0$; then it holds that

$$
C\left(A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right)=A_{1} \cdots A_{n} \mid\left(A_{1} \vee \cdots \vee A_{n}\right)
$$

In ([36]), where also the relationship with conditional objects ([22]) has been studied, the previous formula has been obtained by a suitable inductive reasoning, by showing that:
(i) $C\left(A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}\right)=\left(E_{0} \vee \cdots \vee E_{n-1}\right) \mid\left(A_{1} \vee \cdots \vee A_{n-1}\right)$,
where $E_{0}=A_{1} \cdots A_{n}, E_{1}=A_{1}^{c} A_{2} \cdots A_{n}, \ldots, E_{n-2}=A_{1}^{c} \cdots A_{n-2}^{c} A_{n-1} A_{n}, E_{n-1}=A_{1}^{c} \cdots A_{n-1}^{c}$;
(ii) then

$$
\begin{gather*}
\mathcal{C}\left(A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right)=C\left[\left(E_{1} \vee \cdots \vee E_{n}\right)\left(\left(A_{1} \vee \cdots \vee A_{n-1}\right), A_{1} \mid A_{n}\right]=\right. \\
=A_{1} \cdots A_{n} \mid\left(A_{1} \vee \cdots \vee A_{n}\right) \tag{22}
\end{gather*}
$$

Of course, for any given derangement (a permutation with no fixed point) $\left(i_{1}, i_{2}, \ldots, i_{n}\right)$ of $(1,2, \ldots, n)$, we have

$$
\mathcal{C}\left(A_{i_{1}}\left|A_{1}, \ldots, A_{i_{n-1}}\right| A_{n-1}, A_{i_{n}} \mid A_{n}\right)=\mathcal{C}\left(A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right)
$$

that is, the $n$-conditional $A_{1} \dashv \vdash A_{2} \dashv \vdash \cdots \nvdash A_{n}$ can be represented as the quasi conjunction of the conditional events $A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}$, or equivalently as the quasi conjunction of the conditional events $A_{i_{1}}\left|A_{1}, \ldots, A_{i_{n-1}}\right| A_{n-1}, A_{i_{n}} \mid A_{n}$. In particular for $\left(i_{1}, i_{2}, \ldots, i_{n}\right)=(n, 1,2, \ldots, n-1)$ we have

$$
\begin{equation*}
C\left(A_{1}\left|A_{2}, \ldots, A_{n-1}\right| A_{n}, A_{n} \mid A_{1}\right)=C\left(A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right) . \tag{23}
\end{equation*}
$$

As a consequence, we can immediately obtain the probabilistic interpretation of Loop rule ([54]). Given $n$ logically independent events $A_{1}, A_{2}, \ldots, A_{n}$, Loop rule is the following one:

$$
\begin{equation*}
A_{1} \vdash A_{2}, A_{2} \vdash A_{3}, \cdots, A_{n} \vdash A_{1} \Longrightarrow A_{1} \vdash A_{n} . \tag{24}
\end{equation*}
$$

In [54] it has also been proved that, for every $i, j=1,2, \ldots, n$, it holds that

$$
\begin{equation*}
A_{1} \vdash A_{2}, A_{2} \vdash A_{3}, \cdots, A_{n} \vdash A_{1} \Longrightarrow A_{i} \vdash A_{j} . \tag{25}
\end{equation*}
$$

### 7.2. Probabilistic Aspects on Loop Rule

In our probabilistic approach, formula (25), which generalizes formula (24), can be obtained by the following steps:

- given any p-consistent family of conditional events $\mathcal{F}$, from Corollary 1 it holds that $\mathcal{F}$ p-entails $\mathcal{C}(\mathcal{F})$;
- defining $\mathcal{F}=\left\{A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right\}$, it can be checked that $\mathcal{F}$ is p-consistent; then, for every $i, j=1,2, \ldots, n$, by $22 \mathcal{C} C \mathcal{F}) \subseteq A_{i} \mid A_{j}$; hence $C(\mathcal{F})$ p-entails $A_{i} \mid A_{j}$; moreover, $\mathcal{F}$ p-entails $C(\mathcal{F})$ and then $\mathcal{F}$ p-entails $A_{i} \mid A_{j}$.
Remark 7. By Definition 5 and formulas (23) and (25), for any given derangement $\left(i_{1}, i_{2}, \ldots, i_{n}\right)$ of $(1,2, \ldots, n)$, we obtain the following inference rule (Generalized Loop)

$$
\begin{equation*}
\left\{A_{2}\left|A_{1}, \ldots, A_{n}\right| A_{n-1}, A_{1} \mid A_{n}\right\} \underset{p}{{ }_{p}^{p}}\left\{A_{i_{1}}\left|A_{1}, \ldots, A_{i_{n-1}}\right| A_{n-1}, A_{i_{n}} \mid A_{n}\right\} \tag{26}
\end{equation*}
$$

The Loop rule has been studied by a direct probabilistic reasoning in [36], by exploiting a suitable probabilistic condition named Császár's condition, studied in the framework of an axiomatic approach to probability in [20]. This condition in a particular case reduces to the third axiom of conditional probabilities. A numerical inference rule named generalized Bayes theorem, connected with Császár's condition and with Loop rule, has been studied in [4]; see also [5, 23]. Below, we reconsider an example introduced in [36] to illustrate the generalized Loop rule and p-entailment of $n$-conditionals.

Example 4. Five friends, Linda, Janet, Steve, George, and Peter, have been invited to a party. We define the events: $A_{1}=" L i n d a$ goes to the party", $\ldots, A_{5}=$ Peter goes to the party; moreover, we assume that $A_{1}, \ldots, A_{5}$ are logically independent. We consider the following knowledge base: \{"if Linda goes to the party, then Janet will do the same", ..., "if George goes to the party, then Peter will do the same", "if Peter goes to the party, then Linda will do the same" $\}$. Then, for the associated (p-consistent) family of conditional events $\mathcal{F}=\left\{A_{2}\left|A_{1}, \ldots, A_{5}\right| A_{4}, A_{1} \mid A_{5}\right\}$, we have

$$
C(\mathcal{F})=A_{1} A_{2} \cdots A_{5} \mid\left(A_{1} \vee A_{2} \vee \cdots \vee A_{5}\right)=A_{1} \nvdash A_{2} \dashv \vdash \cdots \nvdash A_{5} .
$$

By generalized Loop rule, for every derangement $\left(i_{1}, \ldots, i_{5}\right)$ of $(1, \ldots, 5)$, it holds that

$$
\left\{A_{2}\left|A_{1}, \ldots, A_{5}\right| A_{4}, A_{1} \mid A_{5}\right\} \underset{p}{{ }_{p}^{2}}\left\{A_{i_{1}}\left|A_{1}, \ldots, A_{i_{4}}\right| A_{4}, A_{i_{5}} \mid A_{5}\right\}
$$

For any given subset $\left\{B_{1}, \ldots, B_{n}\right\} \subset\left\{A_{1}, \ldots, A_{5}\right\}, n=2,3,4$, we have $B_{1} \dashv \vdash \cdots \nvdash B_{n}=B_{1} \cdots B_{n} \mid\left(B_{1} \vee \cdots \vee B_{n}\right)$. This $n$-conditional is associated with the conditional assertion "if at least one of $n$ given friends among Linda, Janet, Steve, George, and Peter, goes to the party, then all n friends will go to the party". We have

$$
A_{1} \cdots A_{5}\left|\left(A_{1} \vee \cdots \vee A_{5}\right) \subseteq B_{1} \cdots B_{n}\right|\left(B_{1} \vee \cdots \vee B_{n}\right)
$$

therefore $A_{1} \dashv \vdash \cdots \not A_{5}$ p-entails $B_{1} \dashv \vdash \cdots \nvdash B_{n}$. Finally, as $\mathcal{F}$ p-entails $\mathcal{C}(\mathcal{F})$, we have that for every subset $\left\{B_{1}, \ldots, B_{n}\right\}, n=2,3,4$, the family $\mathcal{F}$ p-entails the $n$-conditional $B_{1} \dashv \vdash \cdots \not B_{n}$.

## 8. Conclusions

In this paper we have examined probabilistic concepts connected with the inference rules Quasi And, Quasi Or, Or, and generalized Loop. These are linked with Adams' probabilistic analysis of conditionals, and play an important role in applications to nonmonotonic reasoning, to the psychology of uncertain reasoning and to semantic web. We have considered, in a coherence-based setting, the extensions of a given probability assessment on $n$ conditional events to their quasi conjunction and quasi disjunction, by also examining some cases of logical dependencies. In our probabilistic analysis we have shown that the lower and upper probability bounds computed in the different cases coincide with some well known t-norms and t-conorms: minimum, product, Lukasiewicz and Hamacher t-norms, and their dual $t$-conorms. We have shown that, for the Or rule, the quasi conjunction and quasi disjunction of the premises are equal. Moreover, they coincide with the conclusion of the rule. We have identified the relationships among coherence, inclusion relation and p-entailment. Finally, we have considered biconditional events and we have introduced
the notion of $n$-conditional event, by obtaining a probabilistic interpretation for a generalized Loop rule. In Appendix C we give explicit expressions for the Hamacher t-norm and t-conorm in the unitary hypercube $[0,1]^{k}$. As a "take home message", the results obtained in our coherence-based probabilistic approach can be exploited in all researches in nonmonotonic reasoning, as made for instance in [38, 51, 60, 61]. Future work should deepen the theoretical aspects and applications which connect conditional probability with t-norms and t-conorms, in relation to inference patterns in nonmonotonic reasoning. In particular, the representation of probability bounds for the conditional conclusions of some inference patterns involving conditionals in terms of t -norms and t -conorms is a topic that could be expanded. Finally, a relevant topic for further research concerns the study of more general definitions for the logical operations of conjunction and disjunction among conditionals. Such new logical operations should be defined in a way such that the usual probabilistic properties be preserved. Some results on this topic have been given in [43].

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## Appendix A. t-norms and t-conorms.

We recall below the notions of $t$-norm and $t$-conorm (see [48, 52, 53]).
Definition 9. A $t$-norm is a function $T:[0,1]^{2} \longrightarrow[0,1]$ which satisfies, for all $x, y, z \in[0,1]$, the following four axioms:

| (T1) | $T(x, y)=T(y, x)$, | (commutativity) |
| :--- | :--- | :--- |
| (T2) | $T(x, T(y, z))=T(T(x, y), z)$, | (associativity) |
| (T3) | $T(x, y) \leq T(x, z)$ whenever $y \leq z$, | (monotonicity) |
| (T4) | $T(x, 1)=x$. | (boundary condition) |

We recall below some basic t-norms, namely, the minimum $T_{M}$ (which is the greatest t -norm), the product $T_{P}$, the Lukasiewicz t-norm $T_{L}$ :

$$
T_{M}(x, y)=\min (x, y), \quad T_{P}(x, y)=x \cdot y, \quad T_{L}(x, y)=\max (x+y-1,0)
$$

We also recall that the Hamacher t -norm $T_{\lambda}^{H}$, with parameter $\lambda \in[0, \infty]$, is

$$
T_{\lambda}^{H}(x, y)= \begin{cases}T_{D}(x, y), & \lambda=\infty,  \tag{A.1}\\ 0, & \lambda=0 \text { and }(x, y)=(0,0) \\ \overline{\lambda+(1-\lambda)(x+y-x y)}, & \text { otherwise },\end{cases}
$$

where the t-norm $T_{D}(x, y)$ (drastic product) is defined as

$$
T_{D}(x, y)= \begin{cases}0, & (x, y) \in[0,1)^{2} \\ \min (x, y), & \text { otherwise }\end{cases}
$$

In particular, the Hamacher t-norm $T_{1}^{H}$ is the product t-norm $T_{p}$.
Definition 10. A $t$-conorm is a function $S:[0,1]^{2} \longrightarrow[0,1]$ which satisfies, for all $x, y, z \in[0,1],(T 1)-(T 3)$ and

$$
\begin{equation*}
S(x, 0)=x \tag{S4}
\end{equation*}
$$

(boundary condition)
T-conorms can be equivalently introduced as dual operations of t-norms. A function $S:[0,1]^{2} \longrightarrow[0,1]$, is a t -conorm if and only if there exists a t -norm $T$ such that for all $(x, y) \in[0,1]^{2}$ either one of the two equalities holds: $S(x, y)=1-T(1-x, 1-y), T(x, y)=1-S(1-x, 1-y)$. Then, the dual t-conorm of $T_{M}$ is the maximum $S_{M}$, i.e. $S_{M}(x, y)=\max (x, y)$. The dual t -conorm of $T_{P}$ is the probabilistic sum $S_{P}$, i.e.

$$
S_{P}(x, y)=1-(1-x)(1-y)=x+y-x \cdot y
$$

The Lukasiewicz t-conorm, which is the dual t-conorm of $T_{L}$, is

$$
S_{L}(x, y)=\min (x+y, 1)
$$

Moreover, the Hamacher t-conorm $S_{\lambda}^{H}$ with parameter $\lambda \in[0, \infty]$, which is the dual t-conorm of $T_{\lambda}^{H}$, is

$$
S_{\lambda}^{H}(x, y)= \begin{cases}S_{D}(x, y), & \lambda=\infty  \tag{A.2}\\ 1, & \lambda=0 \text { and } x=y=1 \\ \frac{x+y-x y-(1-\lambda) x y}{1-(1-\lambda) x y}, & \text { otherwise }\end{cases}
$$

where the t -conorm $S_{D}(x, y)$ (drastic sum) is defined as

$$
S_{D}(x, y)= \begin{cases}1, & (x, y) \in(0,1]^{2} \\ \max (x, y), & \text { otherwise }\end{cases}
$$

In particular, the Hamacher t-conorm $S_{1}^{H}$ is the probabilistic $\operatorname{sum} S_{p}$.

## Appendix B. t-norms and t-conorms in $[0,1]^{k}$.

We recall that since t -norms and t -conorms are associative they can be easily extended in a unique way to a $k$ ary operation for arbitrary integer $k \geq 2$ by induction (see [46, 48, 53]). Let $T$ be a t-norm (introduced as a binary operator), for any integer $k \geq 2$ the extension of $T$ is defined as

$$
T\left(p_{1}, p_{2}, \ldots, p_{k}\right)= \begin{cases}T\left(T\left(p_{1}, \ldots, p_{k-1}\right), p_{k}\right), & \text { if } k>2 \\ T\left(p_{1}, p_{2}\right), & \text { if } k=2\end{cases}
$$

Let $S$ be a t-conorm (introduced as a binary operator), for any integer $k \in \mathbb{N} \bigcup\{0\}$ the extension of $S$ is defined as

$$
S\left(p_{1}, p_{2}, \ldots, p_{k}\right)= \begin{cases}S\left(S\left(p_{1}, \ldots, p_{k-1}\right), p_{k}\right), & \text { if } k>2 \\ S\left(p_{1}, p_{2}\right), & \text { if } k=2\end{cases}
$$

If $(T, S)$ is a pair of mutually dual t -norms and t -conorms, then

$$
\begin{gathered}
S\left(p_{1}, \ldots, p_{k}\right)=1-T\left(1-p_{1}, \ldots, 1-p_{k}\right) \\
T\left(p_{1}, \ldots, p_{k}\right)=1-S\left(1-p_{1}, \ldots, 1-p_{k}\right) .
\end{gathered}
$$

Finally, we recall that

$$
\begin{aligned}
& T_{M}\left(p_{1}, \ldots, p_{k}\right)=\min \left(p_{1}, \ldots, p_{k}\right), \quad S_{M}\left(p_{1}, \ldots, p_{k}\right)=\max \left(p_{1}, \ldots, p_{k}\right), \\
& T_{p}\left(p_{1}, \ldots, p_{k}\right)=p_{1} \cdots p_{k}, \quad S_{p}\left(p_{1}, \ldots, p_{k}\right)=1-\left(1-p_{1}\right) \cdots\left(1-p_{k}\right), \\
& T_{L}\left(p_{1}, p_{2}, \ldots, p_{k}\right)=\max \left(p_{1}+p_{2}+\ldots+p_{k}-(k-1), 0\right), \\
& S_{L}\left(p_{1}, p_{2}, \ldots, p_{k}\right)=\min \left(p_{1}+p_{2}+\ldots+p_{k}, 1\right) .
\end{aligned}
$$

## Appendix C. Hamacher t-norm and t-conorm in [0, $\mathbf{1}]^{k}$

In this appendix, by using the notion of additive generator, we give self contained constructions of the extensions of the Hamacher t-norm and t-conorm with $\lambda=0$ to $[0,1]^{k}$.
We recall the notion of an additive generator (if any) of a $t$-norm ([52, 53]).
Definition 11. An additive generator $t:[0,1] \longrightarrow[0, \infty]$ of a $t$-norm $T$ is a strictly decreasing function which is also right continuous in 0 and satisfies $t(1)=0$, such that for all $(x, y) \in[0,1]^{2}$ we have

$$
t(x)+t(y) \in \operatorname{Ran}(t) \cup[t(0), \infty] \text { and } T(x, y)=t^{-1}(t(x)+t(y))
$$

where $\operatorname{Ran}(t)=\{t(x): x \in[0,1]\}$ and $t^{-1}$ is the pseudo inverse of $t$.
If $t$ is an additive generator of some t -norm $T$, then we have

$$
\begin{equation*}
T\left(p_{1}, p_{2}, \ldots, p_{k}\right)=t^{-1}\left(t\left(p_{1}\right)+t\left(p_{2}\right)+\ldots+t\left(p_{k}\right)\right) . \tag{C.1}
\end{equation*}
$$

We first observe that: if $(x=0, y=0)$, then $T_{0}^{H}(x, y)=0$; if $(x=0, y>0)$ or $(x>0, y=0)$, then $T_{0}^{H}(x, y)=$ $\frac{x y}{x+y-x y}=0$; if $(x>0, y>0)$, then

$$
T_{0}^{H}(x, y)=\frac{x y}{x+y-x y}=\frac{x y}{x(1-y)+y(1-x)+x y}=\frac{1}{\frac{1-x}{x}+\frac{1-y}{y}+1}>0 .
$$

Thus, $T_{0}^{H}$ can be equivalently redefined as

$$
T_{0}^{H}(x, y)= \begin{cases}0, & (x=0) \vee(y=0)  \tag{C.2}\\ \frac{1}{\frac{1-x}{x}+\frac{1-y}{y}+1}, & (x \neq 0) \wedge(y \neq 0)\end{cases}
$$

We have (see also [46, 52])

Proposition 3. Let $T_{0}^{H}$ be the Hamacher t -norm with $\lambda=0$. Given an integer $k \geq 2$, the extension of $T_{0}^{H}$ to $[0,1]^{k}$ is

$$
T_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right)= \begin{cases}0, & p_{i}=0 \text { for at least one } i,  \tag{C.3}\\ \frac{1}{\sum_{i=1}^{k} \frac{1-p_{i}}{p_{i}}+1}, & p_{i}>0 \text { for } i=1, \ldots, k\end{cases}
$$

Proof. We observe that, considering the function $t:[0,1] \longrightarrow[0,+\infty]$ defined as $t(x)=\frac{1-x}{x}$, with the convention that $t(0)=\lim _{x \rightarrow 0^{+}} \frac{1-x}{x}=+\infty$, it holds $t^{-1}(s)=\frac{1}{1+s}$, if $s \in[0, \infty]$, with $t^{-1}(+\infty)=0$. Then, by applying the conventions $\frac{1}{\infty}=0, \frac{1}{0}=\infty$ and recalling C. 2 , for every $(x, y) \in[0,1]^{2}$ we have

$$
t^{-1}(t(x)+t(y))=\frac{1}{1+\frac{1-x}{x}+\frac{1-y}{y}}=T_{0}^{H}(x, y)
$$

As the function $t(x)=\frac{1-x}{x}$ is the additive generator of $T_{0}^{H}$, we have

$$
T_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right)=t^{-1}\left(\sum_{i=1}^{k} t\left(p_{i}\right)\right)=t^{-1}\left(\sum_{i=1}^{k} \frac{1-p_{i}}{p_{i}}\right)=\frac{1}{1+\sum_{i=1}^{k} \frac{1-p_{i}}{p_{i}}}
$$

Now, we observe that: if $x=1$ and $y=1$, then $S_{0}^{H}(x, y)=1$; if $(x=1, y<1)$ or $(x<1, y=1)$, then $S_{0}^{H}(x, y)=\frac{x+y-2 x y}{1-x y}=1$; if $x<1$ and $y<1$ we have

$$
\begin{aligned}
& S_{0}^{H}(x, y)=\frac{x+y-2 x y}{1-x y}=\frac{x(1-y)+y(1-x)}{x(1-y)+y(1-x)+(1-x)(1-y)}= \\
& \frac{\frac{x}{(1-x)}(1-x)(1-y)+\frac{y}{(1-y)}(1-x)(1-y)}{\frac{x}{1-x)}(1-x)(1-y)+\frac{1}{(1-y)}(1-x)(1-y)+(1-x)(1-y)}=\frac{\frac{x}{\frac{1}{1}-x}+\frac{y}{(1-y)}}{\frac{1}{1-x)}+\frac{1}{(1-y)}+1}<1 .
\end{aligned}
$$

Thus, the Hamacher t-conorm $S_{0}^{H}:[0,1]^{2} \longrightarrow[0,1]$ can be equivalently redefined as

$$
S_{0}^{H}(x, y)= \begin{cases}1, & (x=1) \vee(y=1)  \tag{C.4}\\ \frac{\frac{x}{(1-x)}+\frac{y}{1(--y)}}{(1-x)}+\frac{1}{1(1-y)}, 1 & (x<1) \wedge(y<1)\end{cases}
$$

By observing that $S\left(p_{1}, p_{2}, \ldots, p_{k}\right)=1-T\left(1-p_{1}, 1-p_{2}, \ldots, 1-p_{k}\right)$, it immediately follows
Proposition 4. Let $S_{0}^{H}$ be the Hamacher t-conorm with $\lambda=0$. Given an integer $k \geq 2$, for any vector $\left(p_{1}, p_{2}, \ldots, p_{k}\right) \in$ $[0,1]^{k}$ it holds that

$$
S_{0}^{H}\left(p_{1}, p_{2}, \ldots, p_{k}\right)= \begin{cases}1, & p_{i}=1 \text { for at least one } i,  \tag{C.5}\\ \frac{\sum_{i=1}^{k} \frac{p_{i}}{1-p_{i}}}{\sum_{i=1}^{k} \frac{p_{i}}{1-p_{i}}+1}, & p_{i}<1 \text { for } i=1, \ldots, k\end{cases}
$$

Remark 8. We observe that the Hamacher t-norm $T_{0}^{H}$ and Hamacher t -conorm $S_{0}^{H}$ coincide, respectively for $\alpha=1$ and $\alpha=-1$, with the Dombi operator defined as ([21]):

$$
o\left(p_{1}, \ldots, p_{k}\right)=\frac{1}{1+\left(\sum_{i=1}^{k}\left(\frac{1-p_{i}}{p_{i}}\right)^{\alpha}\right)^{\frac{1}{\alpha}}}
$$


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[^1]:    ${ }^{1}$ The coincidence between the upper bound $u$ and Hamacher t-conorm $S_{0}^{H}(x, y)$ was noticed by Didier Dubois.

[^2]:    ${ }^{2}$ The representation of a biconditional event as a quasi conjunction was noticed in a private communication between A. Fugard and A. Gilio (January 2010).

