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Convergence Proof of an Enhanced Particle Swarm Optimisation Method Integrated with Evolutionary Game Theory

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Abstract

This paper proposes an enhanced Particle Swarm Optimisation (PSO) algorithm and examines its performance. In the proposed PSO approach, PSO is combined with Evolutionary Game Theory to improve convergence. One of main challenges of such stochastic optimisation algorithms is the difficulty in the theoretical analysis of the convergence and performance. Therefore, this paper analytically investigates the convergence and performance of the proposed PSO algorithm. The analysis results show that convergence speed of the proposed PSO is superior to that of the Standard PSO approach. This paper also develops another algorithm combining the proposed PSO with the Standard PSO algorithm to mitigate the potential premature convergence issue in the proposed PSO algorithm. The combined approach consists of two types of particles, one follows Standard PSO and the other follows the proposed PSO. This enables exploitation of both diversification of the particles' exploration and adaptation of the search direction.

Keywords: Particle Swarm Optimisation, Evolutionary Game Theory, local optimality, convergence proof, convergence speed

1. Introduction

Over the last few decades, many scientists have been inspired by the modelling of social interactions of animals to solve NP-hard optimisation problems. Although the communication among the different agents in social interactions is limited to an exchange of basic information, it results in a very effective team work. Particle Swarm Optimisation (PSO) is one of most well known and established approaches based on this concept. The aim of the original PSO method, which was proposed by Kennedy and Eberhart was to reproduce this social interaction among agents in order to solve non-linear continuous optimisation problems [10] [11]. It is known that PSO not only provides efficient and satisfactory solutions like other meta-heuristic methods [8] [9] [21], but also achieves accurate results comparable

to other search algorithms, such as Genetic Algorithms (GA) [1] [5] [16], for the problems involving unconstrained continuous functions and also more complex and highly constrained problems [7].

Since the performance of the PSO method heavily depends on the three weighting parameters determining interactions between the particles, how to set these parameters has been widely investigated. Shi and Eberhart studied the role played by the inertia weight and the maximum velocity in the conventional PSO algorithm [22]. Later, Meissner *et. al* devised one other way to select the coefficients [18]. The concept presented in [18] is based on optimisation of the free parameters of PSO by having swarms within a swarm. They used an empirical determination of the optimised coefficients by using Neural Networks. Note that classical and state-of-the-art PSO algorithms were comprehensively discussed in recent survey papers like [6, 29].

When designing a new optimisation method, like most of the stochastic algorithms, the convergence remains one of the key questions. The PSO initially proposed by Kennedy and Eberhart in [10] was proved to be a stagnant algorithm, *i.e.*, it does not guarantee the convergence to a local optimal solution [20]. Despite many studies investigating the convergence to a stable point of the PSO [26, 27, 4, 20, 25], optimality of this point was not clearly established until recent investigations from Bergh and Engelbrecht in [28]. In their study, an adapted PSO-based method, called Guaranteed Convergence PSO (GCPSO), was proposed and its local convergence was analytically proved.

Despite the key modification proving the convergence of the GCPSO approach, its relative convergent speed to other methods wasn't investigated. This paper was initiated from the idea that it might be possible to design a method that guarantees convergence to the local optimal solution while improving the convergence speed of the original algorithm. Based on this idea, the we developed an enhanced PSO approach which integrates the Evolutionary Game Theory (EGT) with PSO [12, 13, 15, 14]. Although the performance of this approach has been investigated via numerical simulations based on different sets of benchmark problems, there was no theoretical analysis performed.

This paper proposes a new PSO algorithm where EGT is integrated with Standard PSO (SPSO 2011), which was developed by Clerc in [3]. As the proposed PSO algorithm combines EGT with PSO, it is named Evolutionary Game based Particle Swarm Optimiser (EGPSO). The proposed algorithm maps an Evolutionary Stable Strategy (ESS) obtained from EGT to three coefficients determining the motion of the particles in PSO. Note that integrating EGT with PSO was also proposed in our previous studies [12, 13, 15, 14]. However, the mapping in this paper is newly proposed in order to have theoretical guarantees on the local convergence and superiority on the convergence characteristics.

The performance and convergence properties of the EGPSO algorithm proposed in this paper are theoretically examined. The analysis results show that the proposed EGPSO algorithm guarantees the convergent trajectory of particles and the convergent point is indeed a local optimum. Moreover, it is theoretically proved that the proposed

EGPSO guarantees the convergence speed superiority over SPSO 2011. Note that performance analysis of the algorithm based on numerical simulations is not subject of this study, but of future investigation. The analysis of the local convergence of the proposed algorithm is based on the investigation of Solis and Wets in [23] on the convergence of the stochastic methods like random search techniques.

One of the potential issues with the EGPSO algorithms is that the convergence speed superiority might result in premature convergence to a local optimal solution. To alleviate this convergence issue, this paper also proposes a Combined-EGPSO (C-EGPSO) algorithm where there exist two types of particles, one following the EGPSO algorithm developed in this paper and the other following the SPSO 2011. Since EGPSO provides good convergence characteristics and SPSO 2011 enables efficient exploration of the solution space, the proposed C-EGPSO could leverage intensification and diversification of the search.

The rest of the paper is organised as follows. The next section introduces the mathematical conditions to obtain a convergent algorithm, then the required background to understand the basement of the proposed approach. Section 4 investigates deterministic aspects of the particles' trajectory, before proving local optimality of EGPSO. Then, in Section 6, a novel algorithm based on the investigation of the previous sections is proposed.

2. Mathematical theory of convergence

It is open difficult to prove or disprove properties of stochastic optimisers such as PSO due to their stochastic nature. The convergence of stochastic search algorithms have been thoroughly investigated by Solis and Wets [23]. In their study, conditions for global and local convergence of search algorithms are obtained. Recently, using these conditions, Bergh and Engelbrecht examined the convergence of PSO and proved the local convergence of their proposed PSO, called GCPSO [28]. As this paper extensively uses definitions in [23, 28] to prove the local convergence of the proposed PSO algorithm, we will reproduce relevant definitions below for convenience.

A minimisation problem can be defined as:

Definition 1. *For a measurable function $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $S \subseteq \mathbb{R}^n$, the objective is to determine the point $x \in S$ which minimises f on S where S is a convex set and the image of S by f is also convex, since f is a convex function.*

In Definition 1, S is the search space and f denotes the performance index function. This paper considers this minimisation problem.

2.1. The conditions for local convergence

The local convergence for an algorithm depends on two main conditions. The first one is called the **algorithm condition** which checks whether the algorithm can sample a point in the optimality region. The optimality region

can be defined as the set of points around the optimum of the solution space in which the fitness of the particle is lesser (or greater in the case of maximisation problem) than the optimal solution value plus a strictly positive ε. The second condition is the **convergence condition** that describes the ability of the algorithm to have a non-zero probability to get closer to the optimality region or sample a point within this region at each iteration. Showing that the algorithm satisfies the **algorithm condition** and **convergence condition** is sufficient to prove that the algorithm at least converges to a local minimum.

2.1.1. Algorithm condition

Let D be the mapping that combines the new position of one particle $X(t+1)$ with the current best obtained solution X_g. The optimality region can be defined as:

$$R_\varepsilon = \{x \in S \,|\, f(x) < X^* + \varepsilon\} \tag{1}$$

where X^* denotes the optimal solution of f on S and $\varepsilon > 0$. Note that, as this paper investigates the local convergence, the optimal region R_ε is the region around any local optimum.

Cond1. *The mapping* $D : S \times \mathbb{R}^n \longrightarrow S$ *should satisfy* $f(D(X_g, X(t+1)) \leq f(X_g)$ *and if* $X(t+1) \in S$*, then* $f(D(X_g, X(t+1))) \leq f(X(t+1))$*, where* $t \in \mathbb{N}$.

The condition on the mapping simply stipulates that the solution proposed at current iteration is not worse than the previous ones. In other terms, at each iteration, if the new obtained fitness is better than the previous one, then the new one replaces the best fitness. Otherwise, it remains unchanged.

2.1.2. Convergence condition:

Before introducing the convergence condition, let us first define the distance between a point x and a set Z.

Definition 2. *The distance from a point x to a non-empty set Z is defined such as:*

$$dist(x, Z) = \inf_{z \in Z} (dist\,(x, z)) \tag{2}$$

Now, the convergence condition is described as follows:

Cond2. *Based on the definition of the optimal region* R_ε*, the convergence to a local optimum is obtained by the sufficient condition: for all* $X(t) \in S$*, there exist* $\delta \in \mathbb{R} > 0$ *and* $0 < \eta \in \mathbb{R} < 1$ *such that*

$$\mu_t\,(dist(X(t+1), R_\varepsilon) \leq dist(X(t), R_\varepsilon) - \delta \;\; OR \;\; X(t) \in R_\varepsilon) \geq \eta \tag{3}$$

where μ_t represents a probability measure and $X(t)$ denotes the solution proposed by the optimisation algorithm at the iteration t.

This implies that an algorithm can be considered as an optimisation algorithm if at each iteration there exists a non-zero probability for the particle to get closer to the optimal region by a minimum distance δ, or if x is already in the optimality region with a probability greater or equal than η.

Theorem for Local Search algorithm. The local search algorithm is defined as any search algorithm that guarantees the local convergence, that is local optimality. The condition which a local search algorithm should hold is given in the following theorem.

Theorem 1. *Suppose f is a measurable function, S denotes a measurable subset of $\mathbb{R}^n$, and the convergence condition and the algorithm condition are satisfied. Then, considering the sequence $\{x_t\}_{t=0}^{\infty}$ generated by the algorithm, the following condition holds:*

$$\lim_{t\to\infty} \mu_t(x_t \in R_\varepsilon) = 1 \tag{4}$$

The proof of this theorem can be found in [23].

In order to prove that the proposed EGPSO algorithm is a local search algorithm, this paper will show that it holds Theorem 1 by following analysis steps:

- Prove that the proposed EGPSO satisfies the algorithm condition is satisfied
- Prove that the convergence condition is also satisfied in the EGPSO algorithm by showing:
  - The particles' trajectory is convergent and a particle in any position converges into a convex basin.
  - The particles will keep moving closer to the global best particle.
  - There always exists a non-zero probability to sample a point that improves the quality of the global best particle.

3. EGPSO Algorithm

This section will introduce the EGPSO algorithm developed in this paper. Since EGPSO is based on the SPSO 2011 [3] and EGT, we will first briefly introduce the SPSO 2011 and EGT. Then, this section will describe how the proposed EGPSO combines the two approaches.

3.1. The Standard Particle Swarm Optimiser

The PSO method is based on the modelling of social animal behaviours and the simplicity of their interactions to make up an efficient collective intelligence. This method is now well known for its ability to solve numerous optimisation problems (uni-modal, multi-modal, hybrid problems, etc.). Since this paper investigates the convergence speed of a method based on the SPSO 2011, this section gives brief descriptions of the SPSO 2011 which Clerc designed in [3]. First, the particle definition and also the denotation of the required variables are given, and then the moving rules of the particles are explained.

3.1.1. Description of the SPSO 2011

Let $X_i(t) \in S$ be the position of the i^{th} particle in a population of N particles in a feasible solution space $S \in \mathbb{R}^\Delta$ for $\Delta \in \mathbb{N}$ at time step t. $X_i(t)$ can be represented as $X_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{i\Delta}(t)]$ where $x_{id}(t) \in \mathbb{R}$ for $d = 1, 2, \ldots, \Delta$. The velocity of this particle is denoted as $V_i(t) = [v_{i1}(t), v_{i2}(t), \ldots, v_{i\Delta}(t)]$, $v_{id}(t) \in \mathbb{R}$. Each particle is communicating with its neighbourhood accordingly to a communication network named topology. This network (or topology) plays an important role in the convergence speed and the exploration. For more details about the role of topology, the reader is referred to the studies carried out by R. Mendes in [19] and M. Clerc in [2]. $X_g(t) = [x_{g1}(t), x_{g2}(t), \ldots, x_{g\Delta}(t)]$ denotes the best position among all the informants in the topology up to time step t. Each particle has a memory to save its personal best position, which is defined as the best explored solution up to time step t by itself. This personal best position is denoted as $X_p(t) = [x_{p1}(t), x_{p2}(t), \ldots, x_{p\Delta}(t)]$.

The position of one particle at the time step $t + 1$ is obtained from the three previously described components: the current velocity $V_i(t)$, the personal best position $X_p(t)$ and the global best position $X_g(t)$ among the informants of the particles. Let G denote the iso-barycentre of the particles $X_i(t)$, $\phi_1 X_p(t)$ and $\phi_2 X_g(t)$, where ϕ_1 and ϕ_2 denote two positive real coefficients. The coordinates of the barycentre G can be obtained as:

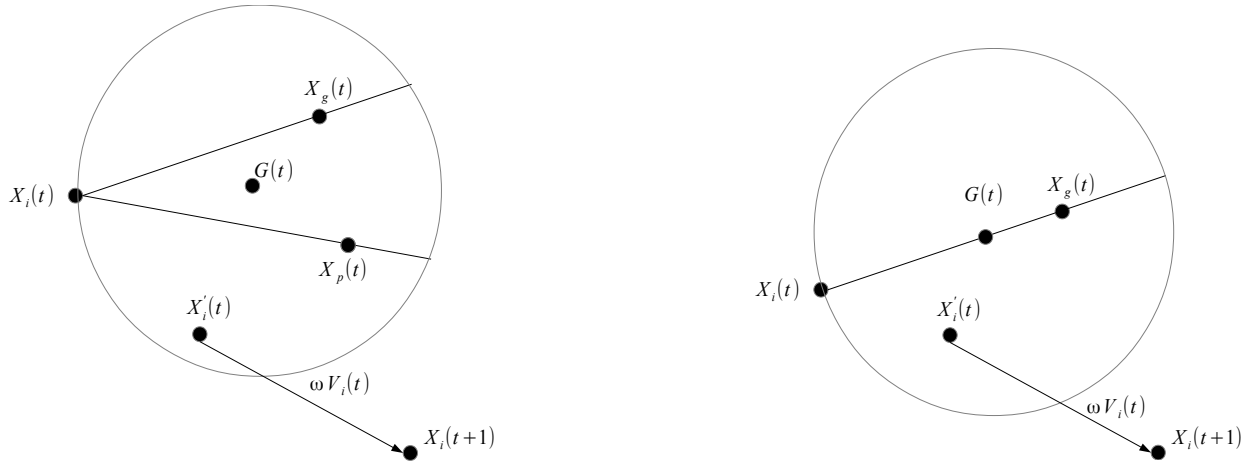
$$G = \frac{X_i(t) + \left[\Big(X_i(t) + \phi_1(X_p(t) - X_i(t))\Big) + \Big(X_i(t) + \phi_2(X_g(t) - X_i(t))\Big)\right]}{3} \tag{5}$$

Then, a point $X_i'(t)$ is randomly drawn in the Hypersphere:

$$H\left(G, \|G - X_i(t)\|\right)$$

which is centred on the barycentre G with a radius equal to $\|G - X_i\|$. This results in the velocity update equation:

$$V_i(t + 1) = \omega V_i(t) + X_i'(t) - X_i(t) \tag{6}$$



(a) Any particle (b) Best particle

Figure 1. Moving rule of the particles. Subfigure 1(a) shows an example of possible sampling for a particle of the swarm that is not the best of its neighbourhood. On the other hand, Subfigure 1(b) shows a possible sampling of a particle that is the best of its neighbourhood. Note that in this second case, the position $X_p(t)$ is ignored and the computation of $G(t)$ only depends on $X_i(t)$ and $X_g(t)$

where ω denotes a real coefficient representing the inertia of the particle. $X_i'(t)$ represents the randomly drawn point in the Hypersphere $H(G, \|G - X_i\|)$ at the instant t.

The position update equation, i.e. the moving rule, is given by:

$$X_i(t+1) = \omega V_i(t) + X_i'(t) \tag{7}$$

Note that in the case where the personal best of a particle is the global best in the neighbourhood, its motion equation is slightly modified. Since the particle positions $X_p(t)$ and $X_g(t)$ are the same, indifferently one of them can be ignored. Then G becomes:

$$G = \frac{X_i(t) + \Big(X_i(t) + \phi_2(X_g(t) - X_i(t))\Big)}{2} \tag{8}$$

Figure 1 shows the moving rules for the particle $X_i(t)$.

Clerc proposes the following values for the three parameters [3]:

$$\begin{cases} \omega &= \dfrac{1}{2\ln(2)} \\ \phi_1 &= \dfrac{1}{2} + \ln(2) \\ \phi_2 &= \dfrac{1}{2} + \ln(2) \end{cases} \tag{9}$$

It is worth to note that in the study proposed by Clerc [3], there is no distinction between ϕ_1 and ϕ_2. However,

since the objective of this study is to optimise the search direction, it was natural to use each of these two parameters as a lever and consider them independently.

3.2. The proposed EGPSO

Since both EGT and PSO are dealing with population evolution, it was natural to mix those two approaches into one to leverage the advantages of the two approaches and to develop an integrated method to efficiently solve optimisation problems. In the classic approach defined in [3], the coefficients are set to specific values, and the initial study of Shi and Eberhart in [22] shows the influence of the coefficients to direct either the exploration or the convergence of the swarm to the optimal solution. The proposed method will enable adaptation of these coefficients using EGT in order to improve the convergence property.

In order to optimise the coefficients ω, ϕ_1 and ϕ_2, the proposed PSO approach uses EGT. In this way, EGPSO is able to determine the optimal ratio of each strategy which is then used to determine the optimal coefficients. For the details of the basic principle of the EGT, the reader is referred to [17, 24]. Each particle is using its own previous experience to fill the payoff matrix, then the Evolutionary Stable Strategy (ESS) provides the optimal ratios that a particle must follow to optimise its exploration and to improve the convergence characteristics.

EGPSO introduces an arbitrarily minimum radius of the hypersphere H, which is described in Section 3.1.1, in order to make the proposed algorithm non-stagnated. If H is reduced to a point, it means that only the current position can be sampled. As consequence, there is no longer exploration and risk of stagnation. This condition alone does not guarantee the non-stagnation, but is necessary. The details on this part will be given in Section 5.3.2. Note that Bergh in [28] also used this update in order to guarantee the non-stagnation of the particles.

3.2.1. The swarm organisation

The number of the particles and the topology of the particles' network should be carefully selected reflecting characteristics of optimal problems. In the EGPSO algorithm, the topology of the communication network among particles is defined the same as that of SPSO 2011 [3]. Each particle informs itself and has a probability $p = \left(1 - (N - 1/N)^K\right)$ to be an informant for the other particles. N denotes the total number of particles and K represents the maximum number of neighbours for one particle, so each particle is informed by at most K other particles, with $K \in [0, N] \cap \mathbb{N}$.

3.2.2. The particle movement

Like SPSO 2011, the state of one particle at the step $t + 1$ is obtained from the three components: the current velocity $V(t)$, its own memory $X_p(t)$ and the best position $X_g(t)$ among the informants of the particles. The notations in this Section are consistent with the ones used in Section 3.1.1. Note that the subscript i on the current velocity is

omitted to simplify the notation. From now on, the notations of the current velocity and position without the subscript will be used unless necessary for clarification.

Movement of each particle is conditioned by the position of the barycentre G and its inertia. Since G depends on the coefficients ϕ_1 and ϕ_2, the search direction can be tuned by adjusting three parameters, ϕ_1, ϕ_2, and ω. Note that the search direction is one of key parameters determining the performance of optimisation algorithms. Therefore, appropriate selection of the three parameters can significantly improve the performance of the SPSO 2011 method. How to optimally adapt these parameters will be described in the next section.

3.2.3. Determination of the coefficients

In order to determine the parameters ω, ϕ_1 and ϕ_2 of SPSO 2011 in an optimal way, we propose to use EGT. This section describes the integration of the EGT within SPSO 2011.

Evolutionary Game Theory. EGT appeared initially in a biologic context. The need to model the evolution phenomena led to the use of mathematical theory of the games to explain the strategic aspect of the evolution. Over the last few decades, the EGT has attracted great attention from the economists, sociologists, social scientists, as well as the philosophers. Although the EGT found its origin in biologic science, such an expansion to different fields can be explained by three facts. First of all, the notion of evolution has to be understood as the change of beliefs and norms over time. Secondly, the modelling of strategies change provides a social aspect which matches exactly the social system interactions. Finally, it was important to dynamically model the interactions within a population, which was one of the missing elements of the classic game theory. As in this former domain, the EGT deals with the equilibrium which is a key point in both of the theories. Here the equilibrium point is called the evolutionary stable strategy. The principle of the EGT is not only based on the strategy performance obtained by itself, but also the performance obtained in the presence of the other strategies.

Evolutionary Stable Strategy An Evolutionary Stable Strategy (ESS) is a strategy such that, if all members of a population adopt it, then no mutant strategy could invade the population under the influence of natural selection. Assume we have a mixed population consisting of mostly p^* individuals (agents playing optimal strategy p^*) with a few individuals using strategy p. Then, the strategy distribution in the population is:

$$(1-\varepsilon)p^* + \varepsilon p$$

where $\varepsilon > 0$ is the small frequency of p users in the population. Let the fitness, i.e. payoff of an individual using strategy q in this mixed population, be

$$\pi(q, (1-\varepsilon)p^* + \varepsilon p).$$

Then, an interpretation of Maynard Smith's requirement [17] for p^* to be an ESS is that, for all $p \neq p^*$,

$$\pi(p, (1-\varepsilon)p^* + \varepsilon p) > \pi(p^*, (1-\varepsilon)p^* + \varepsilon p), \quad \forall \text{ "sufficiently small" } \varepsilon > 0$$

for agents minimising their fitness.

Replicator dynamics A common way to describe strategy interactions is to utilise matrix games. Matrix games are described using notations as follows:

- e_i is the i^{th} unit line vector for $i = 1, ..., m$ where m is the number of pure strategies.
- $A_{ij} = \pi(e_i, e_j)$ is the $m \times m$ payoff matrix.
- $\delta^m \equiv \{p = (p_1, ..., p_m) \mid p_1 + ... + p_m = 1,\ 0 \leq p_i \leq 1\}$ is the set of mixed strategies (probability distributions over the pure strategies e_i).
- $\pi(p, q) = p \cdot Aq^T$ is the payoff of agents playing strategy p facing agents playing strategy q.

Another interpretation of $\pi(p, q)$ is the fitness of a large population of agents playing pure strategies (p describing the agent proportion in each behaviour inside a population) with respect to a large q population.

The replicator equation (RE) is an Ordinary Differential Equation expressing the difference between the fitness of a strategy and the average fitness in the population. Lower payoffs (agents are minimizers) bring faster reproduction in accordance with the Darwinian natural selection process.

$$\dot{p}_i = -p_i(e_i \cdot Ap^T - p \cdot Ap^T)$$

RE for $i = 1, ..., m$ describes the evolution of strategy frequencies p_i. Moreover, for every initial strategy distribution $p(0) \in \delta^m$, there is an unique solution $p(t) \in \delta^m$ for all $t \geq 0$ that satisfies the replicator equation. The replicator equation is the most widely used evolutionary dynamics. It was introduced for matrix games by Taylor and Jonker [24].

The proposed EGPSO combines the EGT principle in determination of the three coefficients of PSO. Based on the EGT principle, the analogy between PSO and EGT is made in Table 1. Each particle is a player having three available

Table 1. Analogy between PSO and EGT

EGT		Analogy in our method
Population	$\longrightarrow$	Swarm
Individual	$\longrightarrow$	Particle
Strategies	$\longrightarrow$	Follow X_p, X_g, V
Payoff matrix	$\longrightarrow$	Mean of the performance obtained by following a specific strategy

strategies (follow his memory, the best neighbour, or moving using only the inertia). These strategies are denoted by e_1, e_2 and e_3. Then, the payoff matrix is defined by:

$$\Pi = \begin{pmatrix} \pi(e_1) & \frac{\pi(e_1)-\pi(e_2)}{2} & \frac{\pi(e_1)-\pi(e_3)}{2} \\ \frac{\pi(e_2)-\pi(e_1)}{2} & \pi(e_2) & \frac{\pi(e_2)-\pi(e_3)}{2} \\ \frac{\pi(e_3)-\pi(e_1)}{2} & \frac{\pi(e_3)-\pi(e_2)}{2} & \pi(e_3) \end{pmatrix} \tag{10}$$

where $\pi(e_i)$, $i \in \{1,\ 2,\ 3\}$ denotes the payoff that a particle can get by following only the strategy i.

This payoff is obtained from the history of the particle. For each particle and at each time step of the algorithm, the mean of the obtained fitness is weighted by the Evolutionary Stable Strategy (ESS). This ESS represents the ratio of each strategy when a stable equilibrium point is reached. In this paper it will be denoted by:

$$E(t) = [Q_1(t), Q_2(t), Q_3(t)] \in\ [0,1]^3 \tag{11}$$

s. t.:

$$(\forall t \in \mathbb{N}) \left(\sum_{i=1}^{3} Q_i(t) = 1 \right) \tag{12}$$

The numerical value of $\pi(e_i) \in \mathbb{R}$, $i \in \{1,\ 2,\ 3\}$ is obtained from the previous experience of the considered particle. If $E(t) = (Q_i(t))$, $i \in \{1,\ 2,\ 3\}$ denotes the evolutionary stable strategy (ESS) in Equation (11) obtained from a given payoff matrix Π, the value of the payoff $\pi(e_i)$ for the strategy i is obtained using:

$$\pi(e_i) = \frac{1}{t} \sum_{k=1}^{t} Q_i(k-1) f(X(k-1)) \tag{13}$$

The first term $\frac{1}{t}$ represents the mean value over the time. The second term, $\sum_{k=1}^{t} Q_i(k-1) f(X(k-1))$ is the sum of the rate of the strategy obtained from the ESS computation multiplied by the obtained cost of the particle X. The

choice of these payoffs can be explained by the fact that it represents the mean value of the obtained gain by following a strategy. Since this strategy can sometimes be mixed, the $Q_i(t)$ term enables to weight the involvement of each strategy in the performance of the particle move.

Once the payoff matrix is filled, the Replicator Dynamic equation is applied to a fictitious population in order to obtain the associated ESS. This ESS is composed by the 3 ratios of each strategy. These ratios will be then used in determination of the coefficients ω, ϕ_1 and ϕ_2. The choice to use these coefficients in the proposed method can be explained by the fact that they represent a stable direction to follow. The stability feature of the ESS is used to face the possible irregularity of the solution space and make the exploration remain unaffected by some local optima. Since the particles are following a sort of instinct developed by their previous experiences they will be able to adapt the 'shape' of the solution space in a certain degree to optimise the search direction.

Mapping Mapping in EGPSO associates the ESS with the three coefficient. This mapping becomes paramount in EGPSO because it determines the convergence properties and performance of the algorithm. This paper proposes two separate sets of mapping depending on the type of particles: the best and non-best particles. For the particle whose personal best is not the global best, mapping from the ESS to each coefficient is a scale-valued function:

$$\begin{cases} M_1 : [0,1] \longrightarrow [0, \omega_{SPSO}] \\ M_2, M_3 : [0,1] \longrightarrow [0,3] \end{cases} \tag{14}$$

The proposed mapping for the non-best particles is given by:

$$\begin{cases} M_1\left(E(t)\right) &= \left(1 - \dfrac{\sigma_{ESS}}{\sigma_{MAX}}\right)\omega_{SPSO} \\ M_2\left(E(t)\right) &= 3Q_2 \\ M_3\left(E(t)\right) &= -3Q_2 + 3 \end{cases} \tag{15}$$

where ω_{SPSO} is the inertia coefficient of the SPSO 2011 algorithm. Here, σ_{MAX} denotes the maximum value for standard deviation of a set of three values that are bounded in $[0, 1]$ and σ_{ESS} represents the standard deviation of the triplet (Q_1, Q_2, Q_3). In the proposed framework, the maximum value of σ_{MAX} is, for example, the standard deviation of the set $\{1, 0, 0\}$ and is approximatively equal to 0.5774.

For the particle of which personal best is identical to the global best, mapping from the ESS to each coefficient is

also a scale-valued function:

$$\begin{cases} M_1 : [0,1] \longrightarrow [0, \omega_{SPSO}] \\ M_2, M_3 : [0,1] \longrightarrow [0,2] \end{cases} \tag{16}$$

In the proposed EGPSO algorithm, these scale-valued functions for the best particles are proposed as:

$$\begin{cases} M_1\left(E(t)\right) &= \left(1 - \dfrac{\sigma_{ESS}}{\sigma_{MAX}}\right)\omega_{SPSO} \\ M_2\left(E(t)\right) &= 2Q_2 \\ M_3\left(E(t)\right) &= -M_2\left(E(t)\right) + 2Q_2 + 2 \end{cases} \tag{17}$$

This mapping is the main difference of the EGPSO proposed in this paper from our previous PSO algorithm in [12, 13, 15, 14]. After thorough investigations, the mapping is designed in a way guaranteeing the local convergence and improving convergence characteristics.

3.2.4. Flowchart of the proposed approach

The algorithm of EGPSO is summarised in the following pseudo-code:

initialisation of the swarm;
while *not exitCondition* **do**
  for *each particle* **do**
    get history of the particle;
    $\Pi \leftarrow$ compute the payoff matrix;
    $E(t) = [Q_1(t), Q_2(t), Q_3(t)] \leftarrow$ solve the replicator equation to obtain the ESS;
    $[\omega, \phi_1, \phi_2] \leftarrow [M_1(E(t)), M_2(E(t)), M(E(t))]$;
    $X_g \leftarrow$ compute the position of X_g using ω, ϕ_1, ϕ_2 as the PSO parameters;
    $X(t+1) \leftarrow$ compute the next position by sampling a random point within the hypersphere H centred on X_g with radius the euclidean distance between $X(t)$ and X_g;
    update the history of the particle;
    if $f(X(t+1)) <$ *personal best* **then**
      personal best $\leftarrow X(t+1)$;
    end
    if $f(X(t+1)) <$ *global best* **then**
      global best $\leftarrow X(t+1)$;
    end
  end
  check exitCondition;
end
return the best particle X_g;

Algorithm 1: Pseudo-code for EGPSO

In the flowchart of Algorithm 1, the following steps are applied:

Initialisation:

Swarm All the particles' positions are randomly drawn in the solution space.

Personal best The personal best position is set as the initial position since no move was performed previously.

Global best A first comparison is done using the initial positions of the neighbouring particles. Then, the global best particles can be identified.

Exit Conditions: The algorithm stops if the maximal number of iterations or maximal allowed computation time is reached.

4. Analysis on the trajectory of the proposed EGPSO

In this section, the trajectory followed by the particles is investigated in order to understand the role played by the coefficients ω, ϕ_1 and ϕ_2 described in Section 3.1.1 and Section 3.2.3. Since SPSO 2011 uses two different moving rules depending on whether or not the particle is the best of its neighbourhood and EGPSO uses the same motion equations as those in SPSO 2011, the two trajectories will be investigated. Note that in this framework, we will only investigate the deterministic trajectory where $X_i'(t) = G$.

4.1. Trajectory of general particles

This section investigates the case where the particle is not the best of its neighbourhood. So, $X_p(t)$ and $X_g(t)$ are distinct. The barycentre is obtained using Equation (5).

4.1.1. Closed form of the trajectory

Since PSO in its general definition is an iterative algorithm, it can therefore be written in a recursive way by developing Equation (7):

$$X(t+1) = \left(\omega + 1 - \frac{\phi_1 + \phi_2}{3}\right)X(t) - \omega X(t-1) + \frac{\phi_1 Xp + \phi_2 Xg}{3} \tag{18}$$

In order to obtain the closed form solution, let us rewrite Equation (18) into a matrix-vector product notation as:

$$\begin{bmatrix} X(t+1) \\ X(t) \\ 1 \end{bmatrix} = \begin{bmatrix} \omega + 1 - \frac{\phi_1+\phi_2}{3} & -\omega & \frac{\phi_1 X_p + \phi_2 X_g}{3} \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X(t) \\ X(t-1) \\ 1 \end{bmatrix} \tag{19}$$

The characteristic equation of Equation (19) is given by:

$$(1-\lambda)\left(\lambda^2 - \lambda(\omega + 1 - \phi) + \omega\right) = 0 \tag{20}$$

where $\phi := (\phi_1 + \phi_2)/3$. The trivial root of Equation (20) is $\lambda = 1$ and the other two roots are obtained as:

$$\lambda_1 = \frac{1 + \omega - \phi - \gamma}{2} \tag{21}$$

$$\lambda_2 = \frac{1 + \omega - \phi + \gamma}{2} \tag{22}$$

where

$$\gamma = \sqrt{(1 + \omega - \phi)^2 - 4\omega} \tag{23}$$

Here, both λ_1 and λ_2 are eigenvalues of the matrix in Equation (19). Therefore, we have:

$$(19) \Longleftrightarrow X(t) = K_1 + K_2\lambda_1^t + K_3\lambda_2^t \tag{24}$$

where $t \in \mathbb{N}$ represents the exponent.

The constants K_1, K_2 and K_3 can be obtained using the initial conditions on the position and velocity at $t = 0$. The velocity at $t = 0$ specifies the position at $t = 1$, $X(1)$ and then $X(2)$ in Equation (18) is obtained as:

$$X(2) = \left(\omega + 1 - \frac{\phi_1 + \phi_2}{3}\right)X(1) - \omega X(0) + \frac{\phi_1 Xp + \phi_2 Xg}{3} \tag{25}$$

From Equation (24), we have:

$$\begin{bmatrix} X(2) \\ X(1) \\ X(0) \end{bmatrix} = \begin{bmatrix} 1 & \lambda_1^2 & \lambda_2^2 \\ 1 & \lambda_1 & \lambda_2 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} \tag{26}$$

By solving the linear system described by Equation (26), the constants K_1, K_2 and K_3 can be obtained as:

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} \dfrac{X_p\phi_1 + X_g\phi_2}{\phi_1 + \phi_2} \\ \dfrac{\lambda_2(X(0) - X(1)) - X(1) + X(2)}{\gamma(\lambda_1 - 1)} \\ -\dfrac{\lambda_1(X(0) - X(1)) - X(1) + X(2)}{\gamma(\lambda_2 - 1)} \end{bmatrix} \tag{27}$$

The aim now is to provide proof that the trajectory can be either divergent or convergent depending on the values of λ_1 and λ_2. This important point is apparent from Equation (24). Since the parameters ω, ϕ_1 and ϕ_2 determine the values of λ_1 and λ_2, how to choose them requires further investigation and will be thus discussed in the subsequent section.

4.1.2. Investigation on the λ_1 and λ_2

Let us examine the set of parameters that guarantees a convergent trajectory. From Equation (24), the following cases can be classified:

- Max $\{|\lambda_1|, |\lambda_2|\} < 1$, which means the trajectory will be convergent
- Min $\{|\lambda_1|, |\lambda_2|\} > 1$, which implies the trajectory will be divergent
- $|\lambda_1| = |\lambda_2| = 1$, the trajectory is neither convergent nor divergent, but circulates around a position that depends on the coefficients K_1, K_2 and K_3.

It is obvious that the trajectory will be convergent if and only if Max $\{|\lambda_1|, |\lambda_2|\} < 1$. Therefore, this paper investigates the range of values of the coefficients, ω, ϕ_1 and ϕ_2, which satisfies this relation.

From Equations (21), (22) and (23), it appears that there could exist values for which λ_1 and λ_2 are complex numbers. The condition that makes λ_1 and λ_2 complex conjugates is given in following Lemma.

Lemma 1. *For a particle X updating its position using Equation (7), λ_1 and λ_2 are complex conjugates iff:*

$$\begin{cases} \phi \in \left](\omega + 1) - 2\sqrt{\omega},\ (\omega + 1) + 2\sqrt{\omega}\right[\\ \omega \geq 0 \end{cases} \tag{28}$$

In other words,

$$\begin{cases} (\phi_1 + \phi_2) \in \left]3(\omega + 1) - 6\sqrt{\omega},\ 3(\omega + 1) + 6\sqrt{\omega}\right[\\ \omega \geq 0 \end{cases} \tag{29}$$

Proof.

$$\gamma \in \mathbb{C} \Longleftrightarrow (1 + \omega - \phi)^2 - 4\omega < 0 \tag{30}$$

If $\omega < 0$, it is obvious that $\gamma \notin \mathbb{C}$. The second degree polynomial in ϕ can be solved using classical approach:

$$(30) \Longleftrightarrow \begin{cases} \phi \in \left](\omega + 1) - 2\sqrt{\omega},\ (\omega + 1) + 2\sqrt{\omega}\right[\\ \omega \geq 0 \end{cases} \tag{31}$$

As $\phi = (\phi_1 + \phi_2)/3$, we have

$$\gamma \in \mathbb{C} \Longleftrightarrow \begin{cases} (\phi_1 + \phi_2) \in \left]3(\omega + 1) - 6\sqrt{\omega},\ 3(\omega + 1) + 6\sqrt{\omega}\right[\\ \omega \geq 0 \end{cases} \tag{32}$$

□

In Lemma 1, *iff* denotes *if and only if*. Note that the absolute value of any complex number can be computed using:

$$(\forall z \in \mathbb{C}) \left(|z| = \sqrt{\Re(z)^2 + \Im(z)^2} \right) \tag{33}$$

where $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of the complex number z, respectively.

Now let us find the conditions on ϕ and ω guaranteeing the convergent trajectory. It is trivial that the particle will have a convergent trajectory *iff* $\text{Max}\,\{|\lambda_1|, |\lambda_2|\} < 1$.

Lemma 2. *For the defined particles' motion in Equation (7), the trajectory of the particle will converge to the point K_1 represented in Equation (27) iff:*

$$\begin{cases} \phi_1 + \phi_2 > 0 \\ \dfrac{1}{6}(\phi_1 + \phi_2) - 1 < \omega < 1 \end{cases} \tag{34}$$

Proof. To find the convergent regions, we need to find the regions holding $\text{Max}\ \{|\lambda_1|, |\lambda_2|\} < 1$.

For $\gamma \in \mathbb{C}$, Equation (28) holds and we have the following equality:

$$\begin{aligned} \text{Max}\ \{|\lambda_1|, |\lambda_2|\} &= |\lambda_1| = |\lambda_2| \\ &= \sqrt{\frac{(1+\omega-\phi)^2 - \left(\sqrt{(1+\omega-\phi)^2 - 4\omega}\right)^2}{4}} \end{aligned} \tag{35}$$

Therefore, we have

$$\begin{aligned} (\text{Max}\ \{|\lambda_1|, |\lambda_2|\} < 1) &\Longleftrightarrow \left(\sqrt{\frac{(1+\omega-\phi)^2 - \left(\sqrt{(1+\omega-\phi)^2 - 4\omega}\right)^2}{4}} < 1 \right) \\ &\Longleftrightarrow \left(\sqrt{\omega} < 1 \right) \end{aligned}$$

Hence, for $\gamma \in \mathbb{C}$, it is clear that

$$(\text{Max}\ \{|\lambda_1|, |\lambda_2|\} < 1) \Longleftrightarrow \begin{cases} 0 \le \omega < 1 \\ (\phi_1 + \phi_2) \in \left]3(\omega+1) - 6\sqrt{\omega},\ 3(\omega+1) + 6\sqrt{\omega}\right[\end{cases} \tag{36}$$

For $\gamma \in \mathbb{R}$, it is obvious that

$$\begin{cases} \phi \in \mathbb{R}, \quad \text{for } \omega < 0 \\ \phi \le (1+\omega) - 2\sqrt{\omega} \quad \text{or} \quad \phi \ge (1+\omega) + 2\sqrt{\omega}, \quad \text{for } \omega \ge 0 \end{cases} \tag{37}$$

And, the condition (Max $\{|\lambda_1|, |\lambda_2|\} < 1$) holds, *iff*

$$-1 < \frac{1 + \omega - \phi \pm \sqrt{(1 + \omega - \phi)^2 - 4\omega}}{2} < 1 \tag{38}$$

Hence:

$$-3 - \omega + \phi < \pm\sqrt{(1 + \omega - \phi)^2 - 4\omega} < 1 - \omega + \phi \tag{39}$$

As $\gamma \in \mathbb{R}$, it is clear that:

$$(39) \Longleftrightarrow \begin{cases} -3 - \omega + \phi < -\sqrt{(1 + \omega - \phi)^2 - 4\omega}, \\ \sqrt{(1 + \omega - \phi)^2 - 4\omega} < 1 - \omega + \phi \end{cases} \tag{40}$$

Therefore, for $\gamma \in \mathbb{R}$

$$(38) \Longleftrightarrow -3 < \omega - \phi < 1, \quad 2\omega - \phi + 2 > 0, \quad \phi > 0 \tag{41}$$

Considering both the cases of $\gamma \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ together, we have

$$(\text{Max } \{|\lambda_1|, |\lambda_2|\} < 1) \iff \begin{cases} \phi > 0 \\ 0.5\phi - 1 < \omega < 1 \end{cases} \tag{42}$$

Since $\phi := (\phi_1 + \phi_2)/3$, it is clear that

$$(\text{Max } \{|\lambda_1|, |\lambda_2|\} < 1) \iff \begin{cases} \phi_1 + \phi_2 > 0 \\ \frac{1}{6}(\phi_1 + \phi_2) - 1 < \omega < 1 \end{cases}$$

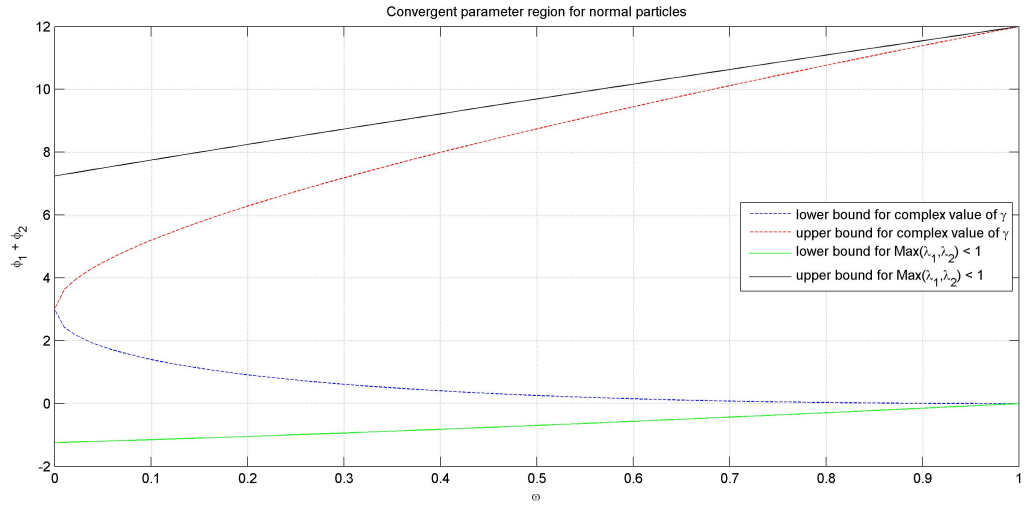
□

Figure 2(a) shows both regions. First the region in which λ_1 and λ_2 are complexes. Second, the region that satisfies $|\lambda_1| = |\lambda_2| < 1$. Note that the first region is included in the second one for $\omega < 1$. From Equation (24), the relation Max $\{|\lambda_1|, |\lambda_2|\} < 1$ implies that the trajectory of the particle $X(t)$ is convergent and will converge to the point $K_1 = \dfrac{\phi_1 X_p + \phi_2 Xg}{\phi_1 + \phi_2}$.

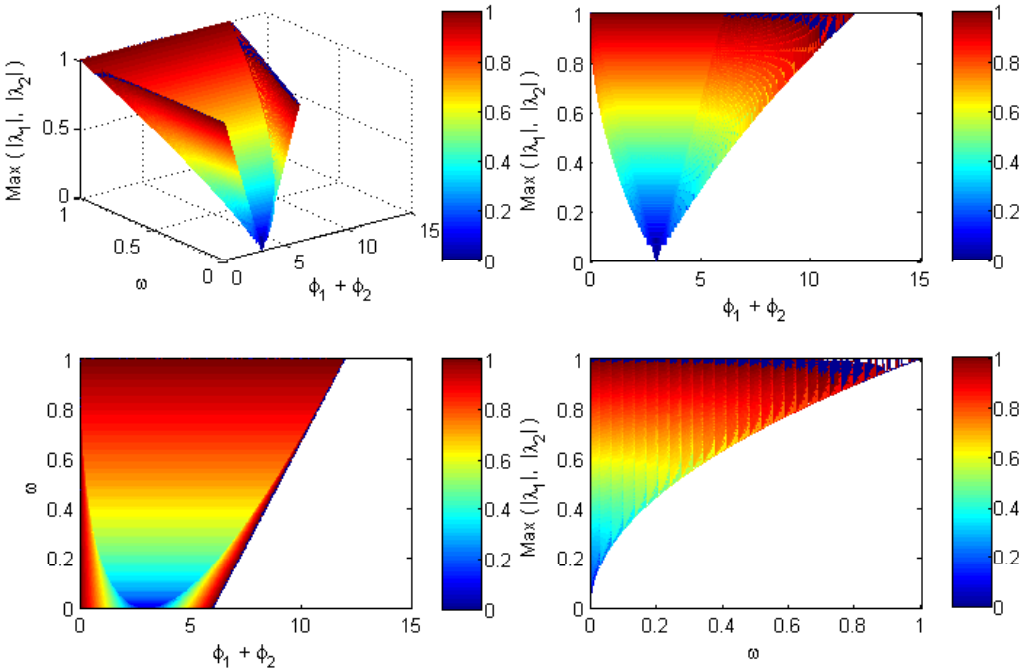
Indeed,

$$|\lambda_1| = |\lambda_2| < 1 \Longrightarrow \lim_{t \to +\infty} |\lambda_1|^t = \lim_{t \to +\infty} |\lambda_2|^t = 0 \tag{43}$$

Thus, it is proved that for any choice of the parameters that fall into this convergent region, the trajectory will be convergent. Note that within this region, the absolute value of λ_1 and λ_2 varies. Figure 2(b) shows the heat-map of



(a) Area into which λ_1 and λ_2 have a absolute value lesser than 1. Therefore, whatever the combination of the parameters ω, ϕ_1 and ϕ_2 the trajectory will be always convergent.



(b) Heat-map of Max $\{|\lambda_1|, |\lambda_2|\}$. Each graph is a projection of the upper left graph on the axis.

Figure 2. Convergent parameter region and heat map of the absolute value of λ_1 and λ_2

the absolute value of Max $\{|\lambda_1|, |\lambda_2|\}$.

4.2. Trajectory of the best particle in the neighbourhood

This section investigates the case where the personal best becomes identical to the global best in the neighbourhood. The motion equation of the particle will be therefore obtained by computing first the barycentre G using Equation (8) and then applying the motion equation shown in Equation (7).

4.2.1. Closed form of the trajectory

Following the similar steps as in Section 4.1, the closed form solution is obtained from the iterative form:

$$X(t+1) = \left(\omega + 1 - \frac{\phi_2}{2}\right) X(t) - \omega X(t-1) + \frac{\phi_2}{2} X_g \tag{44}$$

Leading to:

$$X(t) = K'_1 + K'_2 \lambda'^t_1 + K'_3 \lambda'^t_2 \tag{45}$$

with,

$$K'_1 = X_g \tag{46}$$

$$\lambda'_1 = \frac{(1 + \omega - \phi') - \gamma'}{2} \tag{47}$$

$$\lambda'_2 = \frac{(1 + \omega - \phi') + \gamma'}{2} \tag{48}$$

where $\phi' := \phi_2/2$ and

$$\gamma' = \sqrt{(1 + \omega - \phi')^2 - 4\omega} \tag{49}$$

Like in the previous section, it will be necessary to investigate the values of $|\lambda_1|$ and $|\lambda_2|$ in order to determine the convergence conditions for the particles.

4.2.2. Investigation on the λ'_1 and λ'_2

This section examines the set of parameters that guarantees a convergent trajectory when $X_p(t) = X_g(t)$. From Equation (45), it is clear that the trajectory will be convergent *iff* Max $\{|\lambda_1|, |\lambda_2|\} < 1$. Therefore, the parameters that satisfy this relation are investigated in order to determine the bounds of the coefficients, in this case only ω and ϕ_2.

Lemma 3. *For a particle X updating its position using Equation (7), λ_1 and λ_2 are complex conjugates iff:*

$$\phi_2 \in \left]\, 2(\omega + 1) - 4\sqrt{\omega},\ 2(\omega + 1) + 4\sqrt{\omega} \,\right[\tag{50}$$

Proof. Following the same process as for Lemma 1, λ'_1 and λ'_2 are complex conjugates *iff*:

$$\phi' \in \left](\omega + 1) - 2\sqrt{\omega},\ (\omega + 1) + 2\sqrt{\omega}\right[\tag{51}$$

Since $\phi' := \phi_2/2$, it is clear that

$$\phi_2 \in \left]2(\omega + 1) - 4\sqrt{\omega},\ 2(\omega + 1) + 4\sqrt{\omega}\right[\tag{52}$$

□

Lemma 4. *Following the particles' motion in Equation (7), the particle will have a convergent trajectory to the point K'_1 defined in Equation (46) iff:*

$$\begin{cases} \phi_2 > 0 \\ \dfrac{\phi_2}{4} - 1 < \omega < 1 \end{cases} \tag{53}$$

Proof. Note that, as shown from Equation (45), it is trivial that the particle $X(t)$ is converging to $K'_1 = X_g$ *iff* the condition Max $\left\{|\lambda'_1|, |\lambda'_2|\right\} < 1$ holds.

Following the same derivation procedures in Lemma 2, we have

$$(\text{Max } \{|\lambda'_1|, |\lambda'_2|\} < 1) \iff \begin{cases} \phi' > 0 \\ 0.5\phi' - 1 < \omega < 1 \end{cases} \tag{54}$$

As $\phi' := \phi_2/2$, it is clear that

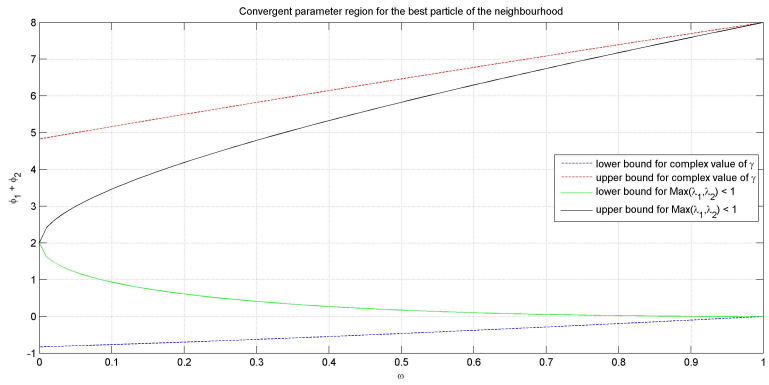
$$(\text{Max } \{|\lambda'_1|, |\lambda'_2|\} < 1) \iff \begin{cases} \phi_2 > 0 \\ \dfrac{\phi_2}{4} - 1 < \omega < 1 \end{cases}$$

□

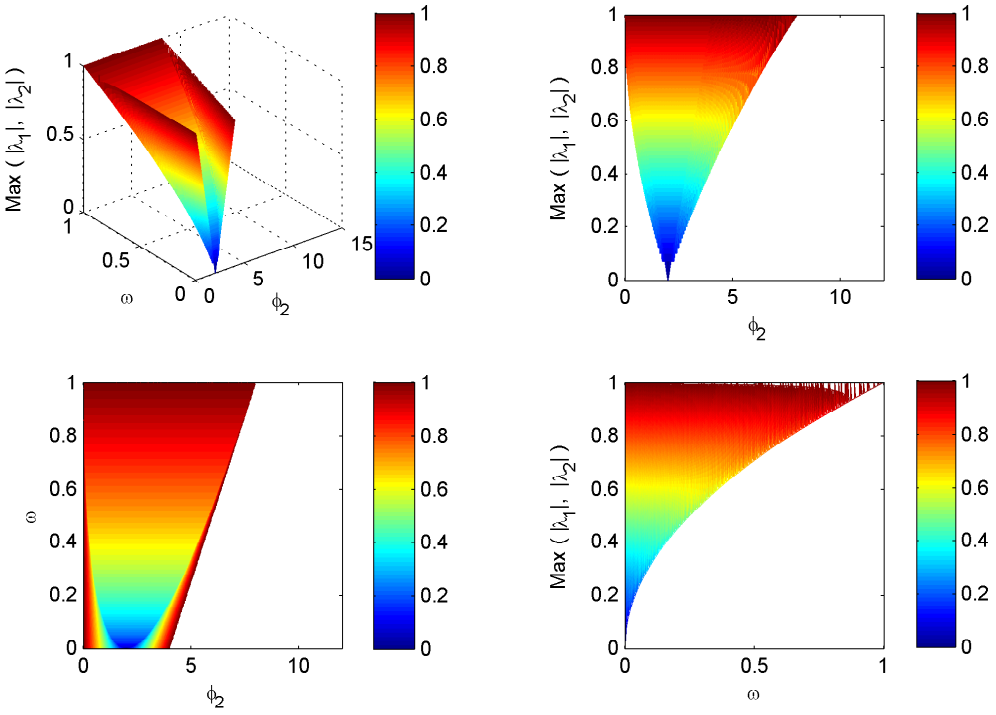
Figure 3(b) shows the heat-map of the absolute value of Max $\left\{|\lambda'_1|, |\lambda'_2|\right\}$.

4.3. A guaranteed faster convergence

In the previous section, the region of the values of λ_1 and λ_2 that guarantees the convergence of the trajectory were investigated. This section will first show that the parameters chosen by Clerc in [3] is consistent with this analysis. Then, it will be proved that the parameters ω, ϕ_1 and ϕ_2 determined by the EGPSO algorithm proposed in this paper



(a) Area into which λ'_1 and λ'_2 have a absolute value lesser than 1. Therefore, whatever the combination of the parameters ω, ϕ_1 and ϕ_2 the trajectory will be always convergent. Note that this region slightly differs from the one obtained in the case of a particle that is not the best in the neighbourhood.



(b) Heat-map of Max $\left\{|\lambda'_1|, |\lambda'_2|\right\}$. Each graph is a projection of the upper left graph on the axis.

Figure 3. Convergent parameter region and heat map of the absolute value of λ_1 and λ_2

guarantees the convergent trajectory, as well as its convergent speed is superior to that of the original SPSO 2011 algorithm.

Considering the deterministic trajectory of the particles, the moving rules still are the same as SPSO 2011 since the barycentre G is identical to that in SPSO 2011. However, as mentioned in Section 3.2.3, the proposed EGPSO redefines the way to determine the values of the three coefficients ω, ϕ_1 and ϕ_2 in order not only to guarantee the convergent trajectory, but also to optimise the search direction using the EGT. The analysis in this section will be divided into two parts depending on the types of the particles: the particle that is not the best in the neighbourhood and the best particles among neighbours.

4.3.1. Non-best particles

In the two previous subsections the deterministic trajectory of the particles was described. For convenience the results are recalled hereafter:

$$X(t) = K_1 + K_2\lambda_1^t + K_3\lambda_2^t$$

with,

$$\begin{cases} \lambda_1 = \dfrac{1+\omega-\phi-\gamma}{2} \\ \lambda_2 = \dfrac{1+\omega-\phi+\gamma}{2} \end{cases}$$

$$\begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix} = \begin{bmatrix} \dfrac{X_p\phi_1 + X_g\phi_2}{\phi_1+\phi_2} \\ \dfrac{\lambda_2(X(0)-X(1)) - X(1) + X(2)}{\gamma(\lambda_1 - 1)} \\ -\dfrac{\lambda_1(X(0)-X(1)) - X(1) + X(2)}{\gamma(\lambda_2 - 1)} \end{bmatrix}$$

where

$$\gamma = \sqrt{\left(1+\omega-\frac{\phi_1+\phi_2}{3}\right)^2 - 4\omega}$$

Since the proposed EGPSO algorithm adapts the determination of the coefficients, ω, ϕ_1 and ϕ_2, to optimise the search direction, an essential point to be verified is whether or not the triplet $[\omega, \phi_1, \phi_2]$ falls into the convergent parameter region. Note that, from Equation (24), the trajectory will be convergent if and only if Max $\{|\lambda_1|, |\lambda_2|\} < 1$. In order to define a mapping that holds this convergence condition, the values of $|\lambda_1|$ and $|\lambda_2|$ have to be investigated.

Lemma 5. *Let X_1 and X_2 be two particles with the same initial conditions. Suppose that the coefficients α_1, α_2, β_1 and β_2 are two pairs of complex conjugates satisfying the convergence condition given in Lemma 2. Then, the motion*

equations of the particles are given by:

$$\begin{cases} X_1(t) = K_1 + K_2\alpha_1^t + K_3\alpha_2^t \\ X_2(t) = K_1 + K_2\beta_1^t + K_3\beta_2^t \end{cases} \tag{55}$$

If $|\alpha_1| = |\alpha_2| < |\beta_1| = |\beta_2|$*, the convergence speed of the particle* X_1 *is superior to that of the particle* X_2*.*

Proof. As both the pairs of α_1 and α_2, and β_1 and β_2 satisfy the convergence condition in Lemma 2, both X_1 and X_2 are converging to K_1. Moreover, it is clear that smaller the absolute value is, the faster the particle $X(t)$ will converge to K_1 from Equation (24). Therefore, if $|\alpha_1| = |\alpha_2| < |\beta_1| = |\beta_2|$, X_1 converges to K_1 faster than X_2. □

The convergence speed comparison between this EGPSO approach and the SPSO 2011 for non-best particles is given in following Lemma.

Theorem 2. *The non-best particles following the EGPSO mapping given in Equation (15) have the convergent trajectory. Furthermore, the convergence speed of the EGPSO algorithm is superior to that of the SPSO 2011 algorithm if the three parameters follow the EGPSO mapping defined in Equation (15).*

Proof. As $\phi_1 + \phi_2 = 3$ and $\omega \in [0, \omega_{SPSO}]$ where $\omega_{SPSO} < 1$, these three parameters fall into the convergent region defined in Equation (34). From Lemma 2, the non-best particles in EGPSO are convergent. Moreover, since these parameters hold Equation (29), from Lemma 1, λ_1 and λ_2 are complex conjugate to each other. Therefore, the proof will be completed by showing that $|\lambda_{EGPSO}| := |\lambda_1|_{EGPSO} < |\lambda_{SPSO}| := |\lambda_1|_{SPSO}$ according to Lemma 5.

Since λ_1 and λ_2 in the EGPSO algorithm are complex conjugates, their magnitudes are obtained as

$$\begin{aligned} |\lambda_1|_{EGPSO} = |\lambda_2|_{EGPSO} &= \sqrt{\frac{\left(1 + \omega_{EGPSO} - \frac{\phi_1+\phi_2}{3}\right)^2 - \left(\sqrt{(1 + \omega_{EGPSO} - \frac{\phi_1+\phi_2}{3})^2 - 4\omega}\right)^2}{4}} \\ &= \sqrt{\omega_{EGPSO}} \\ &= \sqrt{\left(1 - \frac{\sigma_{ESS}}{\sigma_{MAX}}\right)\omega_{SPSO}} \end{aligned} \tag{56}$$

Since

$$0 \leq \left(1 - \frac{\sigma_{ESS}}{\sigma_{MAX}}\right) < 1 \tag{57}$$

it is clear that $\sqrt{\omega_{EGPSO}} < \sqrt{\omega_{SPSO}}$.

Note that the λ_1 and λ_2 in the SPSO 2011 are also complex conjugates and their magnitudes are given as:

$$|\lambda_1|_{SPSO} = |\lambda_2|_{SPSO} = \sqrt{\omega_{SPSO}} \simeq 0.8493 \tag{58}$$

Therefore, it is obvious that $|\lambda_{EGPSO}| < |\lambda_{SPSO}|$. □

Note that, if λ_1 and λ_2 are complex conjugates, the smaller their absolute values are, the faster the particle $X(t)$ will converge to the point K_1 as defined in Equation (24). However, if an absolute value is reduced to 0, the particle will "jump" directly to the desired position without exploration of the surroundings. This might lead to a premature convergence and is not an efficient global search strategy. That explains why a trade-off is required.

The idea to weight ω_{SPSO} by $\left(1 - \frac{\sigma_{ESS}}{\sigma_{MAX}}\right)$ for the inertia component in the EGPSO mapping, i.e. $M_1(E(t))$, comes from the principle that a pure strategy has a high standard deviation. Therefore, the strategy related to the convergent point is likely dominant over the other two which implies that it could be beneficial to go "fast" to the convergent point rather than exploring around the convergent point to check if another convergent point can be found. However, for a mixed strategy in which all strategies have the similar ratio means, the solution space could be better explored by having slower convergence speed to the convergent point. Indeed, whereas an absolute value close to zero enables a fast convergence, an absolute value close to ω_{SPSO} enables the particles to go around the convergent position to explore the solution space.

4.3.2. Best particle

Let us now investigate the deterministic trajectory of the best particle in the neighbourhood. For convenience, the analysis results in Section 4.2 are recalled hereafter:

$$X(t) = K'_1 + K'_2\lambda_1'^t + K'_3\lambda_2'^t$$

with,

$$\lambda'_1 = \frac{(1 + \omega - \phi') - \gamma'}{2}$$

$$\lambda'_2 = \frac{(1 + \omega - \phi') + \gamma'}{2}$$

$$\begin{bmatrix} K'_1 \\ K'_2 \\ K'_3 \end{bmatrix} = \begin{bmatrix} X_g \\ \dfrac{\lambda'_2(X(0) - X(1)) - X(1) + X(2)}{\gamma'(\lambda'_1 - 1)} \\ -\dfrac{\lambda'_1(X(0) - X(1)) - X(1) + X(2)}{\gamma'(\lambda'_2 - 1)} \end{bmatrix}$$

where $\phi' := \phi_2/2$ and

$$\gamma' = \sqrt{(1 + \omega - \phi')^2 - 4\omega}$$

The mapping from the ESS to the coefficients for the best particle is given in Equation (17).

Lemma 6. *Suppose that two pairs of the coefficients* $\{\alpha'_1, \alpha'_2\}$ *and* $\{\beta'_1, \beta'_2\}$ *are complex conjugate pairs and they hold the convergence condition shown in Lemma 4. The motion equations of the particles* X_1 *and* X_2 *are represented as:*

$$\begin{cases} X_1(t) = K'_1 + K'_2\alpha'^t_1 + K'_3\alpha'^t_2 \\ X_2(t) = K'_1 + K'_2\beta'^t_1 + K'_3\beta'^t_2 \end{cases} \tag{59}$$

Then, if $|\alpha'_1| = |\alpha'_2| < |\beta'_1| = |\beta'_2|$*, the particle* X_1 *converges to the point* K'_1 *faster than the particle* X_2*.*

Proof. As both the two pairs meet the convergence condition given in Lemma 4, both X_1 and X_2 are converging to K'_1. If $|\alpha'_1| = |\alpha'_2| < |\beta'_1| = |\beta'_2|$, it is then trivial that the convergence speed of X_1 is superior to that of X_2. □

Theorem 3. *The trajectory of the best particles following the EGPSO mapping given in Equation (17) is convergent. Moreover, the convergence speed of the best particles in the EGPSO algorithm is superior to that in the SPSO 2011 algorithm if the particle follows the mapping shown in Equation (17).*

Proof. Following the mapping defined for the best particles in Equation (17), we have $\phi_2 = 2$ and $\omega \in [0, \omega_{SPSO}]$ which fall inside the convergent region defined in Equation (53). Therefore, from Lemma 4, the best particles have the convergent trajectory. These parameters also hold Equation (29) in Lemma 3 which implies that λ_1 and λ_2 are complex conjugates. Therefore, like in Theorem 2, we can complete the proof by showing $|\lambda'_{EGPSO}| := |\lambda'_1|_{EGPSO} < |\lambda'_{SPSO}| := |\lambda'_1|_{SPSO}$ according to Lemma 6.

The absolute values of the complex conjugates λ'_1 and λ'_2 are given by

$$|\lambda'_1|_{EGPSO} = \sqrt{\omega_{EGPSO}} \tag{60}$$

From Equation (57), it is clear that $\omega_{EGPSO} < \omega_{SPSO}$. Note that the three coefficients in SPSO 2011 are defined in Equation (9). Then, it is clear that λ'_1 and λ'_2 in the EGPSO algorithm are complex conjugates and their magnitudes are smaller than $\sqrt{\omega_{SPSO}} \simeq 0.8493$. Hence, it is trivial that $|\lambda'_{EGPSO}| < |\lambda'_{SPSO}|$. □

4.4. Conclusion

This section used the closed form solution of the trajectory in order to investigate the values of ω, ϕ_1 and ϕ_2 that guarantee a convergent trajectory. It is proved that the particle tends to converge as the time becomes infinity as long as the set of parameters ω, ϕ_1 and ϕ_2 satisfy Equation (34) or Equation (53) depending on type of particles. Furthermore, it was proved that the proposed set of parameters ω, ϕ_1 and ϕ_2 in the EGPSO algorithm guarantees a faster convergence than the SPSO 2011. Note that it hasn't been proved if the convergent point is optimal.

The investigation on convergence of the trajectory does not guarantee the optimality of EGPSO by itself, but it guarantees that from any position $X \in S$, the particle has the ability to converge within the convergent basin since it was assumed in Section 2 that the solution space S and its image by the function f are convex.

Note that these investigations are similar to the studies presented in [23] and [27], but they have been further updated for rigorous mathematical analysis of the proposed EGPSO.

5. Local convergence proof of EGPSO

In the previous section, the trajectory of the particles was investigated in order to prove that whatever the position of a particle X on the solution space S, it will always converge to the barycentre of the particles X_g and X_p. Since X_g and X_p are located in the convex convergent basin, it will always be possible to enter in this basin.

In most stochastic algorithms, it is hard to prove whether or not the algorithm converges to the optimum solution or at least to a satisfying solution in the optimal region. Solis and Wets in [23] investigated the convergence properties of stochastic methods and especially of the pure random search algorithm. They studied under which conditions the method can be either considered as a global search algorithm or a local search one. Later, Bergh and Engelbrecht investigated the PSO and proved that the initial PSO proposed by Clerc and Kennedy have a non-zero probability to stagnate on a point that is not optimal [28]. In this section, the local convergence property of the EGPSO algorithm is investigated for unimodal problems.

In Section 2 the mathematical conditions to have a local search algorithm were described. This paper proposes to prove that EGPSO is a local search algorithm. As the first step, this paper showed that the particles composing the swarm in EGPSO are convergent. Since we cannot have any guarantee about the optimality of the convergence point, the non-stagnation property of EGPSO is used to prove that the particles in the swarm keep moving. Then, the existence of a non-degenerated sampling hypersphere H will be investigated. This section ends with the proof that all the conditions for local convergence are satisfied in the proposed EGPSO algorithm.

For convenience, some notations and definitions from [23] and [28] are reused in the subsequent sections.

5.1. Mathematical model

As stated, only unimodal minimisation problems are considered in this paper, since the local convergence of the proposed algorithm is investigated.

Let us consider a swarm of particles in which X_0 represents the worst particle. In a minimisation problem, the

worst particle means that its fitness defined by the function f is the largest in the swarm, so X_0 is defined as

$$X_0 = \underset{X_i}{\operatorname{argmax}}\{f(X_i)\}, \quad \text{for } 1 \leq i \in \mathbb{N} \leq N \tag{61}$$

where N is the number of particles in the swarm.

From this particle, the convex compact set L_0 can be defined as the convex compact set in which all the points have a value smaller than or equal to $f(X_0)$:

$$L_0 = \{X \in S \,:\, f(X) \leq f(X_0)\} \tag{62}$$

5.2. Algorithm condition

First, the algorithm condition can be validated by using the mapping used in EGPSO to combine a current particle with a new sampled particle. In the EGPSO algorithm, this mapping D can be expressed as:

$$D(X_g(t), X_i(t)) = \begin{cases} X_g(t) & , \text{ if } f(g(X_i(t))) \geq f(X_g(t)) \\ g(X_i(t)) & , \text{ if } f(g(X_i(t))) < f(X_g(t)) \end{cases} \tag{63}$$

The function g is defined as the moving equation of EGPSO. In other words, the function g denote the link between $X(t)$ and $X(t+1)$ as explained in Section 3. Therefore, the obtained sequence $\{f(X_g(t))\}_{t=0}^{\infty}$ is monotone. Hence, the necessary algorithm condition described in Section 2.1.1 is satisfied.

5.3. Convergence condition

In order to prove that the proposed EGPSO satisfies the convergence condition, this paper investigates several points. First, the non-stagnation property of EGPSO is proved. Then, it is shown that at each iteration there exists a non-zero probability to decrease the fitness of X_g. This, with the non-stagnation property, verifies the convergence condition, **Cond2**, defined in Section 2.1.2.

5.3.1. Convergent trajectory

By the definition of L_0, it can be assessed that X_p and X_g are in L_0. Moreover, in Section 4.3.1, it was proved that the setting of the parameters ω, ϕ_1 and ϕ_2 using EGT guarantees that the particles' trajectory is always convergent to a barycentre of the particles X_p and X_g. Although the optimality of this point cannot be stated from this proof, it allows us at least to conclude that, from any position $X \in S$, the particle can enter in the convex convergence basin L_0 where X_p and X_g are located.

5.3.2. The non-stagnation of the algorithm

The stagnation of an algorithm can be defined as the state in which the particles can no longer improve their fitness since the velocity of the particle converges to zero. As a consequence, the sequence $\{f(X_g(t))\}_{t=0}^{\infty}$ is monotone and constant since the particles are no longer exploring the solution space.

The non-stagnation property can be achieved by imposing an arbitrarily minimum radius to the sampling hypersphere H. In this case, since the particle will be always exploring the surrounding of the explored area and the velocity will never becomes null, the stagnation is prevented . Note that Bergh used the same method in order to guarantee the non-stagnation of the particles in [28].

5.3.3. Existence of non-degenerated sampling hypersphere

In this section, it will be shown that EGPSO can sample a new point arbitrarily close to X_g and X_p, hence in L_0. In Section 4, it was proved that the trajectory is converging to the point K_1 described in Equation (27). Since L_0 is convex, then any point along the line connecting X_p to X_g is in L_0. From Equation (24), it is trivial that a new point arbitrarily closer to K_1 can be sampled, hence in L_0. Note that in the case where the particle was the best particle in the neighbourhood, Equation (45) proved that the convergence point will be X_g itself, and will be also in L_0.

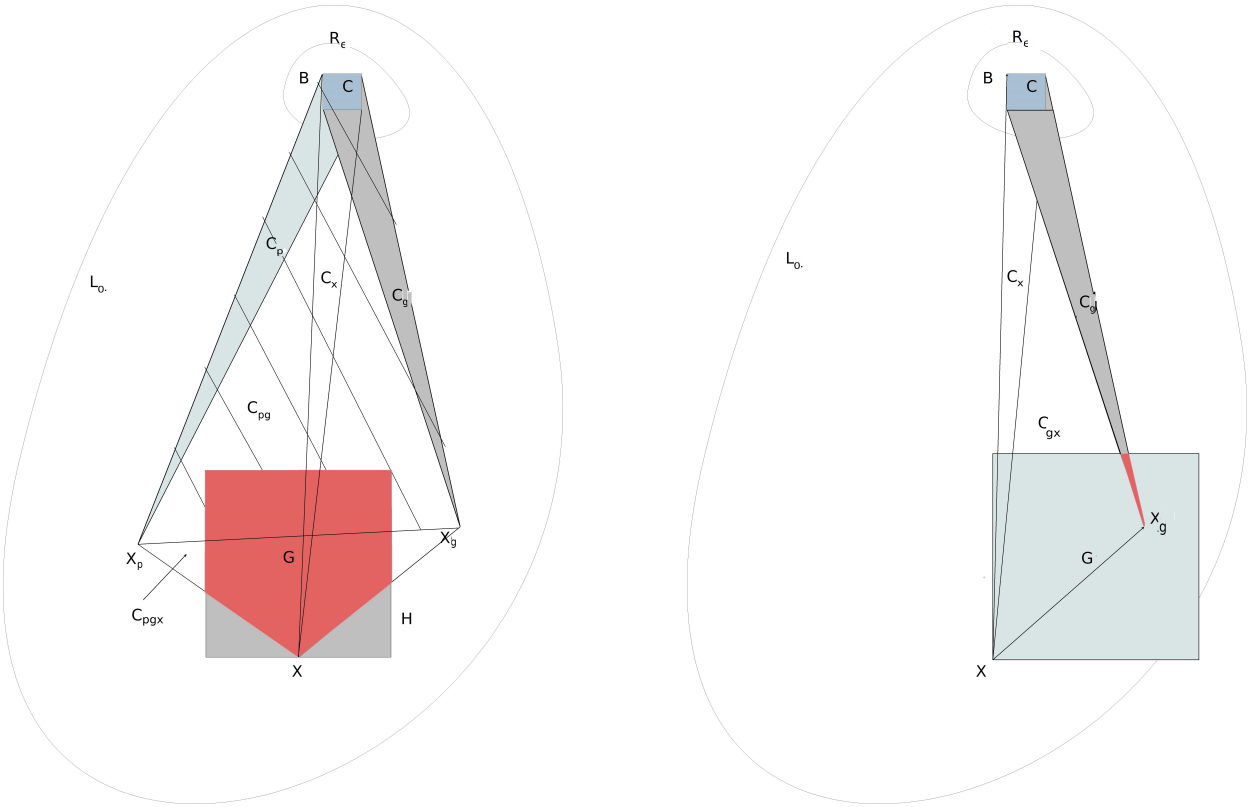
Hence, it is proved that there always exists a non-degenerated sampling hypersphere that guarantees that a new point can be sampled arbitrarily close to either K_1 or K_1', which is in L_0. The local convergence condition can now be investigated in order to finalise this local convergence proof.

5.3.4. Verification of the convergence condition

It is proved that from any position in S the particle will converge in the convergence basin L_0, and that the proposed EGPSO is a non-stagnant algorithm. Hence, proving that there always exists a non-zero probability to decrease the fitness of the best particle will be sufficient to guarantee that the algorithm will converge to the optimal region R_ε with a probability of 1, as described by the convergence condition, **Cond2**.

In order to prove this local convergence, two cases need to be investigated: the first case for the non-best particles and the second for the best particles.

Let S be a non-empty compact set. Then, L_0 is also compact since $L_0 \subset S$. The optimality region R_ε include the optimal point X^*. Let B be the ball centred in c and included in R_ε. The set C_g denotes the convex subset that includes R_ε and the best particle, X_g. Another set is obtained with the best obtained position X_p and is denoted by C_p. The cone C_x has the same definition with the current particle position. The convex hull C_{pg} covers B, C_g and C_p. Finally, C_{pgx} is the convex volume that includes B and passes by X_p, X_g and X.



(a) Possible configuration for sampling a new point. The best particle of the swarm is denoted by X_g, the best obtained position of the particle is X_p. The sampling volume admits G as its centre and is coloured in grey, the current particle X lies on the border of the sampling volume. The convex hulls C_p, C_g and C_{pg} (hatched volume), C_x and C_{pgx} are defined in Section 5.3.4.

(b) Possible configuration for sampling a new point. The best particle of the swarm is denoted by X_g. The sampling volume is denoted by H and admits G as its centre, the current particle X lies on the border of the sampling volume H. The red region shows the intersection of H with C_g that guarantees an improvement of the solution quality.

Figure 4. Probability to improve the fitness of the best particle in the neighbourhood

Local convergence of general particles. In the presented case, the improvement of the fitness function cannot be guaranteed since only the region delimited by the convex hull C_g guarantees to obtain a better solution than X_g itself. Since there is no intersection between the sampling hypersphere H and C_g, it can be immediately concluded that there exist some initial conditions where the convergence condition cannot be always satisfied. Thus, it is necessary to prove that, when the particle is the best in the neighbourhood, it is always guaranteed that the intersection of H and C_g is never reduced to an empty set. This is a sufficient condition to meet the convergence condition.

Figure 4(a) shows a possible configuration that represents this case. Note that for any point $Y \in C_x$ and $X \neq Y$, the following relation will be always satisfied: $f(Y) < f(X)$ in a minimisation problem. This can be generalised to any of the aforementioned convex sets.

Local convergence of best neighbourhood's particles. The same analysis can be done for the best particle. Since the moving rule slightly differs from the one used for the other type of particles, it has to be investigated whether a particle, which becomes the best in the neighbourhood, guarantees convergence to the optimal region with a probability of 1.

Following the same reasoning of the first case, the aim is to prove that there exists a non-empty intersection hypervolume between H and C_g. From the definition of H, the centre of the hypersphere G is the barycentre of X and a point located between $X + (X_g + \rho)$ and $X + 2(X_g - X)$, which is shown in Figure 5. Here, ρ denotes an arbitrary small real value. Note that the hypersphere H is the same as defined in Section 3. C_g denotes the cone of edge X and basis B', the unit ball. Thus, by guaranteeing that $X_g \in H$ there always exists a non-empty intersection between the sampling hypersphere H and the convex set C_g. Note that it is guaranteed that any point $Y \in C_g$ satisfies $f(Y) < f(X_g)$. Figure 4(b) shows this case.

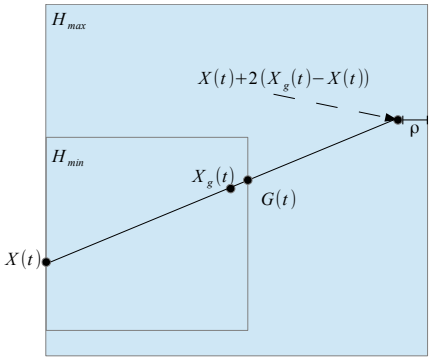


Figure 5. Minimal and maximal sampling hyperspheres to sample a point from the current particle position and the best obtained position for the best particle in the neighbourhood. The ρ value guarantees that the best obtained position X_g will always be located inside the sampling hypersphere H.

Quantification of the improvement In order to prove that there exists a strictly positive real number of μ guaranteeing the convergence condition, it is proposed to quantify the value of μ function under initial conditions.

Figure 6 displays the worst-case approach. This case is where the intersection volume between the sampling hypersphere H and C_g is minimal. It happens when the output of the EGT promotes the dominant use of the inertia over the use of the social component. Thus, the probability to sample a point that is within the volume $C_g \cap H$ can be expressed as:

$$\mu\Big(X(t+1) \in (C_g \cap H)\Big) = \frac{V(C \cap H)}{V(H)} \tag{64}$$

where μ denotes the probability measure, V the volume, $X(t+1)$ the sampled particle at the next iteration.

Thus, geometric rules yield:

$$V(H) = (2(\text{dist}(G, X_g) + \rho))^{\Delta} \tag{65}$$

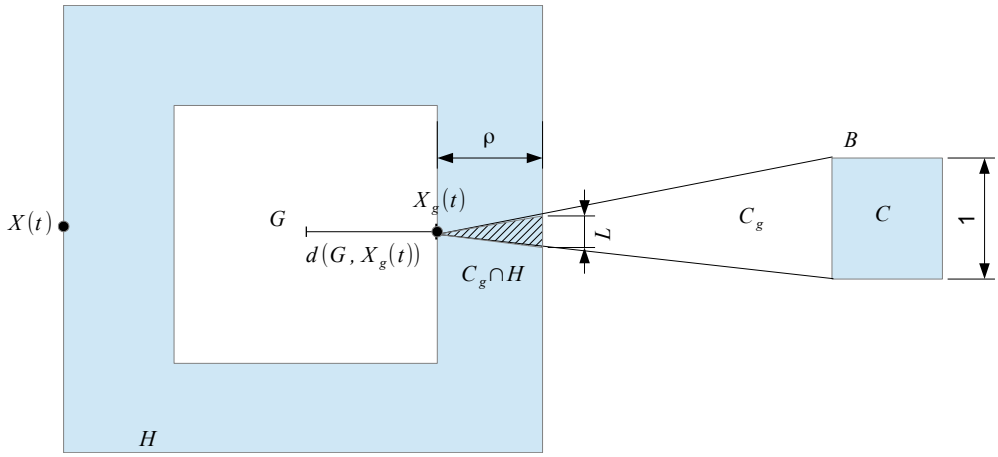


Figure 6. Resolution of the issue generated when there does not exist intersection between the sampling volume H and C, the cone of edge X and basis B.

where Δ denotes the dimension of the problem. And, we have

$$V(C \cap H) = \frac{1}{\Delta} L\rho \tag{66}$$

where L denotes the area of the intersection surface between H and C_g.

Knowing that Thales' theorem can be applied in dimension Δ, the following equality is obtained:

$$\frac{L}{1} = \frac{\rho^{\Delta}}{\text{dist}(X, B)} \tag{67}$$

Thus,

$$V(C_g \cap H) = \frac{1}{\Delta} \frac{\rho^{\Delta+1}}{\text{dist}(X, B)} \tag{68}$$

Then,

$$\mu\left(X(t+1) \in (C_g \cap H)\right) = \frac{1}{\Delta} \frac{\rho^{\Delta+1}}{\text{dist}(X, B)} \frac{1}{(2(\text{dist}(G, X_g) + \rho))^{\Delta}} \tag{69}$$

$$= \frac{\rho^{\Delta+1}}{\Delta \text{dist}(X, B) 2^{\Delta} (\text{dist}(G, X_g) + \rho)^{\Delta}} \tag{70}$$

Figure 7 shows the evolution of μ in function of the value of ρ and $\text{dist}(G, X_g)$. Here the dimension Δ was arbitrary chosen such as $\Delta = 10$. Note that $\text{dist}(X, B)$ is unknown since it represents the distance from the current position to the optimal region which is unknown. As it can be considered as a simple multiplying coefficient, it is fixed to an arbitrary value for the visualisation of μ in Figure 7.

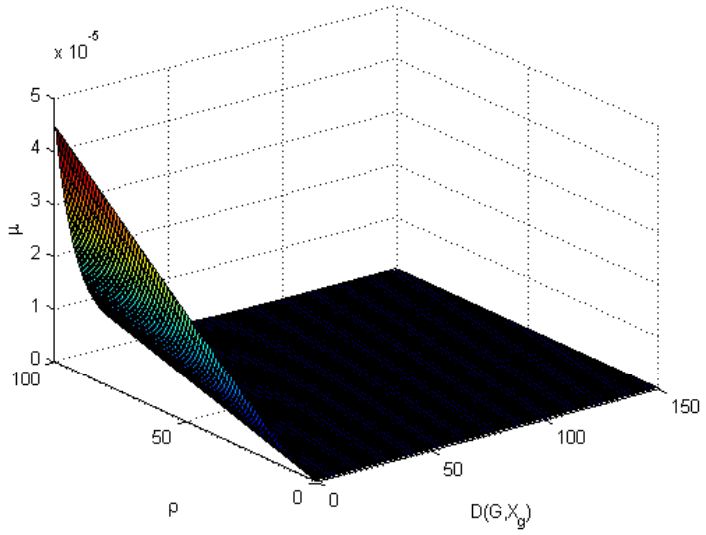


Figure 7. Value of the probability measure to sample a particle with guaranteed improvement.

As shown in Figure 7, the probability measure μ increases as the value of dist(G, X_g) becomes small and ρ increases. This observation can be explained from the fact that when a particle X is arbitrary close to the best particle position X_g, if the radius of the hypersphere H increases, then the delimited hyper surface $V(C_g \cap H)$ will also increase.

5.4. Conclusion

Corresponding to the analysis steps defined in Section 2.1, this section proved that the proposed EGPSO is a locally convergent algorithm by showing:

- The algorithm condition is satisfied (Algorithm Condition).
- The convergence condition is also satisfied (Convergence Condition):
  - The particles' trajectory is convergent and a particle in any position converges into the convex basin L_0.
  - The particles will never stop moving since the sampling hypersphere radius is always greater than an arbitrarily value.
  - There always exists a non-degenerated hypersphere guaranteeing sampling of a new point arbitrarily close to X_g and thus in L_0.
  - There always exists a non-zero probability to sample a point that improves the quality of X_g.

Note that when the optimisation problem is multimodal, then the algorithm will be convergent to a local optimum under the condition that all the communicating particles are within the same convex region of the solution space.

6. The combination of SPSO 2011 with EGPSO: C-EGSPSO

In Section 4, the trajectories of the particles in the proposed EGPSO and SPSO 2011 were investigated. The analysis validated that the three coefficients ω, ϕ_1 and ϕ_2 leveraging the EGT concepts guarantee the convergent trajectory and provide superiority on the convergence speed over the SPSO 2011. However, the superiority on the convergence speed does not guarantee efficient exploration, i.e. global search, of the solution space. When an optimisation algorithm has fast convergence, it often converges to a local minimum solution. As in all the heuristic methods, it would be beneficial to make a trade-off between the convergence speed and the efficient exploration of the solution space.

To cope with this premature convergence issue, this paper proposes to use the EGPSO algorithm, as well as the SPSO 2011. Following this proposition, it is expected to leverage the advantages of both EGPSO and SPSO 2011 algorithms, that is the capability to intensify the local search, as well as to efficiently explore the solution space. As the proposed algorithm combines the two approaches, it is named Combined-EGPSO (C-EGPSO).

The C-EGPSO algorithm consists of two types of particles: the first type follows the SPSO 2011 algorithm and the other uses the EGPSO algorithm. Since the SPSO 2011 algorithm presents an efficient exploration capability and EGPSO can dramatically decrease the computation time, the proposed algorithm could be able to avoid local minima and decrease the computational time.

6.1. The integration of EGPSO within SPSO 2011

The main idea of mixing the role played by the particles (either SPSO 2011 or EGPSO) is to ensure the diversification by using the SPSO 2011 method and at the same time to improve the convergence properties by using EGPSO. Therefore, a ratio between the two types of particles must be defined in order to assign a task to each particle. Note that in order to avoid to have two different swarms within the swarm, each particle can either follow SPSO 2011 or EGPSO and the role of each particle is recomputed at each time step.

In order to get the benefit from EGPSO which improves the convergence speed, without suffering a premature convergence, the determination of the ratio is based on the random selection of the particle's role. There are two possible roles: use the proposed EGPSO algorithm or use the classical SPSO 2011 described in Section 3.1.1. The role assignment is simply based on a random process, which means that a particle has a probability p_R to follow EGPSO process, and $1 - p_R$ to follow SPSO 2011. Thus, by this combination of the two roles, it is expected to be able to improve the convergence speed and avoid the swarm to be trapped in a local minimum. This will allow EGPSO to exploit both the intensification and diversification of the search.

The diagram 8 describes the principle of the proposed approach in the setting of optimal PSO coefficients.

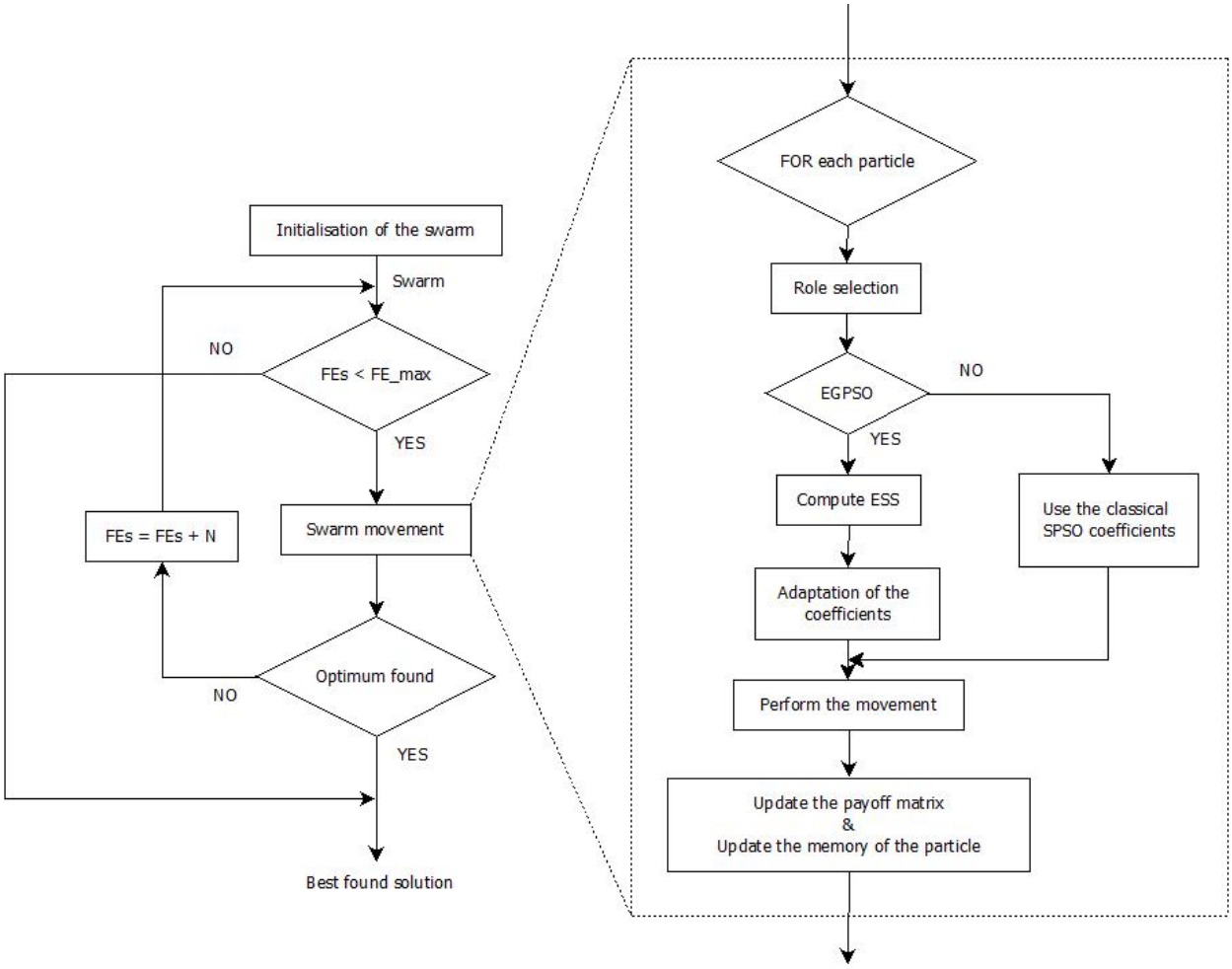


Figure 8. Diagram of the proposed method. The values FEs and FE_{max} denote the current number of cost function calls and the maximum of authorised calls respectively.

Note that, as this study focuses on the theoretical analysis of the proposed EGPSO algorithm, the performance analysis and comparison using numerical simulations are not subject of this study. However, empirical performance analysis and comparison will be followed and discussed in a following study.

7. Conclusion

In this paper, an enhanced SPSO 2011 algorithm was proposed. The presented method was based on the combination of EGT and SPSO 2011 designed by Clerc in [3], so named EGPSO. Integrating the two approaches was first proposed in our previous studies [12, 13, 15, 14] and its performance was empirically investigated. However, theoretical guarantee on the local optimality and the convergence properties have not been performed. This paper proposed new mapping from the ESS to the three PSO coefficients. This mapping is developed in a way thoeretically guaranteeing the local convergence of the EGPSO algorithm and its superiority on the convergence speed over SPSO 2011. As superior convergence speed likely results in the premature convergence issue, C-EGPSO is also proposed. To enable both the intensification and diversification of the search, the C-EGPSO algorithm consists of two different types of particles, one using the EGPSO concept and the other following the SPSO 2011 algorithm.

Note that the performance analysis and comparison using numerical simulations are not subject of this paper. However, it would be important to investigate the performance of EGPSO not only in unimodal functions, but also in complex multimodal functions to examine its performance. Moreover, it would be worth to compare the performance of the proposed EGPSO with the well known state-of-the-art algorithms. Therefore, rigorous performance analysis and comparison based on numerical simulations will be carried out in a forthcoming study.

There exists some space for further improvement of the proposed EGPSO algorithm, especially C-EGPSO. Each particle randomly decides which role it will play between SPSO 2011 and EGPSO with a probability empirically determined. For future work, how to determine this probability needs to be examined to clearly understand the effect of this probability and how the swarm behaviour changes with respect to the probability. In the same way, the probability to guarantee an improvement of the particle fitness could be subject of future study.

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Convergence proof of an enhanced particle swarm optimisation method integrated with evolutionary game theory

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