# Relational Division in Rank-Aware Databases 

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#### Abstract

We present a survey of existing approaches to relational division in rank-aware databases, discuss issues of the present approaches, and outline generalizations of several types of classic division-like operations. We work in a model which generalizes the Codd model of data by considering tuples in relations annotated by ranks, indicating degrees to which tuples in relations match queries. The approach utilizes complete residuated lattices as the basic structures of degrees. We argue that unlike the classic model, relational divisions are fundamental operations which cannot in general be expressed by means of other operations. In addition, we compare the existing and proposed operations and identify those which are faithful counterparts of universally quantified queries formulated in relational calculi. We introduce Pseudo Tuple Calculus in the ranked model which is further used to show mutual definability of the various forms of divisions presented in the paper.


## 1 Introduction

In this paper, we present a survey and new results in the area of divisionlike operations in rank-aware relational models of data. In particular, we are interested in models which allow imperfect matches of queries in addition to the usual precise yes/no matches of queries. By an "imperfect match" we mean a situation where given record in a database does not match a query in the usual sense but the record is sufficiently close to a (hypothetical) record that matches the query exactly. In many situations, it is desirable to include records with imperfect matches in the result of a query and introduce scores which indicate the degrees to which the records match the given query. For instance, in a database of products, we may query for products with price equal to $\$ 1,200$.

[^0]In the traditional understanding, a product sold for $\$ 1,198$ does not match the query. Nevertheless, we may want to include such product in the result and annotate it with a high score indicating that the product matches the query "almost perfectly" but not fully. In fact, reasoning with imperfect matches is inherent to human thinking and human perception of concepts like the proximity of values. Rank-aware databases [29, 31] and related models of data aim at such reasoning with imperfect matches and are concerned with its formalisation, analysis, and implementation in computer database systems.

Our investigation of division-like operations is motivated by the fact that in most of the existing rank-aware approaches to databases, discussion of such operations is either completely omitted or focuses only on particular Codd-style divisions. Indeed, compared to operations like projections and joins, the current rank-aware approaches pay little or no attention to division-like operations.

There seem to be two reasons for the absence of discussions of divisions in rank-aware models: First, a proposed rank-aware model simply omits divisions because its authors do not consider such an operation important. Second, the authors of a rank-aware model expect a division-like operation to be definable by the remaining operations in a similar way as in the classic relational model of data. We argue that neither of the points is tenable and divison-like operations deserve our attention:

1) Divisions are important Division-like operations are considered in relational query systems in order to express queries which take form of particular categorical propositions. It is well understood that classic relational queries of the form "some $\varphi$ is $\psi$ " can be expressed by means of combinations of projections and natural joins which are known as semijoins. Analogous queries can also be considered in rank-aware approaches with the same meaning except for the fact that the results of queries are annotated by scores. Naturally, one should expect to be able to formulate queries of the form of categorical proposition "all $\varphi$ are $\psi "$ in a rank-aware model. In the classic model, such queries are expressed by division-like operations. In addition, some variants of the classic relational division have a close relationship to the notions of containment (subsethood) of relations. From this viewpoint, one should expect that containments and divisions in a rank-aware model should both be defined and related as in the classic model. Note that division-like operations are also interesting from the data-analytical point of view. For instance, concept-forming operators in formal concept analysis [24] can be seen as particular relational divisions.
2) Divisions in rank-aware models are fundamental operations If a rank-aware model contains operations of difference (relational minus), projection, and natural join, one may argue that a Codd-style division [13] is a definable operation in the ranked model in much the same way as it is definable in the classic model. While in the classic model, reasonable division-like operations can indeed be derived, we show further in the paper that this assumption cannot be universally adopted in rank-aware models. Technically, the operation can be defined as in the ordinary case but in many cases it lacks the basic properties of "reasonable division" and no longer is a faithful representation of queries of the form of categorical propositions "all $\varphi$ are $\psi$ ". As a matter of fact, we argue in the paper that suitable variants of divisions (or equivalent formalisms) should be included as fundamental operations in rank-aware models.

In this paper we focus on divisions from the perspective of a relational model which can be seen as a generalization of the Codd [13] model of data from the point of view of residuated structures of degrees. The basic idea of the model is that tuples in relations are annotated by scores indicating degrees to which tuples match queries analogously as in [21, 22], cf. also 31 introducing RankSQL and a survey paper [29]. Our model differs in how we approach the structures of scores and, consequently, the underlying logic of imperfect matches. We use structures of degrees which are recognized by fuzzy logics in the narrow sense [11, 12, 23, 26, 27] and the principle of truth functionality because our intention is to develop the model so that particular issues handled in the model (like querying and data dependencies) can be analyzed in terms of logical deduction in the narrow sense. This is in contrast with various approaches that appeared earlier [6, 9, 7, 19] and utilized techniques from fuzzy sets (in the wide sense) where the connection to residuated structures of degrees is not so strict. We argue in the paper that the role of residuated structures is crucial for a sound treatment of division-like operations.

Our paper is organized as follows. In Section 2, we recall basic notions of our model. In Section 3, we survey existing and propose new approaches to division operations in the classic as well as in the graded setting. In Section 4 we introduce a query language called Pseudo Tuple Calculus (PTC) that enables us to reason about the operations with ease. Finally, in Section 5, we utilize PTC to derive further observations on the mutual definability of the division operations described in the paper.

## 2 Relational Model Based on Residuated Structures

In this section, we present a survey of utilized notions from residuated structures of degrees and fuzzy relational systems. Furthermore, we introduce the basic notions of the generalized relational model of data and its relational algebra [3].

### 2.1 Structures of Degrees

We use complete residuated lattices as structures of degrees which represent scores assigned to tuples and indicating degrees to which tuples match queries. A residuated lattice [2, 23, 27] is a general algebra 34] of the form

$$
\begin{equation*}
\mathbf{L}=\langle L, \wedge, \vee, \otimes, \rightarrow, 0,1\rangle \tag{1}
\end{equation*}
$$

such that $\langle L, \wedge, \vee, 0,1\rangle$ is a bounded lattice 5 with 0 and 1 being the least and the greatest element of $L$, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e., $\otimes$ is commutative, associative, and $a \otimes 1=1 \otimes a=a$ for each $a \in L$ ); $\otimes$ (a multiplication) and $\rightarrow$ (a residuum) satisfy the adjointness property:

$$
\begin{equation*}
a \otimes b \leq c \text { iff } a \leq b \rightarrow c \tag{2}
\end{equation*}
$$

for each $a, b, c \in L$ where $\leq$ is the order induced by the lattice structure of $\mathbf{L}$ (i.e., $a \leq b$ iff $a=a \wedge b$ ). A residuated lattice (1) is called complete if its lattice part is a complete lattice, i.e., if $L$ contains infima (greatest lower bounds) and suprema (least upper bounds) of arbitrary subsets of $L$. The multiplication $\otimes$ and its adjoint residuum $\rightarrow$ can be seen as general aggregation functions which interpret general "conjunction" and "implication" of scores, respectively. That is, if a tuple matches query $Q_{1}$ with a score $a_{1}$ and it also matches query $Q_{2}$ with a score $a_{2}$, then $a_{1} \otimes a_{2}$ may be interpreted as the score to which the tuple matches the composed conjunctive query " $Q_{1}$ and $Q_{2}$." This way the aggregation function is understood in [21. In a similar way, $a_{1} \rightarrow a_{2}$ may be interpreted as the score to which the tuple matches the composed conditional query "if $Q_{1}$ then $Q_{2}$."

A typical choice of a complete residuated lattice $\mathbf{L}$ is a structure given by a left-continuous triangular norm [30]. That is, $L=[0,1]$ (real unit interval), $\wedge$ and $\vee$ are minimum and maximum (in which case the induced $\leq$ is the genuine ordering of reals), and $\otimes$ is a left-continuous triangular norm. The left-continuity of $\otimes$ ensures there is a residuum $\rightarrow$ satisfying (2) which is in addition uniquely given by

$$
\begin{equation*}
a \rightarrow b=\bigvee\{c \in L \mid a \otimes c \leq b\} \tag{3}
\end{equation*}
$$

In words, (3) says that $a \rightarrow b$ is the supremum of all $c \in L$ such that $a \otimes c \leq b$ (it can be shown that $a \rightarrow b$ is in fact the greatest $c \in L$ satisfying such property).

From pragmatic standpoints, the most important complete residuated lattices are exactly those on the real unit interval given by continuous triangular norms. All such structures can be obtained by constructing ordinal sums [2, 27, 30] of (isomorphic copies of) three basic pairs of multiplications (and their corresponding residua): $a \otimes b=\max (a+b-1,0)$ (Eukasiewicz multiplication), $a \otimes b=\min (a, b)$ (Gödel or minimum multiplication), $a \otimes b=a \cdot b$ (Goguen or product multiplication).

Remark 1. The role of residuated lattices as general structure of truth degrees in truth-functional logics has been recognized by Goguen 25]. Important logics based on subclasses of residuated lattices include Höhle's monoidal logic [28], Basic Logic [27], and Monoidal T-norm Logic [20]. Note that the truth-functionality is a crucial property which is not present in other models which also involve ranks like the probabilistic extensions of the Codd model, see [14]. In fact, the probabilistic databases tackle completely different issues and deal with uncertain data which is not our case because the approaches we discuss here deal with certain data and imperfect matches of queries.

An important aspect of the relational model which is relevant to our paper is that the classic relational model is based on the classic predicate logic [15]. As a result, finite relations (informally represented by "data tables") are used to represent both the base data and results of queries. In fact, database instances (i.e., collections of relations interpreting relational symbols/variables) can be seen as predicate structures [32, predicate formulas can be seen as queries, and their interpretation in database instances corresponds to query evaluation. Thus, the structures of truth values of the classical predicate logic-the Boolean algebras, are vital for the model and, loosely speaking, determine laws that hold in the relational model.

The model we use in this paper can be seen as a relational model of data which results from the classic one by replacing the Boolean algebras with complete residuated lattices. This change has, of course, its implications. First, we shift from structures with only yes/no matches to structures which allow us to work with general (intermediate) degrees-this is a desirable property for development of a rank-aware model. Second, some laws that hold in the classic model are no longer valid (e.g., tertium non datur). The second point shall be understood as virtue of the model rather than a vice - note that Basic Logic extended by tertium non datur collapses into the classical logic [27]. In fact,
there are no proper fuzzy logics which satisfy tertium non datur. Our rationale for using (complete) residuated lattices as the structures of degrees is that they represent more general structures than the Boolean algebras which allow us to deal with intermediate degrees and are still reasonably strong.

Remark 2. Let us note that logics based on residuated lattices are used to reason about general scores. If 0 and 1 are used as the only scores, the logic collapses into the classic Boolean logic which is a desirable property. Also, the structures and operations of the generalized model can be implemented inside the classic relational model using the ordinary notions of relations on relation schemes and additional operations with relations.

### 2.2 Attributes, Types, and Ranked Data Tables

In this section, we present our counterpart to the classic relations on relation schemes. We utilize the following notions. We denote by $Y$ a (infinite denumerable) set of attributes, any finite subset $R \subseteq Y$ is called a relation scheme. For each attribute $y \in Y$ we consider its type $D_{y}$ which is understood as the admissible set of values of the attribute $y$, see [17] (note that in earlier literature, types are called domains, cf. [13]). In the paper, we do not refer to types explicitly, i.e., whenever we introduce an attribute, we tacitly consider its type and for simplicity we assume that attributes with the same name have the same type.

We utilize the usual set-theoretic representation of tuples: A direct product $\prod_{y \in R} D_{r}$ of an $R$-indexed system $\left\{D_{y} \mid y \in R\right\}$ is a set of all maps

$$
\begin{equation*}
r: R \rightarrow \bigcup_{y \in R} D_{y} \tag{4}
\end{equation*}
$$

such that $r(y) \in D_{y}$ for each $y \in R$. If $R \subseteq Y$ is finite, then each $r \in \prod_{y \in R} D_{y}$ is called a tuple on relation scheme $R, r(y)$ is called the $y$-value of $r$. For brevity, $\prod_{y \in R} D_{y}$ is denoted by $\operatorname{Tupl}(R)$. For $S \subseteq R$ and $r \in \operatorname{Tupl}(R)$, we denote by $r(S)$ the projection of $r$ onto $S$, i.e., $r(S) \subseteq r$ such that $\langle y, d\rangle \in r(S)$ for some $d \in D_{y}$ iff $y \in S$. In particular, $r(\emptyset) \in \operatorname{Tupl}(\emptyset)=\{\emptyset\}$, i.e., $\emptyset$ is the only tuple on the empty relation scheme. Moreover, if $r \in \operatorname{Tupl}(R), s \in \operatorname{Tupl}(S)$, and $r(R \cap S)=s(R \cap S)$, we call the set-theoretic union $r \cup s$ the join of tuples $r$ and $s$ and denote it by $r s$.

The relations (on relation scheme $R$ ) which appear in the classic model are finite subsets of $\operatorname{Tupl}(R)$. Technically, such subsets can be identified with indicator functions which assign 1 to finitely many tuples from $\operatorname{Tupl}(R)$ (to those belonging to the relation) and 0 otherwise. Our counterpart to relations on
relation schemes result by considering such indicator functions with codomains being the set of degrees from complete residuated lattices.

Definition 1. Let $\mathbf{L}$ be a complete residuated lattice, $R$ be a relation scheme. A ranked data table on relation scheme (shortly, an RDT) is any map of the form $\mathcal{D}: \operatorname{Tupl}(R) \rightarrow L$ such that $\{r \in \operatorname{Tupl}(R) \mid \mathcal{D}(r)>0\}$, called the answer set of $\mathcal{D}$, is finite. The degree $\mathcal{D}(r)$ is called the score of $r$ in $\mathcal{D}$.

Remark 3. (a) Important special cases of RDTs are represented by RDTs on the empty relation scheme. Recall that in the classic model [17, there are only two relations on $\emptyset$, namely the empty relation on $\emptyset$ (called TABLE_DUM in [17]) and the relation on $\emptyset$ containing the empty tuple (called TABLE_DEE). In our case, all RDTs on the empty scheme are maps of the form $\mathcal{D}:\{\emptyset\} \rightarrow L$, i.e., they are uniquely given by the degree $\mathcal{D}(\emptyset) \in L$, i.e., by the degree which is assigned to $\emptyset$ (the empty tuple) by $\mathcal{D}$. Because of this correspondence, for each degree $a \in L$, we define $a_{\emptyset}: \operatorname{Tupl}(\emptyset) \rightarrow L$ as the RDT such that $a_{\emptyset}(\emptyset)=a$. Hence, in addition to TABLE_DUM ( $0_{\emptyset}$ in our notation) and TABLE_DEE ( $1_{\emptyset}$ in our notation) our model admits general DEE-like RDTs for every $a \in L$, leaving $0_{\emptyset}$ and $1_{\emptyset}$ as two borderline cases. As it is argued in [16], special cases of divisions which involve TABLE_DUM and TABLE_DEE are important and have been often neglected in various approaches to division, which in consequence led to divisions with undesirable properties. In our case, the DEE-like tables $a_{\emptyset}$ play analogous important role and shall be taken into account.
(b) RDTs on non-empty relation schemes can be depicted analogously as classic relations on non-empty relation schemes by two-dimensional data tables with columns corresponding to attributes and rows corresponding to tuples. In addition, each row in the table is annotated by the score of the tuple represented by the row (tuples with zero scores are not shown in the table).
(c) If $\mathcal{D}(r) \in\{0,1\}$ for all $r \in \operatorname{Tupl}(R)$, we call $\mathcal{D}$ non-ranked. Clearly, non-ranked RDTs are in a one-to-one correspondence with (finite) relations on relation schemes in the usual sense. A particular case of a non-ranked table is $0_{R}$ called the empty table and satisfying $0_{R}(r)=0$ for all $r \in \operatorname{Tupl}(R)$.

### 2.3 Relational Operations

By virtue of the close connection to logics based on residuated structures of degrees, the rank-aware model we consider admits two basic types of domain independent query systems [3]. First, a system based on evaluating predicate formulas. Second, a system consisting of relational operations which has the same expressive power as the former one. The relational divisions considered in
this paper are particular (fundamental or derived) relational operations. In this subsection, we recall a fragment of the relational operations we need to cope with divisions.

For $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ on the same relation scheme $R$, we define $\mathcal{D}_{1} \cap \mathcal{D}_{2}$ (intersection) and $\mathcal{D}_{1} \cup \mathcal{D}_{2}$ (union) by

$$
\begin{align*}
& \left(\mathcal{D}_{1} \cap \mathcal{D}_{2}\right)(r)=\mathcal{D}_{1}(r) \wedge \mathcal{D}_{2}(r),  \tag{5}\\
& \left(\mathcal{D}_{1} \cup \mathcal{D}_{2}\right)(r)=\mathcal{D}_{1}(r) \vee \mathcal{D}_{2}(r), \tag{6}
\end{align*}
$$

for all $r \in \operatorname{Tupl}(\mathrm{R})$. In words, $\cap$ and $\cup$ are defined componentwise using the lattice operations $\wedge$ and $\vee$ in $\mathbf{L}$.

The natural join in our model is introduced as follows. If $\mathcal{D}_{1}$ is an RDT on relation scheme $R \cup S$ and $\mathcal{D}_{2}$ is an RDT of relation scheme $S \cup T$ such that $R \cap S=R \cap T=S \cap T=\emptyset$ (i.e., $R, S$, and $T$ are pairwise disjoint), then the natural join of $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ is an RDT on relation scheme $R \cup S \cup T$ denoted by $\mathcal{D}_{1} \bowtie \mathcal{D}_{2}$ and defined by

$$
\begin{equation*}
\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)(r s t)=\mathcal{D}_{1}(r s) \otimes \mathcal{D}_{2}(s t) \tag{7}
\end{equation*}
$$

for each $r \in \operatorname{Tupl}(R), s \in \operatorname{Tupl}(S)$, and $t \in \operatorname{Tupl}(T)$. Hence, $\otimes$ in $\mathbf{L}$ acts as a conjunctive aggregator which generalizes the classic conjunction appearing in the definition of ordinary natural join of relations. If $\mathcal{D}$ is an RDT on $R$, the projection of $\mathcal{D}$ onto $S \subseteq R$ is denoted by $\pi_{S}(\mathcal{D})$ and defined by

$$
\begin{equation*}
\left(\pi_{S}(\mathcal{D})\right)(s)=\bigvee_{t \in \operatorname{Tupl}(R \backslash S)} \mathcal{D}(s t) \tag{8}
\end{equation*}
$$

for each $s \in \operatorname{Tupl}(S)$. Using projections of tuples onto $S$, we may write (8) equivalently as $\left(\pi_{S}(\mathcal{D})\right)(s)=\bigvee\{\mathcal{D}(r) \mid r(S)=s\}$. Since $\otimes$ is distributive over $\bigvee$, we may introduce a semijoin of $\mathcal{D}_{1}$ on $R$ and $\mathcal{D}_{2}$ on $S$ as $\pi_{R}\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)$ or equivalently as $\mathcal{D}_{1} \bowtie \pi_{R \cap S}\left(\mathcal{D}_{2}\right)$ and we denote it $\mathcal{D}_{1} \ltimes \mathcal{D}_{2}$.

Analogously as in the classic model, semijoins in our model are important since they allow us to algebraically express existential queries of the form of categorical propositions "some $\varphi$ is $\psi$ " or, in the database terminology [17, "some tuples from $\mathcal{D}_{1}$ are matching tuples in $\mathcal{D}_{2}$ ".

Remark 4. (a) One may check that if all arguments to the above-mentioned operations are non-ranked, then the results of relational operations coincide with the results of the classic relational operations of union, intersection, natural join, and projection [13, 17].
(b) Let us comment on the role of the general suprema in 8). In predicate logics based on residuated structures of degrees [10], general suprema are used
to interpret existentially quantified formulas. In a more detail, for a formula of the form $(\exists x) \varphi$, its truth degree $\|(\exists x) \varphi\|_{\mathbf{M}, v}$ in the $\mathbf{L}$-structure $\mathbf{M}$ under the evaluation $v$ of object variables is defined as the supremum of all truth degrees $\|\varphi\|_{\mathbf{M}, w}$ where $w(y)=v(y)$ for each variable $y$ such that $y \neq x$. Put in words, $\|(\exists x) \varphi\|_{\mathbf{M}, v}$ is the least upper bound of all degrees to which $\varphi$ is true in $\mathbf{M}$ considering $x$ as a variable which can be assigned any value from the universe of $\mathbf{M}$. Note that if $\mathbf{L}$ is the two-element Boolean algebra, this interpretation coincides exactly with the usual interpretation of existentially quantified formulas and, in particular, $\|(\exists x) \varphi\|_{\mathbf{M}, v}=1$ iff there is $w$ such that $\|\varphi\|_{\mathbf{M}, w}=1$ and $w(y)=v(y)$ for all $y \neq x$ (i.e., $x$ can be assigned a value which makes $\varphi$ true in $\mathbf{M})$. Now, since projections are relational operations which express queries formulated by existentially quantified formulas in relational calculi, 88) is defined in terms of $\bigvee$. In words, $\left(\pi_{S}(\mathcal{D})\right)(s)$ is a degree to which there is a tuple in $\mathcal{D}$ whose projection onto $S$ equals to $s$.

## 3 Existing and New Approaches to Division

In this section, we review several classic approaches to division which appeared in the literature on database systems, present their rank-aware counterparts, and comment on their relationship to the existing rank-aware or fuzzy approaches in databases. The section is structured into subsections which roughly follow the structure of [16] which is arguably the best comparison of division-like operations from the point of view of the relational model of data.

In this section, whenever we say that (a relation or an RDT) $\mathcal{D}$ is on scheme $R S$, we mean that it is defined on the scheme $R \cup S$ such that $R \cap S=\emptyset$.

### 3.1 Codd-style Division

Historically, the Codd division is the initial operation in the family of divisionlike operations. Its initial purpose was technical-to ensure completeness of the relational algebra with respect to the relational calculus which allows us to express queries involving universal quantification. Strictly speaking, its presence in the relational algebra is not necessary since in the classical logic, universally quantified formulas of the from $(\forall x) \varphi$ can be replaced by formulas $\pi(\exists x) \pi \varphi$, i.e., universal quantifiers are expressible by means of negations and existential quantification. Thus, the division is considered as a derived operation which is expressed by means of set-theoretic difference (relational counterparts to negations) and projections (relational counterparts to existential quantification).

Namely, for a relation $\mathcal{D}_{1}$ on $R S$ and relation $\mathcal{D}_{2}$ on $S$, the Codd division $\mathcal{D}_{1} \div$ Codd $\mathcal{D}_{2}$ may be introduced [16] as

$$
\begin{equation*}
\mathcal{D}_{1} \div \operatorname{Codd} \mathcal{D}_{2}=\pi_{R}\left(\mathcal{D}_{1}\right) \backslash \pi_{R}\left(\left(\pi_{R}\left(\mathcal{D}_{1}\right) \bowtie \mathcal{D}_{2}\right) \backslash \mathcal{D}_{1}\right) \tag{9}
\end{equation*}
$$

where $\pi_{R}, \bowtie$, and $\backslash$ denote the usual projection, natural join (cross join in this particular case), and set-theoretic difference, respectively. The survey chapter [16] identifies several epistemic issues of (9). The most important are:
(i) Unlike semijoins, (9) is restricted to relations on particular schemes, i.e., the operation cannot be performed with relations on arbitrary schemes which makes it less general (and less useful).
(ii) The meaning of (9) does not faithfully correspond to the categorical proposition "all $\varphi$ are $\psi$ ". If $\varphi$ is $s \in \mathcal{D}_{2}$ and $\psi$ is $r s \in \mathcal{D}_{1}$, then

$$
\begin{equation*}
(\forall s)\left(s \in \mathcal{D}_{2} \Rightarrow r s \in \mathcal{D}_{1}\right) \tag{10}
\end{equation*}
$$

is true for all $r \in \operatorname{Tupl}(R)$ provided that $\mathcal{D}_{2}$ is empty. In contrast, the result of (9) is always a subset of $\pi_{R}\left(\mathcal{D}_{1}\right)$. Hence, in general, the meaning of (9) is "any $r$ in $\pi_{R}\left(\mathcal{D}_{1}\right)$ such that $r s \in \mathcal{D}_{1}$ for all $s \in \mathcal{D}_{2}$ " rather than "any $r$ such that $r s \in \mathcal{D}_{1}$ for all $s \in \mathcal{D}_{2}$ ", cf. [16]. As a consequence, (9) is equivalent to

$$
\begin{equation*}
\mathcal{D}_{1} \div \mathcal{D}_{2}=\left\{r \in \pi_{R}\left(\mathcal{D}_{1}\right) \mid \text { for all } s \in \mathcal{D}_{2}, \text { we have } r s \in \mathcal{D}_{1}\right\} \tag{11}
\end{equation*}
$$

where $\pi_{R}\left(\mathcal{D}_{1}\right)$ can be seen as the range for the division.
By a direct generalization of (9) in rank-aware approaches, we inherit both the issues. In addition, it is questionable how to handle $\backslash$ in the presence of scores. One way to go is to consider $\left(\mathcal{D}_{1} \backslash \mathcal{D}_{2}\right)(r)$ to be the degree to which $r$ is in $\mathcal{D}_{1}$ and is not in $\mathcal{D}_{2}$ and express the negation using $\rightarrow$ and 0 , i.e.,

$$
\begin{equation*}
\left(\mathcal{D}_{1} \backslash \mathcal{D}_{2}\right)(r)=\mathcal{D}_{1}(r) \otimes\left(\mathcal{D}_{2}(r) \rightarrow 0\right) \tag{12}
\end{equation*}
$$

Although $\mathcal{D}_{1} \backslash \mathcal{D}_{2}$ is always finite, it does not fulfill basic properties one would expect for a difference. For instance, $\mathcal{D}_{1} \backslash \mathcal{D}_{2}=0_{R}$ does not imply $\mathcal{D}_{1} \subseteq \mathcal{D}_{2}$ in general. Alternatively, one may introduce $\backslash$ as an independent fundamental connective in $\mathbf{L}$ and induce the difference of RDTs componentwise analogously as $\cap$ or $\cup$. For instance, one may use commutative doubly-residuated lattices 33 ] with \ being adjoint to a non-idempotent disjunction. Note that differencelike operations with relations (with scores) in the database literature are often
defined analogously as $(12)$, usually on $L=[0,1]$ with $\otimes$ being the minimum and $\rightarrow$ being the Łukasiewicz implication [8]. The general issue with graded style-versions of (9) is that universal quantifier (interpreted by infima in $\mathbf{L}$ ) is not definable using the existential one (interpreted by suprema in $\mathbf{L}$ ).

Most common truth-functional approaches [6, 9, 7, 19] that can be found in literature on rank-aware extensions generalize 10 by putting

$$
\begin{equation*}
\left(\mathcal{D}_{1} \div \mathcal{D}_{2}\right)(r)=\bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{2}(s) \rightarrow \mathcal{D}_{1}(r s)\right) \tag{13}
\end{equation*}
$$

for all $r \in \operatorname{Tupl}(R)$ provided that $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are RDTs on schemes $R S$ and $S$, respectively. In our setting, $\bigwedge$ is the operation of infimum in $\mathbf{L}$, and $\rightarrow$ is the residuum in $\mathbf{L}$. The above-cited approaches often use a fixed scale of degrees (with $L=[0,1]$ ) with $\rightarrow$ being a general truth function of implication. In addition to $\rightarrow$ which are adjoint to $\otimes$ (so-called R-implications), the approaches use S-implications [9]. We do not want to endorse this concept here because of its marginal role in fuzzy logics in the narrow sense, see [26] and the soundness issues regarding S-implications.

Remark 5. Observe that since $r \in \operatorname{Tupl}(R)$, 13) solves issue (ii) but this is at the expense of losing domain independence. Indeed, if $R$ contains an attribute which has a type consisting of infinitely many values then the result $\mathcal{D}_{1} \div$ $\mathcal{D}_{2}$ defined by 13 is infinite which is highly undesirable property from the database viewpoint - if a materialization of $\mathcal{D}_{1} \div \mathcal{D}_{2}$ is necessary in order to evaluate a compound query involving the division, the evaluation cannot be performed (in finitely many steps). Probably because of this issue, some of the graded approaches cited above use 13 assuming that $\left(\pi_{R}\left(\mathcal{D}_{1}\right)\right)(r)>0$ which, unfortunately, introduces (ii) again.

In our previous work [4], we have used a fundamental domain-dependent division operation which is sufficient to establish the equivalence between a domain-dependent relational algebra and a domain relational calculus. Recently, we have proposed a domain independent variant [3] with explicit range which is used to establish the equivalence between a domain-independent relational algebra and a domain relational calculus with range declarations. The operation is defined as follows.

Let $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ be RDTs on $R S, S$, and $R$, respectively. Then, a division $\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}$ of $\mathcal{D}_{1}$ by $\mathcal{D}_{2}$ which ranges over $\mathcal{D}_{3}$ is an RDT on $R$ defined by

$$
\begin{equation*}
\left(\mathcal{D}_{1} \div \dot{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{3}(r) \otimes\left(\mathcal{D}_{2}(s) \rightarrow \mathcal{D}_{1}(r s)\right)\right), \tag{14}
\end{equation*}
$$

for each $r \in \operatorname{Tupl}(R)$. Clearly, $\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2} \subseteq \mathcal{D}_{3}$. In addition, 14) possesses further desirable properties. For instance, if $\mathcal{D}_{3}$ is non-ranked, then $\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}$ is
the greatest among all $\mathcal{D} \subseteq \mathcal{D}_{3}$ such that $\mathcal{D} \bowtie \mathcal{D}_{2} \subseteq \mathcal{D}_{1}$. Furthermore, if $R=\emptyset$ and $\mathcal{D}_{3}=1_{\emptyset}$ (see Remark 3), then (14) becomes (the relational representation of) the subsethood degree of $\mathcal{D}_{2}$ in $\mathcal{D}_{1}$, see [2]. Also, the definition eliminates (ii) and is domain independent.

### 3.2 Date's Small Divide (Original and Generalized)

In order to overcome issue (ii), Date (see [16] and the references therein) proposed a Small Divide operation. Consider the following relations on relation schemes: $\mathcal{D}_{1}$ on $R$ (called the dividend), $\mathcal{D}_{2}$ on $S$ (called the divisor), $\mathcal{D}_{3}$ on $R S$ (called the mediator). Then, the original version of Small Divide [16] is

$$
\begin{align*}
\mathcal{D}_{1} \div \dot{\mathcal{D}}_{3} \mathcal{D}_{2} & =\mathcal{D}_{1} \backslash \pi_{R}\left(\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right) \backslash \mathcal{D}_{3}\right) \\
& =\left\{r \in \mathcal{D}_{1} \mid \text { for all } s \in \mathcal{D}_{2}, \text { we have } r s \in \mathcal{D}_{3}\right\} . \tag{15}
\end{align*}
$$

A graded generalization of $\sqrt{15}$ is

$$
\begin{equation*}
\left(\mathcal{D}_{1} \div \dot{\text { gsdo }}_{\mathcal{D}_{3}}^{\mathcal{D}_{2}}\right)(r)=\mathcal{D}_{1}(r) \otimes \bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{2}(s) \rightarrow \mathcal{D}_{3}(r s)\right) \tag{16}
\end{equation*}
$$

with $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ being RDTs on $R, S$, and $R S$, respectively. The graded variant of the Small Divide and 14 are equivalent under the following conditions:

Theorem 2. If $\mathbf{L}$ is prelinear or divisible, then $\mathcal{D}_{3} \div{ }^{\mathcal{D}_{1}} \mathcal{D}_{2}=\mathcal{D}_{1} \div{ }_{\text {gsdo }}^{\mathcal{D}_{3}} \mathcal{D}_{2}$.
Proof. Either of prelinearity or divisibility ensures that $a \otimes(b \wedge c)=(a \otimes b) \wedge(a \otimes c)$ for all $a, b, c \in L$, see [2, 20, 27]. In addition, since $\mathcal{D}_{2}$ and $\mathcal{D}_{3}$ are finite, in both (16) and (14) the infimum is computed using only finitely many degrees other than 1 , i.e., the claim follows by distributivity of $\otimes$ over infima of finitely many degrees which are pairwise distinct.

Note that analogous observation holds if $\mathbf{L}$ is arbitrary and $\mathcal{D}_{1}$ is non-ranked.
Remark 6. The previous observation has two important consequences: In the mainstream fuzzy logics (based on prelinear residuated lattices), graded Small Divide and (14) are equivalent. In particular, if $\mathbf{L}$ is the two-element Boolean algebra, the ranked model becomes the classic one, i.e., this observation pertains to the classic relational model.

In order to cope with issue $(i)$, the original Small Divide has been further extended to accomodate relations on more general schemes. Namely, for $\mathcal{D}_{1}$ on
$R T, \mathcal{D}_{2}$ on $S U$, and $\mathcal{D}_{3}$ on $R S V$, Date introduced [16] a general form of Small Divide as follows:

$$
\begin{equation*}
\mathcal{D}_{1} \div \dot{\mathcal{D}}_{3 d} \mathcal{D}_{2}=\mathcal{D}_{1} \bar{\ltimes}\left(\left(\pi_{R}\left(\mathcal{D}_{1}\right) \bowtie \pi_{S}\left(\mathcal{D}_{2}\right)\right) \bar{\ltimes} \mathcal{D}_{3}\right) \tag{17}
\end{equation*}
$$

where $\bar{\ltimes}$ denotes the semidifference, i.e., $\mathcal{D} \bar{\ltimes} \mathcal{D}^{\prime}=\mathcal{D} \backslash\left(\mathcal{D} \ltimes \mathcal{D}^{\prime}\right)$. By moment's reflection, we derive that

$$
\begin{equation*}
\mathcal{D}_{1} \div{ }_{\mathrm{sd}}^{\mathcal{D}_{3}} \mathcal{D}_{2}=\left\{r t \in \mathcal{D}_{1} \mid \text { for all } s \in \pi_{S}\left(\mathcal{D}_{2}\right), \text { we have } r s \in \pi_{R S}\left(\mathcal{D}_{3}\right)\right\} \tag{18}
\end{equation*}
$$

We may therefore introduce the following operation in the graded setting

$$
\begin{equation*}
\left(\mathcal{D}_{1} \div{ }_{\mathrm{gsd}}^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r t)=\mathcal{D}_{1}(r t) \otimes \bigwedge_{s \in \operatorname{Tupl}(S)}\left(\left(\pi_{S}\left(\mathcal{D}_{2}\right)\right)(s) \rightarrow\left(\pi_{R S}\left(\mathcal{D}_{3}\right)\right)(r s)\right) \tag{19}
\end{equation*}
$$

provided that $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ are RDTs on $R T, S U, R S V$, respectively. As in the classic setting, $\div$ gsd eliminates both the issues $(i)$ and (ii) mentioned earlier.

### 3.3 Todd-style Division

An alternative approach to eliminate issue $(i)$ is the division proposed by Todd, cf. 16. Written directly in the set notation,

$$
\begin{equation*}
\mathcal{D}_{1} \div{ }_{\text {Todd }} \mathcal{D}_{2}=\left\{r t \in \mathcal{U} \mid \text { for all } s \in \operatorname{Tupl}(S): \text { if } s t \in \mathcal{D}_{2}, \text { then } r s \in \mathcal{D}_{1}\right\} \tag{20}
\end{equation*}
$$

where $\mathcal{U}=\pi_{R}\left(\mathcal{D}_{1}\right) \bowtie \pi_{T}\left(\mathcal{D}_{2}\right)$. Unfortunately, $\dot{\text { Todd }}^{\text {Tod }}$ and its direct rank-aware generalizations inherit the issue ( $i i$ ). This is caused by the fact that the ranges for $r$ and $t$ in are considered to be the projections of $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$, respectively. Interestingly, if $\mathcal{U}$ is considered to be the set of all tuples on $R T$, the graded generalization becomes

$$
\begin{equation*}
\left(\mathcal{D}_{1} \div \text { gTodd } \mathcal{D}_{2}\right)(r t)=\bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{2}(s t) \rightarrow \mathcal{D}_{1}(r s)\right) \tag{21}
\end{equation*}
$$

which is the Kohout-Bandler superproduct composition [1, 2] of fuzzy relations $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ (in this order). As in the case of 13 ) $\div$ gTodd is domain dependent, i.e., even if $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ are finite, the result of may be infinite which is an undesirable property.

### 3.4 Date's Great Divide

In the same spirit as the Small Divide has been proposed to eliminate the issues of the classic Codd division, the Great Divide has been proposed by Date [16]
to deal with the issues of the Todd division. Again, we may assume two variants of the operation - the original one and the generalized one. For illustration, we focus here only on the original variant, the generalized one can be obtained in much the same way as in the case of the Small Divide.

According to [16], for relations $\mathcal{D}_{1}$ on $R$ (called the dividend), $\mathcal{D}_{2}$ on $T$ (called the divisor), $\mathcal{D}_{3}$ on $R S$ (called the first mediator), and $\mathcal{D}_{4}$ on $S T$ (called the second mediator), we put

$$
\begin{equation*}
\mathcal{D}_{1} \div{ }_{\mathrm{gdo}}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}=\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right) \bar{\ltimes}\left(\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \bar{\ltimes} \mathcal{D}_{3}\right) . \tag{22}
\end{equation*}
$$

The definition 22 can be equivalently expressed in the set notation as follows:

$$
\begin{equation*}
\mathcal{D}_{1} \div{ }_{\text {gdo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}=\left\{r t \in \mathcal{U} \mid \text { for all } s \in \operatorname{Tupl}(S): \text { if } s t \in \mathcal{D}_{4}, \text { then } r s \in \mathcal{D}_{3}\right\} \tag{23}
\end{equation*}
$$

where $\mathcal{U}=\mathcal{D}_{1} \bowtie \mathcal{D}_{2}$. Based on 23 , we may introduce a graded variant $\div$ ggdo of the original Great Divide as follows

$$
\begin{align*}
\left(\mathcal{D}_{1} \div{ }_{\text {ggdo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}\right)(r t) & =\mathcal{D}_{1}(r) \otimes \mathcal{D}_{2}(t) \otimes \bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{4}(s t) \rightarrow \mathcal{D}_{3}(r s)\right) \\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)(r t) \otimes \bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{4}(s t) \rightarrow \mathcal{D}_{3}(r s)\right) \tag{24}
\end{align*}
$$

with $\mathcal{D}_{1}, \mathcal{D}_{2}, \mathcal{D}_{3}$, and $\mathcal{D}_{4}$ being RDTs on $R, T, R S$, and $S T$, respectively. Loosely speaking, 24 can be seen as a domain-independent variant of the Kohout-Bandler superproduct composition whose range is limited to the natural join of $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$.

Analogously as in the classic case, the graded Great Divide is more general than the graded Small Divide. In particular, $\div$ gsdo can be seen as $\div$ ggdo with the divisior being the RDT $1_{\emptyset}$ on the empty relation scheme:

Corollary 3. We have $\mathcal{D}_{1} \div{ }_{\text {gsdo }}^{\mathcal{D}_{3}} \mathcal{D}_{2}=\mathcal{D}_{1} \div{ }_{\text {ggdo }}^{\mathcal{D}_{3}, \mathcal{D}_{2}} 1_{\emptyset}$.
As we have already mentioned, (24) can be generalized in a similar way as 19 to handle RDTs on more general relational schemes.

### 3.5 Darwen's Divide

Later, Darwen [16] proposed another division-like operation which is now commonly called Darwen's Divide. This operation is defined similarly as Date's Great Divide but it does not impose any requirements on the relation schemes of its arguments.

The definition is as follows [16]. For relations $\mathcal{D}_{1}$ on $R_{1}$ (called the dividend), $\mathcal{D}_{2}$ on $R_{2}$ (called the divisor), $\mathcal{D}_{3}$ on $R_{3}$ (called the first mediator), and $\mathcal{D}_{4}$ on
$R_{4}$ (called the second mediator), we put

$$
\begin{equation*}
\mathcal{D}_{1} \div{ }_{\mathrm{ddo}}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}=\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right) \bar{\ltimes}\left(\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \bar{\ltimes} \mathcal{D}_{3}\right) . \tag{25}
\end{equation*}
$$

Note that the relation scheme of result of Darwen's Divide is $R_{1} \cup R_{2}$ since $R_{1}$ and $R_{2}$ are arbitrary and might have some attributes in common.

In the proof of the set notation of Darwen's Divide we utilize the following lemma.

Lemma 4. Consider relations $\mathcal{D}_{1}$ on $R S$ and $\mathcal{D}_{2}$ on $S T$ such that $R, S, T$ are pairwise disjoint $(R \cap S=R \cap T=S \cap T=\emptyset)$. For every tuple $r \in \operatorname{Tupl}(R)$ and $s \in \operatorname{Tupl}(S)$ we have $r s \in \mathcal{D}_{1} \bar{\propto} \mathcal{D}_{2}$ iff

$$
\begin{equation*}
r s \in \mathcal{D}_{1} \wedge \neg(\exists t \in \operatorname{Tupl}(T)) s t \in \mathcal{D}_{2} \tag{26}
\end{equation*}
$$

or equivalently

$$
\begin{equation*}
r s \in \mathcal{D}_{1} \wedge \neg\left(\exists s^{\prime} t \in \operatorname{Tupl}(S T)\right)\left((r s)(S)=\left(s^{\prime} t\right)(S) \wedge s^{\prime} t \in \mathcal{D}_{2}\right), \tag{27}
\end{equation*}
$$

where $s^{\prime} \in \operatorname{Tupl}(S)$ and $t \in \operatorname{Tupl}(T)$.
Proof. The first part follows directly from the definition of semidifference:

$$
\begin{aligned}
r s \in \mathcal{D}_{1} \bar{\ltimes} \mathcal{D}_{2} & \Longleftrightarrow r s \in \mathcal{D}_{1} \backslash \pi_{R S}\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right) \\
& \Longleftrightarrow r s \in \mathcal{D}_{1} \wedge \neg\left(r s \in \mathcal{D}_{1} \wedge s \in \pi_{S}\left(\mathcal{D}_{2}\right)\right) \\
& \Longleftrightarrow \underbrace{\left(r s \in \mathcal{D}_{1} \wedge \neg r s \in \mathcal{D}_{1}\right)}_{\text {always false }} \vee\left(r s \in \mathcal{D}_{1} \wedge \neg s \in \pi_{S}\left(\mathcal{D}_{2}\right)\right) \\
& \Longleftrightarrow r s \in \mathcal{D}_{1} \wedge \neg(\exists t \in \operatorname{Tupl}(T)) s t \in \mathcal{D}_{2} .
\end{aligned}
$$

The rest follows from the fact that $R, S, T$ are pairwise disjoint and $(r s)(S)=$ $\left(s^{\prime} t\right)(S)$ is equivalent to $s=s^{\prime}$.

To simplify the notation, for two tuples $r_{1} \in \operatorname{Tupl}\left(R_{1}\right)$ and $r_{2} \in \operatorname{Tupl}\left(R_{2}\right)$ we denote by $r_{1} \ell r_{2}$ the fact that $r_{1}$ and $r_{2}$ are joinable $\left(r_{1}\left(R_{1} \cap R_{2}\right)=r_{2}\left(R_{1} \cap R_{2}\right)\right)$.

Let us note that the Lemma 4 can be applied to relations on arbitrary schemes. For relations $\mathcal{D}_{1}$ on $R_{1}$ and $\mathcal{D}_{2}$ on $R_{2}$ it suffices to put $R=R_{1} \backslash R_{2}, S=$ $R_{1} \cap R_{2}$ and $T=R_{2} \backslash R_{1}$. Obviously, relation schemes $R, S, T$ defined in this manner are pairwise disjoint and it holds that $R_{1}=R \cup S$ and $R_{2}=S \cup T$. Now for $r_{1} \in \operatorname{Tupl}\left(R_{1}\right)$ using (27) we have $r_{1} \in \mathcal{D}_{1} \bar{\aleph} \mathcal{D}_{2}$ iff

$$
\begin{equation*}
r_{1} \in \mathcal{D}_{1} \wedge \neg\left(\exists r_{2} \in \operatorname{Tupl}\left(R_{2}\right)\right)\left(r_{1} \oint r_{2} \wedge r_{2} \in \mathcal{D}_{2}\right) \tag{28}
\end{equation*}
$$

To put (28) in words, tuple $r_{1}$ belongs to the result of semidifference of $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ (in this order) iff $r_{1}$ belongs to $\mathcal{D}_{1}$ and there is no tuple $r_{2}$ from $\mathcal{D}_{2}$ that is joinable with $r_{1}$.

Theorem 5. Consider relations $\mathcal{D}_{1}$ on $R_{1}, \mathcal{D}_{2}$ on $R_{2}, \mathcal{D}_{3}$ on $R_{3}$, and $\mathcal{D}_{4}$ on $R_{4}$. The definition can be equivalently expressed in the set notation as follows:

$$
\begin{align*}
& \mathcal{D}_{1} \div{ }_{\mathrm{ddo}}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}=  \tag{29}\\
& \quad\left\{r_{1} r_{2} \in \mathcal{U} \mid \text { for all } r_{4} \in \mathcal{D}_{4}: \text { if } r_{1} r_{2} \chi r_{4}, \text { then there is } r_{3} \in \mathcal{D}_{3}: r_{1} r_{4} \chi r_{3}\right\},
\end{align*}
$$

where $\mathcal{U}=\mathcal{D}_{1} \bowtie \mathcal{D}_{2}$.
Proof. First, the fact that $\mathcal{D}_{1} \div{ }_{\text {ddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2} \subseteq \mathcal{D}_{1} \bowtie \mathcal{D}_{2}$ follows directly from the definition of semidifference.

For brevity, in the following proof we will denote the join of $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ by $\mathcal{U}=\mathcal{D}_{1} \bowtie \mathcal{D}_{2}$. Now, let $r_{1} \in \operatorname{Tupl}\left(R_{1}\right)$ and $r_{2} \in \operatorname{Tupl}\left(R_{2}\right)$ be joinable tuples. Using (28) we have

$$
\begin{aligned}
& r_{1} r_{2} \in \mathcal{D}_{1} \div \dot{\mathcal{D}}_{3}, \mathcal{D}_{4} \mathcal{D}_{2} \\
\Longleftrightarrow & r_{1} r_{2} \in\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right) \bar{\ltimes}\left(\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \bar{\ltimes} \mathcal{D}_{3}\right) \\
\Longleftrightarrow & r_{1} r_{2} \in \mathcal{U} \wedge \pi\left(\exists r^{\prime} \in \operatorname{Tupl}\left(R_{1} \cup R_{4}\right)\right)\left(r_{1} r_{2} \nmid r^{\prime} \wedge r^{\prime} \in\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \bar{\ltimes} \mathcal{D}_{3}\right)
\end{aligned}
$$

The tuple $r^{\prime} \in \operatorname{Tupl}\left(R_{1} \cup R_{4}\right)$ can be seen as a join of tuples $r^{\prime}=r_{1}^{\prime} r_{4}$, where $r_{1}^{\prime} \in \operatorname{Tupl}\left(R_{1}\right)$ and $r_{4} \in \operatorname{Tupl}\left(R_{4}\right)$ such that $r_{1}^{\prime} \gamma r_{4}$. We can replace the $\left(\exists r^{\prime} \in \operatorname{Tupl}\left(R_{1} \cup R_{4}\right)\right)$ with $\left(\exists r_{1}^{\prime} \in \operatorname{Tupl}\left(R_{1}\right)\right)\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right)$ and additional constraint that ensures joinability of $r_{1}^{\prime}$ and $r_{4}$.

It is easy to see that $r_{1} r_{2}$ is joinable with $r^{\prime}$ if and only if $r_{1} r_{2}$ is joinable with all "components" of $r^{\prime}$ (here with both $r_{1}^{\prime}$ and $r_{4}$ ). Symbolically, we have $r_{1} r_{2} \ell r^{\prime}$ iff $r_{1} r_{2} \ell r_{1}^{\prime} \wedge r_{1} r_{2} \ell r_{4}$. Since both $r_{1}, r_{1}^{\prime} \in \operatorname{Tupl}\left(R_{1}\right)$, the first condition $r_{1} r_{2} \ell r_{1}^{\prime}$ is equivalent to $r_{1}=r_{1}^{\prime}$. Furthermore, second condition $r_{1} r_{2} \ell r_{4}$ implies $r_{1} \ell r_{4}$.

Continuing the proof and applying (28) to the second semidifference we have

$$
\begin{aligned}
& r_{1} r_{2} \in \mathcal{D}_{1} \div{ }_{\text {ddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2} \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge \neg\left(\exists r^{\prime} \in \operatorname{Tupl}\left(R_{1} \cup R_{4}\right)\right)\left(r_{1} r_{2} \gamma r^{\prime} \wedge r^{\prime} \in\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \bar{\ltimes} \mathcal{D}_{3}\right) \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge \neg\left(\exists r_{1}^{\prime} \in \operatorname{Tupl}\left(R_{1}\right)\right)\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right) \\
& \left(r_{1}^{\prime} \varnothing r_{4} \wedge r_{1}^{\prime}=r_{1} \wedge r_{1} r_{2} \ell r_{4} \wedge r_{1}^{\prime} r_{4} \in\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \ltimes \mathcal{D}_{3}\right) \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge \pi\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right)\left(r_{1} r_{2} \Varangle r_{4} \wedge r_{1} r_{4} \in\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{4}\right) \bar{\ltimes} \mathcal{D}_{3}\right) \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge \neg\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right) \\
& \left(r_{1} r_{2} \ell r_{4} \wedge r_{1} r_{4} \in \mathcal{D}_{1} \bowtie \mathcal{D}_{4} \wedge \pi\left(\exists r_{3} \in \operatorname{Tupl}\left(R_{3}\right)\right)\left(r_{1} r_{4} \curlywedge r_{3} \wedge r_{3} \in \mathcal{D}_{3}\right)\right)
\end{aligned}
$$

Now, $r_{1} r_{4} \in \mathcal{D}_{1} \bowtie \mathcal{D}_{4}$ is equivalent to $r_{1} \in \mathcal{D}_{1} \wedge r_{4} \in \mathcal{D}_{4}$ provided that $r_{1}$ is joinable with $r_{4}$, but this is ensured by $r_{1} r_{2} \ell r_{4}$. Furthermore, $r_{1} \in \mathcal{D}_{1}$ does
not depend on the existence of $r_{4}$ and can be taken outside the scope of the quantifier. We get

$$
\begin{aligned}
& r_{1} r_{2} \in \mathcal{D}_{1} \div{ }_{\text {ddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2} \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge \neg\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right) \\
& \left(r_{1} r_{2} \ell r_{4} \wedge r_{1} r_{4} \in \mathcal{D}_{1} \bowtie \mathcal{D}_{4} \wedge \pi\left(\exists r_{3} \in \operatorname{Tupl}\left(R_{3}\right)\right)\left(r_{1} r_{4} \curlywedge r_{3} \wedge r_{3} \in \mathcal{D}_{3}\right)\right) \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge \neg\left(r_{1} \in \mathcal{D}_{1} \wedge\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right)\right. \\
& \left(r_{1} r_{2} \ell r_{4} \mathbb{\wedge} r_{4} \in \mathcal{D}_{4} \mathbb{\wedge} \pi\left(\exists r_{3} \in \operatorname{Tupl}\left(R_{3}\right)\right)\left(r_{1} r_{4} \ell r_{3} \wedge r_{3} \in \mathcal{D}_{3}\right)\right) \\
& \Longleftrightarrow \overbrace{\left(r_{1} r_{2} \in \mathcal{U} \wedge \pi r_{1} \in \mathcal{D}_{1}\right)}^{\text {always false }} \vee\left(r_{1} r_{2} \in \mathcal{U} \wedge \pi\left(\exists r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right)\right. \\
& \left.\left(r_{1} r_{2} \ell r_{4} \wedge r_{4} \in \mathcal{D}_{4} \wedge \rightarrow\left(\exists r_{3} \in \operatorname{Tupl}\left(R_{3}\right)\right)\left(r_{1} r_{4} \ell r_{3} \wedge r_{3} \in \mathcal{D}_{3}\right)\right)\right) \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge\left(\forall r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right) \\
& \left(\neg\left(r_{1} r_{2} \ell r_{4} \wedge r_{4} \in \mathcal{D}_{4}\right) \vee \neg \rightarrow\left(\exists r_{3} \in \operatorname{Tupl}\left(R_{3}\right)\right)\left(r_{1} r_{4} \ell r_{3} \wedge r_{3} \in \mathcal{D}_{3}\right)\right) \\
& \Longleftrightarrow r_{1} r_{2} \in \mathcal{U} \wedge\left(\forall r_{4} \in \operatorname{Tupl}\left(R_{4}\right)\right) \\
& \left(\left(r_{1} r_{2} \gamma r_{4} \wedge r_{4} \in \mathcal{D}_{4}\right) \Rightarrow\left(\exists r_{3} \in \operatorname{Tupl}\left(R_{3}\right)\right)\left(r_{1} r_{4} \gamma r_{3} \wedge r_{3} \in \mathcal{D}_{3}\right)\right),
\end{aligned}
$$

which concludes the proof.
Now, based on 29, we may introduce a graded variant $\div$ gddo of the Darwen's Divide as follows

$$
\begin{align*}
& \left(\mathcal{D}_{1} \div{ }_{\text {gddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right)= \\
& =\mathcal{D}_{1}\left(r_{1}\right) \otimes \mathcal{D}_{2}\left(r_{2}\right) \otimes \bigwedge_{\substack{r_{4} \in \operatorname{Tupl}\left(R_{4}\right) \\
r_{1} r_{2} \backslash r_{4}}}\left(\mathcal{D}_{4}\left(r_{4}\right) \rightarrow \bigvee_{\substack{r_{3} \in \operatorname{Tupl}\left(R_{3}\right) \\
r_{1} r_{4} \backslash r_{3}}} \mathcal{D}_{3}\left(r_{3}\right)\right) \\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right) \otimes \bigwedge_{\substack{r_{4} \in \operatorname{Tupl}^{\prime}\left(R_{4}\right) \\
r_{1} r_{2} \searrow r_{4}}}\left(\mathcal{D}_{4}\left(r_{4}\right) \rightarrow \underset{\substack{r_{3} \in \operatorname{Tupl}^{\prime}\left(R_{3}\right) \\
r_{1} r_{4} \searrow r_{3}}}{ } \mathcal{D}_{3}\left(r_{3}\right)\right) \tag{30}
\end{align*}
$$

The condition of joinability is not necessary and can be avoided. We can put $R_{4 \backslash 12}=R_{4} \backslash\left(R_{1} \cup R_{2}\right), R_{4 \cap 12}=R_{4} \cap\left(R_{1} \cup R_{2}\right), R_{3 \backslash 14}=R_{3} \backslash\left(R_{1} \cup R_{4}\right)$ and $R_{3 \cap 14}=R_{3} \cap\left(R_{1} \cup R_{4}\right)$. Obviously, it holds that $R_{4 \backslash 12} \cap R_{4 \cap 12}=\emptyset$ and $R_{4 \backslash 12} \cup R_{4 \cap 12}=R_{4}$. The same holds for $R_{3 \backslash 14}$ and $R_{3 \cap 14}$. Now, denote by $r_{12}^{\ell_{4}}=\left(r_{1} r_{2}\right)\left(R_{4 \cap 12}\right)$ the projection of tuple $r_{1} r_{2}$ onto $R_{4 \cap 12}$ (i.e. onto common attributes of $R_{4}$ and $R_{1} \cup R_{2}$. Considering $r_{4}^{\prime} \in \operatorname{Tupl}\left(R_{4 \backslash 12}\right)$ we get $r_{4}=r_{12}^{\ell 4} r_{4}^{\prime}$. Observe, that tuples $r_{12}^{\chi 4}$ and $r_{4}^{\prime}$ are always joinable since $R_{4 \backslash 12} \cap R_{4 \cap 12}=\emptyset$. We have expressed the tuple $r_{4}$ without any need for joinability condition and we can remove the condition from the infimum operation.

We can now proceed to the condition in supremum. Note, that

$$
r_{1} r_{4}=r_{1} r_{12}^{\bigvee 4} r_{4}^{\prime}=r_{1}\left(r_{1} r_{2}\right)\left(R_{4 \cap 12}\right) r_{4}^{\prime}=r_{1}\left(r_{2}\right)\left(R_{4 \cap 2}\right) r_{4}^{\prime}=r_{1} r_{2}^{\ell 4} r_{4}^{\prime}
$$

Again, by $r_{14}^{\bigvee 3}=\left(r_{1} r_{4}\right)\left(R_{3 \cap 14}\right)=\left(r_{1} r_{2}^{\bigvee 4} r_{4}^{\prime}\right)\left(R_{3 \cap 14}\right)$ we denote the projection of the tuple in question onto $R_{3 \cap 14}$. For $r_{3}^{\prime} \in \operatorname{Tupl}\left(R_{3 \backslash 14}\right)$ we get $r_{3}=r_{14}^{\searrow 3} r_{3}^{\prime}$. Using similar argument, $r_{14}^{\gamma 3}$ and $r_{3}^{\prime}$ are always joinable.

Putting both observations together we finally get

$$
\begin{align*}
& \left(\mathcal{D}_{1} \div{ }_{\text {gddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right)= \\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right) \otimes \bigwedge_{\substack{r_{4} \in \operatorname{Tupl}\left(R_{4}\right) \\
r_{1} r_{2} \emptyset r_{4}}}(\mathcal{D}_{4}\left(r_{4}\right) \rightarrow \underbrace{}_{\substack{r_{3} \in \operatorname{Tupl}\left(R_{3}\right) \\
r_{1} r_{4} \backslash r_{3}}} \mathcal{D}_{3}\left(r_{3}\right)) \\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right) \otimes \bigwedge_{r_{4}^{\prime} \in \operatorname{Tupl}\left(R_{4 \backslash 12}\right)}\left(\mathcal{D}_{4}\left(r_{12}^{\bigvee 4} r_{4}^{\prime}\right) \rightarrow \bigvee_{r_{3}^{\prime} \in \operatorname{Tupl}\left(R_{3 \backslash 14}\right)} \mathcal{D}_{3}\left(r_{14}^{\bigvee 3} r_{3}^{\prime}\right)\right)  \tag{31}\\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right) \otimes \bigwedge_{r_{4}^{\prime} \in \operatorname{Tupl}\left(R_{4 \backslash 12}\right)}\left(\mathcal{D}_{4}\left(r_{12}^{\bigvee 4} r_{4}^{\prime}\right) \rightarrow \pi_{R_{3 \cap 14}}\left(\mathcal{D}_{3}\right)\left(r_{14}^{\bigvee 3}\right)\right) \tag{32}
\end{align*}
$$

Graded Date's Great and Small Divide can be easily expressed by the graded version of Darwen's Divide in the following way.

Theorem 6. For relations on schemes that conform to requirements for Great Divide, precisely for relations $\mathcal{D}_{1}$ on $R, \mathcal{D}_{2}$ on $T, \mathcal{D}_{3}$ on $R S$, and $\mathcal{D}_{4}$ on $S T$, we have

$$
\mathcal{D}_{1} \div{ }_{\text {ggdo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}=\mathcal{D}_{1} \div{ }_{\text {gddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2} .
$$

Proof. For relations $\mathcal{D}_{1}$ on $R, \mathcal{D}_{2}$ on $T, \mathcal{D}_{3}$ on $R S$ and $\mathcal{D}_{4}$ on $S T$, we have

$$
\begin{aligned}
\left(\mathcal{D}_{1}\right. & \left.\div{ }_{\text {gddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}\right)(r t)= \\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)\left(r_{1} r_{2}\right) \otimes \bigwedge_{r_{4}^{\prime} \in \operatorname{Tupl}(S T \backslash(R \cup T))}\left(\mathcal{D}_{4}\left(r_{12}^{\bigotimes 4} r_{4}^{\prime}\right) \rightarrow \pi_{R S \cap(R \cup S T)}\left(\mathcal{D}_{3}\right)\left(r_{14}^{\bigvee 3}\right)\right) \\
& =\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right)(r t) \otimes \bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{4}(s t) \rightarrow \mathcal{D}_{3}(r s)\right) \\
& =\left(\mathcal{D}_{1} \div{ }_{\text {ggdo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}\right)(r t) .
\end{aligned}
$$

Corollary 7. For relations on schemes that conform to requirements for Small Divide, precisely for relations $\mathcal{D}_{1}$ on $R, \mathcal{D}_{2}$ on $S$ and $\mathcal{D}_{3}$ on $R S$, we have $\mathcal{D}_{1} \div{ }_{\text {gsdo }}^{\mathcal{D}_{3}} \mathcal{D}_{2}=\mathcal{D}_{1} \div{ }_{\text {gddo }}^{\mathcal{D}_{3}, \mathcal{D}_{2}} 1_{\emptyset}$.

## 4 Pseudo Tuple Relational Calculus

In this section, we present a query language we use in this paper for easier reasoning about the relational algebra operations. The Pseudo Tuple Calculus (shortly, PTC) is similar to the ordinary tuple calculus, however, it provides more convenient way to reason about relational algebra expressions in the presence of scores. In the next section we use the PTC to show mutual relationships among the division operations.

### 4.1 PTC-expressions and their evaluation

Every PTC-expression $\mathcal{T}\left(\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right)$ of Pseudo Tuple Calculus is associated with a finite set of free tuple variables $r_{1}, \ldots, r_{n}$ that appear in the PTCexpression. For each tuple variable $\mathfrak{r}_{i}$ we consider its relation scheme $R_{i}$. We assume that tuple variables with the same name have the same relation scheme. The relation scheme $R_{\mathcal{T}}$ of PTC-expression $\mathcal{T}\left(\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right)$ is given by the union of relation schemes of the tuple variables $R_{\mathcal{T}}=\bigcup_{i=1}^{n} R_{i}$.

Since we do not utilize any disjunctive operations in this paper we define here only a fragment of the Pseudo Tuple Calculus without the corresponding disjunctive expressions. For the same reason we omit the treatment of restrictions as well.

### 4.1.1 Syntax of PTC-expressions

The PTC-expressions are defined inductively as follows.

1. if $E$ is a relational algebra expression (shortly, RA-expression) on relation scheme $R$ and $\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}$ are tuple variables on $R_{1}, \ldots, R_{n}$ such that $R=$ $\bigcup_{i=1}^{n} R_{i}$, then $E\left(\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right)$ is an (atomic) PTC-expression on relation scheme $R$.

In order to keep our notation simple, we abbreviate finite sets of tuple variables as $\mathbf{r}=\left\{\mathfrak{r}_{1}, \ldots, r_{n}\right\}$ and their corresponding relation schemes as $R_{\mathbf{r}}=\bigcup_{i=1}^{n} R_{i}$. In the simplified notation, the (atomic) PTC-expression $E\left(\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right)$ becomes $E(\mathbf{r})$.
2. if $\mathcal{T}_{1}\left(\mathbf{r}_{1}\right)$ and $\mathcal{T}_{2}\left(\mathbf{r}_{2}\right)$ are PTC-expressions on $R_{\mathbf{r}_{1}}$ and $R_{\mathbf{r}_{2}}$ respectively, then $\left(\mathcal{T}_{1}\left(\mathbf{r}_{1}\right) \circ \mathcal{T}_{2}\left(\mathbf{r}_{2}\right)\right)\left(\mathbf{r}_{1} \cup \mathbf{r}_{2}\right)$ is PTC-expression on $R_{\mathbf{r}_{1}} \cup R_{\mathbf{r}_{2}}$, where $\circ$ is one of the following symbols $\otimes, \wedge, \rightarrow$. Note that $\mathbf{r}_{1} \cup \mathbf{r}_{2}$ is well-defined since we assume that tuple variables with the same name have the same relation scheme.

To simplify notation, we do not have to explicitly mention the set $\mathbf{r}_{1} \cup \mathbf{r}_{2}$ since it can be easily deduced from the form of the subexpressions. Thus, the above mentioned PTC-expression becomes $\mathcal{T}_{1}\left(\mathbf{r}_{1}\right) \circ \mathcal{T}_{2}\left(\mathbf{r}_{2}\right)$. In more complex expressions we utilize outer parentheses to avoid ambiguity in the usual way.
3. if $\mathcal{T}(\mathbf{r})$ is PTC-expression on $R_{\mathbf{r}}$ then $(\nabla \mathcal{T}(\mathbf{r}))(\mathbf{r})$ and $(\Delta \mathcal{T}(\mathbf{r}))(\mathbf{r})$ are PTC-expressions on $R_{\mathbf{r}}$. In simplified notation we have $\nabla \mathcal{T}(\mathbf{r})$ and $\Delta \mathcal{T}(\mathbf{r})$.
4. if $\mathcal{T}\left(\mathbf{r}_{1} \cup \mathbf{r}_{2}\right)$ is PTC-expression on $R_{\mathbf{r}_{1}} \cup R_{\mathbf{r}_{2}}$ such that $R_{\mathbf{r}_{1}} \cap R_{\mathbf{r}_{2}}=\emptyset$ then $\left(\bigvee_{\mathbf{r}_{1}} \mathcal{T}\left(\mathbf{r}_{1} \cup \mathbf{r}_{2}\right)\right)\left(\mathbf{r}_{\mathbf{2}}\right)$ and $\left(\bigwedge_{\mathbf{r}_{1}} \mathcal{T}\left(\mathbf{r}_{1} \cup \mathbf{r}_{2}\right)\right)\left(\mathbf{r}_{\mathbf{2}}\right)$ are PTC-expressions on $R_{\mathbf{r}_{2}}$.
For aesthetic reasons we will denote the set $\mathbf{r}_{1} \cup \mathbf{r}_{2}$ by $\mathbf{r}_{1}, \mathbf{r}_{2}$. In the simplified notation we get $\bigvee_{\mathbf{r}_{1}} \mathcal{T}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ and $\bigwedge_{\mathbf{r}_{1}} \mathcal{T}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$.

### 4.1.2 Semantics of PTC-expressions

The evaluation of PTC-expressions is based on the notion of a database instance $\mathcal{D}$. Loosely speaking, a database instance assigns appropriate relations to relation symbols from a database scheme - database instance can be seen as a snapshot of all base relations that we have in some database. Relations in database naturally change in time, however the database instance is fixed as it reflects the state of the database in a given point of time. We tacitly assume that the database scheme is clear from the context.

Furthermore, we utilize the notion of extended active domains. First, we define the active domain $\operatorname{adom}(y, \mathcal{D})$ for the given attribute $y$ and relation $\mathcal{D}$ as a projection of $\mathcal{D}$ onto $y$ where all tuples with non-zero scores have their score set to one. We denote by $\mathcal{D}_{i}^{y}(i \in I)$ relations from the given database instance $\mathcal{D}$ whose relation schemes contain the attribute $y\left(\{y\} \subseteq R_{i}\right)$. The extended active domain $\operatorname{eadom}{ }^{\mathcal{D}}(y)$ for the given database instance $\mathcal{D}$ and attribute $y$ is defined as

$$
\operatorname{eadom}^{\mathcal{D}}(y)=\bigcup_{i \in I} \operatorname{adom}\left(y, \mathcal{D}_{i}^{y}\right)
$$

For the entire relation scheme $R=\left\{y_{1}, \ldots, y_{n}\right\}$ we define the extended active domain as

$$
\operatorname{eadom}_{R}^{\mathcal{D}}=\operatorname{eadom}^{\mathcal{D}}\left(y_{1}\right) \bowtie \cdots \bowtie \operatorname{eadom}^{\mathcal{D}}\left(y_{n}\right)
$$

It is easy to see that the $e a d o m_{R}^{\mathcal{D}}$ contains every tuple on relation scheme $R$ that can be built from all values of respective domains that are available in the database instance in question. The extended active domain can be seen as a
finite universe of tuples for the given database instance and relation scheme if we do not allow introduction of new domain values (by singleton relations).

Remark 7. As an aside, let us mention that it is easy to modify the definition of extended active domain to incorporate new values introduced by singleton relations. Since RA-expressions are finite, the number of new values is finite as well. Before obtaining extended active domain eadom ${ }_{R}^{\mathcal{D}}$ by joining all extended active domains eadom $\left(y_{i}\right)$ for attributes $y_{i} \in R(i \in\{1, \ldots, n\})$ it suffices to unify each eadom $\left(y_{i}\right)$ with a (finite) set of new values whose domain coincides with the domain of attribute $y_{i}$.

Now, we define the evaluation of PTC-expressions in database instances. Suppose we have the database instance $\mathcal{D}$ and a PTC-expression $\mathcal{T}(\mathbf{r})$, where $\mathbf{r}=\left\{\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right\}$ such that each tuple variable $\mathfrak{r}_{i}$ is on relation scheme $R_{i}$. By evaluating $\mathcal{T}(\mathbf{r})$ in $\mathcal{D}$ we obtain a relation $\mathcal{T}^{\mathcal{D}}$ on relation scheme $R=\bigcup_{i=1}^{n} R_{i}$. For any tuple $r \notin \operatorname{eadom} m_{R}^{\mathcal{D}}$ we put $\mathcal{T}^{\mathcal{D}}(r)=0$. In other words, the relation $\mathcal{T}^{\mathcal{D}}$ may contain only tuples from $\operatorname{eadom}_{R}^{\mathcal{D}}$. For each tuple $r \in e a d o m_{R}^{\mathcal{D}}$ we define its score in the relation $\mathcal{T}^{\mathcal{D}}$ as follows.

Any tuple $r \in e a d o m_{R}^{\mathcal{D}}$ induces a valuation of the tuple variables $\mathbb{r}_{1}, \ldots, \mathfrak{r}_{n}$ from the PTC-expression. The valuation assigns each variable $⿷_{i}$ the projection of tuple $r$ onto the relation scheme $R_{i}$ of the variable in question, symbolically $\left\|\mathrm{r}_{i}\right\|_{r}=r\left(R_{i}\right)$. We denote the join of valuated tuple variables $\left\|\mathrm{r}_{1}\right\|_{r} \cdots\left\|\mathrm{r}_{n}\right\|_{r}$ as $\|\mathbf{r}\|_{r}$. It is easy to see that $\|\mathbf{r}\|_{r}=r$. In general, for a set of tuple variables $\mathbf{r}^{\prime}$ such that $\mathbf{r}^{\prime} \subseteq \mathbf{r}$ with relation scheme $R_{\mathbf{r}^{\prime}} \subseteq R$ it holds that $\left\|\mathbf{r}^{\prime}\right\|_{r}=r\left(R_{\mathbf{r}^{\prime}}\right)$. We define the score $\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)$ of tuple $\|\mathbf{r}\|_{r}$ in the relation $\mathcal{T}^{\mathcal{D}}$ as follows. According to the form of PTC-expression we distinguish the following cases

1. if $\mathcal{T}(\mathbf{r})$ is $E(\mathbf{r})$, we first evaluate the RA-expression $E$ in the database instance $\mathcal{D}$ according to RA-expression evaluation rules (3) and denote the resulting relation as $E^{\mathcal{D}}$, then we set $\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=E^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)$,
2. if $\mathcal{T}(\mathbf{r})$ is $\mathcal{T}_{1}\left(\mathbf{r}_{1}\right) \circ \mathcal{T}_{2}\left(\mathbf{r}_{2}\right)$, where $\circ$ is on of the following symbols $\otimes, \wedge, \rightarrow$, and $\mathbf{r}=\mathbf{r}_{1} \cup \mathbf{r}_{2}$, first we get the scores $\mathcal{T}_{1}^{\mathcal{D}}\left(\left\|\mathbf{r}_{1}\right\|_{r}\right)$ and $\mathcal{T}_{2}^{\mathcal{D}}\left(\left\|\mathbf{r}_{2}\right\|_{r}\right)$ with the valuation induced by $r$. Then we set $\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=\mathcal{T}_{1}^{\mathcal{D}}\left(\left\|\mathbf{r}_{1}\right\|_{r}\right) \circ \mathcal{T}_{2}^{\mathcal{D}}\left(\left\|\mathbf{r}_{2}\right\|_{r}\right)$, where $\circ$ is one of the following operations $\otimes, \wedge, \rightarrow$.
3. if $\mathcal{T}(\mathbf{r})$ is $\nabla \mathcal{T}^{\prime}(\mathbf{r})$ or $\Delta \mathcal{T}^{\prime}(\mathbf{r})$ we get the score $\mathcal{T}^{\prime \mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)$.

If $\mathcal{T}(\mathbf{r})$ is $\nabla \mathcal{T}^{\prime}(\mathbf{r})$ we set

$$
\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)= \begin{cases}1 & \text { if } \mathcal{T}^{\prime \mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)>0 \\ 0 & \text { otherwise }\end{cases}
$$

If $\mathcal{T}(\mathbf{r})$ is $\Delta \mathcal{T}^{\prime}(\mathbf{r})$ we set

$$
\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)= \begin{cases}1 & \text { if } \mathcal{T}^{\prime \mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=1 \\ 0 & \text { otherwise }\end{cases}
$$

4. if $\mathcal{T}(\mathbf{r})$ is $\bigvee_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ or $\bigwedge_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ where $\mathbf{r}=\mathbf{r}_{\mathbf{2}}$, first we get the scores $\mathcal{T}^{\prime \mathcal{D}}\left(\left\|\mathbf{r}_{\mathbf{1}} \cup \mathbf{r}_{\mathbf{2}}\right\|_{r r^{\prime}}\right)$ with the valuation induced by the join of tuples $r$ and $r^{\prime}$ for every $r^{\prime} \in \operatorname{eadom} m_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}$. Note that the tuples $r$ and $r^{\prime}$ are always joinable as the relation schemes $R_{\mathbf{r}_{1}}$ and $R_{\mathbf{r}_{2}}=R$ are disjoint (from the definition of PTC-expression). Since eadom ${\underset{R_{r_{1}}}{\mathcal{D}}}^{\text {in }}$ is finite, we obtain a finite set of scores $\left\{\mathcal{T}^{\prime \mathcal{D}}\left(\left\|\mathbf{r}_{\mathbf{1}} \cup \mathbf{r}_{\mathbf{2}}\right\|_{r r^{\prime}}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\}$.
If $\mathcal{T}(\mathbf{r})$ is $\bigvee_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ we set

$$
\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=\bigvee\left\{\mathcal{T}^{\prime \mathcal{D}}\left(\left\|\mathbf{r}_{\mathbf{1}} \cup \mathbf{r}_{\mathbf{2}}\right\|_{r r^{\prime}}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\}
$$

If $\mathcal{T}(\mathbf{r})$ is $\bigwedge_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ we set

$$
\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=\bigwedge\left\{\mathcal{T}^{\prime \mathcal{D}}\left(\left\|\mathbf{r}_{1} \cup \mathbf{r}_{\mathbf{2}}\right\|_{r r^{\prime}}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\}
$$

### 4.1.3 Splitting principle

Consider a PTC-expression $\mathcal{T}(\ldots, r, \ldots)$ such that the tuple variable $\mathfrak{r}$ is on relation scheme $R$. If we replace the tuple variable $\mathbb{r}$ with two (or more) fresh tuple variables $\mathfrak{r}_{1}, \mathfrak{r}_{2}$ on $R_{1}$ and $R_{2}$ such that $R_{1} \cup R_{2}=R$, we obtain a PTCexpression $\mathcal{T}^{\prime}\left(\ldots, r_{1}, r_{2}, \ldots\right)$ that differs only in the set of free variables. Despite being different on the syntactic level it is straightforward to see that for any database instance $\mathcal{D}$ we have $\mathcal{T}^{\mathcal{D}}=\mathcal{T}^{\prime \mathcal{D}}$.

From the semantic point of view, we are free to "split" free tuple variables and "join" them back without changing the meaning of the PTC-expression. We call this "the splitting principle".

### 4.2 Equivalence of PTC and Relational Algebra

In this section we show that the Pseudo Tuple Calculus and Relational Algebra are equivalent. First, observe that if we evaluate any RA-expression $E$ on relation scheme $R$ in a database instance $\mathcal{D}$, the relation $E^{\mathcal{D}}$ may contain tuples from eadom $\mathcal{D}_{R}^{\mathcal{D}}$ only, since eadom $\mathcal{D}_{R}^{\mathcal{D}}$ consists of all tuples that can possibly be built from the values available in the database instance, i.e. we have

$$
\begin{equation*}
E^{\mathcal{D}}(r)=0 \text { whenever } r \notin \operatorname{eadom}_{R}^{\mathcal{D}} \tag{33}
\end{equation*}
$$

Using this observation we can easily prove the following theorem.

Theorem 8. For any $R A$-expression $E$ on relation scheme $R$ there is a PTCexpression $\mathcal{T}(\mathbf{r})$ on $R$ such that for any database instance $\mathcal{D}$ we have $E^{\mathcal{D}}(r)=$ $\mathcal{T}^{\mathcal{D}}(r)$ for all $r \in \operatorname{Tupl}(R)$.

Proof. Since any RA-expression is directly an (atomic) PTC-expression we can take $E(\mathbb{r})$ with a single tuple variable $\mathbb{r}$ on relation scheme $R$ as the sought PTC-expression $\mathcal{T}(\mathbf{r})$. From the definition of PTC-expression evaluation and the observation (33) we conclude that $E^{\mathcal{D}}(r)=\mathcal{T}^{\mathcal{D}}(r)$ holds for all $r \in \operatorname{Tupl}(R)$.

It follows that the Pseudo Tuple Calculus is at least as powerful as the Relational Algebra. Before proving the converse theorem we need one more observation. Recall that the relation $\operatorname{eadom}_{R}^{\mathcal{D}}$ plays an important role in PTCexpression evaluation as it serves the purpose of an implicit range (or universe) for evaluation. Since evaluation of RA-expressions is unconstrained and takes all tuples in account we need to be able to construct a RA-expression $\mathcal{E}_{R}$ that will evaluate to $\operatorname{eadom}{ }_{R}^{\mathcal{D}}$ and will act as an explicit range for evaluation of RA-expressions.

It is easy to see that the active domain $\operatorname{adom}(y, \mathcal{D})$ for the given attribute $y$ and relation $\mathcal{D}$ can be computed by evaluating the RA-expression $\mathcal{A}_{y}(\mathbb{D})=$ $\pi_{\{y\}}(\nabla \mathbb{D})$ in database instance $\mathcal{D}$, where $\mathbb{D}$ is a relation symbol evaluated to relation $\mathcal{D}$ by the database instance. For the extended active domain eadom ${ }^{\mathcal{D}}(y)$ for an attribute $y$ the RA-expression is $\mathcal{E}_{y}=\bigcup_{i \in I} \mathcal{A}_{y}\left(\mathbb{D}_{i}^{y}\right)$, where $\mathbb{D}_{i}^{y}$ are relation symbols whose relation scheme contains attribute $y$ and the database instance $\mathcal{D}$ interprets each relation symbol $\mathbb{D}_{i}^{y}$ as relation $\mathcal{D}_{i}^{y}$. Finally, we get the extended active domain eadom ${ }_{R}^{\mathcal{D}}$ for scheme $R=\left\{y_{1}, \ldots, y_{n}\right\}$ by evaluating $\mathcal{E}_{R}=\mathcal{E}_{y_{1}} \bowtie$ $\cdots \bowtie \mathcal{E}_{y_{n}}$ in database instance $\mathcal{D}$. In other words we have $\operatorname{eadom}_{R}^{\mathcal{D}}=\mathcal{E}_{R}^{\mathcal{D}}$.

Remark 8. As an aside, if we use the modified definition of extended active domain that allows introduction of new values by singleton relations, we need to modify the previous definition of $\mathcal{E}_{y}$ to reflect the extended meaning of eadom ${ }_{R}^{\mathcal{D}}$. The definition becomes $\mathcal{E}_{y}=\bigcup_{i \in I}\left(\mathcal{A}_{y}\left(\mathbb{D}_{i}^{y}\right)\right) \cup \bigcup_{j=1}^{n}\left[y: \mathfrak{c}_{j}\right]$, where $\mathbb{D}_{i}^{y}$ are relation symbols whose relation scheme contains attribute $y$ and $\mathfrak{c}_{j}$ are symbols denoting new values from the domain of attribute $y$ such that the database instance $\mathcal{D}$ interprets each relation symbol $\mathbb{D}_{i}^{y}$ as relation $\mathcal{D}_{i}^{y}$ and each symbol $\mathfrak{c}_{j}$ as the new value $\mathfrak{c}_{j}^{\mathcal{D}}$ from the respective domain.

Theorem 9. For any PTC-expression $\mathcal{T}(\mathbf{r})$ with $\mathbf{r}=\left\{\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right\}$, where the tuple variables $\mathbb{r}_{i}$ are on relation schemes $R_{i}$, there is a $R A$-expression $F$ on relation scheme $R=\bigcup_{i=1}^{n} R_{i}$ such that for any database instance $\mathcal{D}$ we have $F^{\mathcal{D}}(r)=\mathcal{T}^{\mathcal{D}}(r)$ for all $r \in \operatorname{Tupl}(R)$.

Proof. The theorem is proved by induction on the complexity of the PTCexpression. In each step, we show the RA-expression $F$ that forms the counterpart to the PTC-expression $\mathcal{T}(\mathbf{r})$ in question. Furthermore, we show that the results of evaluating both RA- and PTC-expression coincide, i.e. the relations $F^{\mathcal{D}}$ and $\mathcal{T}^{\mathcal{D}}$ have the same relation scheme and contain the same tuples. Recall that for a set of tuple variables $\mathbf{r}^{\prime}$ on the relation scheme $R_{\mathbf{r}^{\prime}}$ and a tuple $r \in \operatorname{Tupl}(R)$ such that $R_{\mathbf{r}^{\prime}} \subseteq R$ we have $\left\|\mathbf{r}^{\prime}\right\|_{r}=r\left(R_{\mathbf{r}^{\prime}}\right)$.

Let us have a PTC-expression $\mathcal{T}(\mathbf{r})$, where $\mathbf{r}=\left\{\mathfrak{r}_{1}, \ldots, \mathfrak{r}_{n}\right\}$ such that each tuple variable $\mathfrak{r}_{i}$ is on relation scheme $R_{i}$. The relation scheme of the relation $\mathcal{T}^{\mathcal{D}}$ is $R=\bigcup_{i=1}^{n} R_{i}$. We obtain the equivalent RA-expression $F$ as follows.

1. If $\mathcal{T}(\mathbf{r})$ is $E(\mathbf{r})$, the sought RA-expression $F$ is $E$.

Since the relation scheme of $F$ is $R$, the relations $\mathcal{T}^{\mathcal{D}}$ and $F^{\mathcal{D}}$ have the same relation scheme. For any tuple $r \in e a d o m_{R}^{\mathcal{D}}$ we have

$$
\mathcal{T}^{\mathcal{D}}(r)=\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=E^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=F^{\mathcal{D}}(r)
$$

from the definition of PTC-expression evaluation.
From (33) it follows that for all tuples $r \notin \operatorname{eadom}{ }_{R}^{\mathcal{D}}$ we have $F^{\mathcal{D}}(r)=0$. Together, we have $\mathcal{T}^{\mathcal{D}}(r)=F^{\mathcal{D}}(r)$ for all tuples $r \in \operatorname{Tupl}(R)$.
2. If $\mathcal{T}(\mathbf{r})$ is $\mathcal{T}_{1}\left(\mathbf{r}_{1}\right) \circ \mathcal{T}_{2}\left(\mathbf{r}_{2}\right)$, where $\circ$ is on of the following symbols $\otimes, \wedge, \rightarrow$, $\mathbf{r}=\mathbf{r}_{1} \cup \mathbf{r}_{2}$ and $R=R_{\mathbf{r}_{1}} \cup R_{\mathbf{r}_{2}}$, then from the induction hypothesis we have RA-expressions $E_{1}$ on relation scheme $R_{\mathbf{r}_{1}}$ and $E_{2}$ on relation scheme $R_{\mathbf{r}_{2}}$ corresponding to PTC-subexpressions $\mathcal{T}_{1}\left(\mathbf{r}_{1}\right)$ and $\mathcal{T}_{2}\left(\mathbf{r}_{2}\right)$, respectively, such that $\mathcal{T}_{1}^{\mathcal{D}}\left(r_{1}\right)=E_{1}^{\mathcal{D}}\left(r_{1}\right)$ and $\mathcal{T}_{2}^{\mathcal{D}}\left(r_{2}\right)=E_{2}^{\mathcal{D}}\left(r_{2}\right)$ for all $r_{1} \in \operatorname{Tupl}\left(R_{1}\right)$ and $r_{2} \in \operatorname{Tupl}\left(R_{2}\right)$.

According to the symbol $\circ$ we distinguish three cases:
(a) If $\circ$ is $\otimes$, then we put $F=E_{1} \bowtie E_{2}$.

The relation scheme of $F$ is $R_{\mathbf{r}_{1}} \cup R_{\mathbf{r}_{2}}$ as required. We have

$$
\begin{aligned}
\mathcal{T}^{\mathcal{D}}(r) & =\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right) \\
& =\mathcal{T}_{1}^{\mathcal{D}}\left(\left\|\mathbf{r}_{1}\right\|_{r}\right) \otimes \mathcal{T}_{2}^{\mathcal{D}}\left(\left\|\mathbf{r}_{2}\right\|_{r}\right) \\
& =\mathcal{T}_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \otimes \mathcal{T}_{2}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right) \\
& =E_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \otimes E_{2}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right) \\
& =\left(E_{1} \bowtie E_{2}\right)^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right) r\left(R_{\mathbf{r}_{2}}\right)\right) \\
& =F^{\mathcal{D}}(r)
\end{aligned}
$$

for all tuples $r \in \operatorname{eadom}{ }_{R}^{\mathcal{D}}$.
For any tuple $r \notin \operatorname{eadom}{ }_{R}^{\mathcal{D}}$ either or both of $r\left(R_{\mathbf{r}_{1}}\right) \notin \operatorname{eadom}{ }_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}$ and $r\left(R_{\mathbf{r}_{2}}\right) \notin \operatorname{eadom}{\underset{R_{\mathbf{r}_{2}}}{\mathcal{D}}}^{\mathcal{D}^{2}}$ must hold, otherwise we would arrive at contradiction. Without loss of generality let us assume that $r\left(R_{\mathbf{r}_{1}}\right) \notin$ eadom ${\underset{R_{\mathbf{r}_{1}}}{\mathcal{D}}}_{\mathcal{D}}$. Then we have $\mathcal{T}_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right)=0$ and from the induction hypothesis we also have $E_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right)=0$. From the properties of $\otimes$ we conclude that $F^{\mathcal{D}}(r)=0$ for $r \notin \operatorname{eadom}_{R}^{\mathcal{D}}$.
(b) If $\circ$ is $\wedge$, then we put $F=\left(E_{1} \bowtie \mathcal{E}_{R_{\mathrm{r}_{2}}}\right) \cap\left(E_{2} \bowtie \mathcal{E}_{R_{\mathrm{r}_{1}}}\right)$.

Since both $E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}$ and $E_{2} \bowtie \mathcal{E}_{R_{\mathbf{r}_{1}}}$ are on relation scheme $R_{\mathbf{r}_{1}} \cup$ $R_{\mathbf{r}_{2}}$, the RA-expression $F$ is well-defined and its relation scheme matches the relation scheme of the PTC-expression.
Now, observe that for any tuple $r \in e a d o m_{R}^{\mathcal{D}}$ the following holds

$$
E_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right)=E_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \otimes \underbrace{\operatorname{eadom}{\underset{R_{r_{2}}}{\mathcal{D}}}_{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right)}_{=1}=\left(E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}\right)^{\mathcal{D}}(r)
$$

Dually, it holds for $E_{2}^{\mathcal{D}}$ as well. To put the in words, we can "extend" the relation scheme of some relation without changing the scores of tuples in this relation. Hence, we have

$$
\begin{aligned}
\mathcal{T}^{\mathcal{D}}(r) & =\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right) \\
& =\mathcal{T}_{1}^{\mathcal{D}}\left(\left\|\mathbf{r}_{1}\right\|_{r}\right) \wedge \mathcal{T}_{2}^{\mathcal{D}}\left(\left\|\mathbf{r}_{2}\right\|_{r}\right) \\
& =\mathcal{T}_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \wedge \mathcal{T}_{2}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right) \\
& =E_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \wedge E_{2}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right) \\
& =\left(E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}\right) \mathcal{D}^{\mathcal{D}}(r) \wedge\left(E_{2} \bowtie \mathcal{E}_{R_{\mathbf{r}_{1}}}\right)^{\mathcal{D}}(r) \\
& =\left(\left(E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}\right) \cap\left(E_{2} \bowtie \mathcal{E}_{R_{\mathbf{r}_{1}}}\right)\right)^{\mathcal{D}}(r) \\
& =F^{\mathcal{D}}(r)
\end{aligned}
$$

for all tuples $r \in \operatorname{eadom}_{R}^{\mathcal{D}}$.
For any tuple $r \notin$ eadom $_{R}^{\mathcal{D}}$, use the same argument as for the case with $\otimes$ concluding that $F^{\mathcal{D}}(r)=0$ for $r \notin \operatorname{eadom}{ }_{R}^{\mathcal{D}}$.
(c) If $\circ$ is $\rightarrow$, then we put $F=\left(E_{1} \bowtie \mathcal{E}_{R_{\mathrm{r}_{2}}}\right) \rightarrow \mathcal{E}_{R}\left(E_{2} \bowtie \mathcal{E}_{R_{\mathrm{r}_{1}}}\right)$.

Since all $E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}, E_{2} \bowtie \mathcal{E}_{R_{\mathrm{r}_{1}}}$, and $\mathcal{E}_{R}$ are on relation scheme $R_{\mathbf{r}_{1}} \cup R_{\mathbf{r}_{2}}$, the RA-expression $F$ is well-defined and its relation scheme matches the relation scheme of the PTC-expression.
Observe that since $\mathcal{T}^{\mathcal{D}}(r)>0$ only for tuples $r \in \operatorname{eadom}_{R}^{\mathcal{D}}$ and $\mathcal{E}_{R}^{\mathcal{D}}(r)=\operatorname{eadom}_{R}^{\mathcal{D}}(r)=1$ for any tuple $r \in \operatorname{eadom}_{R}^{\mathcal{D}}$, it holds that
$\mathcal{T}^{\mathcal{D}}(r)=\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes \mathcal{T}^{\mathcal{D}}(r)$. Using previous observations we have

$$
\begin{aligned}
\mathcal{T}^{\mathcal{D}}(r) & =\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes \mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right) \\
& =\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes\left(\mathcal{T}_{1}^{\mathcal{D}}\left(\left\|\mathbf{r}_{1}\right\|_{r}\right) \rightarrow \mathcal{T}_{2}^{\mathcal{D}}\left(\left\|\mathbf{r}_{2}\right\|_{r}\right)\right) \\
& =\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes\left(\mathcal{T}_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \rightarrow \mathcal{T}_{2}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right)\right) \\
& =\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes\left(E_{1}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{1}}\right)\right) \rightarrow E_{2}^{\mathcal{D}}\left(r\left(R_{\mathbf{r}_{2}}\right)\right)\right) \\
& =\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes\left(\left(E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}\right) \mathcal{D}^{(r)} \rightarrow\left(E_{2} \bowtie \mathcal{E}_{R_{\mathbf{r}_{1}}}\right) \mathcal{D}^{\mathcal{D}}(r)\right) \\
& =\left(\left(E_{1} \bowtie \mathcal{E}_{R_{\mathbf{r}_{2}}}\right) \rightarrow \mathcal{E}_{R}\left(E_{2} \bowtie \mathcal{E}_{R_{\mathbf{r}_{1}}}\right)\right)^{\mathcal{D}}(r) \\
& =F^{\mathcal{D}}(r)
\end{aligned}
$$

for all tuples $r \in \operatorname{eadom}_{R}^{\mathcal{D}}$. For any tuple $r \notin \operatorname{eadom}_{R}^{\mathcal{D}}$ we have $\mathcal{E}_{R}^{\mathcal{D}}(r)=0$. From the properties of $\otimes$ we conclude that $F^{\mathcal{D}}(r)=0$ for $r \notin$ eadom $_{R}^{\mathcal{D}}$.
3. If $\mathcal{T}(\mathbf{r})$ is $\nabla \mathcal{T}^{\prime}(\mathbf{r})$ or $\Delta \mathcal{T}^{\prime}(\mathbf{r})$, then from the induction hypothesis we have a RA-expression $E$ on relation scheme $R$ corresponding to PTCsubexpression $\mathcal{T}^{\prime}(\mathbf{r})$, such that $\mathcal{T}^{\prime \mathcal{D}}(r)=E^{\mathcal{D}}(r)$ for all $r \in \operatorname{Tupl}(R)$.
We put $F=\nabla E$ or $F=\Delta E$, respectively.
In both cases, the relation scheme of $F$ is $R$ as required. Assuming that the symbol $\square$ denotes $\nabla$ or $\Delta$ we have

$$
\mathcal{T}^{\mathcal{D}}(r)=\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=\square \mathcal{T}^{\prime \mathcal{D}}\left(\|\mathbf{r}\|_{r}\right)=\square \mathcal{T}^{\mathcal{D}}(r)=\square E^{\mathcal{D}}(r)=F^{\mathcal{D}}(r),
$$

for all $r \in$ eadom $_{R}^{\mathcal{D}}$.
For any tuple $r \notin \operatorname{eadom}_{R}^{\mathcal{D}}$ we have $\mathcal{T}^{\boldsymbol{\mathcal { D }}}(r)=0$ and from the induction hypothesis we also have $E^{\mathcal{D}}(r)=0$. From the definition of $\nabla$ or $\Delta$ we conclude that $F^{\mathcal{D}}(r)=0$ for $r \notin$ eadom $_{R}^{\mathcal{D}}$.
4. If $\mathcal{T}(\mathbf{r})$ is $\bigvee_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ or $\bigwedge_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$ where $\mathbf{r}=\mathbf{r}_{\mathbf{2}}, R=R_{\mathbf{r}_{2}}$ and $R_{\mathbf{r}_{1}} \cap R_{\mathbf{r}_{2}}=\emptyset$, then from the induction hypothesis we have a RA-expression $E$ on relation scheme $R_{\mathbf{r}_{1}} \cup R$ corresponding to the PTC-subexpression $\mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$, such that $\mathcal{T}^{\prime \mathcal{D}}\left(r r^{\prime}\right)=E^{\mathcal{D}}\left(r r^{\prime}\right)$ for all $r \in \operatorname{Tupl}(R)$ and $r^{\prime} \in$ $\operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)$. Note that tuples $r$ and $r^{\prime}$ are always joinable since the relation schemes $R_{\mathbf{r}_{1}}$ and $R$ are disjoint.
We distinguish two cases:
(a) if $\mathcal{T}(\mathbf{r})$ is $\bigvee_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$, then we put $F=\pi_{R}(E)$.

The relation scheme of $F$ is $R$ as required. We have

$$
\begin{aligned}
\mathcal{T}^{\mathcal{D}}(r) & =\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right) \\
& =\bigvee\left\{\mathcal{T}^{\mathcal{D}}\left(\left\|\mathbf{r}_{1} \cup \mathbf{r}_{2}\right\|_{r r^{\prime}}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigvee\left\{\mathcal{T}^{\prime \mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{eadom}\right. \\
& =\bigvee\left\{E_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& \left.=\bigvee\left\{r^{\prime}\right)\left|r^{\prime} \in \operatorname{eadom} m_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\left(r r^{\prime}\right)\right| r^{\prime} \in \operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)\right\} \\
& =\left(\pi_{R}(E)\right)^{\mathcal{D}}(r)=F^{\mathcal{D}}(r)
\end{aligned}
$$

for all $r \in$ eadom $_{R}^{\mathcal{D}}$.
 cannot change the score $\bigvee\left\{E^{\mathcal{D}}\left(r r^{\prime}\right)\right\}$ since for $r^{\prime} \notin \operatorname{eadom}{\underset{R_{r_{1}}}{\mathcal{D}}}^{R_{r_{1}}}$ we have $\mathcal{T}^{\prime \mathcal{D}}\left(r r^{\prime}\right)=0$ and thus $E^{\mathcal{D}}\left(r r^{\prime}\right)=0$ for any $r \in \operatorname{Tupl}(R)$. Furthermore, for any $a \in L$ it holds that $a \vee 0=a$. Hence, we have

$$
\bigvee\left\{E^{\mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{eadom}{\underset{R_{r_{1}}}{\mathcal{D}}}_{\mathcal{D}}\right\}=\bigvee\left\{E^{\mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)\right\}
$$

Now we show that $F^{\mathcal{D}}(r)=0$ for all $r \notin \operatorname{eadom}_{R}^{\mathcal{D}}$. For any $r \notin$ eadom $m_{R}^{\mathcal{D}}$ we have $E^{\mathcal{D}}\left(r r^{\prime}\right)=0$ and thus $F^{\mathcal{D}}(r)=\bigvee\{0,0, \ldots\}=0$.
(b) if $\mathcal{T}(\mathbf{r})$ is $\bigwedge_{\mathbf{r}_{1}} \mathcal{T}^{\prime}\left(\mathbf{r}_{1}, \mathbf{r}_{2}\right)$, we put $F=E \div \mathcal{E}_{R} \mathcal{E}_{R_{\mathbf{r}_{1}}}$.

The relation scheme of $F$ is $R$ as required. We have

$$
\begin{aligned}
\mathcal{T}^{\mathcal{D}}(r) & =\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right) \\
& =\bigwedge\left\{\mathcal{T}^{\prime \mathcal{D}}\left(\left\|\mathbf{r}_{\mathbf{1}} \cup \mathbf{r}_{\mathbf{2}}\right\|_{r r^{\prime}}\right) \mid r^{\prime} \in \text { eadom } R_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigwedge\left\{\mathcal{T}^{\prime \mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigwedge\left\{E^{\mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{eadom}{ }_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigwedge\left\{\mathcal{E}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\left(r^{\prime}\right) \rightarrow E^{\mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)\right\} \\
& =\bigwedge\left\{\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes\left(\mathcal{E}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\left(r^{\prime}\right) \rightarrow E^{\mathcal{D}}\left(r r^{\prime}\right)\right) \mid r^{\prime} \in \operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)\right\} \\
& =\left(E \div \dot{\mathcal{E}}_{R} \mathcal{E}_{R_{\mathbf{r}_{1}}}\right)^{\mathcal{D}}(r)=F^{\mathcal{D}}(r)
\end{aligned}
$$

for all $r \in$ eadom $_{R}^{\mathcal{D}}$.
 cannot change the final score of $r$, since for any $r^{\prime} \notin \operatorname{eadom}{\underset{R_{r_{1}}}{\mathcal{D}}}_{\mathcal{D}}$ we have $\mathcal{E}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\left(r^{\prime}\right) \rightarrow E^{\mathcal{D}}\left(r r^{\prime}\right)=1$ and it holds that $a \wedge 1=a$ for any $a \in L$.
For any tuple $r \notin$ eadom $_{R}^{\mathcal{D}}$ we have $\mathcal{E}_{R}^{\mathcal{D}}(r)=0$. Hence, we have $F^{\mathcal{D}}(r)=\bigwedge\{0,0, \ldots\}=0$ for $r \notin$ eadom $_{R}^{\mathcal{D}}$.

Observe that instead of using (14) we can alternatively use Date's Small Divide and put $F^{\prime}=\mathcal{E}_{R} \div{ }_{\text {gsdo }}^{E} \mathcal{E}_{R_{\mathrm{r}_{1}}}$ since it holds that

$$
\begin{aligned}
\mathcal{T}^{\mathcal{D}}(r) & =\mathcal{T}^{\mathcal{D}}\left(\|\mathbf{r}\|_{r}\right) \\
& =\bigwedge\left\{\mathcal{T}^{\prime \mathcal{D}}\left(\left\|\mathbf{r}_{\mathbf{1}} \cup \mathbf{r}_{2}\right\|_{r r^{\prime}}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigwedge\left\{\mathcal{T}^{\prime \mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigwedge\left\{E^{\mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{eadom}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\right\} \\
& =\bigwedge\left\{\mathcal{E}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\left(r^{\prime}\right) \rightarrow E^{\mathcal{D}}\left(r r^{\prime}\right) \mid r^{\prime} \in \operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)\right\} \\
& =\mathcal{E}_{R}^{\mathcal{D}}(r) \otimes \bigwedge\left\{\left(\mathcal{E}_{R_{\mathbf{r}_{1}}}^{\mathcal{D}}\left(r^{\prime}\right) \rightarrow E^{\mathcal{D}}\left(r r^{\prime}\right)\right) \mid r^{\prime} \in \operatorname{Tupl}\left(R_{\mathbf{r}_{1}}\right)\right\} \\
& =\left(\mathcal{E}_{R} \div{ }_{\text {gsdo }} \mathcal{E}_{R_{\mathbf{r}_{1}}}\right)^{\mathcal{D}}(r)=F^{\prime \mathcal{D}}(r)
\end{aligned}
$$

for all $r \in \operatorname{eadom}_{R}^{\mathcal{D}}$.
For any tuple $r \notin$ eadom $_{R}^{\mathcal{D}}$ we have $\mathcal{E}_{R}^{\mathcal{D}}(r)=0$. From the properties of $\otimes$ we conclude that $F^{\prime \mathcal{D}}(r)=0$ for any $r \notin \operatorname{eadom}_{R}^{\mathcal{D}}$.

## 5 More on Relationships of Division Operations

In this section we use the Pseudo Tuple Calculus (PTC) to show further relationships of the division operations presented in this paper. We utilize the PTC in the following way. Let us have an relational operation op that accepts input relations $\mathcal{D}_{1}, \ldots, \mathcal{D}_{n}$ on relation schemes $R_{1}, \ldots, R_{n}$ and its output relation is on relation scheme $R$. For the input relations we consider relation symbols $\mathbb{D}_{1}, \ldots, \mathbb{D}_{n}$ on the respective relation schemes $R_{1}, \ldots, R_{n}$. Note that the relation symbols are themselves RA-expressions. Now using the relation symbols we construct a PTC-expression $\mathcal{T}(\mathbf{r})$ on $R$ that is semantically equivalent to the operation in question. By semantical equivalence we mean that if we evaluate the PTC-expression $\mathcal{T}(\mathbf{r})$ in a database instance $\mathcal{D}$ that maps the relation symbols to the input relations, i. e. we have $\mathbb{D}_{1}^{\mathcal{D}}=\mathcal{D}_{1}, \ldots, \mathbb{D}_{n}^{\mathcal{D}}=\mathcal{D}_{n}$, we get that

$$
o p\left(\mathcal{D}_{1}, \ldots, \mathcal{D}_{n}\right)(r)=\mathcal{T}^{\mathcal{D}}(r)
$$

for all $r \in \operatorname{Tupl}(R)$. Note that this construction does not depend on the actual content of the input relations. Furthermore we apply the Theorem 9 to transform the PTC-expression $\mathcal{T}(\mathbf{r})$ to an equivalent RA-expression that uses only the fundamental operations of the algebra and obtain the requested relationship.

We give an example to illustrate the notion of semantical equivalence. Consider the division operation defined by $(14)$, i. e., for $\operatorname{RDTs} \mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ on
$R S, S$, and $R$, respectively, the division $\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}$ of $\mathcal{D}_{1}$ by $\mathcal{D}_{2}$ which ranges over $\mathcal{D}_{3}$ is an RDT on $R$ defined by

$$
\left(\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\bigwedge_{s \in \operatorname{Tupl}(S)}\left(\mathcal{D}_{3}(r) \otimes\left(\mathcal{D}_{2}(s) \rightarrow \mathcal{D}_{1}(r s)\right)\right)
$$

for each $r \in \operatorname{Tupl}(R)$. Consider relation symbols $\mathbb{D}_{1}, \mathbb{D}_{2}$ and $\mathbb{D}_{3}$ on $R S, S$, and $R$, respectively. Then the PTC-expression

$$
\mathcal{T}(\mathbb{r})=\bigwedge_{s}\left(\mathbb{D}_{3}(\mathbb{r}) \otimes\left(\mathbb{D}_{2}(\mathbb{s}) \rightarrow \mathbb{D}_{1}(\mathbb{r})\right)\right)
$$

is semantically equivalent to the division operation. More precisely, for a database instance $\mathcal{D}$ such that $\mathbb{D}_{1}^{\mathcal{D}}=\mathcal{D}_{1}, \mathbb{D}_{2}^{\mathcal{D}}=\mathcal{D}_{2}$, and $\mathbb{D}_{3}^{\mathcal{D}}=\mathcal{D}_{3}$ we have

$$
\left(\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\mathcal{T}^{\mathcal{D}}(r)
$$

for all $r \in \operatorname{Tupl}(R)$. Now, we are ready to show the relationships among the division operations.

Theorem 10. Let $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ be RDTs on $R S, S$, and $R$, respectively, and let $\div$ gsdo be Date's Small Divide. For the division operation defined by 14 we have

$$
\left(\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\left(\mathcal{E}_{R}^{\mathcal{D}} \div{ }_{\text {gsdo }}^{E^{\mathcal{D}}} \mathcal{E}_{S}^{\mathcal{D}}\right)(r)
$$

for all $r \in \operatorname{Tupl}(R)$ where

$$
E^{\mathcal{D}}=\mathcal{D}_{3} \bowtie\left(\left(\mathcal{D}_{2} \bowtie \mathcal{E}_{R}^{\mathcal{D}}\right) \rightarrow \mathcal{E}_{R S}^{\mathcal{D}} \mathcal{D}_{1}\right)
$$

and the extended active domains $\mathcal{E}_{R}^{\mathcal{D}}, \mathcal{E}_{S}^{\mathcal{D}}$, and $\mathcal{E}_{R S}^{\mathcal{D}}$ contain tuples built only from the values from relations $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$.

Proof. First, using the relation symbols $\mathbb{D}_{1}, \mathbb{D}_{2}$, and $\mathbb{D}_{3}$, corresponding to the input relations $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$, we construct a PTC-expression $\mathcal{T}(\mathfrak{a})$ that is semantically equivalent to the division operation. We have

$$
\mathcal{T}(\mathbb{r})=\bigwedge_{s}\left(\mathbb{D}_{3}(\mathbb{r}) \otimes\left(\mathbb{D}_{2}(\mathfrak{s}) \rightarrow \mathbb{D}_{1}(\mathbb{r} s)\right)\right)
$$

and it holds that $\left(\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\mathcal{T}^{\mathcal{D}}(r)$ for all $r \in \operatorname{Tupl}(R)$ in any database instance $\mathcal{D}$ that maps the relation symbols to their respective input relations. According to the Theorem 9 there is an equivalent RA-expression $F$ such that $\mathcal{T}^{\mathcal{D}}(r)=F^{\mathcal{D}}(r)$ for all $r \in \operatorname{Tupl}(R)$. The sought RA-expression $F$ is

$$
F=\mathcal{E}_{R} \div{ }_{\text {gsdo }}^{E} \mathcal{E}_{S}
$$

where

$$
E=\mathbb{D}_{3} \bowtie\left(\left(\mathbb{D}_{2} \bowtie \mathcal{E}_{R S}\right) \rightarrow \mathcal{E}_{R S}\left(\mathbb{D}_{1} \bowtie \mathcal{E}_{S}\right)\right)
$$

By evaluating $E$ in the database instance $\mathcal{D}$ that maps the relation symbols to their respective input relations we get a relation

$$
E^{\mathcal{D}}=\mathcal{D}_{3} \bowtie\left(\left(\mathcal{D}_{2} \bowtie \mathcal{E}_{R S}^{\mathcal{D}}\right) \rightarrow \mathcal{E}_{R S}^{\mathcal{D}}\left(\mathcal{D}_{1} \bowtie \mathcal{E}_{S}^{\mathcal{D}}\right)\right)
$$

The most simple database instance that maps the relation symbols to their respective input relations contains just the relations $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$. The relations $\mathcal{E}_{R S}^{\mathcal{D}}, \mathcal{E}_{S}^{\mathcal{D}}$, and $\mathcal{E}_{R}^{\mathcal{D}}$, obtained by evaluating $\mathcal{E}_{R S}, \mathcal{E}_{S}$, and $\mathcal{E}_{R}$, in such database instance therefore contain tuples built only from the values from relations $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ as required.

It can be easily checked that even if the database instance contained more relations and thus the extended active domains $\mathcal{E}_{R S}^{\mathcal{D}}, \mathcal{E}_{S}^{\mathcal{D}}$, and $\mathcal{E}_{R}^{\mathcal{D}}$ contained more tuples built from values from other relations these additional tuples do not change the result of evaluating the RA-expression $F$. It is safe to build the relations $\mathcal{E}_{R S}^{\mathcal{D}}, \mathcal{E}_{S}^{\mathcal{D}}$, and $\mathcal{E}_{R}^{\mathcal{D}}$, only from the values from relations $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$.

From the properties of $\bowtie$ and the fact that $\mathcal{E}_{S}^{\mathcal{D}}$ contains the projection of relation $\mathcal{D}_{1}$ to $S$ and the relation $\mathcal{D}_{2}$, we can further simplify the form of the relation $E^{\mathcal{D}}$ to

$$
E^{\mathcal{D}}=\mathcal{D}_{3} \bowtie\left(\left(\mathcal{D}_{2} \bowtie \mathcal{E}_{R}^{\mathcal{D}}\right) \rightarrow \mathcal{E}_{R S}^{\mathcal{D}} \mathcal{D}_{1}\right)
$$

Putting all things together we have

$$
\left(\mathcal{D}_{1} \div{ }^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\mathcal{T}^{\mathcal{D}}(r)=F^{\mathcal{D}}(r)=\left(\mathcal{E}_{R}^{\mathcal{D}} \div{ }_{\text {gsdo }}^{\mathcal{D}} \mathcal{E}_{S}^{\mathcal{D}}\right)(r)
$$

for all $r \in \operatorname{Tupl}(R)$ with the relations $E^{\mathcal{D}}$ and $\mathcal{E}_{R}^{\mathcal{D}}, \mathcal{E}_{S}^{\mathcal{D}}, \mathcal{E}_{R S}^{\mathcal{D}}$ defined as above.
Theorem 11. Let $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$ be RDTs on $R$, $S$, and $R S$, respectively, and let $\div$ be the division operation defined by $\mathbf{1 4}$. For Date's Small Divide we have

$$
\left(\mathcal{D}_{1} \div{ }_{\text {gsdo }}^{\mathcal{D}_{3}} \mathcal{D}_{2}\right)(r)=\left(\mathcal{D}_{1} \bowtie\left(E^{\mathcal{D}} \div \mathcal{E}_{R}^{\mathcal{D}} \mathcal{E}_{S}^{\mathcal{D}}\right)\right)(r)
$$

for all $r \in \operatorname{Tupl}(R)$ where

$$
E^{\mathcal{D}}=\left(\left(\mathcal{D}_{2} \bowtie \mathcal{E}_{R}^{\mathcal{D}}\right) \rightarrow \mathcal{E}_{R S}^{\mathcal{D}} \mathcal{D}_{3}\right)
$$

and the extended active domains $\mathcal{E}_{R}^{\mathcal{D}}, \mathcal{E}_{S}^{\mathcal{D}}$, and $\mathcal{E}_{R S}^{\mathcal{D}}$ contain tuples built only from the values from relations $\mathcal{D}_{1}, \mathcal{D}_{2}$, and $\mathcal{D}_{3}$.

Proof. Use similar arguments as in the proof of Theorem 10
Theorem 12. Let $\mathcal{D}_{1}, \mathcal{D}_{2}, \mathcal{D}_{3}$, and $\mathcal{D}_{4}$ be RDTs on $R_{1}, R_{2}, R_{3}$, and $R_{4}$, respectively, and let $\div$ be the division operation defined by (14). For Darwen's Divide we have

$$
\left(\mathcal{D}_{1} \div{ }_{\text {gddo }}^{\mathcal{D}_{3}, \mathcal{D}_{4}} \mathcal{D}_{2}\right)(r)=\left(\left(\mathcal{D}_{1} \bowtie \mathcal{D}_{2}\right) \bowtie\left(E^{\mathcal{D}} \div \dot{E}_{R_{1}^{\prime}}^{\mathcal{D}} \mathcal{E}_{R_{2}^{\prime}}^{\mathcal{D}}\right)\right)(r)
$$

for all $r \in \operatorname{Tupl}\left(R_{1} \cup R_{2}\right)$ where

$$
\begin{aligned}
E^{\mathcal{D}} & =\left(\left(\mathcal{D}_{4} \bowtie \mathcal{E}_{R_{3}^{\prime}}^{\mathcal{D}}\right) \rightarrow \mathcal{E}_{R_{4}^{\prime}}^{\mathcal{D}}\left(\pi_{R_{3}^{\prime}}\left(\mathcal{D}_{3}\right) \bowtie \mathcal{E}_{R_{4}}^{\mathcal{D}}\right)\right) \\
R_{1}^{\prime} & =\left(R_{4} \cap\left(R_{1} \cup R_{2}\right)\right) \cup\left(R_{1} \cap R_{3}\right) \\
R_{2}^{\prime} & =R_{4} \backslash\left(R_{1} \cup R_{2}\right) \\
R_{3}^{\prime} & =R_{3} \cap\left(R_{1} \cup R_{4}\right), \\
R_{4}^{\prime} & =R_{4} \cup\left(R_{1} \cap R_{3}\right)
\end{aligned}
$$

and the extended active domains contain tuples built only from the values from relations $\mathcal{D}_{1}, \mathcal{D}_{2}, \mathcal{D}_{3}$, and $\mathcal{D}_{4}$.

Proof. As in the previous proofs, we construct PTC-expression $\mathcal{T}(\mathbf{r})$ that is semantically equivalent to the division operation defined by 32 . Using the relation symbols $\mathbb{D}_{1}, \mathbb{D}_{2}, \mathbb{D}_{3}$, and $\mathbb{D}_{4}$ that correspond to the input relations $\mathcal{D}_{1}, \mathcal{D}_{2}, \mathcal{D}_{3}$, and $\mathcal{D}_{4}$ we get

$$
\mathcal{T}(\mathbf{r})=\left(\mathbb{D}_{1} \bowtie \mathbb{D}_{2}\right)(\mathbf{r}) \otimes \bigwedge_{\mathbf{r}_{\mathrm{b}}^{\prime}}\left(\mathbb{D}_{4}\left(\mathbf{r}_{\mathrm{f}}^{\prime}, \mathbf{r}_{\mathrm{b}}^{\prime}\right) \rightarrow \pi_{R_{3}^{\prime}}\left(\mathbb{D}_{3}\right)\left(\mathbf{r}_{\mathrm{f}}^{\prime \prime}, \mathbf{r}_{\mathrm{b}}^{\prime \prime}\right)\right)
$$

where

- $\mathbf{r}$ is on a relation scheme $R_{1} \cup R_{2}$,
- $\mathbf{r}_{\mathrm{b}}^{\prime}$ is on $R_{2}^{\prime}=R_{4} \backslash\left(R_{1} \cup R_{2}\right)$,
- $\mathbf{r}_{\mathrm{f}}^{\prime}$ is on $R_{4} \cap\left(R_{1} \cup R_{2}\right)$,
- $R_{3}^{\prime}=R_{3} \cap\left(R_{1} \cup R_{4}\right)$,
- $\mathbf{r}_{\mathrm{f}}^{\prime \prime}$ is on $\left(R_{1} \cup\left(R_{2} \cap R_{4}\right)\right) \cap R_{3}^{\prime}$,
- $\mathbf{r}_{\mathrm{b}}^{\prime \prime}$ is on $R_{2}^{\prime} \cap R_{3}^{\prime}$
such that each set of tuple variables contains one tuple variable for each attribute in the relation schema of the corresponding subexpression. For instance, the set of tuple variables $\mathbf{r}$ can be characterized as $\mathbf{r}=\left\{\mathfrak{r}_{y} \mid y \in R_{1} \cup R_{2}\right\}$.

According to the Theorem 9 there is an equivalent RA-expression $F$ such that $\mathcal{T}^{\mathcal{D}}(r)=F^{\mathcal{D}}(r)$ for all $r \in \operatorname{Tupl}(R)$. Again, the database instance $\mathcal{D}$ should map each relation symbol to its corresponding input relation. In order to find the RA-expression $F$, we first find the RA-expression $E$ that corresponds to the PTC-subexpression $\mathbb{D}_{4}\left(\mathbf{r}_{\mathrm{f}}^{\prime}, \mathbf{r}_{\mathrm{b}}^{\prime}\right) \rightarrow \pi_{R_{3}^{\prime}}\left(\mathbb{D}_{3}\right)\left(\mathbf{r}_{\mathrm{f}}^{\prime \prime}, \mathbf{r}_{\mathrm{b}}^{\prime \prime}\right)$. The sought RA-expression is

$$
E=\left(\left(\mathbb{D}_{4} \bowtie \mathcal{E}_{R_{3}^{\prime}}\right) \rightarrow \mathcal{E}_{R_{4}^{\prime}}\left(\pi_{R_{3}^{\prime}}\left(\mathbb{D}_{3}\right) \bowtie \mathcal{E}_{R_{4}}\right)\right)
$$

where $R_{4}^{\prime}=R_{4} \cup R_{3}^{\prime}=R_{4} \cup\left(R_{3} \cap\left(R_{1} \cup R_{4}\right)\right)=R_{4} \cup\left(R_{1} \cap R_{3}\right)$.
Now, we are ready to find the RA-expression $F$ that corresponds to the whole PTC-expression $\mathcal{T}(\mathbf{r})$. We have

$$
F=\left(\mathbb{D}_{1} \bowtie \mathbb{D}_{2}\right) \bowtie\left(E \div \dot{\mathcal{E}}_{R_{1}^{\prime}} \mathcal{E}_{R_{2}^{\prime}}\right)
$$

where $R_{1}^{\prime}=R_{4}^{\prime} \backslash R_{2}^{\prime}=\left(R_{4} \cap\left(R_{1} \cup R_{2}\right)\right) \cup\left(R_{1} \cap R_{3}\right)$. Since it holds that $R_{1}^{\prime} \subseteq\left(R_{1} \cup R_{2}\right)$ the relation scheme of $F$ is $R_{1} \cup R_{2}$ as required.

The rest of the proof is clear.
In the previous chapters, we have already shown that Date's Small Divide is a special case of Date's Great Divide which is in turn a special case of Darwen's Divide. Furthermore, we have shown that if the $\mathbf{L}$ is prelinear or divisible, then there is a simple correspondence between Date's Small Divide and the division operation defined by (14).

In this chapter we have shown that they are equivalent regardless of the properties of $\mathbf{L}$. We have also shown that Darwen's Divide can be expressed by the division operation defined by 14 . As a consequence we get the equivalence of all domain-independent division operations presented in this paper. Furthermore we have an exact way to express one division using the other. Therefore we can summarize the observations as follows:

Corollary 13. All domain-independent division operations presented in this paper are equivalent.

This result solves an open question concerning the relationship of Date's Great Divide and Darwen's Divide in the classic setting, see [16, page 187].

## 6 Conclusion

We have presented a survey of graded generalizations of classic division-like operations in a rank-aware model of data. We have focused on generalizing variants of division-like operations which are neglected by other rank-aware approaches in databases. In our model we assume that is a fundamental operation. Under this assumption, we have shown that all the graded generalizations of the classic division operations we have studied in this paper are derived operations. That is, considering the original graded division (14) as the fundamental division, i.e., including it in the relational algebra, all the other divisions 16, (19), (24) and (32), are derived operations in our model. Furthermore, using the Pseudo Tuple Calculus (PTC), we have shown that the various variants of
the division operations are mutually definable. Interestingly, some of our observations we have made on the general level (considering $\mathbf{L}$ as a general complete residuated lattice) pertain to the classic model-when $\mathbf{L}$ is considered as the two-element Boolean algebra. For instance, we have shown that Date's Great Divide and Darwen's Divide are mutually definable. This result solves an open question that was stated by Date in [17, page 187].

Future research in the area may include considerations on the role of fundamental and derived operations in the model. The fundamental division 14 cannot be dropped without losing the expressive power of the relational algebra since in general we cannot introduce universal quantifiers using the existential ones. On the other hand, there may be ways to simplify the present relational algebra by considering other forms of division-like operations. One way to go is to introduce graded subsethood as a fundamental (graded) comparator of relations, and use analogous techniques as image relations [18] to express the division.

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