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The binomial decomposition of generalized Gini welfare functions, the S-Gini and Lorenzen cases

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Abstract

We consider the binomial decomposition of generalized Gini welfare functions in terms of the binomial welfare functions and the associated binomial inequality indices. We examine in detail the weights of the binomial welfare functions and the coefficients of the associated binomial inequality indices which progressively focus on the poorest sector of the population, and we illustrate the numerical behavior of the binomial welfare functions and inequality indices in relation with a new parametric family of income distributions. The main contribution of the paper is to investigate the analogy between the binomial welfare functions and the S-Gini and Lorenzen parametric families of generalized Gini welfare functions, particularly in the context of the binomial decomposition. Finally, we examine the orness of the parametric S-Gini and Lorenzen families of generalized Gini welfare functions.

Keywords: Ordered weighting functions; Orness; Generalized Gini welfare functions; Atkinson-Kolm-Sen framework; Binomial decomposition; S-Gini welfare functions; Lorenzen welfare functions

1. Introduction

The generalized Gini welfare functions introduced by Weymark [44] and the associated inequality indices in Atkinson-Kolm-Sen's (AKS) framework are related by Blackorby and Donaldson's correspondence formula [10, 11],

$$A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x}) \tag{1}$$

where A denotes a generalized Gini welfare function, G is the associated absolute inequality index, and $\bar{x}$ is the plain mean of the income distribution $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathbb{D}^n$ of a population of $n \geq 2$ individuals, with income domain $\mathbb{D} = [0, \infty)$.

The generalized Gini welfare functions [44] have the form $A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}$ where $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ and, as required by the principle of inequality aversion, $w_1 \geq w_2 \geq \cdots \geq w_n \geq 0$ with $\sum_{i=1}^{n} w_i = 1$. These welfare functions correspond to the Schur-concave class of the ordered weighted averaging (OWA) functions introduced by Yager [47], which in turn correspond [24] to the Choquet integrals associated with symmetric capacities.

In this paper we recall the binomial decomposition of generalized Gini welfare functions due to Calvo and De Baets [17], and Bortot and Marques Pereira [15]. The binomial decomposition can be formulated in terms of two equivalent functional bases, the binomial welfare functions and the Atkinson-Kolm-Sen (AKS) associated binomial inequality indices, according to Blackorby and Donaldson's correspondence formula.

The *binomial welfare functions*, denoted C_j with $j = 1, \dots, n$, have null weights associated with the $j-1$ richest individuals in the population and therefore they are progressively focused on the poorest part of the population. Correspondingly, the associated *binomial inequality indices*, denoted G_j with

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$j = 1, \ldots, n$, have equal weights associated with the $j-1$ richest individuals in the population and therefore they are progressively insensitive to income transfers within the richest part of the population.

The main contribution of the paper is to investigate the analogy between the binomial welfare functions and the S-Gini and Lorenzen parametric families of generalized Gini welfare functions, particularly in the context of the binomial decomposition. Finally, we examine the orness of the parametric S-Gini and Lorenzen families of generalized Gini welfare functions.

The paper is organized as follows. In Section 2 we review the fundamental notions of majorization, income transfers, Shur-convexity / concavity, and generalized Gini welfare functions and inequality indices, for populations of $n \geq 2$ individuals.

In Section 3 we consider the binomial decomposition of generalized Gini welfare functions in terms of the binomial welfare functions C_j, with $j = 1, ..., n$, and the associated binomial inequality indices G_j, with $j = 1, ..., n$. We examine in detail the weights of the binomial welfare functions C_j, with $j = 1, ..., n$, and the coefficients of the associated binomial inequality indices G_j, with $j = 1, ..., n$, which progressively focus on the poorest part of the population, and we illustrate the numerical behavior of the binomial welfare functions and inequality indices in relation with a new parametric family of income distributions.

In Sections 4 and 5 we present the main contribution of the paper, which is an investigation on the analogy between the binomial welfare functions and the S-Gini and Lorenzen families of welfare functions, particularly in the context of the binomial decomposition. The S-Gini family, with a continuous parameter $\delta \in [1, \infty)$, interpolates between the first and the last binomial welfare functions C_1 and C_n, and includes the classical Gini welfare function. On the other hand, the Lorenzen family, with a parameter $l = 1, \ldots, n$, interpolates between the last binomial welfare function C_n and the classical Gini welfare function, which combines the first two binomial welfare functions C_1 and C_2.

Finally, in Section 6 we examine the orness of the parametric S-Gini and Lorenzen families of generalized Gini welfare functions, and in Section 7 we present some conclusive remarks.

2. Generalized Gini welfare functions and inequality indices

In this section we consider populations of $n \geq 2$ individuals with income distributions represented by points $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ over the income domain $\mathbb{D} = [0, \infty)$. We briefly review the fundamental notions of majorization, income transfers, Shur-convexity / concavity, and generalized Gini welfare functions and inequality indices.

We begin by presenting notation and basic definitions regarding averaging functions on the domain $\mathbb{D}^n$, with $n \geq 2$ throughout the text. Comprehensive reviews of averaging functions can be found in Fodor and Roubens [25], Marichal [34], Marichal et al. [35], Calvo et al. [18], Beliakov et al. [6], Torra and Narukawa [43], Mesiar et al. [38], Grabisch et al. [27, 28], and Beliakov et al. [7].

Points in $\mathbb{D}^n$ are denoted $\boldsymbol{x} = (x_1, \ldots, x_n)$, with $\mathbf{1} = (1, \ldots, 1)$, $\mathbf{0} = (0, \ldots, 0)$. For every $x \in \mathbb{D}$, we have $x \cdot \mathbf{1} = (x, \ldots, x)$. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, by $\boldsymbol{x} \geq \boldsymbol{y}$ we mean $x_i \geq y_i$ for every $i = 1, \ldots, n$, and by $\boldsymbol{x} > \boldsymbol{y}$ we mean $\boldsymbol{x} \geq \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{y}$. Given $\boldsymbol{x} \in \mathbb{D}^n$, the increasing and decreasing reorderings of the coordinates of $\boldsymbol{x}$ are indicated as $x_{(1)} \leq \cdots \leq x_{(n)}$ and $x_{[1]} \geq \cdots \geq x_{[n]}$, respectively. In particular, $x_{(1)} = \min\{x_1, \ldots, x_n\} = x_{[n]}$ and $x_{(n)} = \max\{x_1, \ldots, x_n\} = x_{[1]}$. In general, given a permutation σ on $\{1, \ldots, n\}$, we denote $\boldsymbol{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. Finally, the arithmetic mean is denoted $\bar{x} = (x_1 + \cdots + x_n)/n$.

We recall the definition of the majorization relation on $\mathbb{D}^n$ and we discuss the central concept of income transfer following the approach in Marshall and Olkin [36]. We refer the classical results relating majorization, income transfers, and bistochastic transformations, see Marshall and Olkin [36, Ch. 4, Prop. A.1].

The *majorization relation* $\preceq$ on $\mathbb{D}^n$ is defined as follows: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that

$$\boldsymbol{x} \preceq \boldsymbol{y} \quad \text{if} \quad \sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)} \qquad k = 1, \ldots, n \tag{2}$$

where the case $k = n$ is an equality due to $\bar{x} = \bar{y}$. As usual, we write $\boldsymbol{x} \prec \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and not $\boldsymbol{y} \preceq \boldsymbol{x}$, and we write $\boldsymbol{x} \sim \boldsymbol{y}$ if $\boldsymbol{x} \preceq \boldsymbol{y}$ and $\boldsymbol{y} \preceq \boldsymbol{x}$. We say that $\boldsymbol{y}$ majorizes $\boldsymbol{x}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ are indifferent if $\boldsymbol{x} \sim \boldsymbol{y}$.

Another traditional reading, which reverses that of majorization, refers to the concept of Lorenz dominance: we say that $\boldsymbol{x}$ is Lorenz superior to $\boldsymbol{y}$ if $\boldsymbol{x} \prec \boldsymbol{y}$, and we say that $\boldsymbol{x}$ is Lorenz indifferent to $\boldsymbol{y}$ if $\boldsymbol{x} \sim \boldsymbol{y}$.

Given an income distribution $\boldsymbol{x} \in \mathbb{D}^n$, with mean income $\bar{x}$, it holds that $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ since $k\,\bar{x} \geq \sum_{i=1}^{k} x_{(i)}$ for $k = 1, \dots, n$. The majorization is strict, $\bar{x} \cdot \mathbf{1} \prec \boldsymbol{x}$, when $\boldsymbol{x}$ is not a uniform income distribution. In such case, $\bar{x} \cdot \mathbf{1}$ is Lorenz superior to $\boldsymbol{x}$. Moreover, for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$ it holds that $\boldsymbol{x} \preceq (0, \dots, 0, n\bar{x})$, which is strict when $\boldsymbol{x} \neq \mathbf{0}$.

The majorization relation is a partial preorder since, by definition, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ are comparable only when $\bar{x} = \bar{y}$, and $\boldsymbol{x} \sim \boldsymbol{y}$ is obtained if and only if $\boldsymbol{x}$ and $\boldsymbol{y}$ differ by a permutation.

In general, given income distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, it holds that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if there exists a bistochastic matrix $\mathbf{C}$ (non-negative square matrix of order n where each row and column sums to one) such that $\boldsymbol{x} = \mathbf{C}\boldsymbol{y}$. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if and only if the bistochastic matrix $\mathbf{C}$ is not a permutation matrix.

An income transfer is a particular case of bistochastic transformation. Given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we say that $\boldsymbol{x}$ is derived from $\boldsymbol{y}$ by means of an *income transfer* if, for some pair $i, j = 1, \dots, n$ with $y_i \leq y_j$, we have

$$x_i = (1-\varepsilon)\, y_i + \varepsilon y_j \qquad x_j = \varepsilon y_i + (1-\varepsilon)\, y_j \qquad \varepsilon \in [0,1] \tag{3}$$

and $x_k = y_k$ for $k \neq i, j$. The income transfer, from a richer to a poorer individual, concerns an income amount of $\varepsilon(y_j - y_i)$. The income transfer obtains $x_i \leq x_j$ if $\varepsilon \in [0, 1/2]$ and $x_i \geq x_j$ if $\varepsilon \in [1/2, 1]$, with $x_i = x_j$ for $\varepsilon = 1/2$.

In relation with the parameter $\varepsilon \in [0,1]$ the income transfer obtains $\boldsymbol{x} = \boldsymbol{y}$ if $\varepsilon = 0$, and exchanges the relative positions of donor and recipient in the income distribution if $\varepsilon = 1$, in which case $\boldsymbol{x} \sim \boldsymbol{y}$. In the intermediate cases $\varepsilon \in (0,1)$ the income transfer produces an income distribution $\boldsymbol{x}$ which is Lorenz superior to the original $\boldsymbol{y}$, that is $\boldsymbol{x} \prec \boldsymbol{y}$.

In general, given income distributions $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, it holds that $\boldsymbol{x} \preceq \boldsymbol{y}$ if and only if $\boldsymbol{x}$ can be derived from $\boldsymbol{y}$ by means of a finite sequence of income transfers. Moreover, $\boldsymbol{x} \prec \boldsymbol{y}$ if and only if at least one of the income transfers is not a permutation.

Definition 1. *Let $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ be a function. We say that*

1. *A is* monotonic *if $\boldsymbol{x} \geq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is* strictly monotonic *if $\boldsymbol{x} > \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*
2. *A is* idempotent *if $A(x \cdot \mathbf{1}) = \mathbf{x}$, for all $x \in \mathbb{D}$. On the other hand, A is* nilpotent *if $A(x \cdot \mathbf{1}) = \mathbf{0}$, for all $x \in \mathbb{D}$.*
3. *A is* symmetric *if $A(\boldsymbol{x}_\sigma) = A(\boldsymbol{x})$, for any permutation σ on $\{1, \dots, n\}$ and all $\boldsymbol{x} \in \mathbb{D}^n$.*
4. *A is* Schur-convex *if $\boldsymbol{x} \preceq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \leq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is* strictly Schur-convex *if $\boldsymbol{x} \prec \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) < A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*
5. *A is* Schur-concave *if $\boldsymbol{x} \preceq \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) \geq A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$. Moreover, A is* strictly Schur-concave *if $\boldsymbol{x} \prec \boldsymbol{y} \;\Rightarrow\; A(\boldsymbol{x}) > A(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$.*
6. *A is* invariant for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = \mathbf{A}(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is* stable for translations *if $A(\boldsymbol{x} + t \cdot \mathbf{1}) = \mathbf{A}(\boldsymbol{x}) + \mathbf{t}$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*
7. *A is* invariant for dilations *if $A(t \cdot \boldsymbol{x}) = A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, A is* stable for dilations *if $A(t \cdot \boldsymbol{x}) = t\, A(\boldsymbol{x})$, for all $t \in \mathbb{D}$ and $\boldsymbol{x} \in \mathbb{D}^n$.*

Notice that either Schur-convexity or Schur-concavity imply symmetry, since $\boldsymbol{x} \sim \boldsymbol{x}_\sigma \;\Rightarrow\; A(\boldsymbol{x}) = A(\boldsymbol{x}_\sigma)$.

Definition 2. *A function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is an* averaging function *if it is monotonic and idempotent. An averaging function is said to be* strict *if it is strictly monotonic. Note that monotonicity and idempotency implies that $\min(\boldsymbol{x}) \leq A(\boldsymbol{x}) \leq \max(\boldsymbol{x})$, for all $\boldsymbol{x} \in \mathbb{D}^n$.*

Particular cases of averaging functions are weighted averaging (WA) functions, ordered weighted averaging (OWA) functions, and Choquet integrals, which contain WA and OWA functions as special cases.

The *Weighted Averaging (WA) function* associated with the weighting vector $\boldsymbol{w} = (w_1, \dots, w_n) \in [0,1]^n$, with $\sum_{i=1}^{n} w_i = 1$, is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i\, x_i. \tag{4}$$

The *Ordered Weighted Averaging (OWA) function* associated with the weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$, with $\sum_{i=1}^n w_i = 1$, is the averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i \, x_{(i)}. \tag{5}$$

The traditional form of OWA functions as introduced by Yager [47] is instead $A(\boldsymbol{x}) = \sum_{i=1}^n \tilde{w}_i \, x_{[i]}$ where $\tilde{w}_i = w_{n-i+1}$. In [48, 49] the theory and applications of OWA functions are discussed in detail.

The following are two classical results particulary relevant in our framework. The first result regards a form of dominance relation between OWA functions and the associated weighting structures, see for instance Bortot and Marques Pereira [15] and references therein.

Proposition 1. *Consider two OWA functions $A, B : \mathbb{D}^n \longrightarrow \mathbb{D}$ associated with weighting vectors $\boldsymbol{u} = (u_1, \ldots, u_n) \in [0,1]^n$ and $\boldsymbol{v} = (v_1, \ldots, v_n) \in [0,1]^n$, respectively. It holds that $A(\boldsymbol{x}) \leq B(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$ if and only if*

$$\sum_{i=1}^{k} u_i \geq \sum_{i=1}^{k} v_i \qquad \textit{for} \quad k = 1, \ldots, n \tag{6}$$

where the case $k = n$ is an equality due to weight normalization.

The next result regards the relation between the weighting structure of an OWA function and its Schur-convexity or Schur-concavity, see for instance Bortot and Marques Pereira [15].

Proposition 2. *Consider an OWA function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ associated with a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$. The OWA function A is Schur-convex if and only if the weights are non decreasing, $w_1 \leq \cdots \leq w_n$, and A is strictly Schur-convex if and only if the weights are increasing, $w_1 < \cdots < w_n$. Analogously, the OWA function A is Schur-concave if and only if the weights are non increasing, $w_1 \geq \cdots \geq w_n$, and A is strictly Schur-concave if and only if the weights are decreasing, $w_1 > \cdots > w_n$.*

We will now review the basic concepts and definitions regarding generalized Gini welfare functions and inequality indices. Their fundamental properties, which are generally considered to be inherent to the concepts of welfare and inequality, are now accepted as basic axioms for welfare and inequality measures, see Kolm [31, 32]. The crucial axiom in this field is the *Pigou-Dalton transfer principle*, which states that welfare (inequality) measures should be non-decreasing (non-increasing) under income transfers. This axiom translates directly into the properties of Schur-concavity and Schur-convexity in the context of symmetric functions on $\mathbb{D}^n$. In fact, a function is Schur-concave (Schur-convex) if and only if it is symmetric and non-decreasing (non-increasing) under income transfers, see for instance Marshall and Olkin [36].

Definition 3. *An averaging function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is a* welfare function *if it is continuous and Schur-concave. The welfare function is said to be* strict *if it is a strict averaging function which is strictly Schur-concave.*

Due to monotonicity and idempotency, a welfare function is increasing along the diagonal $\boldsymbol{x} = x \cdot \mathbf{1} \in \mathbb{D}^n$, with $x \in \mathbb{D}$ and is non decreasing over $\mathbb{D}^n$. Moreover, notice that Schur-concavity implies symmetry. Due to Schur-concavity, a welfare function ranks any Lorenz superior income distribution with the same mean as $\boldsymbol{x}$ as no worse than $\boldsymbol{x}$, whereas a strict welfare function ranks it as better.

Given a welfare function A, the *uniform equivalent income* $\tilde{x}$ associated with an income distribution $\boldsymbol{x}$ is defined as the income level which, if equally distributed among the population, would generate the same welfare value, $A(\tilde{x} \cdot \mathbf{1}) = A(\boldsymbol{x})$. Due to the idempotency of A, we obtain $\tilde{x} = A(\boldsymbol{x})$ for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$.

The uniform equivalent concept has been originally proposed by Chisini [19] in the general context of averaging functions, see for instance Bennet et al. [9]. In the welfare context the uniform equivalent income has been considered by Atkinson [5], Kolm [30], and Sen [40] and further elaborated by Blackorby and Donaldson [10, 11, 12] and Blackorby et al. [14].

Since $\bar{x} \cdot \mathbf{1} \preceq \boldsymbol{x}$ for any income distribution $\boldsymbol{x} \in \mathbb{D}^n$, Schur-concavity implies $A(\bar{x} \cdot \mathbf{1}) \geq A(\boldsymbol{x})$ and therefore $A(\boldsymbol{x}) \leq \bar{x}$ due to the idempotency of the welfare function. In other words, the mean income $\bar{x}$ and the uniform equivalent income $\tilde{x}$ are related by $0 \leq \tilde{x} \leq \bar{x}$.

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In the AKS framework introduced by Atkinson [5], Kolm [30], and Sen [40], a welfare function which is stable for translations induces an associated absolute inequality index by means of the correspondence formula $A(\boldsymbol{x}) = \bar{x} - G(\boldsymbol{x})$, see Blackorby and Donaldson [11]. The notion of absolute inequality index has been introduced by Kolm [31, 32] and developed by Blackorby and Donaldson [11], Blackorby et al. [14], and Weymark [44]. In the AKS framework, the welfare function and the associated absolute inequality index are said to be *ethical*, see also Sen [42], Blackorby et al. [14], Weymark [44], Blackorby and Donaldson [13], and Ebert [22].

Definition 4. *Given a welfare function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ which is stable for translations, the associated* Atkinson-Kolm-Sen (AKS) absolute inequality index $G : \mathbb{D}^n \longrightarrow \mathbb{D}$ *is defined as*

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) \tag{7}$$

The welfare function properties of A, plus the fact that it is stable for translations, ensure that the associated absolute inequality index G is continuous, nilpotent, Schur-convex, and invariant for translations. The absolute inequality index is said to be strict *if it is strictly Schur-convex.*

Given an income distribution $\boldsymbol{x} \in \mathbb{D}^n$ and its uniform equivalent income $\tilde{x}$ with respect to a welfare function A, the associated absolute inequality index can be written as $G(\boldsymbol{x}) = \bar{x} - \tilde{x}$ and represents the per capita income that could be saved if society distributed incomes equally without any loss of welfare.

In relation with the properties of the majorization relation discussed earlier the following holds: over all income distributions $\boldsymbol{x} \in \mathbb{D}^n$ with the same mean income $\bar{x}$, a welfare function has minimum value $A(0, \ldots, 0, n\bar{x})$, and an absolute inequality index has maximum value $G(0, \ldots, 0, n\bar{x})$.

In the AKS framework, with $A(\mathbf{x}) = \bar{x} - G(\mathbf{x})$, a welfare function A which is stable for both translations and dilations is associated with both absolute and relative inequality indices G and G_R, respectively, with $G(\boldsymbol{x}) = \bar{x}\, G_R(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{D}^n$. For convenience, as the notation suggests, in what follows we will omit the term "absolute" when referring to an absolute inequality index G.

A class of welfare functions which plays a central role in this paper is that of the generalized Gini welfare functions introduced by Weymark [44], see also Mehran [37], Donaldson and Weymark [20, 21], Yaari [45, 46], Ebert [23], Quiggin [39], Ben-Porath and Gilboa [8].

Definition 5. *Given a weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0, 1]^n$, with $w_1 \geq \cdots \geq w_n \geq 0$ and $\sum_{i=1}^n w_i = 1$, the* generalized Gini welfare function *associated with $\boldsymbol{w}$ is the function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ defined as*

$$A(\boldsymbol{x}) = \sum_{i=1}^{n} w_i x_{(i)} \tag{8}$$

and, in the AKS framework, the associated generalized Gini inequality index *is defined as*

$$G(\boldsymbol{x}) = \bar{x} - A(\boldsymbol{x}) = -\sum_{i=1}^{n} \left(w_i - \frac{1}{n} \right) x_{(i)} \,. \tag{9}$$

According to Proposition 2, generalized Gini welfare functions are strict if and only if $w_1 > \cdots > w_n$. Moreover, generalized Gini welfare functions are clearly stable for both translations and dilations. For this reason they have a natural role within the AKS framework and Blackorby and Donaldson's correspondence formula.

The classical Gini welfare function $A^c_G(\boldsymbol{x})$ and the associated classical Gini inequality index $G^c(\boldsymbol{x}) = \bar{x} - A^c_G(\boldsymbol{x})$ are important instances of the generalized Gini AKS framework,

$$A^c_G(\boldsymbol{x}) = \sum_{i=1}^{n} \frac{2(n-i)+1}{n^2}\, x_{(i)} \qquad\qquad G^c(\boldsymbol{x}) = -\sum_{i=1}^{n} \frac{n-2i+1}{n^2}\, x_{(i)} \tag{10}$$

where the coefficients of $A^c(\boldsymbol{x})$ have unit sum, $\sum_{i=1}^n (2(n-i)+1) = n^2$, and the coefficients of $G^c(\boldsymbol{x})$ have zero sum, $\sum_{i=1}^n (n - 2i + 1) = 0$.

This form of the classical Gini inequality index is obtained from the traditional definition as follows,

$$G^c(\boldsymbol{x}) = \frac{1}{2n^2} \sum_{i,j=1}^{n} |x_i - x_j| = \frac{1}{n^2} \sum_{i=2}^{n} \sum_{j=1}^{i-1} (x_{(i)} - x_{(i)}) \tag{11}$$

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which in fact corresponds to the expression of G^c in (10),

$$G^c(\boldsymbol{x}) = -\frac{1}{n^2}\Big((n-1)x_{(1)} + (n-3)x_{(2)} + \cdots + (-n+1)x_{(n)}\Big)\,. \tag{12}$$

Given an income distribution $\boldsymbol{x} \in \mathbb{D}^n$, the so-called Lorenz area measures the deviation from the uniform income distribution and is related with the classical relative Gini inequality index. In a population of $n \geq 2$ individuals, the graphical representation of the classical relative Gini inequality index can be described as follows (we review the traditional derivation).

Introducing the auxiliary functions

$$V(\boldsymbol{x}) = \sum_{i=1}^{n}(x_{(1)} + \cdots + x_{(i)}) \qquad U(\boldsymbol{x}) = \sum_{i=1}^{n}(x_{(i)} + \cdots + x_{(n)}) \tag{13}$$

we obtain $U(\boldsymbol{x}) + V(\boldsymbol{x}) = n(n+1)\,\bar{x}$. Moreover, writing the classical Gini inequality index G^c as in (12), we obtain $n^2 G^c(\boldsymbol{x}) = U(\boldsymbol{x}) - V(\boldsymbol{x})$.

Consider now the area illustrated in Fig. 1, in which the diagonal p_i values correspond to the cumulative income distribution, whereas the q_i values correspond to the actual cumulative income distribution,

$$p_i = \frac{i}{n} \qquad\qquad q_i = \frac{x_{(1)} + \cdots + x_{(i)}}{x_{(1)} + \cdots + x_{(n)}} \tag{14}$$

assuming $\boldsymbol{x} \neq \mathbf{0}$. The *Lorenz area* $L(\boldsymbol{x})$ corresponds to the overall area difference between the two series of

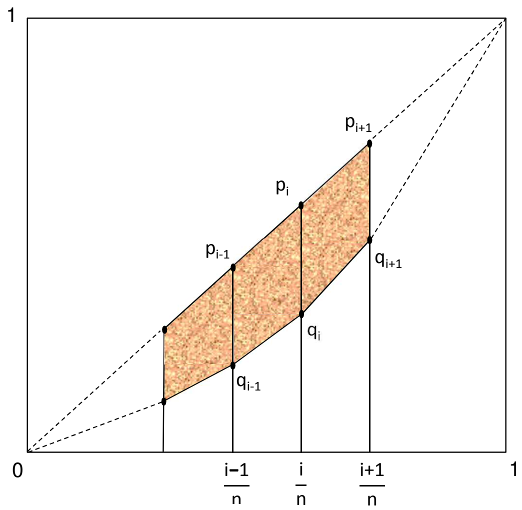

Figure 1: Lorenz area.

vertical trapezia associated with each subinterval $[\frac{i-1}{n}, \frac{i}{n}]$ with $i = 1, \ldots, n$: the higher trapezia bounded by the diagonal line associated with the uniform cumulative income distribution, and the lower trapezia bounded by the piecewise linear curve associated with the actual cumulative income distribution,

$$L(\boldsymbol{x}) = \sum_{i=1}^{n}\frac{p_i + p_{i-1}}{2n} - \sum_{i=1}^{n}\frac{q_i + q_{i-1}}{2n} = \frac{1}{n}\sum_{i=1}^{n}\Big(p_i - q_i\Big)\,. \tag{15}$$

Substituting for p_i, q_i with $i = 1, \ldots, n$ as in (14),

$$\begin{aligned} L(\boldsymbol{x}) &= \frac{1}{n}\sum_{i=1}^{n}\Big(\frac{i}{n} - \frac{x_{(1)} + \cdots + x_{(i)}}{x_{(1)} + \cdots + x_{(n)}}\Big) \\ &= \frac{1}{n^2\bar{x}}\Big(\frac{n(n+1)}{2}\,\bar{x} - V(\boldsymbol{x})\Big) \\ &= \frac{1}{2n^2\bar{x}}\Big(U(\boldsymbol{x}) - V(\boldsymbol{x})\Big) = \frac{1}{2}\,G^c_R(\boldsymbol{x}) \end{aligned} \tag{16}$$

we obtain that the Lorenz area is half the *classical relative Gini inequality index* $G^c_R(\boldsymbol{x}) = G^c(\boldsymbol{x})/\bar{x}$.

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3. The binomial decomposition of generalized Gini welfare functions

In this section we review the binomial decomposition of generalized Gini welfare functions due to Calvo and De Baets [17] and Bortot and Marques Pereira [15]. We examine the weighting structures of the binomial welfare functions C_j, with $j = 1, \ldots, n$, and the associated binomial inequality indices G_j, with $j = 1, \ldots, n$, and we provide numerical and graphical illustrations thereof.

Definition 6. *The* binomial welfare functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, *with* $j = 1, \ldots, n$, *are defined as*

$$C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji}\, x_{(i)} \qquad w_{ji} = \frac{\binom{n-i}{j-1}}{\binom{n}{j}} \qquad j = 1, \ldots, n \tag{17}$$

where the binomial weights w_{ji}, *with* $i, j = 1, \ldots, n$, *are null when* $i + j > n + 1$, *according to the usual convention that* $\binom{p}{q} = 0$ *when* $p < q$, *with* $p, q = 0, 1, \ldots$

Apart from $C_1(\boldsymbol{x}) = \bar{x}$, the binomial welfare functions C_j, with $j = 2, \ldots, n$, have an increasing number of null weights, the $j-1$ weights in correspondence with $x_{(n-j+2)}, \ldots, x_{(n)}$. The weight normalization of the binomial welfare functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$, is due to the column-sum property of binomial coefficients,

$$\sum_{i=1}^{n} \binom{n-i}{j-1} = \sum_{i=0}^{n-1} \binom{i}{j-1} = \binom{n}{j} \qquad j = 1, \ldots, n\,. \tag{18}$$

The binomial welfare functions C_j, with $j = 1, \ldots, n$, are continuous, idempotent, and stable for translations, where the latter two properties follow immediately from $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$. Moreover, the C_j are Schur-concave: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we have that $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ due to Proposition 2 and the fact that $w_{j1} \geq w_{j2} \geq \cdots \geq w_{jn}$, for $j = 1, \ldots, n$.

The following interesting result concerning the cumulative properties of binomial weights is due to Calvo and De Baets [17], see also Bortot and Marques Pereira [15].

Proposition 3. *The binomial weights* $w_{ji} \in [0, 1]$, *with* $i, j = 1, \ldots, n$, *have the following cumulative property,*

$$\sum_{i=1}^{k} w_{j-1,i} \leq \sum_{i=1}^{k} w_{ji} \qquad k = 1, \ldots, n \tag{19}$$

for each $j = 2, \ldots, n$.

If follows, according to Proposition 1, that the binomial OWA functions C_j, with $j = 1, \ldots, n$, satisfy the relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \cdots \geq C_n(\boldsymbol{x}) \geq 0$, for any $\boldsymbol{x} \in \mathbb{D}^n$.

Proposition 4. *Generalized Gini welfare functions* $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ *can be written uniquely as*

$$A(\boldsymbol{x}) = \alpha_1 C_1(\boldsymbol{x}) + \alpha_2 C_2(\boldsymbol{x}) + \cdots + \alpha_n C_n(\boldsymbol{x}) \tag{20}$$

where the coefficients α_j, *with* $j = 1, \ldots, n$, *are subject to the following conditions,*

$$\alpha_1 = 1 - \sum_{j=2}^{n} \alpha_j \geq 0 \tag{21}$$

$$\sum_{j=2}^{n} \left[1 - n \frac{\binom{i-1}{j-1}}{\binom{n}{j}} \right] \alpha_j \leq 1 \qquad i = 2, \ldots, n \tag{22}$$

$$\sum_{j=2}^{n} \frac{\binom{n-i}{j-2}}{\binom{n}{j}}\, \alpha_j \geq 0 \qquad i = 2, \ldots, n\,. \tag{23}$$

The binomial welfare functions constitute therefore a functional basis for the generalized Gini welfare functions, which can be uniquely expressed as $A(\boldsymbol{x}) = \sum_{j=1}^{n} \alpha_j \, C_j(\boldsymbol{x})$ where the coefficients α_j, with $j = 1, \ldots, n$, satisfy the constraints (21)-(22)-(23) one of which is $\sum_{j=1}^{n} \alpha_j = 1$. However, the binomial decomposition does not express a simple convex combination of the binomial welfare functions, as the condition $\alpha_1 + \cdots + \alpha_n = 1$ might suggest. In fact, condition (21) ensures $\alpha_1 \geq 0$ but conditions (22)-(23) allow for negative $\alpha_2, \ldots, \alpha_n$ values.

In the Atkinson-Kolm-Sen (AKS) framework, the binomial welfare functions C_j, with $j = 1, \ldots, n$, are associated with the binomial inequality indices G_j, with $j = 1, \ldots, n$, by means of Blackorby and Donaldson's correspondence formula.

Definition 7. *Consider the binomial welfare functions $C_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $C_j(\boldsymbol{x}) = \sum_{i=1}^{n} w_{ji} x_{(i)}$ for $j = 1, \ldots, n$. The* binomial inequality indices *$G_j : \mathbb{D}^n \longrightarrow \mathbb{D}$, with $j = 1, \ldots, n$, are defined as*

$$G_j(\boldsymbol{x}) = \bar{x} - C_j(\boldsymbol{x}) \qquad j = 1, \ldots, n \tag{24}$$

which means that

$$G_j(\boldsymbol{x}) = -\sum_{i=1}^{n} v_{ji} x_{(i)} = -\sum_{i=1}^{n} \left[w_{ji} - \frac{1}{n} \right] x_{(i)} \qquad j = 1, \ldots, n \tag{25}$$

where the coefficients v_{ji}, with $i, j = 1, \ldots, n$, are equal to $-1/n$ when $i + j > n + 1$, since in such case the binomial weights w_{ji} are null.

The weight normalization of the binomial welfare functions, $\sum_{i=1}^{n} w_{ji} = 1$ for $j = 1, \ldots, n$, implies that $\sum_{i=1}^{n} v_{ji} = 0$ for $j = 1, \ldots, n$.

The binomial inequality indices G_j, with $j = 1, \ldots, n$, are continuous, nilpotent, and invariant for translations, where the latter two properties follow immediately from $\sum_{i=1}^{n} v_{ji} = 0$ for $j = 1, \ldots, n$. Moreover, the G_j are Schur-convex: given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$ with $\bar{x} = \bar{y}$, we have that $\boldsymbol{x} \preceq \boldsymbol{y} \Rightarrow C_j(\boldsymbol{x}) \geq C_j(\boldsymbol{y}) \Rightarrow G_j(\boldsymbol{x}) \leq G_j(\boldsymbol{y})$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{D}^n$, due to the Schur-concavity of the C_j, with $j = 1, \ldots, n$.

In correspondence with the analogous but inverse relations satisfied by the binomial welfare functions, see note after Proposition 3, the binomial inequality indices satisfy the relations $0 = G_1(\boldsymbol{x}) \leq G_2(\boldsymbol{x}) \leq \cdots \leq G_n(\boldsymbol{x}) \leq 1$ for any $\boldsymbol{x} \in \mathbb{D}^n$.

Notice that $C_1(\boldsymbol{x}) = \bar{x}$ and $G_1(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in \mathbb{D}^n$. On the other hand, $C_2(\boldsymbol{x})$ has $n - 1$ positive linearly decreasing weights and one null last weight, and the associated $G_2(\boldsymbol{x})$ has linearly increasing coefficients. In terms of the classical Gini welfare function and inequality index, we have that

$$A^c(\boldsymbol{x}) = \frac{1}{n} \, C_1(\boldsymbol{x}) + \frac{n-1}{n} \, C_2(\boldsymbol{x}) \qquad\qquad G^c(\boldsymbol{x}) = \frac{n-1}{n} \, G_2(\boldsymbol{x}) \, . \tag{26}$$

The only strict binomial welfare function is $C_2(\boldsymbol{x})$ and the only strict binomial inequality index is $G_2(\boldsymbol{x})$. The remaining $C_j(\boldsymbol{x})$, with $j = 3, \ldots, n$, have $n - j + 1$ positive non-linear decreasing weights and $j - 1$ null last weights, and the associated $G_j(\boldsymbol{x})$, with $j = 3, \ldots, n$, have $n - j + 2$ non-linear increasing weights and $j - 1$ equal last weights. These binomial welfare functions and inequality indices are therefore non strict, in the sense that they are insensitive to income transfers involving only the $j - 1$ richest individuals of the population.

We now compute the weights of the binomial welfare functions and the coefficients of the associated binomial inequality indices in dimensions $n = 2, 3, 4, 5, 6, 7, 8$.

In dimensions $n = 2, 3, 4, 5, 6, 7, 8$ the weights $w_{ij} \in [0, 1]$, with $i, j = 1, \ldots, n$, of the binomial welfare functions C_j, with $j = 1, \ldots, n$, and the coefficients $-v_{ij} \in [-(n-1)/n, 1/n]$, with $i, j = 1, \ldots, n$, of the binomial inequality indices G_j, with $j = 1, \ldots, n$, are as follows,

$n = 2$

$$\begin{array}{llll} C_1: & (\frac{1}{2}, \frac{1}{2}) & G_1: & (0, 0) \\ C_2: & (1, 0) & G_2: & (-\frac{1}{2}, \frac{1}{2}) \end{array}$$

$n = 3$

$$\begin{array}{llll} C_1: & (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) & G_1: & (0, 0, 0) \\ C_2: & (\frac{2}{3}, \frac{1}{3}, 0) & G_2: & (-\frac{1}{3}, 0, \frac{1}{3}) \\ C_3: & (1, 0, 0) & G_3: & (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3}) \end{array}$$

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$n = 4$

$$\begin{array}{llll}
C_1: & (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}) & G_1: & (0, 0, 0, 0) \\
C_2: & (\frac{3}{6}, \frac{2}{6}, \frac{1}{6}, 0) & G_2: & (-\frac{3}{12}, -\frac{1}{12}, \frac{1}{12}, \frac{3}{12}) \\
C_3: & (\frac{3}{4}, \frac{1}{4}, 0, 0) & G_3: & (-\frac{2}{4}, 0, \frac{1}{4}, \frac{1}{4}) \\
C_4: & (1, 0, 0, 0) & G_4: & (-\frac{3}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})
\end{array}$$

$n = 5$

$$\begin{array}{llll}
C_1: & (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) & G_1: & (0, 0, 0, 0, 0) \\
C_2: & (\frac{4}{10}, \frac{3}{10}, \frac{2}{10}, \frac{1}{10}, 0) & G_2: & (-\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}) \\
C_3: & (\frac{6}{10}, \frac{3}{10}, \frac{1}{10}, 0, 0) & G_3: & (-\frac{4}{10}, -\frac{1}{10}, \frac{1}{10}, \frac{2}{10}, \frac{2}{10}) \\
C_4: & (\frac{4}{5}, \frac{1}{5}, 0, 0, 0) & G_4: & (-\frac{3}{5}, 0, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}) \\
C_5: & (1, 0, 0, 0, 0) & G_5: & (-\frac{4}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})
\end{array}$$

$n = 6$

$$\begin{array}{llll}
C_1: & (\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) & G_1: & (0, 0, 0, 0, 0, 0) \\
C_2: & (\frac{5}{15}, \frac{4}{15}, \frac{3}{15}, \frac{2}{15}, \frac{1}{15}, 0) & G_2: & (-\frac{5}{30}, -\frac{3}{30}, -\frac{1}{30}, \frac{1}{30}, \frac{3}{30}, \frac{5}{30}) \\
C_3: & (\frac{10}{20}, \frac{6}{20}, \frac{3}{20}, \frac{1}{20}, 0, 0) & G_3: & (-\frac{20}{60}, -\frac{8}{60}, \frac{1}{60}, \frac{7}{60}, \frac{10}{60}, \frac{10}{60}) \\
C_4: & (\frac{10}{15}, \frac{4}{15}, \frac{1}{15}, 0, 0, 0) & G_4: & (-\frac{15}{30}, -\frac{3}{30}, \frac{3}{30}, \frac{5}{30}, \frac{5}{30}, \frac{5}{30}) \\
C_5: & (\frac{5}{6}, \frac{1}{6}, 0, 0, 0, 0) & G_5: & (-\frac{4}{6}, 0, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}) \\
C_6: & (1, 0, 0, 0, 0, 0) & G_6: & (-\frac{5}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})
\end{array}$$

$n = 7$

$$\begin{array}{llll}
C_1: & (\frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}) & G_1: & (0, 0, 0, 0, 0, 0, 0) \\
C_2: & (\frac{6}{21}, \frac{5}{21}, \frac{4}{21}, \frac{3}{21}, \frac{2}{21}, \frac{1}{21}, 0) & G_2: & (-\frac{3}{21}, -\frac{2}{21}, -\frac{1}{21}, 0, \frac{1}{21}, \frac{2}{21}, \frac{3}{21}) \\
C_3: & (\frac{15}{35}, \frac{10}{35}, \frac{6}{35}, \frac{3}{35}, \frac{1}{35}, 0, 0) & G_3: & (-\frac{10}{35}, -\frac{5}{35}, -\frac{1}{35}, \frac{2}{35}, \frac{4}{35}, \frac{5}{35}, \frac{5}{35}) \\
C_4: & (\frac{20}{35}, \frac{10}{35}, \frac{4}{35}, \frac{1}{35}, 0, 0, 0) & G_4: & (-\frac{15}{35}, -\frac{5}{35}, \frac{1}{35}, \frac{4}{35}, \frac{5}{35}, \frac{5}{35}, \frac{5}{35}) \\
C_5: & (\frac{15}{21}, \frac{5}{21}, \frac{1}{21}, 0, 0, 0, 0) & G_5: & (-\frac{12}{21}, -\frac{2}{21}, \frac{2}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}, \frac{3}{21}) \\
C_6: & (\frac{6}{7}, \frac{1}{7}, 0, 0, 0, 0, 0) & G_6: & (-\frac{5}{7}, 0, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}) \\
C_7: & (1, 0, 0, 0, 0, 0, 0) & G_7: & (-\frac{6}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7}, \frac{1}{7})
\end{array}$$

$n = 8$

$$\begin{array}{llll}
C_1: & (\frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}) & G_1: & (0, 0, 0, 0, 0, 0, 0, 0) \\
C_2: & (\frac{7}{28}, \frac{6}{28}, \frac{5}{28}, \frac{4}{28}, \frac{3}{28}, \frac{2}{28}, \frac{1}{28}, 0) & G_2: & (-\frac{7}{56}, -\frac{5}{56}, -\frac{3}{56}, -\frac{1}{56}, \frac{1}{56}, \frac{3}{56}, \frac{5}{56}, \frac{7}{56}) \\
C_3: & (\frac{21}{56}, \frac{15}{56}, \frac{10}{56}, \frac{6}{56}, \frac{3}{56}, \frac{1}{56}, 0, 0) & G_3: & (-\frac{14}{56}, -\frac{8}{56}, -\frac{3}{56}, \frac{1}{56}, \frac{4}{56}, \frac{6}{56}, \frac{7}{56}, \frac{7}{56}) \\
C_4: & (\frac{35}{70}, \frac{20}{70}, \frac{10}{70}, \frac{4}{70}, \frac{1}{70}, 0, 0, 0) & G_4: & (-\frac{105}{280}, -\frac{45}{280}, -\frac{5}{280}, \frac{19}{280}, \frac{31}{280}, \frac{35}{280}, \frac{35}{280}, \frac{35}{280}) \\
C_5: & (\frac{35}{56}, \frac{15}{56}, \frac{5}{56}, \frac{1}{56}, 0, 0, 0, 0) & G_5: & (-\frac{28}{56}, -\frac{8}{56}, \frac{2}{56}, \frac{6}{56}, \frac{7}{56}, \frac{7}{56}, \frac{7}{56}, \frac{7}{56}) \\
C_6: & (\frac{21}{28}, \frac{6}{28}, \frac{1}{28}, 0, 0, 0, 0, 0) & G_6: & (-\frac{35}{56}, -\frac{5}{56}, \frac{5}{56}, \frac{7}{56}, \frac{7}{56}, \frac{7}{56}, \frac{7}{56}, \frac{7}{56}) \\
C_7: & (\frac{7}{8}, \frac{1}{8}, 0, 0, 0, 0, 0, 0) & G_7: & (-\frac{6}{8}, 0, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}) \\
C_8: & (1, 0, 0, 0, 0, 0, 0, 0) & G_8: & (-\frac{7}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8}, \frac{1}{8})
\end{array}$$

The binomial welfare functions C_j, with $j = 1, \ldots, n$, have null weights associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population. Correspondingly, the binomial inequality indices G_j, with $j = 1, \ldots, n$, have equal coefficients associated with the $j-1$ richest individuals in the population and therefore, as j increases from 1 to n, they are progressively insensitive to income transfers within the richest part of the population.

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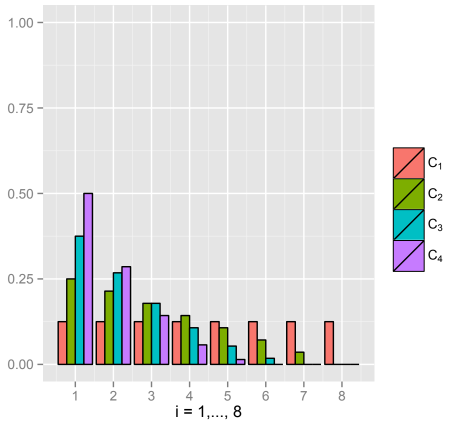
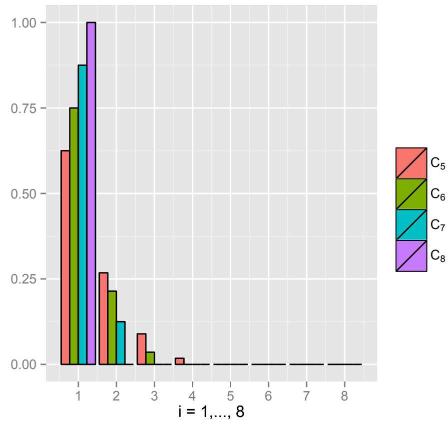
In the case $n = 8$, the weights of the binomial welfare functions C_j, with $j = 1, \ldots, 8$, and the coefficients of the binomial inequality indices G_j, with $j = 1, \ldots, 8$, are graphically represented in Fig. 2 and Fig. 3.

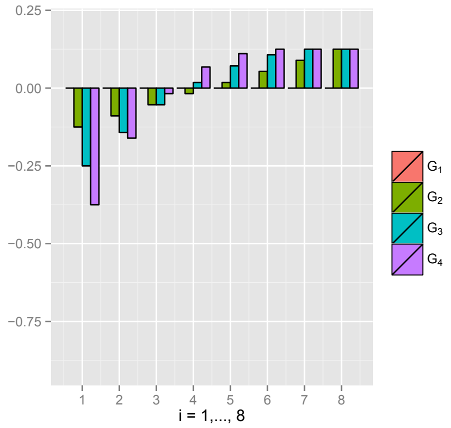


(a) Weights of C_1, C_2, C_3, C_4 with $n = 8$.

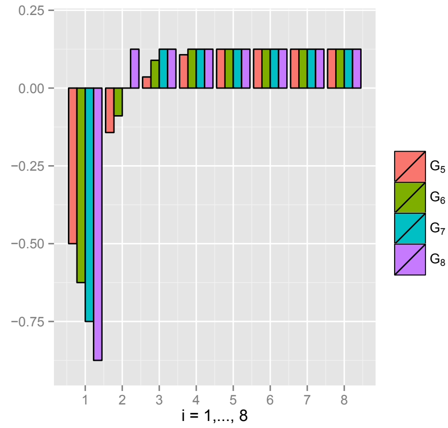


(b) Weights of C_5, C_6, C_7, C_8 with $n = 8$.

Figure 2: Weights of the binomial welfare functions C_j, with $j = 1, \ldots, n$, for $n = 8$.



(a) Coefficients of G_1, G_2, G_3, G_4 with $n = 8$.



(b) Coefficients of G_5, G_6, G_7, G_8 with $n = 8$.

Figure 3: Coefficients of the binomial inequality indices G_j, with $j = 1, \ldots, n$, for $n = 8$.

We now examine the binomial welfare functions and inequality indices in relation with a parametric family of income distributions with $n = 4, 6, 8$. This family of income distributions, each with unit average income, is defined on the basis of the parametric Lorenz curve associated with the generating function

$$f_\beta : [0,1] \to [0,1] \qquad f_\beta(r) = r e^{-\beta(1-r)} \qquad r \in [0,1] \tag{27}$$

where the parameter $\beta \geq 0$ is related with inequality. Fig. 4 provides a graphical illustration of the parametric Lorenz curve for parameter values $\beta = 0, 1, \ldots, 8$.

Consider a population with n individuals. The family of income distributions $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$ with unit average income $\bar{x} = 1$ associated with the parametric Lorenz curve above is given by

$$x_{(i)} = n\Big[f_\beta\Big(\frac{i}{n}\Big) - f_\beta\Big(\frac{i-1}{n}\Big)\Big] \qquad i = 1, \ldots, n. \tag{28}$$

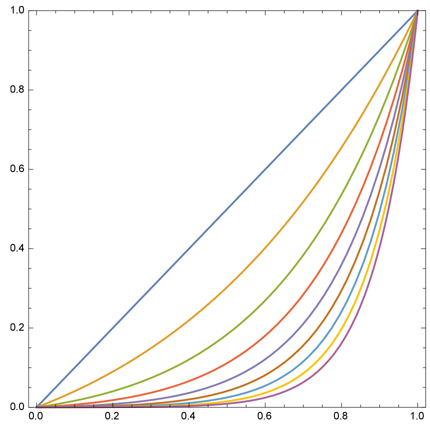


Figure 4: Parametric Lorenz curve for parameter values $\beta = 0, 1, \ldots, 8$.

We now compute the binomial welfare functions and inequality indices in relation to the family of income distributions (28), with $n = 4$, $n = 6$ and $n = 8$ for $\beta = 0, 1, \ldots, 8$.

The pattern of the numerical data in Fig. 5 reflects the dominance relations $\bar{x} = C_1(\boldsymbol{x}) \geq C_2(\boldsymbol{x}) \geq \cdots \geq C_n(\boldsymbol{x}) \geq 0$ and $0 = G_1(\boldsymbol{x}) \leq G_2(\boldsymbol{x}) \leq \cdots \leq G_n(\boldsymbol{x}) \leq \bar{x}$ for the income distributions considered here, with $\boldsymbol{x} \in \mathbb{D}^n$ and $\bar{x} = 1$.

Moreover, considering the parametric Lorenz curve depicted in Fig. 4, the values taken by the binomial welfare functions and inequality indices with $n = 4$, $n = 6$ and $n = 8$ for $\beta = 0, 1, \ldots, 8$, illustrate clearly the effect of the parameter $\beta \geq 0$ in relation with inequality.

4. The single paramether family of generalized Gini welfare functions

The binomial welfare functions and inequality indices bear some analogy with the S-Gini family of welfare functions and inequality indices introduced by Donaldson and Weymark [20], and independently by Kakwani [29] as an extension of a poverty measure proposed by Sen [41], see also Donaldson e Weymark [21], Yitzhaki [50], Bossert [16], Aaberge [1, 2, 3]. The welfare functions of the S-Gini family are of the form

$$A_\delta^S(\boldsymbol{x}) = \sum_{i=1}^{n} \left[\left(\frac{n-i+1}{n} \right)^\delta - \left(\frac{n-i}{n} \right)^\delta \right] x_{(i)} \tag{29}$$

where $\delta \in [1, \infty)$ is an inequality aversion parameter. In analogy with the binomial welfare functions, $A_1^S(\boldsymbol{x}) = \bar{x}$ and $A_2^S(\boldsymbol{x}) = \bar{x} - G^c(\boldsymbol{x})$, where G^c is the classical Gini. As the inequality aversion parameter increases, A_δ^S tends to the limit case $A_\infty^S(\boldsymbol{x}) = x_{(1)}$. In other words, the full range of the inequality aversion parameter, from $\delta = 1$ to $\delta = \infty$, corresponds to a continuous interpolation of the index $j = 1, \ldots, n$, with $A_1^S = C_1$ and $A_\infty^S = C_n$.

In Fig. 6 we depict the parameter value of the S-Gini welfare function whose weight distribution more closely resembles (mean square differences) the one of each binomial welfare function C_j, with $j = 1, \ldots, n$. The S-Gini parameter values indicated in the vertical axis are expressed in the normalized scale $\Delta \in [1, n)$ according to the transformation

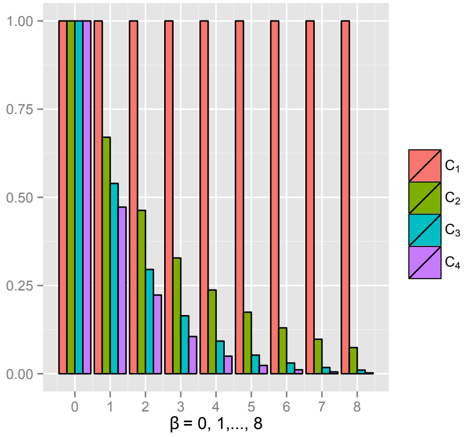
$$\Delta = 1\left(\frac{2}{\delta+1}\right) + n\left(\frac{\delta-1}{\delta+1}\right) \tag{30}$$

so that $\delta = 1$ corresponds to $\Delta = 1$ and $\delta = \infty$ corresponds to $\Delta = n$.

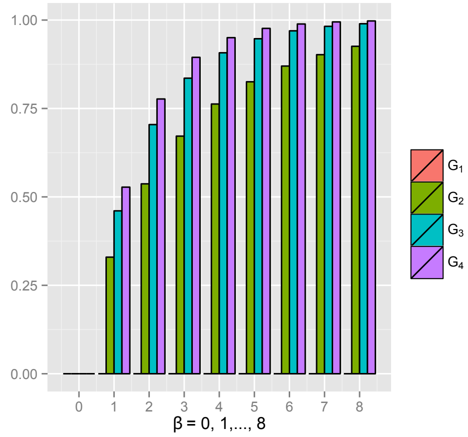
We can see that the normalized parameter value Δ increases very rapidly with respect to the first values of the binomial index $j = 1, ..., n$, a behavior which becomes more pronounced with increasing n.

The S-Gini welfare functions A_δ^S can be written as

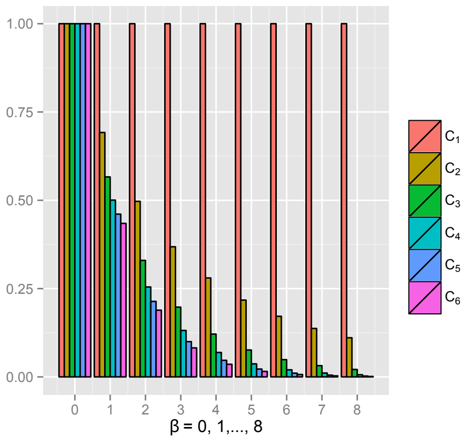
$$A_\delta^S(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^S(\delta)\, x_{(i)} \qquad \delta \in [1, \infty) \tag{31}$$



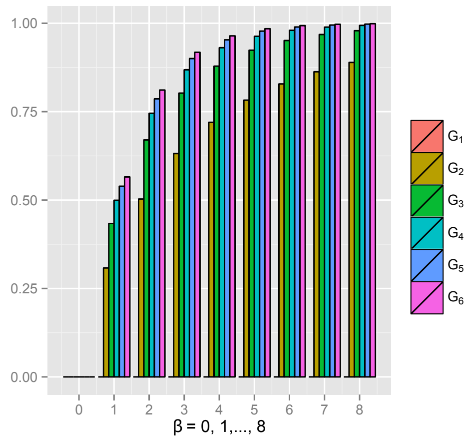
(a) $C_1, \ldots, C_4$ for $\beta = 0, 1, \ldots, 8$.



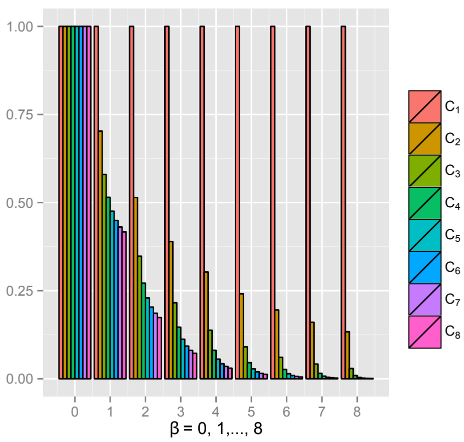
(b) $G_1, \ldots, G_4$ for $\beta = 0, 1, \ldots, 8$.



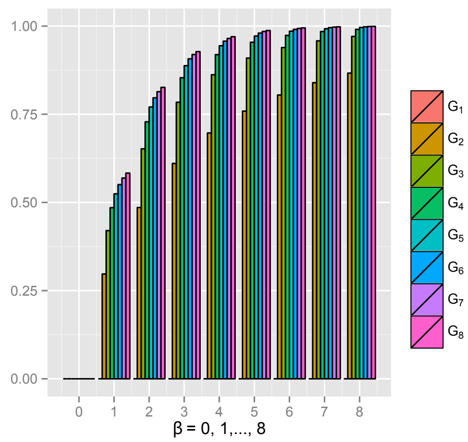
(c) $C_1, \ldots, C_6$ for $\beta = 0, 1, \ldots, 8$.



(d) $G_1, \ldots, G_6$ for $\beta = 0, 1, \ldots, 8$.



(e) $C_1, \ldots, C_8$ for $\beta = 0, 1, \ldots, 8$.



(f) $G_1, \ldots, G_8$ for $\beta = 0, 1, \ldots, 8$.

Figure 5: Values of the binomial welfare functions C_j and binomial inequality indices G_j, with $j = 1, \ldots, n$, for $n = 4, 6, 8$ for Lorenz curve parameter $\beta = 0, 1, \ldots, 8$.

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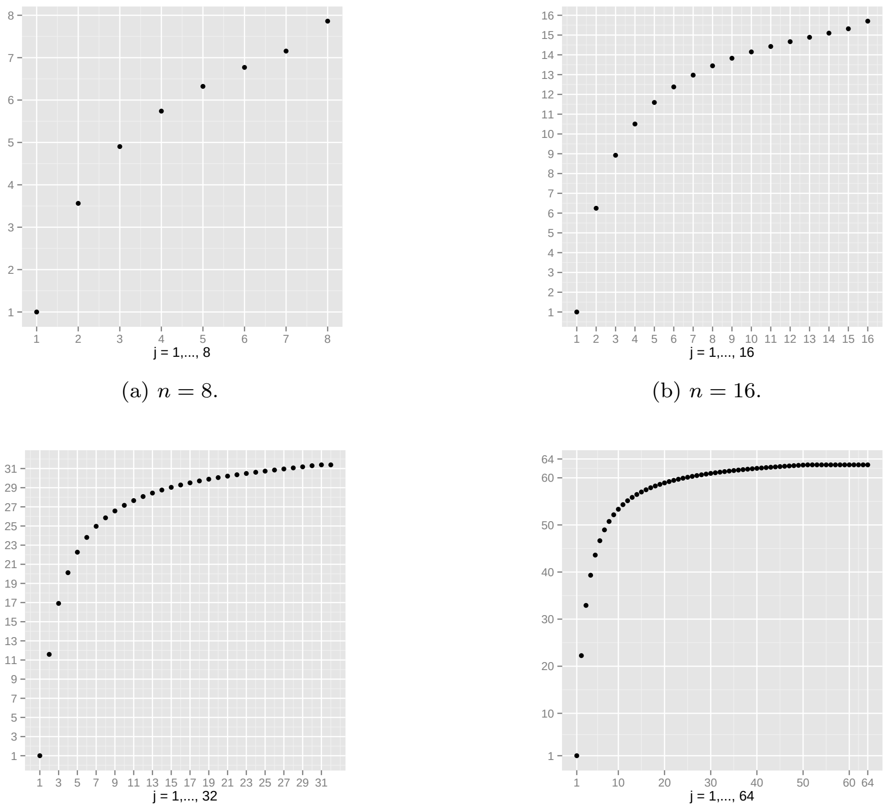

(a) $n = 8$. (b) $n = 16$.

(c) $n = 32$. (d) $n = 64$.

Figure 6: The normalized parameter Δ of the S-Gini welfare function whose weight distribution is closest to the one of each binomial welfare function C_j, with $j = 1, \dots, n$, for $n = 8, 16, 32, 64$.

where $w_i^S(\delta) = ((n-i+1)/n)^\delta - ((n-i)/n)^\delta$ as in (29).

In the context of the binomial decomposition (20), each S-Gini welfare function A_δ^S can be expressed in terms of the binomial welfare functions $C_1, C_2, \dots C_n$ as follows,

$$A_\delta^S(\boldsymbol{x}) = \alpha_1(\delta)C_1(\boldsymbol{x}) + \alpha_2(\delta)C_2(\boldsymbol{x}) + \cdots + \alpha_n(\delta)C_n(\boldsymbol{x}) \qquad \delta \in [1, \infty) \tag{32}$$

which can be written as

$$\sum_{i=1}^n w_i^S(\delta)\, x_{(i)} = \alpha_1(\delta) \sum_{i=1}^n w_{1i}\, x_{(i)} + \alpha_2(\delta) \sum_{i=1}^n w_{2i}\, x_{(i)} + \cdots + \alpha_n(\delta) \sum_{i=1}^n w_{ni}\, x_{(i)} \qquad \delta \in [1, \infty)\,. \tag{33}$$

The expression of the binomial decomposition is unique and therefore, for each value of the parameter $\delta \in [1, \infty)$, we obtain a unique solution $\alpha_1(\delta), \dots, \alpha_n(\delta)$ by solving the linear system

$$\begin{cases} w_1^S(\delta) = w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + \cdots + w_{n-1,1}\alpha_{n-1}(\delta) + w_{n1}\alpha_n(\delta) \\ w_2^S(\delta) = w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + \cdots + w_{n-1,2}\alpha_{n-1}(\delta) + w_{n2}\alpha_n(\delta) \\ \qquad \dots \\ w_n^S(\delta) = w_{1n}\alpha_1(\delta) + w_{2n}\alpha_2(\delta) + \cdots + w_{n-1,n}\alpha_{n-1}(\delta) + w_{nn}\alpha_n(\delta) \end{cases} \tag{34}$$

where the binomial weights w_{ji}, with $i, j = 1, \dots, n$, are as in (17). The linear system reduces to the triangular form

$$\begin{cases} w_1^S(\delta) = w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + \cdots + w_{n-1,1}\alpha_{n-1}(\delta) + w_{n1}\alpha_n(\delta) \\ w_2^S(\delta) = w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + \cdots + w_{n-1,2}\alpha_{n-1}(\delta) \\ \qquad \cdots \\ w_n^S(\delta) = w_{1n}\alpha_1(\delta) \end{cases} \tag{35}$$

due to the fact that the binomial weights w_{ji}, with $i, j = 1, \ldots, n$, are null when $i + j > n + 1$.

Example 1. In the case $n = 4$ the linear system (35) corresponds to

$$\begin{cases} w_1^S(\delta) = w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + w_{31}\alpha_3(\delta) + w_{41}\alpha_4(\delta) \\ w_2^S(\delta) = w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + w_{32}\alpha_3(\delta) \\ w_3^S(\delta) = w_{13}\alpha_1(\delta) + w_{23}\alpha_2(\delta) \\ w_4^S(\delta) = w_{14}\alpha_1(\delta) \end{cases} \tag{36}$$

and admits the unique solution

$$\begin{cases} \alpha_1(\delta) = 4 \cdot \left[1^\delta\right]/4^\delta \\ \alpha_2(\delta) = 6 \cdot \left[2^\delta \ - 2 \cdot 1^\delta\right]/4^\delta \\ \alpha_3(\delta) = 4 \cdot \left[3^\delta \ - 3 \cdot 2^\delta \ + 3 \cdot 1^\delta\right]/4^\delta \\ \alpha_4(\delta) = \left[4^\delta - 4 \cdot 3^\delta \ + 6 \cdot 2^\delta \ - 4 \cdot 1^\delta\right]/4^\delta \end{cases} \tag{37}$$

in which the $\alpha_j(\delta)$, with $j = 1, \ldots, 4$, are given explicitly as functions of the parameter $\delta \in [1, \infty)$, as illustrated in Fig. 7.

Example 2. In the case $n = 6$ the linear system (35) corresponds to

$$\begin{cases} w_1^S(\delta) = w_{11}\alpha_1(\delta) + w_{21}\alpha_2(\delta) + w_{31}\alpha_3(\delta) + w_{41}\alpha_4(\delta) + w_{51}\alpha_5(\delta) + w_{61}\alpha_6(\delta) \\ w_2^S(\delta) = w_{12}\alpha_1(\delta) + w_{22}\alpha_2(\delta) + w_{32}\alpha_3(\delta) + w_{42}\alpha_4(\delta) + w_{52}\alpha_5(\delta) \\ w_3^S(\delta) = w_{13}\alpha_1(\delta) + w_{23}\alpha_2(\delta) + w_{33}\alpha_3(\delta) + w_{43}\alpha_4(\delta) \\ w_4^S(\delta) = w_{14}\alpha_1(\delta) + w_{24}\alpha_2(\delta) + w_{34}\alpha_3(\delta) \\ w_5^S(\delta) = w_{15}\alpha_1(\delta) + w_{25}\alpha_2(\delta) \\ w_6^S(\delta) = w_{16}\alpha_1(\delta) \end{cases} \tag{38}$$

and admits the unique solution

$$\begin{cases} \alpha_1(\delta) = 6 \cdot \left[1^\delta\right]/6^\delta \\ \alpha_2(\delta) = 15 \cdot \left[2^\delta \ - 2 \cdot 1^\delta\right]/6^\delta \\ \alpha_3(\delta) = 20 \cdot \left[3^\delta \ - 3 \cdot 2^\delta \ + 3 \cdot 1^\delta\right]/6^\delta \\ \alpha_4(\delta) = 15 \cdot \left[4^\delta - 4 \cdot 3^\delta \ + 6 \cdot 2^\delta \ - 4 \cdot 1^\delta\right]/6^\delta \\ \alpha_5(\delta) = 6 \cdot \left[5^\delta - 5 \cdot 4^\delta + 10 \cdot 3^\delta \ - 10 \cdot 2^\delta \ + 5 \cdot 1^\delta\right]/6^\delta \\ \alpha_6(\delta) = \left[6^\delta - 6 \cdot 5^\delta + 15 \cdot 4^\delta - 20 \cdot 3^\delta \ + 15 \cdot 2^\delta \ - 6 \cdot 1^\delta\right]/6^\delta \end{cases} \tag{39}$$

in which the $\alpha_j(\delta)$, with $j = 1, \ldots, 6$, are given explicitly as functions of the parameter $\delta \in [1, \infty)$, as illustrated in Fig. 8.

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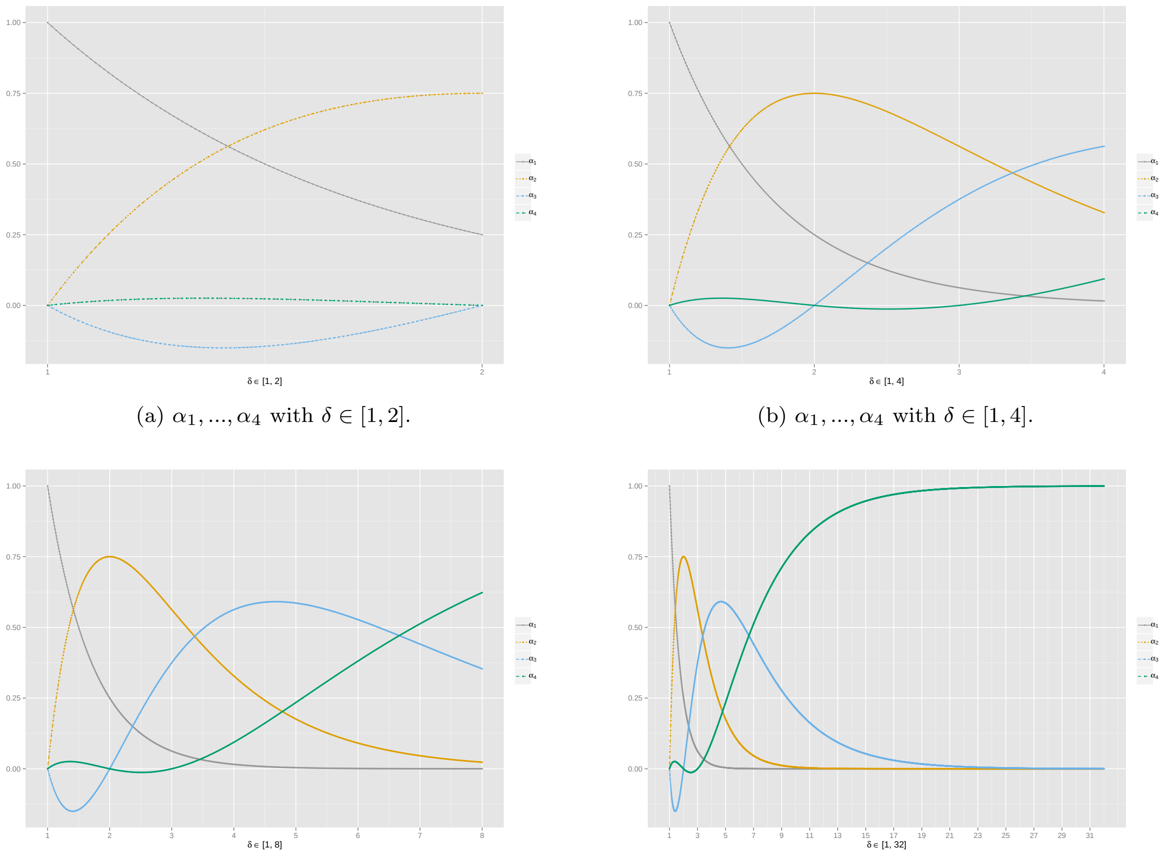

(a) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 2]$.

(b) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 4]$.

(c) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 8]$.

(d) $\alpha_1, ..., \alpha_4$ with $\delta \in [1, 32]$.

Figure 7: Coefficients of the binomial decomposition for $n = 4$.

Example 3. In the case $n = 8$ the linear system (35) admits the unique solution

$$\begin{cases} \alpha_1(\delta) = 8 \cdot \left[1^\delta\right] / 8^\delta \\ \alpha_2(\delta) = 28 \cdot \left[2^\delta \ - 2 \cdot 1^\delta\right] / 8^\delta \\ \alpha_3(\delta) = 56 \cdot \left[3^\delta \ - 3 \cdot 2^\delta \ + 3 \cdot 1^\delta\right] / 8^\delta \\ \alpha_4(\delta) = 70 \cdot \left[4^\delta - 4 \cdot 3^\delta \ + 6 \cdot 2^\delta \ - 4 \cdot 1^\delta\right] / 8^\delta \\ \alpha_5(\delta) = 56 \cdot \left[8^\delta - 5 \cdot 4^\delta + 10 \cdot 3^\delta \ - 10 \cdot 2^\delta \ + 5 \cdot 1^\delta\right] / 8^\delta \\ \alpha_6(\delta) = 28 \cdot \left[6^\delta - 6 \cdot 5^\delta + 15 \cdot 4^\delta - 20 \cdot 3^\delta \ + 15 \cdot 2^\delta \ - 6 \cdot 1^\delta\right] / 8^\delta \\ \alpha_7(\delta) = 8 \cdot \left[7^\delta - 7 \cdot 6^\delta + 21 \cdot 5^\delta - 35 \cdot 4^\delta + 35 \cdot 3^\delta \ - 21 \cdot 2^\delta \ + 7 \cdot 1^\delta\right] / 8^\delta \\ \alpha_8(\delta) = \left[8^\delta - 8 \cdot 7^\delta + 28 \cdot 6^\delta - 56 \cdot 5^\delta + 70 \cdot 4^\delta - 56 \cdot 3^\delta \ + 28 \cdot 2^\delta \ - 8 \cdot 1^\delta\right] / 8^\delta \end{cases} \tag{40}$$

in which the $\alpha_j(\delta)$, with $j = 1, \dots, 8$, are given explicitly as functions of the parameter $\delta \in [1, \infty)$, as illustrated in Fig. 9.

In Fig. 7-9 we depict the solution $\alpha_1(\delta), \dots, \alpha_n(\delta)$ as a function of the parameter $\delta \in [1, \infty)$ in the cases $n = 4, 6, 8$ and for various ranges of the parameter. We observe, as expected, that

- $\alpha_1 = 1$ and $\alpha_2 = \cdots = \alpha_n = 0$ for parameter $\delta = 1$,
- $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n = 1$ for parameter $\delta = \infty$.

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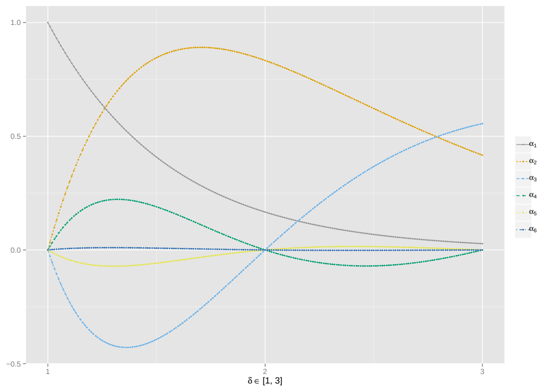
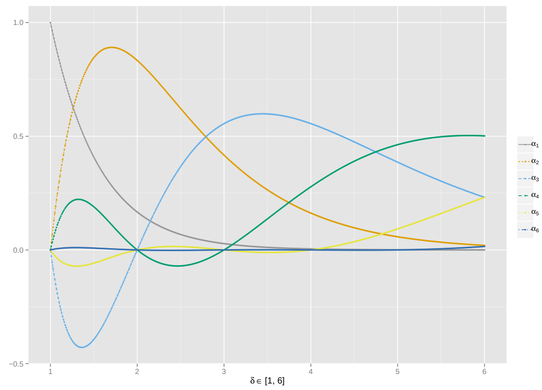
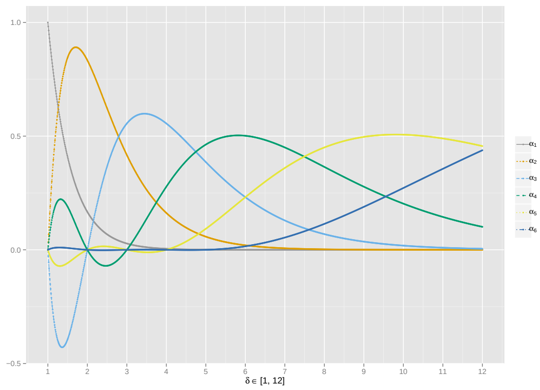


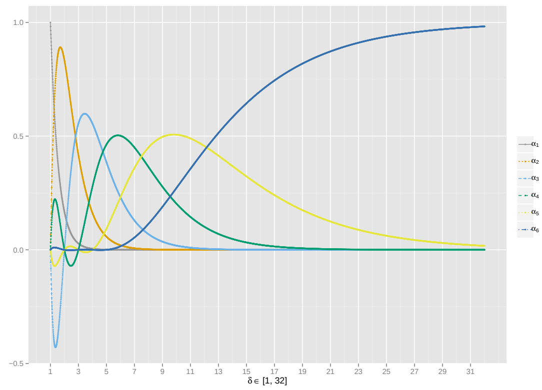
(a) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 3]$.



(b) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 6]$.



(c) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 12]$.



(d) $\alpha_1, ..., \alpha_6$ with $\delta \in [1, 32]$.

Figure 8: Coefficients of the binomial decomposition for $n = 6$.

Moreover, each coefficient $\alpha_j(\delta)$, with $j = 2, \ldots, n$, is null in correspondence with the integer values $\delta = 1, \ldots, j-1$. Finally, each $\alpha_j(\delta)$ shows an oscillating behavior as a function of δ up to $\delta = j-1$, assuming positive and negative values in consecutive unit intervals, and ultimately assuming positive values for $\delta > j-1$. The norm of the negative values is larger in the very first unit intervals of the parameter range.

The welfare functions of the S-Gini family (29) are of the general form

$$A_f(\boldsymbol{x}) = \sum_{i=1}^{n} \left[f\left(\frac{n-i+1}{n}\right) - f\left(\frac{n-i}{n}\right) \right] x_{(i)} \tag{41}$$

where f is a continuous and increasing function on the unit interval, with $f(0) = 0$ and $f(1) = 1$. The power functions with positive integer exponent $f(t) = t^k$, with $k = 1, \ldots, n$, can be seen in relation with the k-additivity of the welfare function, as discussed in Gajdos [26]. In fact we observe in Fig. 7-9 that for $\delta = k$ we have $\alpha_{k+1}(\delta) = \cdots = \alpha_n(\delta) = 0$, as required by k-additivity.

5. The Lorenzen family of generalized Gini welfare functions

An alternative generalization of the classical Gini which again has some analogy with the binomial welfare functions and inequality indices is that proposed by Lorenzen [33], see Weymark [44],

$$A_l^L(\boldsymbol{x}) = \sum_{i=1}^{l} \frac{l+n-2i+1}{nl} x_{(i)} = \sum_{i=1}^{l} \frac{1}{n} x_{(i)} + \sum_{i=1}^{l} \frac{n-2i+1}{nl} x_{(i)} \qquad l = 1, \ldots, n\,. \tag{42}$$

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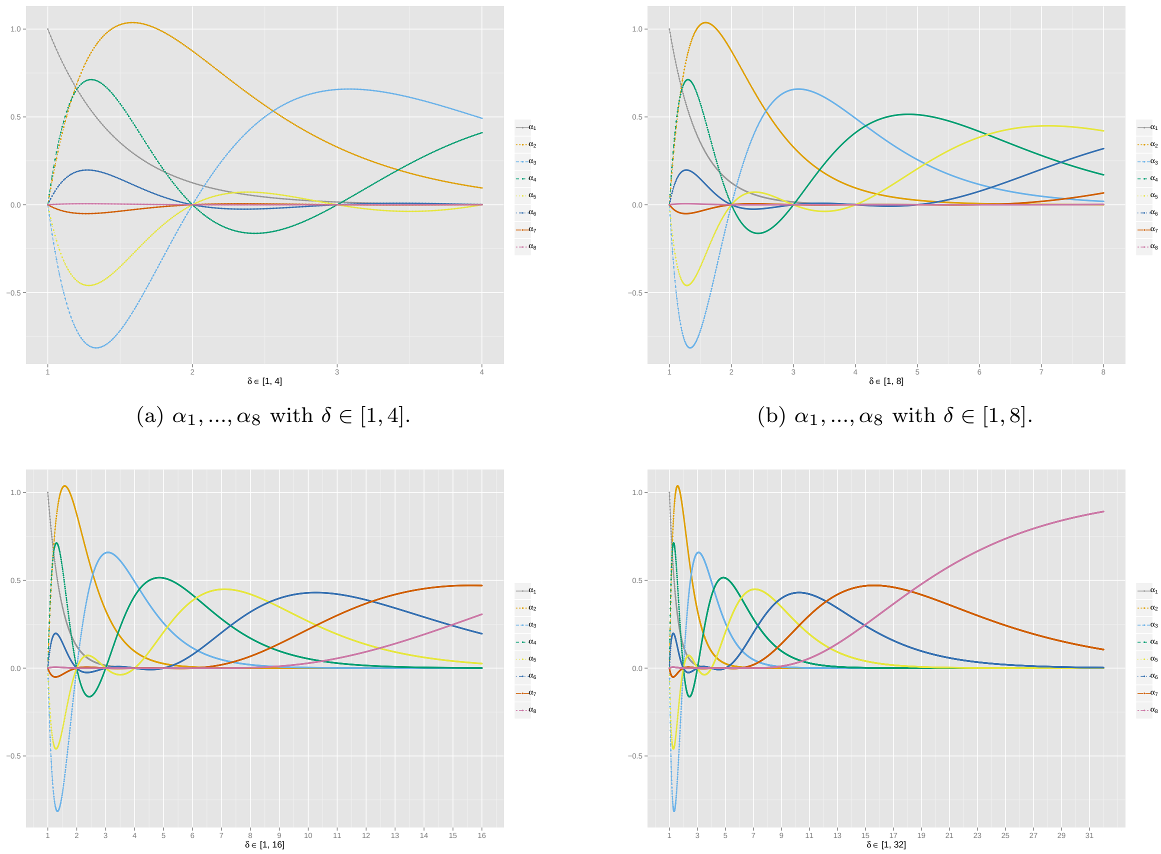

(a) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 4]$. (b) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 8]$.

(c) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 16]$. (d) $\alpha_1, ..., \alpha_8$ with $\delta \in [1, 32]$.

Figure 9: Coefficients of the binomial decomposition for $n = 8$.

The extreme cases are $A_1^L(\boldsymbol{x}) = x_{(1)}$ and $A_n^L(\boldsymbol{x}) = \bar{x} - G^c(\boldsymbol{x}) = A^c(\boldsymbol{x})$, where G^c is the classical Gini. As l increases from 1 to n, the Lorenzen welfare function A_l^L involves only the l poorest individuals in the population, to whom it assigns linearly decreasing positive weights. Analogously, the binomial welfare functions C_{n-j+1}, for $j = 1, \ldots, n-1$, also involve only the j poorest individuals but assign them non-linear binomial weights, from $C_n(\boldsymbol{x}) = x_{(1)}$ to $C_2(\boldsymbol{x}) = A^c(\boldsymbol{x}) - \frac{1}{n-1} G^c(\boldsymbol{x})$, where G^c is the classical Gini and A^c is the associated welfare function.

In Fig. 10 we depict the index value associate with the Lorenzen welfare function whose weight distribution more closely resembles (mean square differences) the one of each binomial welfare function C_j, with $j = 1, \ldots, n$.

We can see that the Lorenzen index l decreases very rapidly with respect to the first values of the binomial index $l = 1, ..., n$, a behavior which becomes more pronounced with increasing n.

The Lorenzen welfare functions A_l^L, with $l = 1, \ldots, n$, can be written as

$$A_l^L(\boldsymbol{x}) = \sum_{i=1}^{n} w_i^L(l)\, x_{(i)} \qquad l = 1, \ldots, n \tag{43}$$

where $w_i^L(l) = (l + n - 2i + 1)/nl$ for $i \leq l$ as in (42), and $w_i^L(l) = 0$ otherwise.

In the context of the binomial decomposition (20), each Lorenzen welfare function A_l^L can be expressed in terms of the binomial welfare functions $C_1, C_2, \ldots C_n$ as follows,

$$A_l^L(\boldsymbol{x}) = \alpha_1(l)C_1(\boldsymbol{x}) + \alpha_2(l)C_2(\boldsymbol{x}) + \cdots + \alpha_n(l)C_n(\boldsymbol{x}) \qquad l = 1, \ldots, n \tag{44}$$

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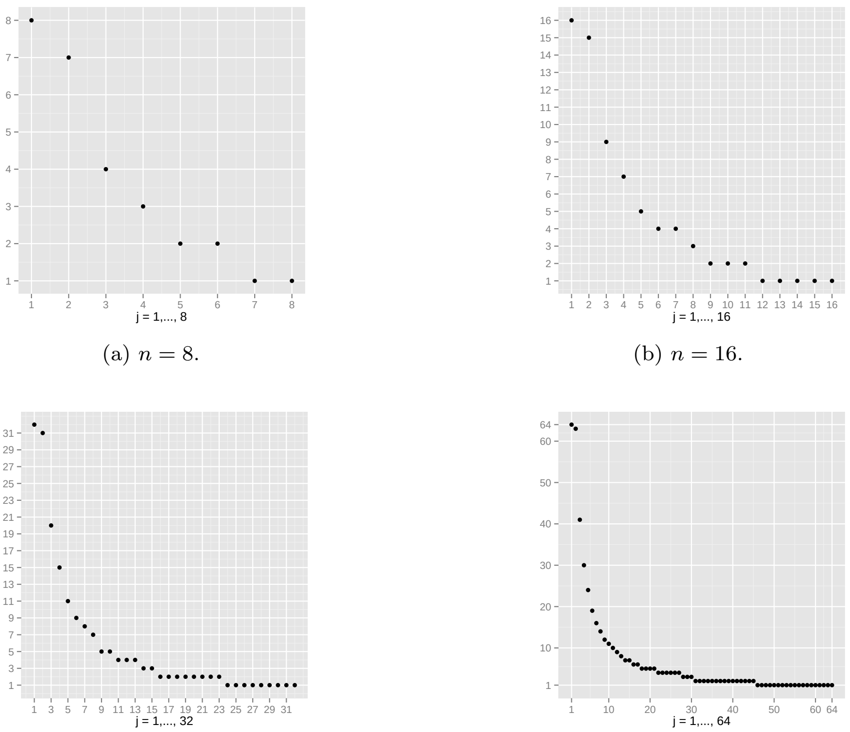

(a) $n = 8$.

(b) $n = 16$.

(c) $n = 32$.

(d) $n = 64$.

Figure 10: The index value $l(j)$ of the Lorenzen welfare function whose weight distribution is closest to the one of each binomial welfare function C_j, with $j = 1, \ldots, n$, for $n = 8, 16, 32, 64$.

which can be written as

$$\sum_{i=1}^{n} w_i^L(l)\, x_{(i)} = \alpha_1(l) \sum_{i=1}^{n} w_{1i}\, x_{(i)} + \alpha_2(l) \sum_{i=1}^{n} w_{2i}\, x_{(i)} + \cdots + \alpha_n(l) \sum_{i=1}^{n} w_{ni}\, x_{(i)} \qquad l = 1, \ldots, n\,. \tag{45}$$

The expression of the binomial decomposition is unique and therefore, for each value of the Lorenzen index $l = 1, \ldots, n$, we obtain a unique solution $\alpha_1(\delta), \ldots, \alpha_n(\delta)$ by solving the triangular linear system

$$\begin{cases} w_1^L(l) = w_{11}\alpha_(l)1 + w_{21}\alpha_2(l) + \cdots + w_{n-1,1}\alpha_{n-1}(l) + w_{n1}\alpha_n(l) \\ w_2^L(l) = w_{12}\alpha_1(l) + w_{22}\alpha_2(l) + \cdots + w_{n-12}\alpha_{n-1}(l) \\ \qquad \cdots \\ w_n^L(l) = w_{1n}\alpha_1(l) \end{cases} \tag{46}$$

where the binomial weights w_{ji}, with $i, j = 1, \ldots, n$, are as in (17).

Example 4. In the case n=4 the linear system (46) corresponds to

$$\begin{cases} w_1^L(l) = w_{11}\alpha_1(l) + w_{21}\alpha_2(l) + w_{31}\alpha_3(l) + w_{41}\alpha_4(l) \\ w_2^L(l) = w_{12}\alpha_1(l) + w_{22}\alpha_2(l) + w_{32}\alpha_3(l) \\ w_3^L(l) = w_{13}\alpha_1(l) + w_{23}\alpha_2(l) \\ w_4^L(l) = w_{14}\alpha_1(l) \end{cases} \tag{47}$$

and admits the unique solution

$$\begin{cases} \alpha_1(l = 1, \ldots, 4) = \left(0, 0, 0, \frac{1}{4}\right) \\ \alpha_2(l = 1, \ldots, 4) = \left(0, 0, 1, \frac{3}{4}\right) \\ \alpha_3(l = 1, \ldots, 4) = \left(0, \frac{3}{2}, 0, 0\right) \\ \alpha_4(l = 1, \ldots, 4) = \left(1, -\frac{1}{2}, 0, 0\right) \end{cases} \tag{48}$$

in which the $\alpha_j(l)$, with $j = 1, \ldots, 4$, are given explicitly as functions of the parameter $l = 1, \ldots, 4$, as illustrated in Fig. 11 (left).

Example 5. In the case n=6 the linear system (46) corresponds to

$$\begin{cases} w_1^L(l) = w_{11}\alpha_1(l) + w_{21}\alpha_2(l) + w_{31}\alpha_3(l) + w_{41}\alpha_4(l) + w_{51}\alpha_5(l) + w_{61}\alpha_6(l) \\ w_2^L(l) = w_{12}\alpha_1(l) + w_{22}\alpha_2(l) + w_{32}\alpha_3(l) + w_{42}\alpha_4(l) + w_{52}\alpha_5(l) \\ w_3^L(l) = w_{13}\alpha_1(l) + w_{23}\alpha_2(l) + w_{33}\alpha_3(l) + w_{43}\alpha_4(l) \\ w_4^L(l) = w_{14}\alpha_1(l) + w_{24}\alpha_2(l) + w_{34}\alpha_3(l) \\ w_5^L(l) = w_{15}\alpha_1(l) + w_{25}\alpha_2(l) \\ w_6^L(l) = w_{16}\alpha_1(l) \end{cases} \tag{49}$$

and admits the unique solution

$$\begin{cases} \alpha_1(l = 1, \ldots, 6) = \left(0, 0, 0, 0, 0, \frac{1}{6}\right) \\ \alpha_2(l = 1, \ldots, 6) = \left(0, 0, 0, 0, 1, \frac{5}{6}\right) \\ \alpha_3(l = 1, \ldots, 6) = \left(0, 0, 0, \frac{5}{2}, 0, 0\right) \\ \alpha_4(l = 1, \ldots, 6) = \left(0, 0, \frac{10}{3}, -\frac{5}{2}, 0, 0\right) \\ \alpha_5(l = 1, \ldots, 6) = \left(0, \frac{5}{2}, -\frac{10}{3}, \frac{5}{4}, 0, 0\right) \\ \alpha_6(l = 1, \ldots, 6) = \left(1, -\frac{3}{2}, 1, -\frac{1}{4}, 0, 0\right) \end{cases} \tag{50}$$

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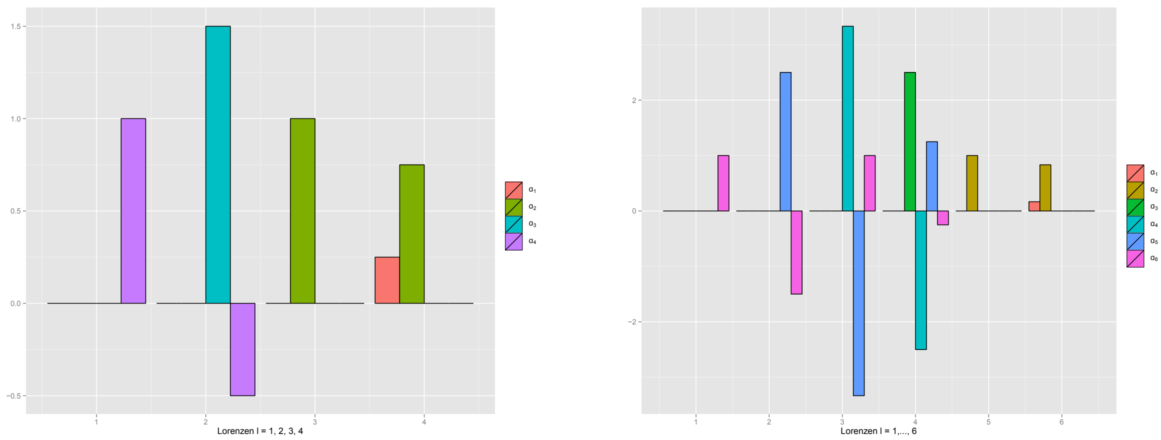
in which the $\alpha_j(l)$, with $j = 1, \ldots, 6$, are given explicitly as functions of the parameter $l = 1, \ldots, 6$, as illustrated in Fig. 11 (right).



(a) $\alpha_1(l), ..., \alpha_4(l)$, with $l = 1, ..., n$, for $n = 4$.

(b) $\alpha_1(l), ..., \alpha_6(l)$, with $l = 1, ..., n$, for $n = 6$.

Figure 11: Coefficients of the binomial decomposition for $n = 4, 6$.

Example 6. In the case n=8 the linear system (46) admits the unique solution

$$\begin{cases} \alpha_1(l = 1, \ldots, 8) = \left(0, 0, 0, 0, 0, 0, 0, \frac{1}{8}\right) \\ \alpha_2(l = 1, \ldots, 8) = \left(0, 0, 0, 0, 0, 0, 1, \frac{7}{8}\right) \\ \alpha_3(l = 1, \ldots, 8) = \left(0, 0, 0, 0, 0, \frac{7}{2}, 0, 0\right) \\ \alpha_4(l = 1, \ldots, 8) = \left(0, 0, 0, 0, 7, -\frac{35}{6}, 0, 0\right) \\ \alpha_5(l = 1, \ldots, 8) = \left(0, 0, 0, \frac{35}{4}, -14, \frac{35}{6}, 0, 0\right) \\ \alpha_6(l = 1, \ldots, 8) = \left(0, 0, 7, -\frac{63}{4}, \frac{63}{5}, -\frac{7}{2}, 0, 0\right) \\ \alpha_7(l = 1, \ldots, 8) = \left(0, \frac{7}{2}, -\frac{28}{3}, \frac{21}{2}, -\frac{28}{5}, \frac{7}{6}, 0, 0\right) \\ \alpha_8(l = 1, \ldots, 8) = \left(1, -\frac{5}{2}, \frac{10}{3}, -\frac{5}{2}, 1, -\frac{1}{6}, 0, 0\right) \end{cases} \tag{51}$$

in which the $\alpha_j(l)$, with $l = 1, \ldots, 8$, are given explicitly as functions of the parameter $l = 1, \ldots, 8$, as illustrated in Fig. 12.

In Fig. 11-12 we depict the solution $\alpha_1(l), \ldots, \alpha_n(l)$ as a function of the Lorenzen index $l = 1, \ldots, n$, in the cases $n = 4, 6, 8$. We observe, as expected, that

- $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n = 1$ for Lorenzen index $l = 1$,
- $\alpha_1 = 1/n$, $\alpha_2 = (n-1)/n$, and $\alpha_3 = \cdots = \alpha_n = 0$ for Lorenzen index $l = n$.

Moreover, we observe that the higher coefficients $\alpha_j(l)$ assume significant an alternating positive and negative values in the upper central range of the Lorenzen index l. The norm of these alternating positive and negative values increases considerably with the population size n.

6. Orness of generalized Gini welfare functions

Generalized Gini welfare functions have non increasing weights, see Definition 5, and therefore their orness [47] takes values in the interval $[0, 1/2]$. In this section we write the orness of a generalized Gini welfare function A in terms of the coefficients α_j, with $j = 1, \ldots, n$, of its binomial decomposition, and we examine the orness of the S-Gini and Lorenzen welfare functions.

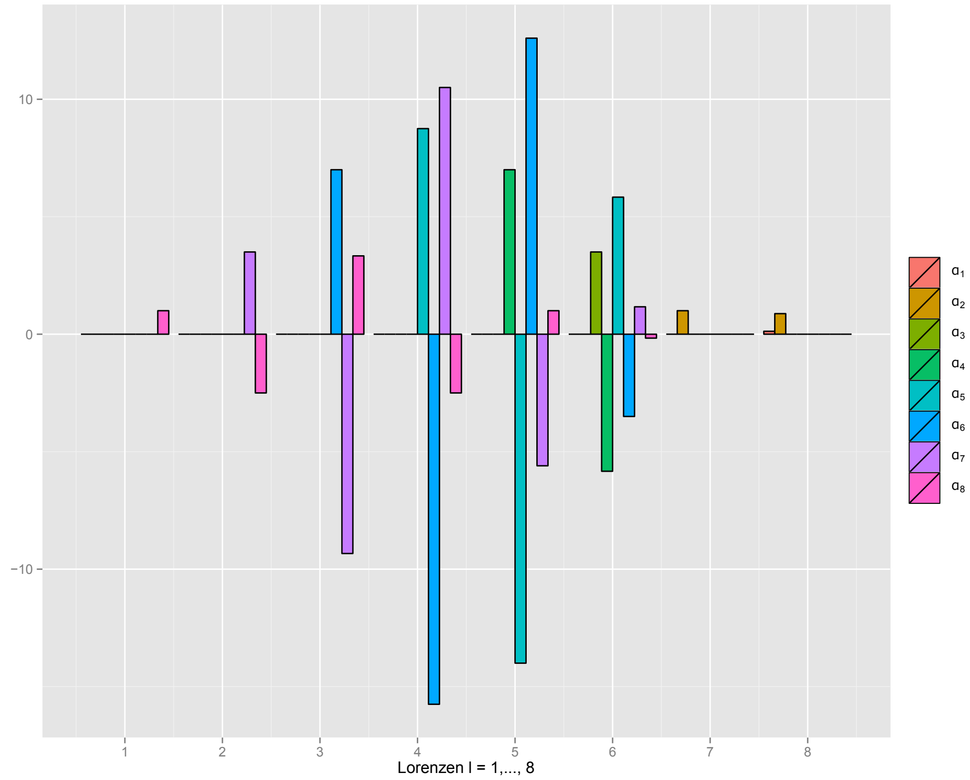


Figure 12: $\alpha_1(l), ..., \alpha_8(l)$, with $j = 1, ..., n$, for $n = 8$.

Definition 8. *Let A be the generalized Gini welfare function associated with the weighting vector $\boldsymbol{w} = (w_1, \ldots, w_n) \in [0,1]^n$. The* orness *of A is defined as*

$$\text{Orness}\,(A) = \frac{1}{n-1}\sum_{i=1}^{n}(i-1)\,w_i\,. \tag{52}$$

The orness of A coincides with the value $A(\boldsymbol{x}_0)$, where $x_i^0 = (i-1)/(n-1)$,

$$\text{Orness}\,(A) = \frac{1-1}{n-1}\,w_1 + \frac{2-1}{n-1}\,w_2 + \cdots + \frac{(n-1)-1}{n-1}\,w_{n-1} + \frac{n-1}{n-1}\,w_n\,. \tag{53}$$

Proposition 5. *In relation with the binomial decomposition, the orness of a generalized Gini welfare function $A : \mathbb{D}^n \longrightarrow \mathbb{D}$ is given by*

$$\text{Orness}\,(A) = \sum_{j=1}^{n}\frac{(n-j)}{(n-1)(j+1)}\,\alpha_j \tag{54}$$

Proof: We begin by showing that the orness of the binomial welfare functions C_j, with $j = 1, \ldots, n$, is given by

$$\text{Orness}\,(C_j) = \frac{n-j}{(n-1)(j+1)} \qquad j = 1, \ldots, n\,. \tag{55}$$

From the definition of C_j (17) and the general definition of orness (52), we have

$$\text{Orness}\,(C_j) = C_j(\boldsymbol{x}_0) = \sum_{i=1}^{n}\frac{\binom{n-i}{j-1}}{\binom{n}{j}}\,\frac{i-1}{n-1} \qquad j = 1, \ldots, n\,. \tag{56}$$

Using the formula

$$\sum_{i=1}^{n}\binom{n-i}{j-1}(i-1) = \binom{n}{j+1} \qquad j = 1, \ldots, n \tag{57}$$

and substituting in (56), we obtain

$$\text{Orness}\,(C_j) = \frac{1}{n-1}\,\frac{\binom{n}{j+1}}{\binom{n}{j}} = \frac{n-j}{(n-1)(j+1)} \qquad j = 1, \ldots, n\,. \tag{58}$$

Notice that the orness of the binomial welfare function is strictly decreasing with respect to $j = 1, \ldots, n$, from $\text{Orness}\,(C_1) = 1/2$ to $\text{Orness}\,(C_n) = 0$.

Considering the binomial decomposition of a generalized Gini welfare function,

$$\text{Orness}\,(A) = A(\boldsymbol{x}_0) = \sum_{j=1}^{n} \alpha_j\, C_j(\boldsymbol{x}_0) = \sum_{j=1}^{n} \frac{(n-j)}{(n-1)(j+1)}\, \alpha_j \tag{59}$$

where we have used that $C_j(\boldsymbol{x}_0) = \text{Orness}\,(C_j)$. □

In the following figures we illustrate the orness of the S-Gini and Lorenzen families of welfare functions in terms of their respective parameters, $\delta \in [1, \infty)$ and $l = 1, \ldots, n$.

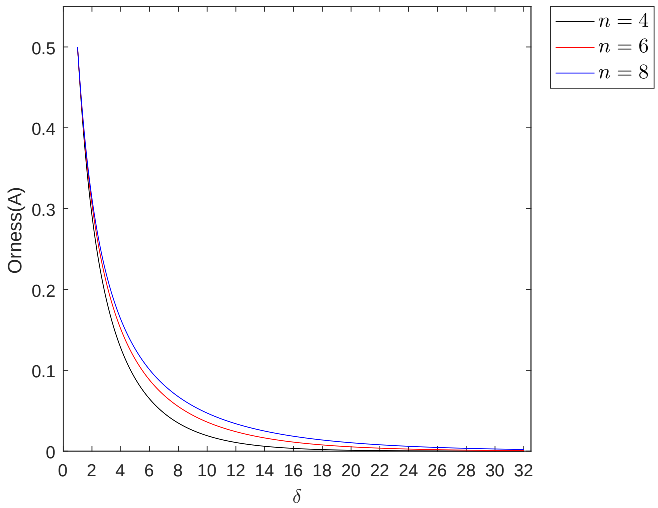
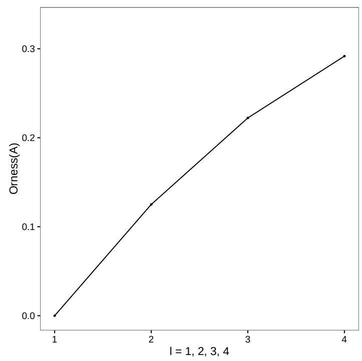
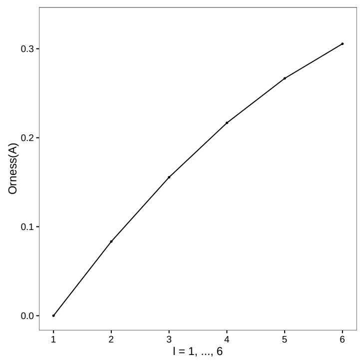


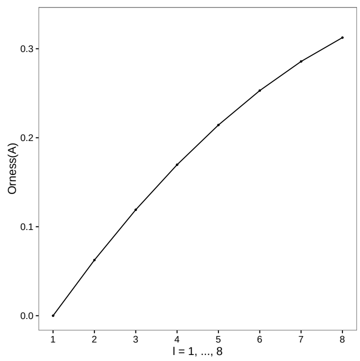
Figure 13: Orness of S-Gini welfare functions for $n = 4, 6, 8$.



(a) Lorenzen orness for $n = 4$.



(b) Lorenzen orness for $n = 6$.



(c) Lorenzen orness for $n = 8$.

Figure 14: Orness of Lorenzen welfare functions for $n = 4, 6, 8$.

The orness values taken by the S-Gini and Lorenzen welfare functions in terms of their respective parameters express the nature of these families of welfare functions. The S-Gini family, with its continuous

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parameter $\delta \in [1, \infty)$, interpolates between the first and the last binomial welfare functions C_1 and C_n, with orness 1/2 and 0, respectively. On the other hand, the Lorenzen family, with its parameter $l = 1, \ldots, n$, interpolates between the last binomial welfare function C_n with null orness and the classical Gini welfare function, whose orness is given by $1/3 - 1/6n$, see for instance Aristondo et al. [4].

7. Conclusions

We consider the binomial decomposition of generalized Gini welfare functions in terms of the binomial welfare functions C_j, with $j = 1, ..., n$, and the associated binomial inequality indices G_j, with $j = 1, ..., n$, for all income distributions $\boldsymbol{x} \in \mathbb{D}^n$.

We illustrate the weights of the binomial welfare functions C_j, with $j = 1, ..., n$, and the coefficients of the associated binomial inequality indices G_j, with $j = 1, ..., n$. The binomial welfare functions C_j, with $j = 1, \ldots, n$, have null weights associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they behave in analogy with poverty measures which progressively focus on the poorest part of the population. Correspondingly, the binomial inequality indices G_j, with $j = 1, \ldots, n$, have equal coefficients associated with the $j - 1$ richest individuals in the population and therefore, as j increases from 1 to n, they are progressively insensitive to income transfers within the richest part of the population.

We introduce a family of income distributions described by a parameter $\beta \geq 0$ related with inequality and we examine the binomial welfare functions and inequality indices with $n = 4, 6, 8$ for $\beta = 0, 1, \ldots, 8$. The data obtained reflects the dominance relations regarding binomial welfare functions and inequality indices and illustrates the effect of the parameter $\beta \geq 0$ in relation with inequality. We illustrate the numerical behavior of the binomial welfare functions and inequality indices in relation with the parametric family of income distributions.

The central and main contribution of the paper, in the context of the binomial decomposition, regards the investigation of the analogy between the binomial welfare functions and two well-known families of generalized Gini welfare functions: the S-Gini and the Lorenzen welfare functions. In particular, the coefficients of the binomial decomposition of the S-Gini and Lorenzen welfare functions in terms of the binomial welfare functions show interesting patterns of behavior. Finally, we have examined the orness of the parametric S-Gini and Lorenzen families of generalized Gini welfare functions.

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