General Decay Lag Anti-synchronization of Multi-weighted Delayed Coupled Neural Networks with Reaction–Diffusion Terms

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Abstract

We propose a new anti-synchronization concept, called general decay lag anti-synchronization, by combining the definitions of decay synchronization and lag synchronization. Novel criteria for the decay lag anti-synchronization of multi-weighted delayed coupled reaction–diffusion neural networks (MWD-CRDNNs) with and without bounded distributed delays are derived by constructing an appropriate nonlinear controller and using the Lyapunov functional method. Moreover, the robust decay lag anti-synchronization of MWD-CRDNNs with and without bounded distributed delays is considered. Finally, two numerical simulations are performed to validate the obtained results.

Keywords: General decay lag anti-synchronization, Bounded distributed delays, Multi-weights, Coupled reaction–diffusion neural networks

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1. Introduction

Recently, coupled neural networks (CNNs) have attracted considerable attention owing to their extensive application in chaos generator design, optimization, pattern recognition, secure communication, etc. [5, 10, 29, 32]. As is well known, one of the most important dynamic phenomena in CNNs is synchronization, which has been extensively studied in recent years [8, 11, 23, 27, 32]. In [8], the $\mathcal{H}_\infty$ synchronization problem of the master and slave structure of second-order neutral master–slave systems with time-varying delays was investigated using the Lyapunov–Krasovskii method in terms of a linear matrix inequality. The synchronization of Markovian CNNs with random coupling strengths and nonidentical node-delays was considered in [27] by designing a novel Lyapunov functional and using certain inequalities. In fact, anti-synchronization is also an interesting phenomenon in the real world. Anti-synchronization is the phenomenon in which the state vectors of a synchronous system have the same absolute values but opposite signs. As reported in [12, 31], anti-synchronization is important in communication systems and laser applications. Hence, it is meaningful to study anti-synchronization of CNNs [12, 13, 16, 31]. Meng and Wang [12] designed an anti-synchronization scheme for a class of delayed chaotic neural networks based on the Halanay inequality and Lyapunov stability theory. Ren et al. [13] investigated the exponential anti-synchronization problem for chaotic delayed neural networks. The anti-synchronization of a class of memristive CNNs was investigated by using randomly occurring control in [16].

In fact, time delay is ubiquitous and inevitable in CNNs owing to, for instance, their finite transmission and switching speed, as well as traffic congestion. Time delay may lead to undesirable results, such as instability and poor performance. Consequently, it is necessary to consider the problems of lag synchronization and lag anti-synchronization [2, 6, 17, 28]. A controller based on the output of neuron cells was constructed by the authors in [2] to study global exponential lag anti-synchronization in switched neural networks with time-varying delays. [6] designed and used a feedback controller to obtain novel results on the finite-time lag synchronization of CNNs. The exponential lag anti-synchronization problem for memristive CNNs was considered in [28]. Additionally, the estimation of the synchronization and anti-synchronization convergence rate is a very interesting and useful subject. However, the convergence rate of the system is difficult to determine in some cases, which prompts the definition of a new type of convergence rate, namely, convergence with general decay [1, 14, 18, 19]. Several new results on the general decay synchronization of delayed neural networks with general activation functions were obtained using a nonlinear feedback controller designed in [1]. Wang et al. [19] considered the decay synchronization of a class of switched CNNs by constructing suitable nonlinear controllers. The decay synchronization of delayed bidirectional associative memory neural networks was studied in [14]. To the best of our knowledge, the decay lag anti-synchronization of CNNs has not yet been studied.

It is worth mentioning that reaction–diffusion was neglected in the aforementioned studies. When electrons propagate in inhomogeneous electromagnetic fields, reaction–diffusion in CNNs is inevitable. Thus, it is crucial to consider reaction–diffusion terms in the study of CNNs, and numerous studies on coupled reaction–diffusion neural networks (CRDNNs) have been conducted [7, 15, 22, 26]. The passivity and synchronization of CRDNNs with multiple time-varying delays were analyzed by impulsive control in [15]. Wang et al. [22] constructed suitable state feedback controllers to study lag $\mathcal{H}_\infty$ synchronization of CRDNNs. However, there are currently no results on decay lag anti-synchronization of CRDNNs.

It is worth noting that the network models used in the majority of the studies above are single-weighted. In practice, numerous existing networks can be represented more precisely by multi-weighted complex dynamic networks (MWCDNs), such as transportation networks, social networks, and communication networks. Multi-weighted CNNs (MWCNNs), which are a special type of MWCDNs, have attracted increasing attention [4, 20, 30]. Based on Lyapunov stability theory and the robust adaptive principle, Zhao et al. [30] investigated the synchronization of MWCNNs with multiple coupled time-varying delays. In [20], sufficient conditions for ensuring finite-time synchronization of MWCNNs were obtained. However, the anti-synchronization of multi-weighted coupled reaction–diffusion neural networks (MWCRDNNs) has not been extensively studied. In [4], the authors were concerned with anti-synchronization and pinning control of MWCNNs with and without reaction–diffusion terms. To the best of our knowledge, the decay lag anti-synchronization of MWCRDNNs has not been investigated.

Accordingly, the principal goal in the present study is to investigate the general decay lag anti-synchronization of MWDCRDNNs. The main contributions of this study are as follows.

(1) The new concept of general decay lag anti-synchronization is presented by generalizing lag synchronization and introducing ψ-type functions.

(2) The decay lag anti-synchronization problem for MWDCRDNNs with and without parametric uncertainties is discussed, and several criteria are established by designing a suitable nonlinear controller and constructing an appropriate Lyapunov functional.

(3) The decay lag anti-synchronization and the robust decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays are discussed and analyzed as well.

The rest of this paper is organized as follows. The definition of ψ-type functions and several lemmas required in the subsequent sections are provided in Section 2. In Section 3, the MWDCRDNN is first presented, after which the decay lag anti-synchronization and robust decay lag anti-synchronization are investigated for this model. In Section 4, the decay lag anti-synchronization and robust decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays are analyzed. Several simulation examples are provided in Section 5 to verify the obtained theoretical results. Finally, this paper is concluded in Section 6.

2. Preliminaries

Definition 2.1. *([25]) If the function $\psi(t)$: $\mathbb{R}_+ \to (0, +\infty)$ satisfies the following conditions:*

1) $\psi(t)$ is nondecreasing and differentiable,

2) $\psi(0) = 1$ and $\psi(+\infty) = +\infty$,

3) $\overline{\psi}(t) := \frac{\dot{\psi}(t)}{\psi(t)}$ is decreasing, and

4) $\forall p, q \geqslant 0,\ \psi(p+q) \leqslant \psi(p)\psi(q)$,

then it is called a ψ-type function.

Lemma 2.1. *(see [9]) Let Ω be a cube $|\mu_r| < \iota_r (r = 1, 2, \cdots, p)$ and let $Z(\mu)$ be a real-valued function belonging to $C^1(\Omega)$ that vanishes on the boundary $\partial\Omega$ of Ω, i.e., $Z(\mu)|_{\partial\Omega} = 0$. Then*

$$\int_{\Omega} Z^2(\mu) d\mu \leqslant \iota_r^2 \int_{\Omega} \left(\frac{\partial Z}{\partial \mu_r} \right)^2 d\mu,$$

where $\mu = (\mu_1, \mu_2, \cdots, \mu_p)^T$.

Lemma 2.2. *([3]) Let $f(h) : [\omega_1, \omega_2] \to \mathbb{R}^p$ $(\omega_1 < \omega_2)$ be a vector function. Then, for any constant matrix $0 < M \in \mathbb{R}^{p \times p}$, we have*

$$(\omega_2 - \omega_1) \int_{\omega_1}^{\omega_2} f^T(h) M f(h) dh \geqslant \left(\int_{\omega_1}^{\omega_2} f(h) dh \right)^T M \left(\int_{\omega_1}^{\omega_2} f(h) dh \right),$$

provided that the integrals above are well defined.

We will use the following notations: $\lambda(\cdot)$ denotes the eigenvalue of the corresponding matrix. For any $e(\mu, t) = (e_1(\mu, t), e_2(\mu, t), \cdots, e_N(\mu, t))^T \in \mathbb{R}^N$, we let $\|e(\cdot, t)\|_2 = \sqrt{\int_{\Omega} \sum_{s=1}^{N} e_s^2(\mu, t) d\mu}$, where $\Omega = \{\mu = (\mu_1, \mu_2, \cdots, \mu_p)^T | \, |\mu_r| < \iota_r, \ r = 1, 2, \cdots, p\} \subset \mathbb{R}^p$ and $(\mu, t) \in \Omega \times \mathbb{R}$.

3. General decay lag anti-synchronization of MWDCRDNNs

3.1. General decay lag anti-synchronization of MWDCRDNNs

In this section, the considered MWDCNN model with reaction–diffusion terms is described by

$$\begin{aligned} \frac{\partial Y_s(\mu, t)}{\partial t} = & - AY_s(\mu, t) + H \triangle Y_s(\mu, t) + Df(Y_s(\mu, t)) + Bg(\widehat{Y_s(\mu, t)}) \\ & + c_1 \sum_{j=1}^{N} M_{sj}^1 \Gamma_1 Y_j(\mu, t) + c_2 \sum_{j=1}^{N} M_{sj}^2 \Gamma_2 Y_j(\mu, t) + \cdots \end{aligned}$$

$$+ c_m \sum_{j=1}^{N} M_{sj}^{m} \Gamma_m Y_j(\mu,t), \quad s = 1, 2, \cdots, N. \tag{1}$$

Here, $Y_s(\mu,t) = (Y_{s1}(\mu,t), Y_{s2}(\mu,t), \cdots, Y_{sn}(\mu,t))^T \in \mathbb{R}^n$ is the state vector of the s-th node. $\mu = (\mu_1, \mu_2, \cdots, \mu_p)^T \in \Omega \subset \mathrm{R}^p$. $A = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in \mathbb{R}^{n\times n} > 0$, $B = (b_{sj})_{n\times n} \in \mathbb{R}^{n\times n}$, and $D = (d_{sj})_{n\times n} \in \mathbb{R}^{n\times n}$ are constant matrices. $\triangle = \sum_{r=1}^{p} \frac{\partial^2}{\partial \mu_r^2}$, $H = \mathrm{diag}(h_1, h_2, \cdots, h_n)$, $h_s > 0$ is the transmission diffusion coefficient, $g(\widehat{Y_s(\mu,t)}) = (g_1(Y_{s1}(\mu, t-\tau(t))), g_2(Y_{s2}(\mu, t-\tau(t))), \cdots, g_n(Y_{sn}(\mu, t-\tau(t))))^T \in \mathbb{R}^n$, $f(Y_s(\mu,t)) = (f_1(Y_{s1}(\mu,t)), f_2(Y_{s2}(\mu,t)), \cdots, f_n(Y_{sn}(\mu,t)))^T \in \mathbb{R}^n$, and $\tau(t)$ is the time-varying delay with $0 \leqslant \tau(t) \leqslant \tau$ and $\dot{\tau}(t) \leqslant \gamma < 1$. $\mathbb{R} \ni c_\kappa > 0$ $(\kappa = 1, 2, \cdots, m)$ is the coupling strength for the κ-th coupling form. $\Gamma_\kappa \in \mathbb{R}^{n\times n} > 0$ $(\kappa = 1, 2, \cdots, m)$ represents the inner coupling matrix for the κth coupling form. $M^\kappa = (M_{sj}^\kappa)_{N\times N} \in \mathbb{R}^{N\times N}$ $(\kappa = 1, 2, \cdots, m)$ expresses the coupling weight in the κ-th coupling form, where M_{sj}^κ is defined as follows: if there exists a connection between node s and node j for the κ-th coupling form, then $M_{sj}^\kappa = M_{js}^\kappa > 0$; otherwise, $M_{sj}^\kappa = M_{js}^\kappa = 0$ $(s \neq j)$. Finally, the diagonal elements of the matrix M^κ are defined as follows:

$$M_{ss}^\kappa = -\sum_{\substack{j=1 \\ j\neq s}}^{N} M_{sj}^\kappa, \ \ s = 1, 2, \cdots, N.$$

For the network (1),

$$\begin{aligned} &Y_s(\mu,t) = \phi_s(\mu,t) \in \mathbb{R}^n, \quad (\mu,t) \in \Omega \times [-\tau, 0], \\ &Y_s(\mu,t) = 0, \quad (\mu,t) \in \partial\Omega \times [-\tau, +\infty), \end{aligned}$$

where $\phi_s(\mu,t)(s = 1, 2, \cdots, N)$ is bounded and continuous on $\Omega \times [-\tau, 0]$.

Remark 1. Recently, CNNs have attracted increasing attention owing to their extensive application in chaos generator design, optimization, secure communication, etc. [5, 10, 21, 29, 32, 33]. Therefore, numerous studies have been conducted on the dynamical behavior of CNNs [8, 11, 23, 27, 32]. Unfortunately, the CNN models considered in these studies have only a single weight. It is well known that several real-world networks, such as social, communication, and transportation networks, can be represented by MWCDNs with multiple node coupling. Furthermore, when electrons propagate in inhomogeneous electromagnetic fields, diffusion is inevitable. For instance, the overall structure and dynamic behavior of cellular neural networks depends heavily not only on the evolution time and location (space) of each variable but also on their interactions, which are derived from the spatial distribution structure of the entire network. Therefore, it is meaningful to discuss a MWDCRDNN model in which the node state varies with time and space.

We consider the network model (1) to be the drive system. Then, the corresponding response system is as follows:

$$\begin{aligned}\frac{\partial W_s(\mu,t)}{\partial t} =& - AW_s(\mu,t) + H\triangle W_s(\mu,t) + Bg(\widehat{W_s(\mu,t)}) + Df(W_s(\mu,t)) \\ &+ \sum_{\kappa=1}^{m}\sum_{j=1}^{N} c_\kappa M_{sj}^{\kappa}\Gamma_\kappa W_j(\mu,t) + u_s(\mu,t), \qquad (2)\end{aligned}$$

where $s = 1, 2, \cdots, N$, $W_s(\mu,t) = (W_{s1}(\mu,t), W_{s2}(\mu,t), \cdots, W_{sn}(\mu,t))^T \in \mathbb{R}^n$ is the state vector of the s-th neuron at time t and in space μ, $u_s(\mu,t) = (u_{s1}(\mu,t), u_{s2}(\mu,t), \cdots, u_{sn}(\mu,t))^T \in \mathbb{R}^n$ is a suitable controller for achieving a certain control objective, and $A,\ H,\ \triangle,\ B,\ D,\ g(\cdot),\ f(\cdot),\ c_\kappa,\ M_{sj}^{\kappa},\ \Gamma_\kappa$ are defined as in system (1).

For the network (2),

$$W_s(\mu,t)=\varphi_s(\mu,t)\in\mathbb{R}^n,\quad (\mu,t)\in\Omega\times[-\tau,0],$$
$$W_s(\mu,t)=0,\quad (\mu,t)\in\partial\Omega\times[-\tau,+\infty),$$

where $\varphi_s(\mu,t)(s=1,2,\cdots,N)$ is bounded and continuous on $\Omega\times[-\tau,0]$.

Assumption 1. *For any $\gamma_1,\gamma_2\in\mathbb{R}$, the functions $g_i(\cdot)$ and $f_i(\cdot)$ $(i=1,2,\cdots,n)$ satisfy*

$$|g_i(\gamma_1)+g_i(\gamma_2)|\leqslant G_i|\gamma_1+\gamma_2|,$$
$$|f_i(\gamma_1)+f_i(\gamma_2)|\leqslant F_i|\gamma_1+\gamma_2|,$$

where $0<G_i\in\mathbb{R}$ and $0<F_i\in\mathbb{R}$. Let $G=\mathrm{diag}(G_1^2,G_2^2,\cdots,G_n^2)\in\mathbb{R}^{n\times n}$ and $F=\mathrm{diag}(F_1^2,F_2^2,\cdots,F_n^2)\in\mathbb{R}^{n\times n}$.

Remark 2. Recently, the anti-synchronization of CNNs has received considerable attention [12, 13, 16, 31]. In these studies, it is common to assume that the activation function satisfies the Lipschitz condition and is odd. Therefore, the functions $g_i(\cdot)$ and $f_i(\cdot)$ $(i=1,2,\cdots,n)$ in the considered network models will be required to satisfy this assumption throughout this paper.

Let $e_s(\mu,t)=W_s(\mu,t)+Y_s(\mu,t-\sigma_s)$. By (1) and (2), one has

$$\begin{aligned}\frac{\partial e_s(\mu,t)}{\partial t}=&-Ae_s(\mu,t)+H\triangle e_s(\mu,t)+Bg(Y_s(\widehat{\mu,t-}\sigma_s))+Bg(\widehat{W_s(\mu,t})) \\ &+Df(Y_s(\mu,t-\sigma_s))+Df(W_s(\mu,t))+u_s(\mu,t) \\ &+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa e_j(\mu,t),\end{aligned}\tag{3}$$

where $\sigma_s\geqslant 0$ $(s=1,2,\cdots,N)$ is the lag delay.

Definition 3.1. *If there exists a constant* $\lambda > 0$ *such that*

$$\limsup_{t\to\infty} \frac{\log\|e(\cdot,t)\|}{\log\psi(t)} \leqslant -\lambda,$$

where $e(\mu,t) = (e_1^T(\mu,t), e_2^T(\mu,t), \cdots, e_N^T(\mu,t))^T$ *and* $\psi(t)$ *is a* ψ*-type function as in Definition 2.1, then the network* (3) *is called* ψ*-type stable, that is, the drive-response systems* (1) *and* (2) *achieve general decay lag anti-synchronization, where* λ *is the convergence rate as* $e(\mu,t) \to 0$*.*

Remark 3. Recently, ψ-type stability for neural networks has attracted considerable attention [14, 18, 19]. It is defined as follows:

$$\limsup_{t\to\infty} \frac{\log\|W_s(\cdot,t) - Y_s(\cdot,t)\|}{\log\psi(t)} \leqslant -\lambda, \;\; s = 1,2,\cdots,N. \tag{4}$$

In this study, the synchronization error $\|W_s(\cdot,t) - Y_s(\cdot,t)\|$ is changed to $\|W_s(\cdot,t)+Y_s(\cdot,t-\sigma_s)\|$, which implies that the state variables of the systems (1) and (2) have the same amplitude but are different in sign and time. Then, we have the following natural generalization of ψ-type stability:

$$\limsup_{t\to\infty} \frac{\log\|W_s(\cdot,t) + Y_s(\cdot,t-\sigma_s)\|}{\log\psi(t)} \leqslant -\lambda, \;\; s = 1,2,\cdots,N.$$

To the best of our knowledge, the concept of decay lag anti-synchronization of MWDCRDNNs in Definition 3.1, which combines decay synchronization and lag synchronization, has not previously been considered.

Before presenting the main results, we state a lemma that is important in the proof. To this end, the following assumption is required.

Assumption 2. *([19]) There exist* $\rho(t) \in C(\mathbb{R}, \mathbb{R}^+)$ *and* $\varepsilon > 0$ *such that*

$$\overline{\psi}(t) \leqslant 1,$$

$$\sup_{t\in[0,\infty)} \int_0^t \psi^{\varepsilon}(\beta)\rho(\beta)d\beta < \infty,$$

where $\overline{\psi}(t)$ *and* $\psi(t)$ *are as in Definition 2.1.*

Lemma 3.1. *([19]) Under Assumption 2, if there exists a differentiable function* $V(t, e_0(t)) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}^+$ *and two constants* $0 < \alpha_1 \in \mathbb{R},\ 0 < \alpha_2 \in \mathbb{R}$ *such that*

$$\begin{aligned}
&(\alpha_1\|e_0(t)\|)^2 \leqslant V(t, e_0(t)),\\
&\dot{V}(t, e_0(t))|_{(3)} + \varepsilon V(t, e_0(t)) \leqslant \alpha_2\rho(t),
\end{aligned}$$

where $e_0(t)$ *is a solution of network* (3)*, and* ε *and* $\rho(t)$ *are as in Assumption ass2, then the system* (3) *is called* ψ*-type stable, that is, the drive-response systems* (1) *and* (2) *achieve general decay lag anti-synchronization. In addition, the convergence rate is* $\frac{\varepsilon}{2}$*.*

In this section, the nonlinear controller in the response system (2) is designed as follows:

$$u_s(\mu, t) = -\ \beta_s \frac{\|e(\cdot, t)\|^2 e_s(\mu, t)}{\|e(\cdot, t)\|^2 + \rho(t)} - q_s e_s(\mu, t),\ s = 1, 2, \cdots, N, \tag{5}$$

where $\mathbb{R} \ni q_s > 0,\ \mathbb{R} \ni \beta_s > 0.$

For convenience, we let $\beta = \max\limits_{1\leqslant s\leqslant N}\{\beta_s\},\ \hat{q} = \mathrm{diag}(q_1, q_2, \cdots, q_N)$, and $\hat{\beta} = \mathrm{diag}(\beta_1, \beta_2, \cdots, \beta_N)$.

Theorem 3.1. *Under Assumptions 1 and 2, the system* (3) *is* ψ*-type stable with convergence rate* $\frac{\varepsilon}{2}$*, i.e., the systems* (1) *and* (2) *achieve general decay lag anti-synchronization if*

$$\Xi_1 = \sum_{\kappa=1}^{m} c_\kappa M^\kappa \otimes \Gamma_\kappa + I_N \otimes P_1 - (\hat{\beta} + \hat{q}) \otimes I_n < 0, \tag{6}$$

$$\Psi_1 = I_N \otimes \big((\varepsilon\tau - 1)I_n + \frac{\varepsilon G}{2(1-\gamma)}\big) < 0, \tag{7}$$

where $P_1 = \frac{1}{2}(DD^T + BB^T + F + \frac{G}{1-\gamma}) + (\tau + \frac{\varepsilon}{2})I_n - A - \sum_{r=1}^{p} \frac{1}{\iota_r^2} H.$

Proof. We construct a Lyapunov functional for network (3) as follows:

$$\begin{aligned} V_1(t) =& \frac{1}{2}\sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu,t)e_s(\mu,t)d\mu + \int_{-\tau}^{0}\int_{t+\rho}^{t}\int_{\Omega} e^T(\mu,h)e(\mu,h)d\mu dh d\rho \\ &+ \frac{1}{2(1-\gamma)}\int_{t-\tau(t)}^{t}\int_{\Omega} e^T(\mu,h)(I_N\otimes G)e(\mu,h)d\mu dh. \end{aligned} \tag{8}$$

Obviously, $(\frac{1}{\sqrt{2}}\|e(\cdot,t)\|)^2 \leqslant V_1(t)$, and it can be deduced from (8) that

$$\begin{aligned} V_1(t) \leqslant& \frac{1}{2}\sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu,t)e_s(\mu,t)d\mu + \tau\int_{t-\tau}^{t}\int_{\Omega} e^T(\mu,h)e(\mu,h)d\mu dh \\ &+ \frac{1}{2(1-\gamma)}\int_{t-\tau}^{t}\int_{\Omega} e^T(\mu,h)(I_N\otimes G)e(\mu,h)d\mu dh \\ =& \int_{t-\tau}^{t}\int_{\Omega} e^T(\mu,h)\Big(I_N\otimes\big(\tau I_n + \frac{G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh \\ &+ \frac{1}{2}\sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu,t)e_s(\mu,t)d\mu. \end{aligned} \tag{9}$$

By calculating the derivative of (8) along the trajectories of the system (3), one obtains

$$\begin{aligned} \dot{V}_1(t) \leqslant& \sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu,t)\Big(- Ae_s(\mu,t) + Bg(Y_s\widehat{(\mu,t-\sigma_s)}) + \sum_{\kappa=1}^{m}\sum_{j=1}^{N} c_\kappa M_{sj}^{\kappa}\Gamma_\kappa e_j(\mu,t) \\ &+ Bg(\widehat{W_s(\mu,t)}) + H\triangle e_s(\mu,t) - \beta_s\frac{\|e(\cdot,t)\|^2 e_s(\mu,t)}{\|e(\cdot,t)\|^2+\rho(t)} + Df(W_s(\mu,t)) \\ &- q_s e_s(\mu,t) + Df(Y_s(\mu,t-\sigma_s))\Big)d\mu - \int_{t-\tau}^{t}\int_{\Omega} e^T(\mu,h)e(\mu,h)d\mu dh \\ &+ \frac{1}{2(1-\gamma)}\int_{\Omega} e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu + \tau\int_{\Omega} e^T(\mu,t)e(\mu,t)d\mu \end{aligned}$$

$$-\frac{1}{2}\int_{\Omega} e^T(\mu, t-\tau(t))(I_N \otimes G)e(\mu, t-\tau(t))d\mu, \tag{10}$$

where $e(\mu, t-\tau(t)) = (e_1^T(\mu, t-\tau(t)), e_2^T(\mu, t-\tau(t)), \cdots, e_N^T(\mu, t-\tau(t)))^T$. By Assumption 1, one obtains

$$\begin{aligned}
&\sum_{s=1}^{N} e_s^T(\mu, t)B(g(Y_s(\widehat{\mu, t-}\sigma_s)) + g(\widehat{W_s(\mu, t}))) \\
\leqslant&\frac{1}{2}\sum_{s=1}^{N}\left(e_s^T(\mu, t)BB^Te_s(\mu, t) + e_s^T(\mu, t-\tau(t))Ge_s(\mu, t-\tau(t))\right) \\
=&\frac{1}{2}e^T(\mu, t)(I_N \otimes (BB^T))e(\mu, t) + \frac{1}{2}e^T(\mu, t-\tau(t))(I_N \otimes G)e(\mu, t-\tau(t)), \quad (11)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{s=1}^{N} e_s^T(\mu, t)D(f(Y_s(\mu, t-\sigma_s)) + f(W_s(\mu, t))) \\
\leqslant&\frac{1}{2}e^T(\mu, t)(I_N \otimes (DD^T + F))e(\mu, t). \qquad (12)
\end{aligned}$$

By Green's formula,

$$\int_{\Omega} e_{sl}(\mu, t)\triangle e_{sj}(\mu, t)d\mu = -\sum_{r=1}^{p}\int_{\Omega}\frac{\partial e_{sl}(\mu, t)}{\partial \mu_r}\frac{\partial e_{sj}(\mu, t)}{\partial \mu_r}d\mu,$$

where $l, j \in \{1, 2, \cdots, n\}$, $s = 1, 2, \cdots, N$. Let $\pi(\mu, t) = (I_N \otimes \sqrt{H})e(\mu, t)$. Furthermore, Lemma 2.1 implies

$$\begin{aligned}
&\sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu, t)H\triangle e_s(\mu, t)d\mu \\
=&-\sum_{r=1}^{p}\sum_{s=1}^{N}\sum_{j=1}^{n}\sum_{l=1}^{n} h_l\int_{\Omega}\frac{\partial e_{sj}(\mu, t)}{\partial \mu_r}\frac{\partial e_{sl}(\mu, t)}{\partial \mu_r}d\mu \\
=&-\sum_{r=1}^{p}\int_{\Omega}\left(\frac{\partial e(\mu, t)}{\partial \mu_r}\right)^T(I_N \otimes H)\frac{\partial e(\mu, t)}{\partial \mu_r}d\mu
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{r=1}^{p}\int_{\Omega}\left(\frac{\partial\pi(\mu,t)}{\partial\mu_r}\right)^T\frac{\partial\pi(\mu,t)}{\partial\mu_r}d\mu\\
&\leqslant -\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega}\pi^T(\mu,t)\pi(\mu,t)d\mu\\
&= -\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega}e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu. \qquad (13)
\end{aligned}
$$

Eqs. (10)–(13) yield

$$
\begin{aligned}
\dot{V}_1(t) \leqslant &\sum_{\kappa=1}^{m}c_\kappa\int_{\Omega}e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu-\int_{t-\tau}^{t}\int_{\Omega}e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\sum_{s=1}^{N}\int_{\Omega}e_s^T(\mu,t)\bigg(-Ae_s(\mu,t)-q_se_s(\mu,t)-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\mu,t)}{\|e(\cdot,t)\|^2+\rho(t)}\bigg)d\mu\\
&+\frac{1}{2}\int_{\Omega}e^T(\mu,t)(I_N\otimes(BB^T))e(\mu,t)d\mu+\tau\int_{\Omega}e^T(\mu,t)e(\mu,t)d\mu\\
&+\frac{1}{2}\int_{\Omega}e^T(\mu,t)(I_N\otimes(DD^T+F))e(\mu,t)d\mu\\
&-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega}e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_{\Omega}e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu\\
=&\sum_{\kappa=1}^{m}c_\kappa\int_{\Omega}e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu-\int_{t-\tau}^{t}\int_{\Omega}e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\sum_{s=1}^{N}\int_{\Omega}e_s^T(\mu,t)\bigg(-Ae_s(\mu,t)-q_se_s(\mu,t)-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\mu,t)}{\|e(\cdot,t)\|^2+\rho(t)}\bigg)d\mu\\
&+\frac{1}{2}\int_{\Omega}e^T(\mu,t)(I_N\otimes(BB^T))e(\mu,t)d\mu+\sum_{s=1}^{N}\beta_s\int_{\Omega}e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\frac{1}{2}\int_{\Omega}e^T(\mu,t)(I_N\otimes(DD^T+F))e(\mu,t)d\mu+\tau\int_{\Omega}e^T(\mu,t)e(\mu,t)d\mu\\
&-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega}e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu-\sum_{s=1}^{N}\beta_s\int_{\Omega}e_s^T(\mu,t)e_s(\mu,t)d\mu
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2(1-\gamma)}\int_\Omega e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu\\
=&\sum_{\kappa=1}^m c_\kappa\int_\Omega e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu-\int_{t-\tau}^t\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\frac{1}{2}\int_\Omega e^T(\mu,t)(I_N\otimes(DD^T+F))e(\mu,t)d\mu+\tau\int_\Omega e^T(\mu,t)e(\mu,t)d\mu\\
&+\frac{1}{2}\int_\Omega e^T(\mu,t)(I_N\otimes(BB^T))e(\mu,t)d\mu+\int_\Omega e^T(\mu,t)(-I_N\otimes A-(\hat{\beta}\\
&+\hat{q})\otimes I_n)e(\mu,t)d\mu-\sum_{r=1}^p\frac{1}{\iota_r^2}\int_\Omega e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu\\
&+\frac{\rho(t)}{\|e(\cdot,t)\|^2+\rho(t)}\sum_{s=1}^N\beta_s\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_\Omega e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu\\
\leqslant&\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^m c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(DD^T+BB^T+F+\frac{G}{1-\gamma})+\tau I_n\\
&-A-\sum_{r=1}^p\frac{1}{\iota_r^2}H\big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\frac{\beta\|e(\cdot,t)\|^2\rho(t)}{\|e(\cdot,t)\|^2+\rho(t)}\\
&-\int_{t-\tau}^t\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh. \qquad (14)
\end{aligned}
$$

By combining (9) and (14), we easily obtain that

$$
\begin{aligned}
\dot{V}_1(t)+\varepsilon V_1(t)\leqslant&\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^m c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(DD^T+BB^T+F+\frac{G}{1-\gamma})\\
&+\tau I_n-A-\sum_{r=1}^p\frac{1}{\iota_r^2}H\big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\frac{\beta\|e(\cdot,t)\|^2\rho(t)}{\|e(\cdot,t)\|^2+\rho(t)}\\
&-\int_{t-\tau}^t\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh+\frac{\varepsilon}{2}\sum_{s=1}^N\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\varepsilon\int_{t-\tau}^t\int_\Omega e^T(\mu,h)\Big(I_N\otimes\big(\tau I_n+\frac{G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^{m} c_\kappa M^\kappa \otimes \Gamma_\kappa + I_N \otimes \big(\frac{1}{2}(DD^T + BB^T + F + \frac{G}{1-\gamma})\\
&+ (\tau + \frac{\varepsilon}{2})I_n - A - \sum_{r=1}^{p}\frac{1}{\iota_r^2}H\big) - (\hat{\beta} + \hat{q}) \otimes I_n\Big] e(\mu,t)d\mu + \beta\rho(t)\\
&+ \int_{t-\tau}^{t}\int_\Omega e^T(\mu,h)\Big(I_N \otimes \big((\varepsilon\tau - 1)I_n + \frac{\varepsilon G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh.
\end{aligned}
$$

By (6) and (7), one obtains

$$
\dot{V}_1(t) + \varepsilon V_1(t) \leqslant \beta\rho(t). \tag{15}
$$

Letting $\alpha_1 = \frac{1}{\sqrt{2}}$, $\alpha_2 = \beta$, we easily obtain that the network (3) is ψ-type stable with convergence rate $\frac{\varepsilon}{2}$. Then, the drive system (1) and the response system (2) achieve general decay lag anti-synchronization. The proof is completed.

Remark 4. Recently, the anti-synchronization of traditional neural networks without reaction–diffusion terms, which are described by ordinary differential equations, has been investigated [12, 13, 16, 31]. However, diffusion effects cannot be prevented in neural networks when electrons move in an asymmetric electromagnetic field. Thus, it is necessary to consider diffusion effects in the study of neural networks. It should be noted that the network models in the majority of the studies discussed above are single-weighted. In practice, numerous networks are represented more precisely by using multiple weights [4, 20, 30]. In this study, we investigate the decay lag anti-synchronization of MWCRDNNs. We note that the network models considered here are described by partial differential equations, which implies that the state of each neuron depends not only on the time variable but also on the space variable. In fact, the main difficulty in extending the existing anti-synchronization results to decay lag anti-synchronization analysis of MWCRDNNs arises from the reaction–diffusion and multi-weighted terms, which cannot be handled by existing techniques used in single-weight CNNs or without reaction–diffusion terms. In addition, the definition of decay lag anti-synchronization and the design of a suitable controller are also important. By using Green's formula, Lemmas 2.1, 2.2, and 3.1, as well as certain inequalities, several sufficient conditions (dependent on the reaction–diffusion and multi-weighted terms) for achieving decay lag anti-synchronization of the considered MWDCRDNNs are established.

3.2. General decay lag anti-synchronization of MWDCRDNNs with parametric uncertainties

It is well known that equipment limitations and external interferences in neural network modeling may bring about bounded parameter deviations. Therefore, the following MWDCRDNN with parametric uncertainties is considered in this section:

$$\begin{aligned}\frac{\partial Y_s(\mu,t)}{\partial t} =& - AY_s(\mu,t) + H\triangle Y_s(\mu,t) + Df(Y_s(\mu,t)) + Bg(\widehat{Y_s(\mu,t)})\\ &+\sum_{\kappa=1}^{m}\sum_{j=1}^{N} c_\kappa M_{sj}^{\kappa}\Gamma_\kappa Y_j(\mu,t), \quad s=1,2,\cdots,N, \end{aligned} \tag{16}$$

where $Y_s(\mu,t),\ \triangle,\ f(\cdot),\ g(\cdot),\ c_\kappa,\ M_{sj}^{\kappa},\ \Gamma_\kappa,\ \kappa=1,2,\cdots,m$ are defined as in Section 3.1, and the parameters $A,\ H,\ D,$ and B vary in certain given ranges as follows:

$$\begin{cases} A_I := \{A = \mathrm{diag}(a_r) : A^- \leqslant A \leqslant A^+, i.e., 0 < a_r^- \leqslant a_r \leqslant a_r^+, \\ \qquad r = 1, 2, \cdots, n, \forall A \in A_I\}, \\ H_I := \{H = \mathrm{diag}(h_r) : H^- \leqslant H \leqslant H^+, i.e., 0 < h_r^- \leqslant h_r \leqslant h_r^+, \\ \qquad r = 1, 2, \cdots, n, \forall H \in H_I\}, \\ D_I := \{D = (d_{rj})_{n\times n} : D^- \leqslant D \leqslant D^+, i.e., d_{rj}^- \leqslant d_{rj} \leqslant d_{rj}^+, r, \\ \qquad j = 1, 2, \cdots, n, \forall D \in D_I\}, \\ B_I := \{B = (b_{rj})_{n\times n} : B^- \leqslant B \leqslant B^+, i.e., b_{rj}^- \leqslant b_{rj} \leqslant b_{rj}^+, r, \\ \qquad j = 1, 2, \cdots, n, \forall B \in B_I\}. \end{cases} \tag{17}$$

For convenience, we define

$$\begin{aligned} \tilde{d}_{rj} &= \max\{|d_{rj}^-|, |d_{rj}^+|\},\ r = 1, 2, \cdots, n,\ j = 1, 2, \cdots, n, \\ \tilde{b}_{rj} &= \max\{|b_{rj}^-|, |b_{rj}^+|\},\ r = 1, 2, \cdots, n,\ j = 1, 2, \cdots, n, \\ \varrho_D &= \sum_{r=1}^{n}\sum_{j=1}^{n} \tilde{d}_{rj}^2,\ \ \varrho_B = \sum_{r=1}^{n}\sum_{j=1}^{n} \tilde{b}_{rj}^2. \end{aligned}$$

We consider the network model (16) to be the drive system. Then, the corresponding response system is

$$\begin{aligned} \frac{\partial W_s(\mu,t)}{\partial t} =& -AW_s(\mu,t) + H\triangle W_s(\mu,t) + Bg(\widehat{W_s(\mu,t)}) + Df(W_s(\mu,t)) \\ &+ \sum_{\kappa=1}^{m}\sum_{j=1}^{N} c_\kappa M_{sj}^{\kappa} \Gamma_\kappa W_j(\mu,t) + u_s(\mu,t), \end{aligned} \tag{18}$$

where $s = 1, 2, \cdots, N,\ W_s(\mu,t),\ \triangle,\ f(\cdot),\ g(\cdot),\ c_\kappa,\ M_{sj}^{\kappa},\ \Gamma_\kappa,\ \kappa = 1, 2, \cdots, m$, and $u_s(\mu,t)$ are defined as in Section 3.1. The ranges of $A,\ H,\ D$, and B are as in (17).

Let $e_s(\mu,t)=W_s(\mu,t)+Y_s(\mu,t-\sigma_s)$. By (16) and (18), one can obtain

$$\begin{aligned}\frac{\partial e_s(\mu,t)}{\partial t}=&-Ae_s(\mu,t)+H\triangle e_s(\mu,t)+Bg(Y_s(\widehat{\mu,t-}\sigma_s))+Bg(\widehat{W_s(\mu,t)})\\&+Df(Y_s(\mu,t-\sigma_s))+Df(W_s(\mu,t))+u_s(\mu,t)\\&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa e_j(\mu,t),\end{aligned}\tag{19}$$

where $\sigma_s\geqslant 0$ $(s=1,2,\cdots,N)$ is the lag delay. The ranges of $A,\ H,\ D$, and B are as in (17).

Definition 3.2. *For all $A\in A_I,\ H\in H_I,\ D\in D_I$, and $B\in B_I$, if there exists a constant $\lambda>0$ such that*

$$\limsup_{t\to\infty}\frac{\log\|e(\cdot,t)\|}{\log\psi(t)}\leqslant-\lambda,$$

where $e(\mu,t)=(e_1^T(\mu,t),e_2^T(\mu,t),\cdots,e_N^T(\mu,t))^T$ and $\psi(t)$ is a ψ-type function as in Definition 2.1, then the network (19) *is called robustly ψ-type stable, that is, the drive-response systems* (16) *and* (18) *achieve general robust decay lag anti-synchronization, where λ is the convergence rate as $e(\mu,t)\to 0$.*

We construct the same nonlinear controller (5) for the response system (18) in this section.

Theorem 3.2. *Under Assumptions 1 and 2, the system* (19) *is robustly ψ-type stable with convergence rate $\frac{\varepsilon}{2}$, that is, systems* (16) *and* (18) *achieve general robust decay lag anti-synchronization for all $A\in A_I,\ H\in H_I,\ D\in D_I$, and $B\in B_I$ if*

$$\Xi_2=\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes P_2-(\hat\beta+\hat q)\otimes I_n<0,\tag{20}$$

$$\Psi_2 = I_N \otimes \big((\varepsilon\tau - 1)I_n + \frac{\varepsilon G}{2(1-\gamma)}\big) < 0, \tag{21}$$

where $P_2 = \frac{1}{2}(\varrho_D I_n + \varrho_B I_n + F + \frac{G}{1-\gamma}) + (\tau + \frac{\varepsilon}{2})I_n - A^- - \sum_{r=1}^{p} \frac{1}{\iota_r^2} H^-$.

Proof. We construct the same Lyapunov functional as in (8) for network (19). Then, one obtains

$$\begin{aligned}
\dot{V}_1(t) \leqslant & \sum_{s=1}^{N} \int_\Omega e_s^T(\mu,t)\Big(- Ae_s(\mu,t) + Bg(Y_s(\widehat{\mu, t-}\sigma_s)) + \sum_{\kappa=1}^{m}\sum_{j=1}^{N} c_\kappa M_{sj}^\kappa \Gamma_\kappa e_j(\mu,t) \\
& + Bg(\widehat{W_s(\mu,t)}) + H\triangle e_s(\mu,t) - \beta_s \frac{\|e(\cdot,t)\|^2 e_s(\mu,t)}{\|e(\cdot,t)\|^2 + \rho(t)} + Df(W_s(\mu,t)) \\
& - q_s e_s(\mu,t) + Df(Y_s(\mu,t-\sigma_s))\Big) d\mu - \int_{t-\tau}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh \\
& + \frac{1}{2(1-\gamma)}\int_\Omega e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu + \tau\int_\Omega e^T(\mu,t)e(\mu,t)d\mu \\
& - \frac{1}{2}\int_\Omega e^T(\mu,t-\tau(t))(I_N\otimes G)e(\mu,t-\tau(t))d\mu.
\end{aligned} \tag{22}$$

By Assumption 1, it easily follows that

$$\begin{aligned}
& \sum_{s=1}^{N} e_s^T(\mu,t)B(g(Y_s(\widehat{\mu, t-}\sigma_s)) + g(\widehat{W_s(\mu,t)})) \\
\leqslant & \frac{1}{2}e^T(\mu,t)(I_N\otimes(BB^T))e(\mu,t) + \frac{1}{2}e^T(\mu,t-\tau(t))(I_N\otimes G)e(\mu,t-\tau(t)) \\
\leqslant & \frac{1}{2}e^T(\mu,t)(I_N\otimes(\varrho_B I_n))e(\mu,t) + \frac{1}{2}e^T(\mu,t-\tau(t))(I_N\otimes G)e(\mu,t-\tau(t)),
\end{aligned} \tag{23}$$

and

$$\begin{aligned}
& \sum_{s=1}^{N} e_s^T(\mu,t)D(f(Y_s(\mu,t-\sigma_s)) + f(W_s(\mu,t))) \\
\leqslant & \frac{1}{2}e^T(\mu,t)(I_N\otimes(DD^T+F))e(\mu,t) \\
\leqslant & \frac{1}{2}e^T(\mu,t)(I_N\otimes(\varrho_D I_n + F))e(\mu,t).
\end{aligned} \tag{24}$$

By (13), one obtains

$$\begin{aligned}
&\sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu,t)H\triangle e_s(\mu,t)d\mu \\
\leqslant & -\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega} e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu \\
\leqslant & -\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega} e^T(\mu,t)(I_N\otimes H^-)e(\mu,t)d\mu.
\end{aligned} \tag{25}$$

Eqs. (22)–(25) imply that

$$\begin{aligned}
\dot{V}_1(t) \leqslant & \sum_{\kappa=1}^{m} c_\kappa \int_{\Omega} e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu - \int_{t-\tau}^{t}\int_{\Omega} e^T(\mu,h)e(\mu,h)d\mu dh \\
& + \sum_{s=1}^{N}\int_{\Omega} e_s^T(\mu,t)\bigg(- A^- e_s(\mu,t) - q_s e_s(\mu,t) - \beta_s \frac{\|e(\cdot,t)\|^2 e_s(\mu,t)}{\|e(\cdot,t)\|^2+\rho(t)}\bigg)d\mu \\
& + \frac{1}{2}\int_{\Omega} e^T(\mu,t)(I_N\otimes(\varrho_B I_n))e(\mu,t)d\mu + \sum_{s=1}^{N}\beta_s\int_{\Omega} e_s^T(\mu,t)e_s(\mu,t)d\mu \\
& + \frac{1}{2}\int_{\Omega} e^T(\mu,t)(I_N\otimes(\varrho_D I_n + F))e(\mu,t)d\mu + \tau\int_{\Omega} e^T(\mu,t)e(\mu,t)d\mu \\
& - \sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega} e^T(\mu,t)(I_N\otimes H^-)e(\mu,t)d\mu - \sum_{s=1}^{N}\beta_s\int_{\Omega} e_s^T(\mu,t)e_s(\mu,t)d\mu \\
& + \frac{1}{2(1-\gamma)}\int_{\Omega} e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu \\
= & \sum_{\kappa=1}^{m} c_\kappa \int_{\Omega} e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu - \int_{t-\tau}^{t}\int_{\Omega} e^T(\mu,h)e(\mu,h)d\mu dh \\
& + \frac{1}{2}\int_{\Omega} e^T(\mu,t)(I_N\otimes(\varrho_D I_n + F))e(\mu,t)d\mu + \tau\int_{\Omega} e^T(\mu,t)e(\mu,t)d\mu \\
& + \frac{1}{2}\int_{\Omega} e^T(\mu,t)(I_N\otimes(\varrho_B I_n))e(\mu,t)d\mu + \int_{\Omega} e^T(\mu,t)(-I_N\otimes A^- - (\hat{\beta} \\
& + \hat{q})\otimes I_n)e(\mu,t)d\mu - \sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega} e^T(\mu,t)(I_N\otimes H^-)e(\mu,t)d\mu
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\rho(t)}{\|e(\cdot,t)\|^2+\rho(t)}\sum_{s=1}^{N}\beta_s\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_\Omega e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu\\
\leqslant&\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(\varrho_D I_n+\varrho_B I_n+F+\frac{G}{1-\gamma})+\tau I_n\\
&-A^- -\sum_{r=1}^{p}\frac{1}{\iota_r^2}H^-\big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\frac{\beta\|e(\cdot,t)\|^2\rho(t)}{\|e(\cdot,t)\|^2+\rho(t)}\\
&-\int_{t-\tau}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh. \qquad (26)
\end{aligned}
$$

Combining (9) and (26), one obtains

$$
\begin{aligned}
\dot{V}_1(t)+\varepsilon V_1(t)\leqslant&\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(\varrho_D I_n+\varrho_B I_n+F+\frac{G}{1-\gamma})\\
&+\tau I_n-A^- -\sum_{r=1}^{p}\frac{1}{\iota_r^2}H^-\big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\beta\rho(t)\\
&-\int_{t-\tau}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh+\frac{\varepsilon}{2}\sum_{s=1}^{N}\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\varepsilon\int_{t-\tau}^{t}\int_\Omega e^T(\mu,h)\Big(I_N\otimes\big(\tau I_n+\frac{G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh\\
\leqslant&\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(\varrho_D I_n+\varrho_B I_n+F+\frac{G}{1-\gamma})\\
&+(\tau+\frac{\varepsilon}{2})I_n-A^- -\sum_{r=1}^{p}\frac{1}{\iota_r^2}H^-\big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\beta\rho(t)\\
&+\int_{t-\tau}^{t}\int_\Omega e^T(\mu,h)\Big(I_N\otimes\big((\varepsilon\tau-1)I_n+\frac{\varepsilon G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh.
\end{aligned}
$$

By (20) and (21),

$$
\dot{V}_1(t)+\varepsilon V_1(t)\leqslant\beta\rho(t).
$$

By letting $\alpha_1=\frac{1}{\sqrt{2}}$, $\alpha_2=\beta$, we easily obtain that the network (19) is robustly ψ-type stable with convergence rate $\frac{\varepsilon}{2}$. Then, the drive system (16) and the response system (18) achieve general robust decay lag anti-synchronization. The proof is completed.

Remark 5. In some situations, environment noise, equipment limitations, and external interferences may result in bounded parameter variation during the network modeling process. In addition, it is not easy to render models with the planed parameter values or ensure that the parameters are constant. Hence, it is meaningful to consider parametric uncertainties; some interesting results have been obtained regarding robust synchronization and robust anti-synchronization of neural networks [12, 24, 25, 30]. In [12], the authors studied the robust anti-synchronization of a class of delayed chaotic neural networks. Unfortunately, the robust decay lag anti-synchronization of MWDCRDNNs has not been studied. In Section 3.2, we investigate the robust decay lag anti-synchronization of MWDCRDNNs with parametric uncertainties, which is one of the main contributions of this study.

4. General decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays

4.1. General decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays

In this section, the following MWDCRDNN with bounded distributed delays is considered:

$$\begin{aligned}\frac{\partial Y_s(\mu,t)}{\partial t} =& - AY_s(\mu,t) + H\triangle Y_s(\mu,t) + Bg(\widehat{Y_s(\mu,t)}) + Df(Y_s(\mu,t)) \\ &+ \sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_\kappa \hat{M}_{sj}^{\kappa}\hat{\Gamma}_\kappa \int_{t-\delta(t)}^{t} Y_j(\mu,h)dh\end{aligned}$$

$$+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa Y_j(\mu,t), \quad s=1,2,\cdots,N. \tag{27}$$

Here, $Y_s(\mu,t)$, A, H, $\triangle$, B, D, $f(\cdot)$, $g(\cdot)$, c_κ, M_{sj}^{κ}, Γ_κ are defined as in Section 3.1. $\delta(t)$ is the distributed delay, which satisfies $0<\delta(t)<\delta$. $\mathbb{R}\ni\hat{c}_\kappa>0$ $(\kappa=1,2,\cdots,m)$ is the coupling strength for the κ-th coupling form. $\hat{\Gamma}_\kappa\in\mathbb{R}^{n\times n}>0$ $(\kappa=1,2,\cdots,m)$ represents the inner coupling matrix for the κ-th coupling form. $\hat{M}^{\kappa}=(\hat{M}_{sj}^{\kappa})_{N\times N}\in\mathbb{R}^{N\times N}$ $(\kappa=1,2,\cdots,m)$ expresses the coupling weight in the κ-th coupling form, where $\hat{M}_{sj}^{\kappa}$ is defined as follows: if there exists a connection between node s and node j for the κ-th coupling form, then $\hat{M}_{sj}^{\kappa}=\hat{M}_{js}^{\kappa}>0$; otherwise, $\hat{M}_{sj}^{\kappa}=\hat{M}_{js}^{\kappa}=0$ $(s\neq j)$. Finally, the diagonal elements of the matrix $\hat{M}^{\kappa}$ are defined as follows:

$$\hat{M}_{ss}^{\kappa}=-\sum_{\substack{j=1\\j\neq s}}^{N}\hat{M}_{sj}^{\kappa},\ s=1,2,\cdots,N.$$

For the network (27),

$$\begin{aligned}&Y_s(\mu,t)=\hat{\phi}_s(\mu,t)\in\mathbb{R}^n, \quad (\mu,t)\in\Omega\times[-\epsilon,0],\\&Y_s(\mu,t)=0, \quad (\mu,t)\in\partial\Omega\times[-\epsilon,+\infty),\end{aligned}$$

where $\epsilon=\max\{\tau,\delta\}$ and $\hat{\phi}_s(\mu,t)(s=1,2,\cdots,N)$ is bounded and continuous on $\Omega\times[-\epsilon,0]$.

We consider the network model (27) to be the drive system. Then, the corresponding response system is

$$\begin{aligned}\frac{\partial W_s(\mu,t)}{\partial t}=&-AW_s(\mu,t)+H\triangle W_s(\mu,t)+Bg(\widehat{W_s(\mu,t)})+Df(W_s(\mu,t))\\&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_\kappa\hat{M}_{sj}^{\kappa}\hat{\Gamma}_\kappa\int_{t-\delta(t)}^{t}W_j(\mu,h)dh+u_s(\mu,t)\end{aligned}$$

$$+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa W_j(\mu,t), \quad s=1,2,\cdots,N, \tag{28}$$

where $W_s(\mu,t),\ A,\ H,\ \triangle,\ B,\ D,\ g(\cdot),\ f(\cdot),\ c_\kappa,\ M_{sj}^{\kappa},\ \Gamma_\kappa,\ u_s(\mu,t)$ are defined as in Section 3.1, and $\delta(t),\ \hat{c}_\kappa,\ \hat{M}_{sj}^{\kappa},\ \hat{\Gamma}_\kappa$ are defined as in (27).

For the network (28),

$$\begin{aligned}
&W_s(\mu,t)=\hat{\varphi}_s(\mu,t)\in\mathbb{R}^n, \quad (\mu,t)\in\Omega\times[-\epsilon,0],\\
&W_s(\mu,t)=0, \quad (\mu,t)\in\partial\Omega\times[-\epsilon,+\infty),
\end{aligned}$$

where $\hat{\varphi}_s(\mu,t)(s=1,2,\cdots,N)$ is bounded and continuous on $\Omega\times[-\epsilon,0]$.

Let $e_s(\mu,t)=W_s(\mu,t)+Y_s(\mu,t-\sigma_s)$. By (27) and (28), one obtains

$$\begin{aligned}
\frac{\partial e_s(\mu,t)}{\partial t}=&-Ae_s(\mu,t)+H\triangle e_s(\mu,t)+Bg(Y_s(\widehat{\mu,t-}\sigma_s))+Bg(\widehat{W_s(\mu,t)})\\
&+Df(Y_s(\mu,t-\sigma_s))+Df(W_s(\mu,t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa e_j(\mu,t)\\
&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_\kappa\hat{M}_{sj}^{\kappa}\hat{\Gamma}_\kappa\int_{t-\delta(t)}^{t}e_j(\mu,h)dh+u_s(\mu,t),
\end{aligned} \tag{29}$$

where $\sigma_s\geqslant 0$ $(s=1,2,\cdots,N)$ is the lag delay.

We construct the same nonlinear controller (5) for the response system (28) in this section.

Remark 6. As there exist many parallel pathways of varying axon size and length, neural networks often have a certain spatial extent. Thus, there may be a distribution of conduction velocities along these pathways or a distribution of propagation delays over a period of time in some cases, which results in certain types of time delays, that is, distributed delays in neural networks. Therefore, it is necessary to consider these delays in the study of anti-synchronization of neural networks; some related papers on the anti-synchronization of neural networks have recently been published [16, 26]. However, the decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays has never been considered. In this section, several decay lag anti-synchronization criteria for MWCRDNNs with bounded distributed delays are derived.

Theorem 4.1. *Under Assumptions 1 and 2, the system* (29) *is ψ-type stable with convergence rate $\frac{\varepsilon}{2}$, that is, systems* (27)*and* (28) *achieve general decay lag anti-synchronization, if*

$$\Xi_3 = \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa \otimes \hat{\Gamma}^\kappa)^2 + \sum_{\kappa=1}^{m} c_\kappa M^\kappa \otimes \Gamma_\kappa + I_N \otimes P_3 - (\hat{\beta}+\hat{q}) \otimes I_n < 0, \quad (30)$$

$$\Psi_3 = I_N \otimes \big((\varepsilon\epsilon + \sum_{\kappa=1}^{m} \frac{\varepsilon \hat{c}_\kappa \delta^2}{2} - 1)I_n + \frac{\varepsilon G}{2(1-\gamma)}\big) < 0, \quad (31)$$

where $P_3 = \frac{1}{2}(DD^T + BB^T + F + \frac{G}{1-\gamma}) + (\frac{\varepsilon}{2} + \epsilon + \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa \delta^2}{2})I_n - A - \sum_{r=1}^{p} \frac{1}{\iota_r^2} H.$

Proof. We construct a Lyapunov functional for the network (29) as follows:

$$\begin{aligned} V_2(t) =& \frac{1}{2}\sum_{s=1}^{N} \int_\Omega e_s^T(\mu, t) e_s(\mu, t) d\mu + \int_{-\epsilon}^{0} \int_{t+\rho}^{t} \int_\Omega e^T(\mu, h) e(\mu, h) d\mu dh d\rho \\ &+ \frac{1}{2(1-\gamma)} \int_{t-\tau(t)}^{t} \int_\Omega e^T(\mu, h)(I_N \otimes G) e(\mu, h) d\mu dh \\ &+ \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa \delta}{2} \int_{t-\delta}^{t} \int_\rho^t \int_\Omega e^T(\mu, h) e(\mu, h) d\mu dh d\rho. \end{aligned} \quad (32)$$

Obviously, $(\frac{1}{\sqrt{2}}\|e(\cdot, t)\|)^2 \leqslant V_2(t)$, and it can be deduced from (32) that

$$V_2(t) \leqslant \frac{1}{2}\sum_{s=1}^{N} \int_\Omega e_s^T(\mu, t) e_s(\mu, t) d\mu + \epsilon \int_{t-\epsilon}^{t} \int_\Omega e^T(\mu, h) e(\mu, h) d\mu dh$$

$$
\begin{aligned}
&+\frac{1}{2(1-\gamma)}\int_{t-\tau}^{t}\int_{\Omega}e^{T}(\mu,h)(I_N\otimes G)e(\mu,h)d\mu dh\\
&+\sum_{\kappa=1}^{m}\frac{\hat{c}_{\kappa}\delta^{2}}{2}\int_{t-\delta}^{t}\int_{\Omega}e^{T}(\mu,t)e(\mu,t)d\mu dh\\
\leqslant&\int_{t-\epsilon}^{t}\int_{\Omega}e^{T}(\mu,h)\bigg(I_N\otimes\big((\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_{\kappa}\delta^{2}}{2})I_n+\frac{G}{2(1-\gamma)}\big)\bigg)e(\mu,h)d\mu dh\\
&+\frac{1}{2}\sum_{s=1}^{N}\int_{\Omega}e_s^{T}(\mu,t)e_s(\mu,t)d\mu. \qquad (33)
\end{aligned}
$$

By calculating the derivative of (32) along the trajectories of the system (29), one obtains

$$
\begin{aligned}
\dot{V}_2(t)\leqslant&\sum_{s=1}^{N}\int_{\Omega}e_s^{T}(\mu,t)\bigg(-Ae_s(\mu,t)+Bg(Y_s(\widehat{\mu,t-\sigma_s}))+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_{\kappa}M_{sj}^{\kappa}\Gamma_{\kappa}e_j(\mu,t)\\
&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_{\kappa}\hat{M}_{sj}^{\kappa}\hat{\Gamma}_{\kappa}\int_{t-\delta(t)}^{t}e_j(\mu,h)dh-\beta_s\frac{\|e(\cdot,t)\|^{2}e_s(\mu,t)}{\|e(\cdot,t)\|^{2}+\rho(t)}+Df(W_s(\mu,t))\\
&-q_se_s(\mu,t)+Df(Y_s(\mu,t-\sigma_s))+Bg(\widehat{W_s(\mu,t)})+H\triangle e_s(\mu,t)\bigg)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_{\Omega}e^{T}(\mu,t)(I_N\otimes G)e(\mu,t)d\mu+\epsilon\int_{\Omega}e^{T}(\mu,t)e(\mu,t)d\mu\\
&-\int_{t-\epsilon}^{t}\int_{\Omega}e^{T}(\mu,h)e(\mu,h)d\mu dh+\sum_{\kappa=1}^{m}\frac{\hat{c}_{\kappa}\delta^{2}}{2}\int_{\Omega}e^{T}(\mu,t)e(\mu,t)d\mu\\
&-\frac{1}{2}\int_{\Omega}e^{T}(\mu,t-\tau(t))(I_N\otimes G)e(\mu,t-\tau(t))d\mu\\
&-\sum_{\kappa=1}^{m}\frac{\hat{c}_{\kappa}\delta}{2}\int_{t-\delta}^{t}\int_{\Omega}e^{T}(\mu,h)e(\mu,h)d\mu dh. \qquad (34)
\end{aligned}
$$

Obviously,

$$
\begin{aligned}
&\sum_{\kappa=1}^{m}\sum_{s=1}^{N}\sum_{j=1}^{N}\hat{c}_{\kappa}\hat{M}_{sj}^{\kappa}\int_{\Omega}e_s^{T}(\mu,t)\hat{\Gamma}_{\kappa}\int_{t-\delta(t)}^{t}e_j(\mu,h)dhd\mu\\
=&\sum_{\kappa=1}^{m}\hat{c}_{\kappa}\int_{\Omega}e^{T}(\mu,t)(\hat{M}^{\kappa}\otimes\hat{\Gamma}^{\kappa})\int_{t-\delta(t)}^{t}e(\mu,h)dhd\mu
\end{aligned}
$$

$$\leqslant \sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega\Big(\int_{t-\delta(t)}^{t}e(\mu,h)dh\Big)^T\Big(\int_{t-\delta(t)}^{t}e(\mu,h)dh\Big)d\mu$$
$$+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega e^T(\mu,t)(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2e(\mu,t)d\mu. \tag{35}$$

Then, by Lemma 2.2, one obtains

$$\begin{aligned}
&\sum_{\kappa=1}^{m}\sum_{s=1}^{N}\sum_{j=1}^{N}\hat{c}_\kappa\hat{M}_{sj}^\kappa\int_\Omega e_s^T(\mu,t)\hat{\Gamma}_\kappa\int_{t-\delta(t)}^{t}e_j(\mu,h)dhd\mu\\
\leqslant&\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega\delta(t)\int_{t-\delta(t)}^{t}e^T(\mu,h)e(\mu,h)dhd\mu\\
&+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega e^T(\mu,t)(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2e(\mu,t)d\mu\\
\leqslant&\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega e^T(\mu,t)(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2e(\mu,t)d\mu\\
&+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta}{2}\int_{t-\delta}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh.
\end{aligned}\tag{36}$$

By (11)–(13) and (34)–(36), one obtains

$$\begin{aligned}
\dot{V}_2(t)\leqslant&\sum_{\kappa=1}^{m}c_\kappa\int_\Omega e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu-\int_{t-\epsilon}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\sum_{s=1}^{N}\int_\Omega e_s^T(\mu,t)\Big(-Ae_s(\mu,t)-q_se_s(\mu,t)-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\mu,t)}{\|e(\cdot,t)\|^2+\rho(t)}\Big)d\mu\\
&-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_\Omega e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2}\int_\Omega e^T(\mu,t)e(\mu,t)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_\Omega e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu-\sum_{s=1}^{N}\beta_s\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\frac{1}{2}\int_\Omega e^T(\mu,t)(I_N\otimes(BB^T))e(\mu,t)d\mu+\frac{1}{2}\int_\Omega e^T(\mu,t)(I_N\otimes(DD^T\\
&+F))e(\mu,t)d\mu+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega e^T(\mu,t)(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2e(\mu,t)d\mu
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{s=1}^{N}\beta_s\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu+\epsilon\int_\Omega e^T(\mu,t)e(\mu,t)d\mu\\
=&\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2}\int_\Omega e^T(\mu,t)e(\mu,t)d\mu+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_\Omega e^T(\mu,t)(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2e(\mu,t)d\mu\\
&+\sum_{\kappa=1}^{m}c_\kappa\int_\Omega e^T(\mu,t)(M^\kappa\otimes\Gamma_\kappa)e(\mu,t)d\mu-\int_{t-\epsilon}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\frac{1}{2}\int_\Omega e^T(\mu,t)(I_N\otimes(DD^T+F))e(\mu,t)d\mu+\epsilon\int_\Omega e^T(\mu,t)e(\mu,t)d\mu\\
&+\frac{1}{2}\int_\Omega e^T(\mu,t)(I_N\otimes(BB^T))e(\mu,t)d\mu+\int_\Omega e^T(\mu,t)(-I_N\otimes A-(\hat{\beta}\\
&+\hat{q})\otimes I_n)e(\mu,t)d\mu-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_\Omega e^T(\mu,t)(I_N\otimes H)e(\mu,t)d\mu\\
&+\frac{\rho(t)}{\|e(\cdot,t)\|^2+\rho(t)}\sum_{s=1}^{N}\beta_s\int_\Omega e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_\Omega e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu\\
\leqslant&\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2+\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(DD^T+BB^T\\
&+F+\frac{G}{1-\gamma})+(\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n-A-\sum_{r=1}^{p}\frac{1}{\iota_r^2}H\big)-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu\\
&+\beta\rho(t)-\int_{t-\epsilon}^{t}\int_\Omega e^T(\mu,h)e(\mu,h)d\mu dh. \qquad (37)
\end{aligned}
$$

By combining (33) and (37), we easily obtain

$$
\begin{aligned}
\dot{V}_2(t)+\varepsilon V_2(t)\leqslant&\varepsilon\int_{t-\epsilon}^{t}\int_\Omega e^T(\mu,h)\bigg(I_N\otimes\big((\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n+\frac{G}{2(1-\gamma)}\big)\bigg)e(\mu,h)d\mu dh\\
&+\int_\Omega e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2+\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(DD^T\\
&+BB^T+F+\frac{G}{1-\gamma})+(\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n-A-\sum_{r=1}^{p}\frac{1}{\iota_r^2}H\big)-(\hat{\beta}
\end{aligned}
$$

$$
\begin{aligned}
&+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\beta\rho(t)-\int_{t-\epsilon}^{t}\int_{\Omega}e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\frac{\varepsilon}{2}\sum_{s=1}^{N}\int_{\Omega}e_s^T(\mu,t)e_s(\mu,t)d\mu\\
\leqslant&\int_{\Omega}e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2+\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(DD^T\\
&+BB^T+F+\frac{G}{1-\gamma})+(\frac{\varepsilon}{2}+\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n-A-\sum_{r=1}^{p}\frac{1}{\iota_r^2}H\big)-(\hat{\beta}\\
&+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\beta\rho(t)+\int_{t-\epsilon}^{t}\int_{\Omega}e^T(\mu,h)\Big(I_N\otimes\big((\varepsilon\epsilon\\
&+\sum_{\kappa=1}^{m}\frac{\varepsilon\hat{c}_\kappa\delta^2}{2}-1)I_n+\frac{\varepsilon G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh.
\end{aligned}
$$

By (30) and (31), one obtains

$$\dot{V}_2(t)+\varepsilon V_2(t)\leqslant\beta\rho(t).$$

Letting $\alpha_1=\frac{1}{\sqrt{2}}$, $\alpha_2=\beta$, one easily obtains that network (29) is ψ-type stable with convergence rate $\frac{\varepsilon}{2}$. Then, the drive system (27) and the response system (28) achieve general decay lag anti-synchronization. The proof is completed.

4.2. General decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays and parametric uncertainties

In this section, we consider the following MWDCRDNN with bounded distributed delays and parametric uncertainties:

$$
\begin{aligned}
\frac{\partial Y_s(\mu,t)}{\partial t}=&-AY_s(\mu,t)+H\triangle Y_s(\mu,t)+Bg(\widehat{Y_s(\mu,t)})+Df(Y_s(\mu,t))\\
&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_\kappa\hat{M}_{sj}^\kappa\hat{\Gamma}_\kappa\int_{t-\delta(t)}^{t}Y_j(\mu,h)dh
\end{aligned}
$$

$$+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa Y_j(\mu,t), \quad s=1,2,\cdots,N, \tag{38}$$

where $Y_s(\mu,t)$, $\triangle$, $g(\cdot)$, $f(\cdot)$, $\delta(t)$, $\hat{c}_\kappa$, $\hat{M}_{sj}^{\kappa}$, $\hat{\Gamma}_\kappa$, c_κ, M_{sj}^{κ}, $\Gamma_\kappa(\kappa=1,2,\cdots,m)$ are defined as in Section 4.1, and the ranges of A, H, B, and D are as in (17).

We consider the network model (38) to be the drive system. Then, the corresponding response system is

$$\begin{aligned}\frac{\partial W_s(\mu,t)}{\partial t}=&-AW_s(\mu,t)+H\triangle W_s(\mu,t)+Bg(\widehat{W_s(\mu,t)})+Df(W_s(\mu,t))\\&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_\kappa\hat{M}_{sj}^{\kappa}\hat{\Gamma}_\kappa\int_{t-\delta(t)}^{t}W_j(\mu,h)dh+u_s(\mu,t)\\&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa W_j(\mu,t), \quad s=1,2,\cdots,N,\end{aligned}\tag{39}$$

where $W_s(\mu,t)$, $\triangle$, $g(\cdot)$, $f(\cdot)$, $\delta(t)$, c_κ, M_{sj}^{κ}, Γ_κ, $u_s(\mu,t)$, $\hat{c}_\kappa$, $\hat{M}_{sj}^{\kappa}$, $\hat{\Gamma}_\kappa(\kappa=1,2,\cdots,m)$ are defined as in Section 4.1, and the ranges of A, H, B, and D are as in (17).

Let $e_s(\mu,t)=W_s(\mu,t)+Y_s(\mu,t-\sigma_s)$. By (38) and (39), one obtains

$$\begin{aligned}\frac{\partial e_s(\mu,t)}{\partial t}=&-Ae_s(\mu,t)+H\triangle e_s(\mu,t)+Bg(Y_s(\widehat{\mu,t-}\sigma_s))+Bg(\widehat{W_s(\mu,t)})\\&+Df(Y_s(\mu,t-\sigma_s))+Df(W_s(\mu,t))+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}c_\kappa M_{sj}^{\kappa}\Gamma_\kappa e_j(\mu,t)\\&+\sum_{\kappa=1}^{m}\sum_{j=1}^{N}\hat{c}_\kappa\hat{M}_{sj}^{\kappa}\hat{\Gamma}_\kappa\int_{t-\delta(t)}^{t}e_j(\mu,h)dh+u_s(\mu,t),\end{aligned}\tag{40}$$

where $\sigma_s\geqslant 0$ $(s=1,2,\cdots,N)$ is the lag delay, and the ranges of A, H, B, and D are as in (17).

We construct the same nonlinear controller (5) for the response system (39) in this section.

Theorem 4.2. *Under Assumptions 1 and 2, the system* (40) *is robustly ψ-type stable with convergence rate $\frac{\varepsilon}{2}$, that is, systems* (38)*and* (39) *achieve general robust decay lag anti-synchronization for all $A \in A_I,\ H \in H_I,\ D \in D_I$, and $B \in B_I$ if*

$$\Xi_4 = \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa}{2} (\hat{M}^\kappa \otimes \hat{\Gamma}^\kappa)^2 + \sum_{\kappa=1}^{m} c_\kappa M^\kappa \otimes \Gamma_\kappa + I_N \otimes P_4 - (\hat{\beta} + \hat{q}) \otimes I_n < 0, \quad (41)$$

$$\Psi_4 = I_N \otimes \big((\varepsilon\epsilon + \sum_{\kappa=1}^{m} \frac{\varepsilon \hat{c}_\kappa \delta^2}{2} - 1) I_n + \frac{\varepsilon G}{2(1-\gamma)} \big) < 0, \quad (42)$$

where $P_4 = \frac{1}{2}(\varrho_D I_n + \varrho_B I_n + F + \frac{G}{1-\gamma}) + (\frac{\varepsilon}{2} + \epsilon + \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa \delta^2}{2}) I_n - A^- - \sum_{r=1}^{p} \frac{1}{\iota_r^2} H^-$.

Proof. We construct the same Lyapunov functional as in (32) for network (29). Then, one obtains

$$\begin{aligned}
\dot{V}_2(t) \leqslant & \sum_{s=1}^{N} \int_\Omega e_s^T(\mu,t) \Big(- A^- e_s(\mu,t) + Bg(Y_s(\widehat{\mu, t-} \sigma_s)) + \sum_{\kappa=1}^{m} \sum_{j=1}^{N} c_\kappa M_{sj}^\kappa \Gamma_\kappa e_j(\mu,t) \\
& + \sum_{\kappa=1}^{m} \sum_{j=1}^{N} \hat{c}_\kappa \hat{M}_{sj}^\kappa \hat{\Gamma}_\kappa \int_{t-\delta(t)}^{t} e_j(\mu,h) dh - \beta_s \frac{\|e(\cdot,t)\|^2 e_s(\mu,t)}{\|e(\cdot,t)\|^2 + \rho(t)} + Df(W_s(\mu,t)) \\
& - q_s e_s(\mu,t) + Df(Y_s(\mu, t-\sigma_s)) + Bg(\widehat{W_s(\mu,t)}) + H \triangle e_s(\mu,t) \Big) d\mu \\
& + \frac{1}{2(1-\gamma)} \int_\Omega e^T(\mu,t)(I_N \otimes G) e(\mu,t) d\mu + \epsilon \int_\Omega e^T(\mu,t) e(\mu,t) d\mu \\
& - \int_{t-\epsilon}^{t} \int_\Omega e^T(\mu,h) e(\mu,h) d\mu dh + \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa \delta^2}{2} \int_\Omega e^T(\mu,t) e(\mu,t) d\mu \\
& - \frac{1}{2} \int_\Omega e^T(\mu, t-\tau(t))(I_N \otimes G) e(\mu, t-\tau(t)) d\mu \\
& - \sum_{\kappa=1}^{m} \frac{\hat{c}_\kappa \delta}{2} \int_{t-\delta}^{t} \int_\Omega e^T(\mu,h) e(\mu,h) d\mu dh. \qquad (43)
\end{aligned}$$

From (23)–(25) and (36), it is easy to derive that

$$\dot{V}_2(t) \leqslant \sum_{\kappa=1}^{m} c_\kappa \int_\Omega e^T(\mu,t)(M^\kappa \otimes \Gamma_\kappa) e(\mu,t) d\mu - \int_{t-\epsilon}^{t} \int_\Omega e^T(\mu,h) e(\mu,h) d\mu dh$$

$$
\begin{aligned}
&+\sum_{s=1}^{N}\int_{\Omega}e_s^T(\mu,t)\bigg(-A^{-}e_s(\mu,t)-q_se_s(\mu,t)-\beta_s\frac{\|e(\cdot,t)\|^2e_s(\mu,t)}{\|e(\cdot,t)\|^2+\rho(t)}\bigg)d\mu\\
&-\sum_{r=1}^{p}\frac{1}{\iota_r^2}\int_{\Omega}e^T(\mu,t)(I_N\otimes H^{-})e(\mu,t)d\mu+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2}\int_{\Omega}e^T(\mu,t)e(\mu,t)d\mu\\
&+\frac{1}{2(1-\gamma)}\int_{\Omega}e^T(\mu,t)(I_N\otimes G)e(\mu,t)d\mu-\sum_{s=1}^{N}\beta_s\int_{\Omega}e_s^T(\mu,t)e_s(\mu,t)d\mu\\
&+\frac{1}{2}\int_{\Omega}e^T(\mu,t)(I_N\otimes(\varrho_BI_n))e(\mu,t)d\mu+\frac{1}{2}\int_{\Omega}e^T(\mu,t)(I_N\otimes(\varrho_DI_n\\
&+F))e(\mu,t)d\mu+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}\int_{\Omega}e^T(\mu,t)(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2e(\mu,t)d\mu\\
&+\sum_{s=1}^{N}\beta_s\int_{\Omega}e_s^T(\mu,t)e_s(\mu,t)d\mu+\epsilon\int_{\Omega}e^T(\mu,t)e(\mu,t)d\mu\\
\leqslant&\int_{\Omega}e^T(\mu,t)\bigg[\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2+\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(\varrho_DI_n+\varrho_BI_n\\
&+F+\frac{G}{1-\gamma})+(\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n-A^{-}-\sum_{r=1}^{p}\frac{1}{\iota_r^2}H^{-}\big)-(\hat{\beta}\\
&+\hat{q})\otimes I_n\bigg]e(\mu,t)d\mu+\beta\rho(t)-\int_{t-\epsilon}^{t}\int_{\Omega}e^T(\mu,h)e(\mu,h)d\mu dh. \qquad (44)
\end{aligned}
$$

Combining (33) and (44), one obtains

$$
\begin{aligned}
\dot{V}_2(t)+\varepsilon V_2(t)\leqslant&\varepsilon\int_{t-\epsilon}^{t}\int_{\Omega}e^T(\mu,h)\bigg(I_N\otimes\big((\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n+\frac{G}{2(1-\gamma)}\big)\bigg)e(\mu,h)d\mu dh\\
&+\int_{\Omega}e^T(\mu,t)\bigg[\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2+\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(\varrho_DI_n\\
&+\varrho_BI_n+F+\frac{G}{1-\gamma})+(\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n-A^{-}-\sum_{r=1}^{p}\frac{1}{\iota_r^2}H^{-}\big)-(\hat{\beta}\\
&+\hat{q})\otimes I_n\bigg]e(\mu,t)d\mu+\beta\rho(t)-\int_{t-\epsilon}^{t}\int_{\Omega}e^T(\mu,h)e(\mu,h)d\mu dh\\
&+\frac{\varepsilon}{2}\sum_{s=1}^{N}\int_{\Omega}e_s^T(\mu,t)e_s(\mu,t)d\mu
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \int_{\Omega} e^T(\mu,t)\Big[\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa}{2}(\hat{M}^\kappa\otimes\hat{\Gamma}^\kappa)^2+\sum_{\kappa=1}^{m}c_\kappa M^\kappa\otimes\Gamma_\kappa+I_N\otimes\big(\frac{1}{2}(\varrho_D I_n\\
&+\varrho_B I_n+F+\frac{G}{1-\gamma})+(\frac{\varepsilon}{2}+\epsilon+\sum_{\kappa=1}^{m}\frac{\hat{c}_\kappa\delta^2}{2})I_n-A^- -\sum_{r=1}^{p}\frac{1}{\iota_r^2}H^-\big)\\
&-(\hat{\beta}+\hat{q})\otimes I_n\Big]e(\mu,t)d\mu+\beta\rho(t)+\int_{t-\epsilon}^{t}\int_{\Omega}e^T(\mu,h)\Big(I_N\otimes\big((\varepsilon\epsilon\\
&+\sum_{\kappa=1}^{m}\frac{\varepsilon\hat{c}_\kappa\delta^2}{2}-1)I_n+\frac{\varepsilon G}{2(1-\gamma)}\big)\Big)e(\mu,h)d\mu dh.
\end{aligned}
$$

By (41) and (42),

$$\dot{V}_2(t)+\varepsilon V_2(t)\leqslant\beta\rho(t).$$

Letting $\alpha_1=\frac{1}{\sqrt{2}}$, $\alpha_2=\beta$, one easily obtains that the network (40) is robustly ψ-type stable with convergence rate $\frac{\varepsilon}{2}$. Then, the drive system (38) and the response system (39) achieve general robust decay lag anti-synchronization. The proof is completed.

Remark 7. From the conditions of Theorems 3.1, 3.2, 4.1, and 4.2, it can be clearly seen that the dimensions of these matrix inequalities depend on the number of nodes in the network and the dimension of each node. When the number and the dimension of the nodes are large, the implementation complexity increases accordingly. In this case, it may be difficult to verify the conditions by using Matlab. Therefore, it is an important and challenging problem to establish more tractable conditions that ensures the decay lag anti-synchronization of the considered MWDCRDNNs, which will be a future research direction.

Remark 8. In this study, several decay lag anti-synchronization and robust decay lag anti-synchronization criteria for MWDCRDNNs with and without bounded distributed delays are derived using certain inequalities and Lyapunov's functional method. To the best of our knowledge, this is the first step towards studying the decay lag anti-synchronization of MWDCRDNNs. However, the time-varying delay $\tau(t)$ in the considered model should satisfy the conditions $0 \leqslant \tau(t) \leqslant \tau$ and $\dot{\tau}(t) \leqslant \gamma < 1$. In recent years, research has been conducted on networks with unbounded time delay, and it is possible to remove the restriction on the derivative of the time delay (i.e., the condition $\dot{\tau}(t) \leqslant \gamma < 1$) by using delay interval decomposition. In future work, it would be interesting to establish some less restrictive decay lag anti-synchronization criteria by adopting these new methods or techniques to remove the constraints on time delay.

5. Numerical Examples

Example 5.1. We consider the following MWDCNN with reaction–diffusion terms:

$$\begin{aligned}\frac{\partial Y_s(\mu,t)}{\partial t} =& - AY_s(\mu,t) + H\triangle Y_s(\mu,t) + Df(Y_s(\mu,t)) + Bg(\widehat{Y_s(\mu,t)})\\ &+ 0.3\sum_{j=1}^{5} M_{sj}^{1}\Gamma_1 Y_j(\mu,t) + 0.4\sum_{j=1}^{5} M_{sj}^{2}\Gamma_2 Y_j(\mu,t)\\ &+ 0.2\sum_{j=1}^{5} M_{sj}^{3}\Gamma_3 Y_j(\mu,t),\end{aligned} \tag{45}$$

where $s = 1,2,\cdots,5$, $f_i(\omega) = \frac{|\omega+1|-|\omega-1|}{8}$, $g_i(\omega) = \frac{|\omega+1|-|\omega-1|}{4}$ $(i = 1,2,3)$, $\Omega = \{\mu| - 1 < \mu < 1\}$, $\Gamma_1 = \mathrm{diag}(0.2, 0.5, 0.3)$, $\Gamma_2 = \mathrm{diag}(0.1, 0.2, 0.3)$, $\Gamma_3 = \mathrm{diag}(0.3, 0.1, 0.4)$, $\tau(t) = \frac{1}{20} - \frac{1}{10}e^{-t}$, $\tau = \frac{1}{20}$, $\gamma = \frac{1}{10}$; the matrices M^1, M^2, and M^3 are chosen as follows:

$$M^1 = \begin{pmatrix} -0.3 & 0 & 0.1 & 0.2 & 0 \\ 0 & -0.6 & 0.3 & 0.2 & 0.1 \\ 0.1 & 0.3 & -0.6 & 0 & 0.2 \\ 0.2 & 0.2 & 0 & -0.6 & 0.2 \\ 0 & 0.1 & 0.2 & 0.2 & -0.5 \end{pmatrix},$$

$$M^2 = \begin{pmatrix} -0.6 & 0.1 & 0 & 0.3 & 0.2 \\ 0.1 & -0.3 & 0 & 0.1 & 0.1 \\ 0 & 0 & -0.1 & 0.1 & 0 \\ 0.3 & 0.1 & 0.1 & -0.5 & 0 \\ 0.2 & 0.1 & 0 & 0 & -0.3 \end{pmatrix},$$

$$M^3 = \begin{pmatrix} -0.5 & 0.2 & 0.1 & 0 & 0.2 \\ 0.2 & -0.4 & 0 & 0.1 & 0.1 \\ 0.1 & 0 & -0.1 & 0 & 0 \\ 0 & 0.1 & 0 & -0.4 & 0.3 \\ 0.2 & 0.1 & 0 & 0.3 & -0.6 \end{pmatrix}.$$

The parameters $A,\ H,\ D,\ B$ in the network (45) may vary as follows:

$$\begin{cases} A_I := \{A = \mathrm{diag}(a_1, a_2, a_3) : 0.4 \leqslant a_1 \leqslant 0.5, 0.5 \leqslant a_2 \leqslant 0.6, \\ \qquad 0.6 \leqslant a_3 \leqslant 0.7\}, \\ H_I := \{H = \mathrm{diag}(h_1, h_2, h_3) : 0.6 \leqslant h_1 \leqslant 0.7, 0.7 \leqslant h_2 \leqslant 0.8, \\ \qquad 0.8 \leqslant h_3 \leqslant 0.9\}, \\ D_I := \{D = (d_{rj})_{3\times 3} : \dfrac{1}{3(r+j)} + 0.02 \leqslant d_{rj} \leqslant \dfrac{1}{3(r+j)} + 0.03\}, \\ B_I := \{B = (b_{rj})_{3\times 3} : \dfrac{1}{4(r+j)} + 0.01 \leqslant b_{rj} \leqslant \dfrac{1}{4(r+j)} + 0.02\}. \end{cases} \tag{46}$$

Apparently, $f_i(\cdot)$ and $g_i(\cdot)$ $(i = 1, 2, 3)$ satisfy Assumption 1 with $F_i = 0.25$ and $G_i = 0.5$, respectively. We consider (45) to be the drive system; then, the corresponding response system is as follows:

$$\begin{aligned}\frac{\partial W_s(\mu,t)}{\partial t} =& - AW_s(\mu,t) + H\triangle W_s(\mu,t) + Bg(\widehat{W_s(\mu,t)}) + Df(W_s(\mu,t)) \\ & + 0.3\sum_{j=1}^{5} M_{sj}^1 \Gamma_1 W_j(\mu,t) + 0.4\sum_{j=1}^{5} M_{sj}^2 \Gamma_2 W_j(\mu,t) + \cdots \\ & + 0.2\sum_{j=1}^{5} M_{sj}^3 \Gamma_3 W_j(\mu,t) + u_s(\mu,t). \end{aligned} \tag{47}$$

The parameters in the controller $u_s(\mu,t)$ defined in (5) are chosen as follows: $\hat{q} = \mathrm{diag}(0.7, 0.8, 0.5, 0.3, 0.4)$, $\hat{\beta} = \mathrm{diag}(0.4, 0.6, 0.2, 0.4, 0.2)$, and $\rho(t) = e^{-0.3t}$. Then, the nonlinear controller (5) takes the following form:

$$\begin{cases} u_1(\mu,t) = - 0.7e_1(\mu,t) - 0.4\dfrac{\|e(\cdot,t)\|^2 e_1(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}}, \\ u_2(\mu,t) = - 0.8e_2(\mu,t) - 0.6\dfrac{\|e(\cdot,t)\|^2 e_2(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}}, \\ u_3(\mu,t) = - 0.5e_3(\mu,t) - 0.2\dfrac{\|e(\cdot,t)\|^2 e_3(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}}, \\ u_4(\mu,t) = - 0.3e_4(\mu,t) - 0.4\dfrac{\|e(\cdot,t)\|^2 e_4(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}}, \\ u_5(\mu,t) = - 0.4e_5(\mu,t) - 0.2\dfrac{\|e(\cdot,t)\|^2 e_5(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}}. \end{cases} \tag{48}$$

The other parameters in (47) are defined as in (45). We choose $\varepsilon = 0.02$, that is, the convergence rate is $\frac{\varepsilon}{2} = 0.01$. For convenience, the lag delays are chosen as $\sigma_s = 0.008$ $(s = 1, 2, \cdots, 5)$. Through a simple operation based on the above parameters using the MATLAB toolbox, one obtains

$$\lambda(\Xi_2) =\{-2.5879, -2.3879, -2.3076, -2.1343, -2.0719, -1.9154, -1.8401,$$

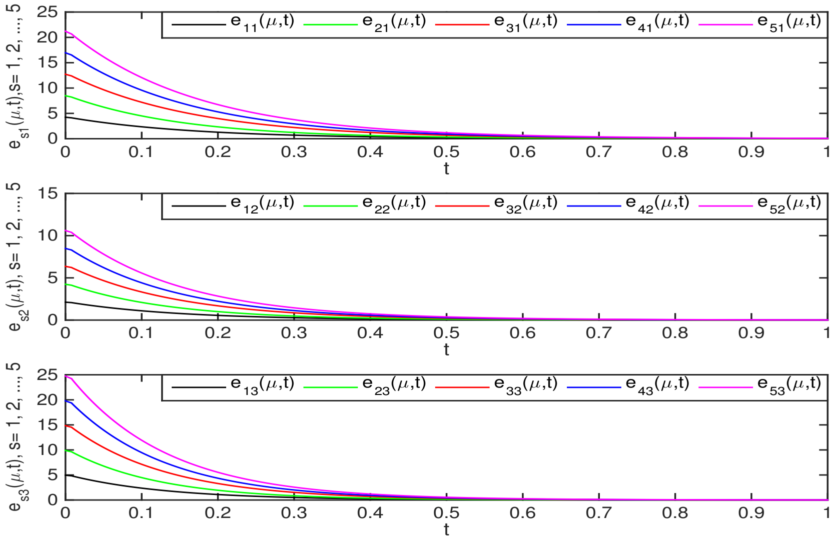

Figure 1: $e_{sj}(\mu,t)$ between systems (45) and (47), $s=1,2,\cdots,5,\ j=1,2,3.$

$$
\begin{aligned}
&-1.8339, -1.7596, -1.7004, -1.6673, -1.5473, -1.4471, -1.4077,\\
&-1.3257\},\\
\lambda(\Psi_2) =& -0.9962,
\end{aligned}
$$

which satisfy the conditions (20) and (21).

By Theorem 3.2, the systems (45) and (47) achieve general robust decay lag anti-synchronization under the nonlinear controller (48). The above simulation result demonstrates the validity of Theorem 3.2 in Section 3. Figure 1 shows the trajectories of the three components of the errors states $(e_s(t), s = 1, 2, \cdots, 5)$ between systems (45) and (47) under the controller (48). It is clear that each component converges to 0 as the time t gradually increases to 1 s, and this state is maintained thereafter.

Example 5.2. We consider the following MWDCRDNN with bounded distributed delays:

$$\begin{aligned}\frac{\partial Y_s(\mu,t)}{\partial t} =& -AY_s(\mu,t)+H\triangle Y_s(\mu,t)+Df(Y_s(\mu,t))+Bg(\widehat{Y_s(\mu,t)})\\&+0.1\sum_{j=1}^{5}\hat{M}^1_{sj}\hat{\Gamma}_1\int_{t-\delta(t)}^{t}Y_j(\mu,h)dh+0.3\sum_{j=1}^{5}M^1_{sj}\Gamma_1Y_j(\mu,t)\\&+0.3\sum_{j=1}^{5}\hat{M}^2_{sj}\hat{\Gamma}_2\int_{t-\delta(t)}^{t}Y_j(\mu,h)dh+0.4\sum_{j=1}^{5}M^2_{sj}\Gamma_2Y_j(\mu,t)\\&+0.2\sum_{j=1}^{5}\hat{M}^3_{sj}\hat{\Gamma}_3\int_{t-\delta(t)}^{t}Y_j(\mu,h)dh+0.2\sum_{j=1}^{5}M^3_{sj}\Gamma_3Y_j(\mu,t),\quad (49)\end{aligned}$$

where $s=1,2,\cdots,5$, $f_i(\omega)=\frac{|\omega+1|-|\omega-1|}{4}$, $g_i(\omega)=\frac{|\omega+1|-|\omega-1|}{8}$ $(i=1,2,3)$, $\Omega=\{\mu|-1<\mu<1\}$, $\Gamma_1=\mathrm{diag}(0.2,0.5,0.3)$, $\Gamma_2=\mathrm{diag}(0.1,0.2,0.3)$, $\Gamma_3=\mathrm{diag}(0.3,0.1,0.4)$, $\hat{\Gamma}_1=\mathrm{diag}(0.4,0.2,0.3)$, $\hat{\Gamma}_2=\mathrm{diag}(0.1,0.5,0.4)$, $\hat{\Gamma}_3=\mathrm{diag}(0.2,0.2,0.4)$, $\tau(t)=\frac{1}{20}-\frac{1}{10}e^{-t}$, $\tau=\frac{1}{20}$, $\gamma=\frac{1}{10}$, $\delta(t)=\frac{1}{10}-\frac{1}{10}e^{-t}$, $\delta=\frac{1}{10}$; the matrices M^1, M^2, M^3, $\hat{M}^1$, $\hat{M}^2$, and $\hat{M}^3$ are chosen as

$$M^1=\begin{pmatrix}-0.3&0&0.1&0.2&0\\0&-0.6&0.3&0.2&0.1\\0.1&0.3&-0.6&0&0.2\\0.2&0.2&0&-0.6&0.2\\0&0.1&0.2&0.2&-0.5\end{pmatrix},$$

$$M^2=\begin{pmatrix}-0.6&0.1&0&0.3&0.2\\0.1&-0.3&0&0.1&0.1\\0&0&-0.1&0.1&0\\0.3&0.1&0.1&-0.5&0\\0.2&0.1&0&0&-0.3\end{pmatrix},$$

$$M^3 = \begin{pmatrix} -0.5 & 0.2 & 0.1 & 0 & 0.2 \\ 0.2 & -0.4 & 0 & 0.1 & 0.1 \\ 0.1 & 0 & -0.1 & 0 & 0 \\ 0 & 0.1 & 0 & -0.4 & 0.3 \\ 0.2 & 0.1 & 0 & 0.3 & -0.6 \end{pmatrix},$$

$$\hat{M}^1 = \begin{pmatrix} -0.5 & 0.2 & 0 & 0.2 & 0.1 \\ 0.2 & -0.6 & 0.3 & 0.1 & 0 \\ 0 & 0.3 & -0.7 & 0.2 & 0.2 \\ 0.2 & 0.1 & 0.2 & -0.7 & 0.2 \\ 0.1 & 0 & 0.2 & 0.2 & -0.5 \end{pmatrix},$$

$$\hat{M}^2 = \begin{pmatrix} -0.5 & 0 & 0.3 & 0.2 & 0 \\ 0 & -0.2 & 0 & 0.1 & 0.1 \\ 0.3 & 0 & -0.6 & 0.1 & 0.2 \\ 0.2 & 0.1 & 0.1 & -0.4 & 0 \\ 0 & 0.1 & 0.2 & 0 & -0.3 \end{pmatrix},$$

$$\hat{M}^3 = \begin{pmatrix} -0.6 & 0.3 & 0.2 & 0.1 & 0 \\ 0.3 & -0.6 & 0.1 & 0 & 0.2 \\ 0.2 & 0.1 & -0.4 & 0 & 0.1 \\ 0.1 & 0 & 0 & -0.4 & 0.3 \\ 0 & 0.2 & 0.1 & 0.3 & -0.6 \end{pmatrix}.$$

The parameters $A,\ H,\ D,\ B$ in the network (49) are defined by (46). Apparently, $f_i(\cdot)$ and $g_i(\cdot)$ $(i = 1, 2, 3)$ satisfy Assumption 1 with $F_i = 0.5$ and $G_i = 0.25$, respectively. We consider (49) to be the drive system; then, the corresponding response system is as follows:

$$\begin{aligned}\frac{\partial W_s(\mu,t)}{\partial t} =& - AW_s(\mu,t) + H\triangle W_s(\mu,t) + Df(W_s(\mu,t)) + Bg(\widehat{W_s(\mu,t)})\\ &+ 0.1\sum_{j=1}^{5}\hat{M}_{sj}^{1}\hat{\Gamma}_1\int_{t-\delta(t)}^{t} W_j(\mu,h)dh + 0.3\sum_{j=1}^{5} M_{sj}^{1}\Gamma_1 W_j(\mu,t)\\ &+ 0.3\sum_{j=1}^{5}\hat{M}_{sj}^{2}\hat{\Gamma}_2\int_{t-\delta(t)}^{t} W_j(\mu,h)dh + 0.4\sum_{j=1}^{5} M_{sj}^{2}\Gamma_2 W_j(\mu,t)\\ &+ 0.2\sum_{j=1}^{5}\hat{M}_{sj}^{3}\hat{\Gamma}_3\int_{t-\delta(t)}^{t} W_j(\mu,h)dh + 0.2\sum_{j=1}^{5} M_{sj}^{3}\Gamma_3 W_j(\mu,t)\\ &+ u_s(\mu,t). \end{aligned} \tag{50}$$

The parameters in the controller $u_s(\mu,t)$ defined in (5) are chosen as follows: $\hat{q} = \mathrm{diag}(0.6, 0.8, 0.2, 0.4, 0.1)$, $\hat{\beta} = \mathrm{diag}(0.4, 0.5, 0.2, 0.1, 0.3)$, and $\rho(t) = e^{-0.3t}$. Then, the nonlinear controller (5) takes the following form:

$$\begin{cases} u_1(\mu,t) = -0.6e_1(\mu,t) - 0.4\dfrac{\|e(\cdot,t)\|^2 e_1(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}},\\ u_2(\mu,t) = -0.8e_2(\mu,t) - 0.5\dfrac{\|e(\cdot,t)\|^2 e_2(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}},\\ u_3(\mu,t) = -0.2e_3(\mu,t) - 0.2\dfrac{\|e(\cdot,t)\|^2 e_3(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}},\\ u_4(\mu,t) = -0.4e_4(\mu,t) - 0.1\dfrac{\|e(\cdot,t)\|^2 e_4(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}},\\ u_5(\mu,t) = -0.1e_5(\mu,t) - 0.3\dfrac{\|e(\cdot,t)\|^2 e_5(\mu,t)}{\|e(\cdot,t)\|^2 + e^{-0.3t}}. \end{cases} \tag{51}$$

The other parameters in (50) are defined as in (49). We choose $\varepsilon = 0.04$, that is, the convergence rate is $\frac{\varepsilon}{2} = 0.02$. For convenience, the lag delays are chosen as $\sigma_s = 0.002$ $(s = 1, 2, \cdots, 5)$. Through a simple operation based on the above parameters by using the MATLAB toolbox, one obtains

$$\lambda(\Xi_4) =\{-1.0422, -1.0761, -1.1873, -1.9750, -1.6756, -2.4221, -2.2289,$$

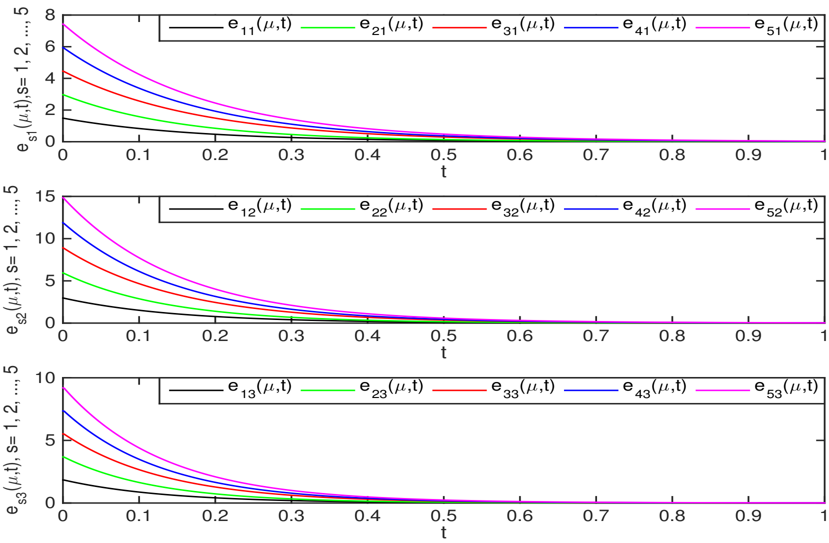

Figure 2: $e_{sj}(\mu, t)$ between systems (49) and (50), $s = 1, 2, \cdots, 5,\ j = 1, 2, 3.$

$$
\begin{aligned}
&- 2.1342, -1.8990, -1.6489, -1.2652, -1.3173, -1.5117, -1.4398, \\
&- 1.4569\}, \\
\lambda(\Psi_4) =& - 0.9945,
\end{aligned}
$$

which satisfy the conditions (41) and (42).

By Theorem 4.2, the systems (49) and (50) achieve general robust decay lag anti-synchronization under the nonlinear controller (51). The above simulation demonstrates the validity of Theorem 4.2 in Section 4. Figure 2 shows the trajectories of the three components of the errors states ($e_s(t), s = 1, 2, \cdots, 5$) between systems (49) and (50) under the controller (51). It is clear that each component tends to 0 as the time t gradually increases to 1 s, and this state is maintained thereafter.

Remark 9. Owing to the difficulty in estimating the convergence rate of a system in practice, a new type of synchronization was proposed, namely decay synchronization. It generalizes traditional synchronization concepts, such as, exponential, asymptotic, and polynomial synchronization [1, 14, 18, 19]. However, these studies ignore the impact of time delay on the network, which may lead to various undesirable results, such as instability and poor performance. For this reason, lag synchronization of neural networks has been extensively studied [2, 6, 17, 28]. In various applications, another interesting phenomenon, that is, anti-synchronization, has been observed in chaotic neural networks [12, 13, 31] and memristive neural networks [16, 28]. Unfortunately, decay lag anti-synchronization has not been considered to date. In this study, the decay lag anti-synchronization of MWDCRDNNs with and without bounded distributed delays was first investigated, and related conditions were derived by introducing the concept of decay lag anti-synchronization and designing an appropriate nonlinear controller.

Remark 10. In this section, the theoretical results were verified using two numerical simulations, in which the parameters need only to satisfy the conditions in the network model. Recently, the problem of decay synchronization and lag synchronization of neural networks has attracted considerable attention [1, 2, 6, 14, 17–19, 28]. In these studies, purely numerical examples were used to validate the derived theoretical results. Accordingly, we adopted the same strategy here and used two numerical examples. It would be interesting to find potential applications in future work.

6. Conclusion

This study was concerned with the general decay lag anti-synchronization of MWDCRDNNs, which combines the concepts of anti-synchronization, decay synchronization, and lag synchronization. Using Lyapunov functionals, certain inequalities, and an appropriate nonlinear controller, we derived sufficient conditions whereby the decay lag anti-synchronization of MWDCRDNNs with and without parametric uncertainties is ensured. Similarly, the decay lag anti-synchronization and the robust decay lag anti-synchronization of MWDCRDNNs with bounded distributed delays were also studied. Finally, several simulations were performed to validate the obtained results.

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Declaration of Interest Statement

None.

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