ANSWERING AN OPEN PROBLEM ON T-NORMS FOR TYPE-2 FUZZY SETS

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ABSTRACT. This paper proves that a binary operation $\star$ on $[0,1]$, ensuring that the binary operation $\curlywedge$ is a t-norm or $\curlyvee$ is a t-conorm, is a t-norm, where $\curlywedge$ and $\curlyvee$ are special convolution operations defined by

$$(f \curlywedge g)(x) = \sup\left\{f(y) \star g(z) : y \triangle z = x\right\},$$
$$(f \curlyvee g)(x) = \sup\left\{f(y) \star g(z) : y \triangledown z = x\right\},$$

for any $f, g \in Map([0,1],[0,1])$, where $\triangle$ and $\triangledown$ are a continuous t-norm and a continuous t-conorm on $[0,1]$, answering negatively an open problem posed in [17]. Besides, some characteristics of t-norm and t-conorm are obtained in terms of the binary operations $\curlywedge$ and $\curlyvee$.

1. INTRODUCTION

In 1975, Zadeh [1] introduced the notion of type-2 fuzzy sets (T2FSs) – that is, fuzzy set with fuzzy sets as truth values (simply, "fuzzy-fuzzy sets") – being an extension of type-1 fuzzy sets (FSs) and interval-valued fuzzy sets (IVFSs), which was also equivalently expressed in different forms by Mendel *et al.* ([2]–[5]). Because the truth values of T2FSs are fuzzy, they are more adaptable to a further study of uncertainty than FSs and have been applied in many studies ([6]–[25]). Mendel [6] summarized some important advances for FSs and T2FSs from 2001 to 2007. Hu and Kwong [7] discussed t-norm operations of T2FSs and obtained a few properties of type-2 fuzzy numbers. For better understanding of T2FSs, Aisbett *et al.* [8] translates their constructs to the language of functions in spaces. Chen and Wang [9] used T2FSs to give a new technique for fuzzy multiple attributes decision making. Sola *et al.* [10] provided a more general perspective for interval T2FSs and showed that IVFSs can be viewed as a special case of interval T2FSs. Ruiz *et al.* [11] obtained two results for join and meet operations for T2FSs with arbitrary secondary memberships. Recently, Wang [12] introduced the notion of conditional fuzzy sets to characterize T2FSs. Then, Wu *et al.* [13] presented a Jaccard similarity measure for general T2FSs, as an extension of the Jaccard similarity measure for FSs and IVFSs.

Being an extension of the logic connective conjunction and disjunctionin classical two-valued logic, triangular norms (t-norms) with the neutral 1 and triangular conorms (t-conorms) with the neutral 0 on the unit interval $I = [0, 1]$ were introduced by Menger [14] in 1942 and by Schweizer and Sklar [15] in 1961, respectively. Because t-norms and t-conorms have a close connection with fuzzy set theory and order related theories, they play an important role in many fields, such as fuzzy set theory [26], fuzzy logic [16], fuzzy systems modeling [27], and probabilistic metric spaces [15]. Walker and Walker [28] extended t-norms and t-conorms to the algebra of truth values of T2FSs. Then, Hernándes *et al.* [17] introduced the notions of t_r-norm and t_r-conorm by adding some "restrictive axioms" (see Definition 2 below) with systematic analysis. In particular, they [17] proved that the following binary operation $\curlywedge$ (resp., $\curlyvee$) on the set of all normal and convex functions constructed by convolution is a t_r-norm (resp., a t_r-conorm). Recently, we proved [29] that the fuzzy metric M of every stationary fuzzy metric space $(X, M, \star)$ is uniformly continuous.

Throughout this paper, let $I = [0, 1]$, $Map(X, Y)$ be the set of all mappings from X to Y, and '$\leq$' denote the usual order relation in the lattice of real numbers. In particular, let $\mathbf{M} = Map(I, I)$ and $\mathbf{L}$ be the set of all normal and convex functions in $\mathbf{M}$.

Definition 1. [24] A *t-norm* on I is a binary operation $\star : I^2 \to I$ satisfying

(T1) (*commutativity/symmetry*) $x \star y = y \star x$ for $x, y \in I$;
(T2) (*associativity*) $(x \star y) \star z = x \star (y \star z)$ for $x, y, z \in I$;
(T3) (*increasing*) $\star$ is increasing in each argument;
(T4) (*neutral element*) $1 \star x = x \star 1 = x$ for $x \in I$.

A binary operation $\star : I^2 \to I$ is a *t-conorm* on I if it satisfies axioms (T1), (T2), and (T3) above; axiom (T4'): $0 \star x = x \star 0 = x$ for $x \in I$.

For any subset B of X, a special fuzzy set χ_B, which is called the *characteristic function* of B, is defined as

$$\chi_B(x) = \begin{cases} 1, & x \in B, \\ 0, & x \in X \setminus B. \end{cases}$$

Definition 2. [17] A binary operation $T : \mathbf{L}^2 \to \mathbf{L}$ is a *t_r-norm* (t-norm according to the restrictive axioms), if

(O1) T is commutative, i.e., $T(f, g) = T(g, f)$ for $f, g \in \mathbf{L}$;
(O2) T is associative, i.e., $T(T(f, g), h) = T(f, T(g, h))$ for $f, g, h \in \mathbf{L}$;
(O3) $T(f, \chi_{\{1\}}) = f$ for $f \in \mathbf{L}$ (neutral element);
(O4) letting $f, g, h \in \mathbf{L}$ such that $g \sqsubseteq h$; then, $T(f, g) \sqsubseteq T(f, h)$ (increasing in each argument);
(O5) $T(\chi_{[0,1]}, \chi_{[a,b]}) = \chi_{[0,b]}$;
(O6) T is closed on $\mathbf{J}$;
(O7) T is closed on $\mathbf{K}$;

where $\mathbf{J}$ is the set of all characteristic functions of the elements of I, and $\mathbf{K}$ is the set of all characteristic functions of the closed subintervals of I, i.e., $\mathbf{J} = \{\chi_{\{x\}} : x \in I\}$, $\mathbf{K} = \{\chi_{[a,b]} : 0 \leq a \leq b \leq 1\}$.

A binary operation $S : \mathbf{L}^2 \to \mathbf{L}$ is a t_r-conorm if it satisfies axioms (O1), (O2), (O4), (O6), and (O7) above; axiom (O3'): $S(f, \chi_{\{0\}}) = f$; and axiom (O5'): $S(\chi_{[0,1]}, \chi_{[a,b]}) = \chi_{[a,1]}$. Axioms (O1), (O2), (O3), (O3'), and (O4) are called "*basic axioms*", and an operation that complies with these axioms will be referred to as *t-norm* or *t-conorm*, respectively.

Convolution as a standard way to combine functions was used to construct operations on $Map(J, [0, 1])$. Let $\circ$ and $\blacktriangle$ be two binary operations defined on X and Y, respectively, and $\blacktriangledown$ be an appropriate operation on Y. Define a binary operation $\bullet$ on the set $Map(X, Y)$ by

$$(f \bullet g)(x) = \blacktriangledown \{f(y) \blacktriangle g(z) : y \circ z = x\}.$$

This method of defining an operation on $Map(X, Y)$ from operations on X and Y is called *convolution*.

Definition 3. [17] Let $\star$ be a binary operation on I, $\triangle$ be a t-norm on I, and $\triangledown$ be a t-conorm on I. Define the binary operations $\curlywedge$ and $\curlyvee : \mathbf{M}^2 \to \mathbf{M}$ as follows: for $f, g \in \mathbf{M}$,

$$(f \curlywedge g)(x) = \sup \left\{ f(y) \star g(z) : y \triangle z = x \right\}, \tag{1.1}$$

and

$$(f \curlyvee g)(x) = \sup \left\{ f(y) \star g(z) : y \triangledown z = x \right\}. \tag{1.2}$$

In 2015, Hernándes *et al.* [17] proposed the following open problem on the binary operations $\curlywedge$ and $\curlyvee$.

Question 4. [17] *Apart from the t-norms, does there exist other binary operation '$\star$' on I such that '$\curlywedge$' and '$\curlyvee$' are, respectively, a t_r-norm and a t_r-conorm on $\mathbf{L}$?*

This paper first gives a negative answer to Question 4, proving that, if a binary operation $\star$ ensures that $\curlywedge$ is a t_r-norm on $\mathbf{L}$ or $\curlyvee$ is a t_r-conorm on $\mathbf{L}$, then $\star$ is a t-norm, i.e., $\star$ satisfies axioms (T1)–(T4). Then, it is proved that the following are equivalent:

(1) $\star$ is a t-norm on I;
(2) $\curlywedge$ is a t_r-norm on $\mathbf{L}$;
(3) $\curlywedge$ is a t-norm on $\mathbf{L}$;
(4) $\curlyvee$ is a t_r-conorm on $\mathbf{L}$;
(5) $\curlyvee$ is a t-conorm on $\mathbf{L}$.

Finally, analogous results on $\triangle$ are presented when the binary operation $\star$ is restricted to be a continuous t-norm.

2. Preliminaries

A *type-1 fuzzy set* A in space X is a mapping from X to I, i.e., $A \in Map(X, I)$, and $A(x)$ is called the *degree of membership* of an element $x \in X$ to the set A. The two sets $\emptyset$ and X are special elements in $Map(X, I)$, with $\emptyset(x) \equiv 0$ and $X(x) \equiv 1$, respectively. A fuzzy set $A \in Map(X, I)$ is *normal* if $\sup\{A(x) : x \in I\} = 1$.

Definition 5. [17] A function $f \in \mathbf{M}$ is *convex* if, for any $x \leq y \leq z$, it holds that $f(y) \geq f(x) \wedge f(z)$.

Definition 6. [30] A *type-2 fuzzy set* A in space X is a mapping

$$A : X \to \mathbf{M},$$

i.e., $A \in Map(X, \mathbf{M})$. For any $x \in X$, $A(x)$ is also called the *degree of membership* of an element $x \in X$ to the set A.

Definition 7. [30] The operations of $\sqcup$ (union), $\sqcap$ (intersection), $\neg$ (complementation) on $\mathbf{M}$ are defined as follows: for any $f, g \in \mathbf{M}$,

$$(f \sqcup g)(x) = \sup\{f(y) \wedge g(z) : y \vee z = x\},$$

$$(f \sqcap g)(x) = \sup\{f(y) \wedge g(z) : y \wedge z = x\},$$

and

$$(\neg f)(x) = \sup\{f(y) : 1 - y = x\} = f(1 - x).$$

From [30], it follows that $\mathbb{M} = (\mathbf{M}, \sqcup, \sqcap, \neg, \chi_{\{0\}}, \chi_{\{1\}})$ does not have a lattice structure, although $\sqcup$ and $\sqcap$ satisfy the De Morgan's laws with respect to the given operation $\neg$.

Walker and Walker [30] introduced the following partial order on $\mathbf{M}$.

Definition 8. [30] $f \sqsubseteq g$ if $f \sqcap g = f$; $f \preceq g$ if $f \sqcup g = g$.

It follows from [30, Proposition 14] that both $\sqsubseteq$ and $\preceq$ are partial orders on $\mathbf{M}$. In [22, 23, 30], it was proved that the subalgebra $\mathbb{L} = (\mathbf{L}, \sqcup, \sqcap, \neg, \chi_{\{0\}}, \chi_{\{1\}})$ is a bounded complete lattice. In particular, $\chi_{\{0\}}$ and $\chi_{\{1\}}$ are the minimum and maximum, respectively.

For $f \in \mathbf{M}$, define f^L and f^R in $\mathbf{M}$ by

$$f^L(x) = \sup\{f(y) : y \le x\},$$

and

$$f^R(x) = \sup\{f(y) : y \ge x\}.$$

Clearly, f^L and f^R are monotonically increasing and decreasing, respectively. The following properties of f^L and f^R are obtained by Walker *et al.* ([22, 23, 30]).

Proposition 9. [30] For $f, g \in \mathbf{M}$,

(1) $f \sqsubseteq g$ if and only if $f^R \wedge g \le f \le g^R$;
(2) $f \preceq g$ if and only if $f \wedge g^L \le g \le f^L$;
(3) $f \le f^L$, $f \le f^R$;
(4) $(f^L)^L = f^L$, $(f^R)^R = f^R$;
(5) $(f^L)^R = (f^R)^L = \sup f$;

Theorem 10. *([22, 23]) Let $f, g \in \mathbf{L}$. Then, $f \sqsubseteq g$ if and only if $g^L \le f^L$ and $f^R \le g^R$.*

Lemma 11. *For $f \in \mathbf{M}$, $f^L(0) = f(0)$ and $f^R(1) = f(1)$.*

Proof. From the definitions of f^L and f^R, this holds trivially. $\square$

3. Answer to the Open Problem

3.1. Commutativity and Associativity of $\star$.

Lemma 12. *Let $\star$ be a t-norm on I. Then, $x \star y = 1$ if and only if $x = y = 1$.*

Lemma 13. *Let $\triangle$ be a continuous t-norm on I and $\star$ be a binary operation on I. Then,*

$$(f \curlywedge g)(1) = f(1) \star g(1).$$

Proof. Since $\triangle$ is a t-norm, from Lemma 12, it follows that

$$(f \curlywedge g)(1) = \sup\{f(y) \star g(z) : y \triangle z = 1\} = f(1) \star g(1).$$

$\square$

Proposition 14. Let $\triangle$ be a continuous t-norm on I and $\star$ be a binary operation on I. Then,

(1) $\curlywedge$ and $\curlyvee$ are commutative on $\mathbf{L}$ if and only if $\star$ is commutative;
(2) If $\curlywedge$ and $\curlyvee$ are associative on $\mathbf{L}$, then $\star$ is associative.

Proof. (1) The sufficiency follows from the proof of [17, Proposition 1]. It remains to prove the necessity. Suppose on the contrary that $\star$ is not commutative. Then, there exist $u, v \in I$ such that $u \star v \neq v \star u$. Choose two functions $f, g \in \mathbf{M}$ as

$$f(x) = (u - 1)x + 1,$$

and

$$g(x) = (v - 1)x + 1,$$

for $x \in I$. It can be verified that $f, g \in \mathbf{L}$, as f and g are decreasing. Since $\curlywedge$ is commutative, applying Lemma 13 yields that

$$u \star v = f(1) \star g(1) = (f \curlywedge g)(1)$$
$$= (g \curlywedge f)(1) = g(1) \star f(1) = v \star u,$$

which is a contradiction. Therefore, $\star$ is commutative.

(2) Suppose on the contrary that $\star$ is not associative. Then, there exist $u, v, w \in I$ such that $u \star (v \star w) \neq (u \star v) \star w$. Choose three functions $f, g, h \in \mathbf{M}$ as

$$f(x) = (u - 1)x + 1,$$

$$g(x) = (v - 1)x + 1,$$

and

$$h(x) = (w - 1)x + 1,$$

for $x \in I$. It can be verified that $u, v, w \in \mathbf{L}$, as f, g, and h are decreasing. Since $\curlywedge$ is associative, applying Lemma 13 yields that

$$u \star (v \star w) = f(1) \star (g \curlywedge h)(1) = (f \curlywedge (g \curlywedge h))(1)$$
$$= ((f \curlywedge g) \curlywedge h)(1) = (f \curlywedge g)(1) \star h(1) = (u \star v) \star w,$$

which is a contradiction. Therefore, $\star$ is associative.

$\square$

FIGURE 1. An illustration diagram of the functions f, g, h.

Remark 15. Similar results to Proposition 14 are obtained by Hernández *et al.* [17] under the assumption that $\curlywedge$ and $\curlyvee$ are commutative or associative on $\mathbf{M}$, which is stronger than the condition in Proposition 14.

3.2. **Neutral Element** 1 **for** $\star$. For any fixed $x \in I$, define $\mathscr{W}_x : I \to I$ by

$$\mathscr{W}_x(t) = \begin{cases} 0, & t \in [0, x), \\ t, & t \in [x, 1], \end{cases}$$

for $t \in I$. It can be verified that $\mathscr{W}_x \in \mathbf{L}$, as $\mathscr{W}_x$ is increasing for $x \in I$.

FIGURE 2. An illustration diagram of the function $\mathscr{W}_x(t)$.

Lemma 16. *Let $\triangle$ be a continuous t-norm on I and $\star$ be a binary operation on I. If $\curlywedge$ is a t-norm on $\mathbf{L}$, then $0 \star x = x \star 0 = 0$ for all $x \in I$.*

Proof. (1) As $\chi_{\{1\}}$ is a neural element, by Lemma 13, one has

$$0 = \chi_{\{0\}}(1) = (\chi_{\{1\}} \curlywedge \chi_{\{0\}})(1)$$
$$= \chi_{\{1\}}(1) \star \chi_{\{0\}}(1) = 1 \star 0.$$

(2) Fix any $x \in (0, 1)$. From $\mathscr{W}_x(t) = (\mathscr{W}_x \curlywedge \chi_{\{1\}})(t) = \sup\{\mathscr{W}_x(y) \star \chi_{\{1\}}(z) : y \triangle z = t\}$, it follows that, for any $t \in (0, x)$,

$$0 = \mathscr{W}_x(t) = \sup\{\mathscr{W}_x(y) \star \chi_{\{1\}}(z) : y \triangle z = t\}. \tag{3.1}$$

Since $\triangle(x, -)$ is continuous on $[0, 1]$, and $\triangle(x, 0) = 0$, $\triangle(x, 1) = x$, it follows from the intermediate value theorem that there exists some $z_1 \in (0, 1)$ such that $\triangle(x, z_1) = x \triangle z_1 = t$. This, together with (3.1), implies that

$$0 \geq \mathscr{W}_x(x) \star \chi_{\{1\}}(z_1) = x \star 0, \text{ i.e., } x \star 0 = 0.$$

(3) Note that $0 = \chi_{\{0\}}(\frac{1}{4}) = (\chi_{\{0\}} \curlywedge \chi_{\{1\}})(\frac{1}{4}) = \sup\{\chi_{\{0\}}(y) \star \chi_{\{1\}}(z) : y \triangle z = \frac{1}{4}\}$. Similarly to the proof of (2), it follows that there exists $y \in (0, 1)$ such that $y \triangle \frac{1}{2} = \frac{1}{4}$. This implies that

$$0 \geq \chi_{\{0\}}(y) \star \chi_{\{1\}}(\frac{1}{2}) = 0 \star 0, \text{ i.e., } 0 = 0 \star 0.$$

Summing up (1)–(3) and the commutativity of $\star$ (Proposition 14), it follows that, for any $x \in [0, 1]$,

$$x \star 0 = 0 \star x = 0.$$

$\square$

Lemma 17. *Let $\triangle$ be a continuous t-norm on I and $\star$ be a binary operation on I. If $\curlywedge$ is a t-norm on $\mathbf{L}$, then $1 \star x = x \star 1 = x$ for all $x \in I$.*

Proof. (1) Since $\chi_{\{1\}}$ is a neural element, from Lemma 13, it follows that

$$\begin{aligned}
1 = \chi_{\{1\}}(1) &= (\chi_{\{1\}} \curlywedge \chi_{\{1\}})(1) \\
&= \chi_{\{1\}}(1) \star \chi_{\{1\}}(1) = 1 \star 1.
\end{aligned}$$

(2) For any fixed $x \in (0, 1)$, $x = \mathscr{W}_x(x) = (\mathscr{W}_x \curlywedge \chi_{\{1\}})(x) = \sup\{\mathscr{W}_x(y) \star \chi_{\{1\}}(z) : y \triangle z = x\}$. For $y, z \in I$ with $y \triangle z = x$, consider the following two cases:

Case 1. If $z = 1$, then $y = x$. This implies that $\mathscr{W}_x(y) \star \chi_{\{1\}}(z) = x \star 1$;

Case 2. If $z < 1$, then $\chi_{\{1\}}(z) = 0$. Applying Lemma 16 gives that

$$\mathscr{W}_x(y) \star \chi_{\{1\}}(z) = 0.$$

Thus,

$$x = \sup\{\mathscr{W}_x(y) \star \chi_{\{1\}}(z) : y \triangle z = x\} = x \star 1.$$

The proof is completed by applying $0 \star 1 = 0$ and the commutativity of $\star$. $\square$

3.3. **Increasing in Each Argument for $\star$.** For any fixed $x \in I$, define $\mathscr{V}_x : I \to I$ by

$$\mathscr{V}_x(t) = (x - 1)t + 1, \quad \forall t \in I.$$

It can be verified that $\mathscr{V}_x \in \mathbf{L}$, as $\mathscr{V}_x$ is decreasing for $x \in I$. Clearly, functions f, g, and h constructed in Proposition 14 satisfy that $f = \mathscr{V}_u$, $g = \mathscr{V}_v$, and $h = \mathscr{V}_w$.

FIGURE 3. An illustration diagram of the function $\mathscr{V}_x(t)$.

Applying the decreasing property of $\mathscr{V}_x$ immediately yields the following result.

Lemma 18. *For any $x \in I$, $\mathscr{V}_x^L \equiv 1$ and $\mathscr{V}_x^R = \mathscr{V}_x$.*

Lemma 19. *For any $x_1, x_2 \in I$ with $x_1 \leq x_2$, $\mathscr{V}_{x_1} \sqsubseteq \mathscr{V}_{x_2}$.*

Proof. Clearly, $\mathscr{V}_{x_1} \leq \mathscr{V}_{x_2}$. Applying Lemma 18 yields that

$$\mathscr{V}_{x_2}^L \leq \mathscr{V}_{x_1}^L,$$

and

$$\mathscr{V}_{x_1}^R \leq \mathscr{V}_{x_2}^R.$$

This, together with Theorem 10, implies that

$$\mathscr{V}_{x_1} \sqsubseteq \mathscr{V}_{x_2}.$$

$\square$

FIGURE 4. An illustration diagram of the function $\mathscr{V}_{x_1}(t)$ and $\mathscr{V}_{x_2}(t)$.

Lemma 20. *Let $\triangle$ be a continuous t-norm on I and $\star$ be a binary operation on I. If $\curlywedge$ is a t-norm on $\mathbf{L}$, then for any $y \in (0,1)$ the functions $\star_y^r$ and $\star_y^l$ are increasing, where $\star_y^r(x) = x \star y$ and $\star_y^l(x) = y \star x$ for $x \in I$.*

Proof. It follows from Proposition 14 that $\star_y^r = \star_y^l$. So, it suffices to prove that $\star_y^r$ is increasing.

For any $0 \leq x_1 \leq x_2 \leq 1$, since $\curlywedge$ is increasing in each argument, from Lemma 19, it follows that

$$\mathscr{V}_{x_1} \curlywedge \mathscr{V}_y \sqsubseteq \mathscr{V}_{x_2} \curlywedge \mathscr{V}_y.$$

In particular, by Theorem 10,

$$(\mathscr{V}_{x_1} \curlywedge \mathscr{V}_y)^R \leq (\mathscr{V}_{x_2} \curlywedge \mathscr{V}_y)^R.$$

This, together with Lemmas 11 and 13, implies that

$$\begin{aligned}
x_1 \star y = \mathscr{V}_{x_1}(1) \star \mathscr{V}_y(1) &= (\mathscr{V}_{x_1} \curlywedge \mathscr{V}_y)(1) \\
&= (\mathscr{V}_{x_1} \curlywedge \mathscr{V}_y)^R(1) \leq (\mathscr{V}_{x_2} \curlywedge \mathscr{V}_y)^R(1) \\
&= (\mathscr{V}_{x_2} \curlywedge \mathscr{V}_y)(1) = \mathscr{V}_{x_2}(1) \star \mathscr{V}_y(1) \\
&= x_2 \star y,
\end{aligned}$$

i.e.,

$$\star_y^r(x_1) = x_1 \star y \leq x_2 \star y = \star_y^r(x_2).$$

Therefore, $\star_y^r$ is increasing. $\qquad\square$

3.4. **Answer to Question 4.**

Theorem 21. *Let $\triangle$ be a continuous t-norm on I and $\star$ be a binary operation on I. If $\curlywedge$ is a t-norm on $\mathbf{L}$, then $\star$ is a t-norm.*

Proof. It follows directly from Proposition 14, and Lemmas 17 and 20. $\qquad\square$

Similarly, the following result can be verified.

Theorem 22. *Let $\triangledown$ be a continuous t-conorm on I and $\star$ be a binary operation on I. If $\curlyvee$ is a t-conorm on $\mathbf{L}$, then $\star$ is a t-norm.*

Remark 23. Theorems 21 and 22 show that a binary operation $\star$ on I, ensuring that $\curlywedge$ is a t-norm (thus a t_r-norm) on $\mathbf{L}$ or $\curlyvee$ is a t-conorm (thus a t_r-conorm) on $\mathbf{L}$, must be a t-norm. This give a negative answer to Question 4.

Combining together Theorems 21, 22, and [17, Proposition 14], one obtains the following.

Theorem 24. *Let $\triangle$ be a continuous t-norm, $\triangledown$ be a continuous t-conorm, and $\star$ be a continuous binary operation on I. Then, the following statements are equivalent:*
(1) $\star$ is a t-norm on I;
(2) $\curlywedge$ is a t_r-norm on $\mathbf{L}$;
(3) $\curlywedge$ is a t-norm on $\mathbf{L}$;
(4) $\curlyvee$ is a t_r-conorm on $\mathbf{L}$;
(5) $\curlyvee$ is a t-conorm on $\mathbf{L}$.

4. Further Study on the Binary Operation $\curlywedge$

Let $\star$ be a continuous t-norm on I and $\triangle$ be a surjective binary operation on I. Define the binary operation $\curlywedge : \mathbf{M}^2 \to \mathbf{M}$ as follows: for $f, g \in \mathbf{M}$,

$$(f \curlywedge g)(x) = \sup\{f(y) \star g(z) : y \triangle z = x\}. \tag{4.1}$$

Here, the surjection assumption on $\triangle$ is necessary, because $(f \curlywedge g)(x)$ is not well defined for every point x in $I \backslash \triangle (I^2)$, if $\triangle$ is not surjective. Motivated by Question 4, a fundamental question is: *Apart from the t-norms, does there exist other binary operation '$\triangle$' on I such that '$\curlywedge$' is a t_r-norm on $\mathbf{L}$?*

This section will also give a negative answer to this question.

Lemma 25. *For $x_1, x_2 \in I$, $\chi_{\{x_1\}} \sqsubseteq \chi_{\{x_2\}}$ if and only if $x_1 \leq x_2$.*

Proof. Firstly, it can be verified that, for any $x \in I$,

$$\chi_{\{x\}}^L(t) = \begin{cases} 0, & t \in [0, x), \\ 1, & t \in [x, 1], \end{cases}$$

and

$$\chi_{\{x\}}^R(t) = \begin{cases} 1, & t \in [0, x], \\ 0, & t \in (x, 1]. \end{cases}$$

This, together with Theorem 10, implies that

$$\begin{aligned} & \chi_{\{x_1\}} \sqsubseteq \chi_{\{x_2\}} \\ \Leftrightarrow \quad & \chi_{\{x_2\}}^L \leq \chi_{\{x_1\}}^L \text{ and } \chi_{\{x_1\}}^R \leq \chi_{\{x_2\}}^R \\ \Leftrightarrow \quad & x_1 \leq x_2. \end{aligned}$$

$\square$

Lemma 26. *Let $\star$ be a continuous t-norm on I and $\triangle$ be a binary operation on I. Then, for any $x_1, x_2 \in I$, $\chi_{\{x_1\}} \curlywedge \chi_{\{x_2\}} = \chi_{\{x_1 \triangle x_2\}}$.*

Proof. Since $\star$ is a continuous t-norm, applying Lemma 12 gives
(a) for $y, z \in I$, $\chi_{\{x_1\}}(y) \star \chi_{\{x_2\}}(z) \in \{0, 1\}$;
(b) $\chi_{\{x_1\}}(y) \star \chi_{\{x_2\}}(z) = 1$ if and only if $y = x_1$ and $z = x_2$.
This, together with $(\chi_{\{x_1\}} \curlywedge \chi_{\{x_2\}})(x) = \sup\{\chi_{\{x_1\}}(y) \star \chi_{\{x_2\}}(z) : y \triangle z = x\}$, implies that

$$\chi_{\{x_1\}} \curlywedge \chi_{\{x_2\}} = \chi_{\{x_1 \triangle x_2\}}.$$

$\square$

Lemma 27. *Let $\star$ be a continuous t-norm on I and $\triangle$ be a binary operation on I. Then,*
(1) $\curlywedge$ is commutative on $\mathbf{L}$ if and only if $\triangle$ is commutative;
(2) If $\curlywedge$ is associative on $\mathbf{L}$, then $\triangle$ is associative.

Proof. (1) The sufficiency holds trivially. It remains to check the necessity.
For $x, y \in I$, since $\curlywedge$ is commutative, it follows from Lemma 26 that

$$\chi_{\{x \triangle y\}} = \chi_{\{x\}} \curlywedge \chi_{\{y\}} = \chi_{\{y\}} \curlywedge \chi_{\{x\}} = \chi_{\{y \triangle x\}},$$

implying that

$$x \triangle y = y \triangle x.$$

Thus, $\triangle$ is commutative.

(2) For $x, y, z \in I$, since $\curlywedge$ is associative, it follows from Lemma 26 that

$$\chi_{(x\triangle y)\triangle z} = \chi_{\{x\triangle y\}} \curlywedge \chi_{\{z\}} = \left(\chi_{\{x\}} \curlywedge \chi_{\{y\}}\right) \curlywedge \chi_{\{z\}}$$
$$= \chi_{\{x\}} \curlywedge \left(\chi_{\{y\}} \curlywedge \chi_{\{z\}}\right) = \chi_{\{x\}} \curlywedge \chi_{\{y\triangle z\}}$$
$$= \chi_{x\triangle(y\triangle z)},$$

implying that

$$(x \triangle y) \triangle x = x \triangle (y \triangle z).$$

Thus, $\triangle$ is associative. $\qquad\square$

Lemma 28. *Let $\star$ be a continuous t-norm on I and $\triangle$ be a binary operation on I. If $\curlywedge$ is a t-norm on $\mathbf{L}$, then $1 \triangle x = x \triangle 1 = x$ for all $x \in I$.*

Proof. For any $x \in I$, since $\chi_{\{1\}}$ is an neutral element, applying Lemma 26 yields that

$$\chi_{\{1\triangle x\}} = \chi_{\{1\}} \curlywedge \chi_{\{x\}} = \chi_{\{x\}} = \chi_{\{x\}} \curlywedge \chi_{\{1\}} = \chi_{\{x\triangle 1\}}.$$

Thus, $1 \triangle x = x = x \triangle 1$. $\qquad\square$

Lemma 29. *Let $\star$ be a continuous t-norm on I and $\triangle$ be a binary operation on I. If $\curlywedge$ is a t-norm on $\mathbf{L}$, then, for any $y \in (0,1)$, the functions $\triangle_y^r$ and $\triangle_y^l$ is increasing, where $\triangle_y^r(x) = x \triangle y$ and $\triangle_y^l(x) = y \triangle x$ for any $x \in I$.*

Proof. It follows from Lemma 27 that $\triangle_y^r = \triangle_y^l$. So, it suffices to prove that $\triangle_y^r$ is increasing.

For $0 \le x_1 \le x_2 \le 1$, applying Lemma 25 follows that

$$\chi_{\{x_1\}} \sqsubseteq \chi_{\{x_2\}}.$$

Since $\curlywedge$ is increasing in each argument, applying Lemma 26 yields that

$$\chi_{\{x_1\triangle y\}} = \chi_{\{x_1\}} \curlywedge \chi_{\{y\}} \sqsubseteq \chi_{\{x_2\}} \curlywedge \chi_{\{y\}} = \chi_{\{x_2\triangle y\}}.$$

This, together with Lemma 25, implies that

$$\triangle_y^R(x_1) = x_1 \triangle y \le x_2 \triangle y = \triangle_y^R(x_2).$$

Therefore, $\triangle_y^R$ is increasing. $\qquad\square$

Combining together Lemmas 27–29 and [17, Proposition 14] immediately yields the following result.

Theorem 30. *Let $\star$ be a continuous t-norm on I and $\triangle$ be a continuous binary operation on I. Then, the following statements are equivalent:*

(1) $\triangle$ is a t-norm on I;
(2) $\curlywedge$ is a t_r-norm on $\mathbf{L}$;
(3) $\curlywedge$ is a t-norm on $\mathbf{L}$.

Similarly, one can obtain the following result.

Theorem 31. *Let $\star$ be a continuous t-norm on I and $\triangle$ be a continuous binary operation on I. Then, the following statements are equivalent:*

(1) $\triangle$ is a t-conorm on I;
(2) $\curlywedge$ is a t_r-conorm on $\mathbf{L}$;
(3) $\curlywedge$ is a t-conorm on $\mathbf{L}$.

5. Conclusion

This paper has further studied the binary operations $\curlywedge$ and $\curlyvee$ defined in (1.1), (1.2) and (4.1) on $\mathbf{L}$. By introducing two special families of functions $\mathscr{W}_x$ and $\mathscr{V}_x$ ($x \in I$), it first proves that, when the continuous t-norm $\triangle$ or continuous t-conorm $\triangledown$ is fixed, the following hold:

(1) $\curlywedge$ is a continuous t_r-norm on $\mathbf{L}$ if and only if $\curlywedge$ is a continuous t-norm on $\mathbf{L}$ if and only if $\star$ is a continuous t-norm;

(2) $\curlyvee$ is a continuous t_r-conorm on $\mathbf{L}$ if and only if $\curlyvee$ is a continuous t-conorm on $\mathbf{L}$ if and only if $\star$ is a continuous t-norm.

In particular, these results negatively answer Question 4. Similarly to Question 4, the case that the binary operation $\triangle$ is fixed (see (4.1)) has been investigated and some analogous results were obtained.

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