Semi-overlap Functions and Novel Fuzzy Reasoning Algorithms with Applications

Xiaohong Zhang^a, Mei Wang^{b,*}, Benjamín Bedregal^c, Mengyuan Li^a, Rong Liang^a

^aSchool of Mathematics and Data Science, Shaanxi University of Science and Technology, Xi'an 710021, Shaanxi, China

Abstract

It is worth noticing that a fuzzy conjunction and its corresponding fuzzy implication can form a residual pair if and only if it is left-continuous. In order to get a more general result related on residual implications that induced by aggregation functions, we relax the definition of general overlap functions, more precisely, removing its right-continuous, and then introduce a new kind of aggregation functions, which called semi-overlap functions. Subsequently, we study some of their related algebraic properties and investigate their corresponding residual implications. Moreover, serval scholars have provided kinds of methods for fuzzy modus ponens (FMP, for short) problems so far, such as Zadeh's compositional rule of inference (CRI, for short), Wang's triple I method (TIM, for short) and the quintuple implication principle (QIP, for short). Compared with CRI and TIM methods, QIP method has some advantages in solving FMP problems, in this paper, we further consider the QIP method for FMP problems and prove that it satisfies the reducibility of multiple-rules fuzzy reasoning. Finally, we propose a new classification algorithm that based on semi-overlap functions and QIP method, which called SO5I-FRC algorithm. Through the comparative tests, the average accuracy of SO5I-FRC algorithm is higher than FARC-HD algorithm. The experimental results indicate that semi-overlap functions and QIP method have certain advantages and a wide range of applications in classification problems.

Keywords: overlap function; semi-overlap function; fuzzy reasoning; quintuple implication principle; classification algorithm

1. Introduction

Overlap functions were introduced in [2] by Bustince, which is a special class of aggregation functions. This concept arise from some problems in information aggregation process, such as image processing [21], decision making [4], community detection problems [14] and classification [10, 11]. As a generalization of overlap functions, Miguel et

Email addresses: zhangxiaohong@sust.edu.cn(X.H. Zhang), wangmeimath@163.com(M. Wang), bedregal@dimap.ufrn.br (B. Bedregal), 201706060120@sust.edu.cn(M.Y. Li), 1137882903@qq.com(R. Liang).

Preprint submitted to . March 2, 2022

^bSchool of Electrical and Control Engineering, Shaanxi University of Science and Technology, Xi'an 710021, Shaanxi, China

^cDepartamento de Informática e Matemática Aplicada, Universidade Federal do Rio Grande do Norte, Campus Universitário s/n, 59072-970 Natal, Brazil

^{*}Corresponding author.

al. [8] introduced the concept of general overlap functions. Notice that the difference between binary general overlap functions and overlap functions is boundary condition [15]. In the definition of overlap functions, the boundary condition is a necessary and sufficient condition, while in the binary general overlap functions, the boundary condition is a sufficient condition. The authors in [8] made an experiment and showed that the general overlap function behave better when calculating the matching degree in some classification problems. There is a close relation between fuzzy conjunction, which is a generalization form of aggregate functions, and fuzzy implication. Fuzzy implication has been widely studied in theory and practice [6]. The most studied generalization is obtained by t-norm, called residual implication. However, Bustince et.al [2, 4] and Fodor et.al [13] have pointed out that the associativity of the t-norm is unnecessary in many applications, such as decision making and classification problems. Therefore, it is necessary to construct the residual implication corresponding to the aggregation function that does not meet the law of association. On the other hand, a fuzzy conjunction and its corresponding fuzzy implication can be form a residual pair if and only if it is left-continuous. In order to get a more general result related on residual implications that induced by aggregation functions, we relax the definition of general overlap functions, more precisely, removing its right-continuous, and then introduce a new kind of aggregation functions, which called semi-overlap functions in the present paper.

The most fundamental inference models of fuzzy reasoning are fuzzy modus ponens(FMP) and fuzzy modus tollens(FMT), which can be respectively expressed as follows:

FMP: for given $A \rightarrow B(\text{rule})$ and $A^*(\text{input})$, calculate $B^*(\text{output})$

FMT: for given $A \rightarrow B(\text{rule})$ and $B^*(\text{input})$, calculate $A^*(\text{output})$

where $A, A^* \in \mathcal{F}(U)$ (the set of all fuzzy subsets on U) and $B, B^* \in \mathcal{F}(V)$ (the set of all fuzzy subsets on V). In order to solve FMP problems, Zadeh [33] proposed compositional rule of inference (CRI, for short). Although this method is simple in calculation, it lacks clear logic sense. Therefore, the basic problem of fuzzy reasoning has attracted great attention and a series of meaningful achievements have been made. Among them, G.J. Wang [29] proposed the full implication triple I method of fuzzy reasoning (TIM, for short). This method effectively improved the CRI method. However, in CRI and TIM methods, the closeness of A and A^* (or B and B^*) are not explicitly used in the process of calculating the consequence, which sometimes makes the computed approximation useless or misleading. In order to characterized the approximation A^* and A (or B and B^*) in the process of fuzzy reasoning, B.K. Zhou et al [31] proposed the quintuple implication principle(QIP, for short). This method efficiently improves CRI and TIM methods. Subsequently, M.X. Luo et al [24] investigate QIP method based on interval-valued fuzzy inference for FMP and FMT problems. In the present paper, we will continue to discuss the QIP method based on semi-overlap functions and their residual implications.

Fuzzy Rule-Based Classification Systems(FRBCSs) [19] are one of the most popular methods in pattern recognition and machine learning. FRBCSs is also used in many practical applications, such as image processing [25] and anomaly intrusion detection [28]. In order to improve the interpretability of classification rules and avoid unnatural boundaries in attribute division, researchers have made different studies on classification systems based on fuzzy association rules [7, 16, 17, 18, 26, 30]. In particular, J. Alcalá-Fdez [1] propose a fuzzy association rule-based classification method for high-dimensional problems (FARC-HD) to obtain an accurate and compact fuzzy rule-based classifier with a low computational

cost. And in [12], based on FARC-HD algorithm, *n*-dimensional overlap functions are used to solve the multi-class classification problems, and good results are obtained. In this paper, based on semi-overlap functions and quintuple implication principle, we propose a new classification algorithm, which is called SO5I-FRC algorithm. And the experimental results show that the average accuracy of classification using SO5I-FRC algorithm is higher than that of FARC-HD algorithm.

The paper is organized as follows. In Section 2, we recall some preliminary concepts related to this paper. In Section 3, we introduce the concept of semi-overlap functions and investigate the residual implication induced by semi-overlap functions. In Section 4, we study the quintuple implication principle and present an algorithm that satisfies the reducibility of multiple-rules fuzzy reasoning. In Section 5, we apply semi-overlap functions and quintuple implication principle to propose a new algorithm, which called SO5I-FRC algorithm. Through comparative experiments, we conclude that the average accuracy of SO5I-FRC algorithm is higher than FARC-HD algorithm.

2. Preliminaries

In this part, we will recall some basic definitions and results, which will be used in the following sections.

2.1. Aggregation functions

Definition 2.1. ([2]) A function $A: [0,1]^2 \to [0,1]$ is said to be an aggregation function if it satisfies the following conditions:

- (A1) A is increasing in each variable;
- (A2) A(0,0) = 0 and A(1,1) = 1.

Definition 2.2. ([2]) A bivariate function $T: [0,1]^2 \to [0,1]$ is said to be a t-norm if it satisfies the following conditions:

```
(T1) T(u, v) = T(v, u);
```

- (T2) T(u, T(v, w)) = T(T(u, v), w);
- (T3) T is increasing;
- (T4) T(u, 1) = u, for any $u \in [0, 1]$.

Moreover, a t-norm *T* is called

- (T5) continuous if it is continuous in both arguments at the same time.
- (T6) positive if T(u, v) = 0, then either u = 0 or v = 0.

Definition 2.3. ([23]) A t-norm T is left-continuous if, for any $u \in [0, 1]$ and $\{v_i | i \in I\} \subseteq [0, 1]$, for a non-empty set of index I, it satisfies:

$$T(u, \sup\{v_i|i \in I\}) = \sup\{T(u, v_i)|i \in I\}.$$

Definition 2.4. ([2, 3]) A mapping $O : [0, 1]^2 \to [0, 1]$ is called an overlap function if it satisfies the following conditions: for any $u, v \in [0, 1]$,

- (O1) O(u, v) = O(v, u);
- (*O*2) O(u, v) = 0 if and only if uv = 0;
- (*O*3) O(u, v) = 1 if and only if uv = 1;
- (O4) O is increasing;
- (O5) O is continuous.

Definition 2.5. ([8]) A *n*-ary function $GO : [0,1]^n \to [0,1]$ is called a general overlap function if the following conditions hold: for all $u_1, u_2, \dots, u_n \in [0,1]$,

- (GO1) GO(u, v) = GO(v, u);
- (GO2) If $\prod_{i=1}^{n} u_i = 0$, then $GO(u_1, u_2, \dots, u_n) = 0$;
- (GO3) If $\prod_{i=1}^{n} u_i = 1$, then $GO(u_1, u_2, \dots, u_n) = 1$;
- (GO4) GO is increasing;
- (GO5) GO is continuous.

Definition 2.6. ([27]) A mapping $QO : [0,1]^2 \to [0,1]$ is called a quasi-overlap function if it satisfies the following conditions: for any $u, v \in [0,1]$,

- (QO1) QO(u, v) = QO(v, u);
- (QO2) QO(u, v) = 0 if and only if uv = 0;
- (QO3) QO(u, v) = 1if and only if uv = 1;
- (QO4) QO is increasing.

2.2. Fuzzy implication and fuzzy conjunction

Definition 2.7. ([22]) A function $C: [0,1]^2 \to [0,1]$ is called a fuzzy conjunction if the following conditions hold:

- (C1) C is increasing;
- (C2) C(1, 1) = 1;
- (C3) C(0,0) = C(0,1) = C(1,0) = 0.

It is easy to know that t-norms, overlap functions, quasi-overlap functions and binary general overlap functions are fuzzy conjunctions.

Definition 2.8. ([22, 32]) A function $I : [0, 1]^2 \to [0, 1]$ is a fuzzy implication if it satisfies the following conditions: for each $u, v, w \in [0, 1]$,

- (I1) If $u \le v$, then $I(v, w) \le I(u, w)$;
- (I2) If $v \le w$, then $I(u, v) \le I(u, w)$;
- (I3) I(0,0) = 1;
- (I4) I(1,1) = 1;
- (I5) I(1,0) = 0.

Let $C: [0,1]^2 \to [0,1]$ be a fuzzy conjunction. Then we can define the function $I_C: [0,1]^2 \to [0,1]$ as follows, for all $u, v \in [0,1]$,

$$I_C(u, v) = \sup\{w \in [0, 1] | C(u, w) \le v\}$$
 (1)

Theorem 2.9. ([22]) Let C be a fuzzy conjunction. Then the function $I_C : [0,1]^2 \to [0,1]$ given by Eq.(1) is a fuzzy implication if and only if C fulfils the condition:

$$C(1, v) > 0, \forall v \in (0, 1].$$
 (2)

If for any $u, v, w \in [0, 1]$, we have

$$C(u, w) \le v \Leftrightarrow I_C(u, v) \ge w.$$
 (**RP**)

Then we say that the fuzzy conjunction C and the function I_C defined in Eq.(1) satisfy the residuation property (**RP**).

Theorem 2.10. ([22]) Let $C : [0,1]^2 \to [0,1]$ be a fuzzy conjunction satisfying the condition (2), and $I_C : [0,1]^2 \to [0,1]$ be a fuzzy implication derived from C. Then the following statements are equivalent:

- (i) C is left-continuous with respect to the second variable;
- (ii) C and I_C satisfy the residuation property (**RP**);
- (iii) $I_C(u, v) = \max\{w \in [0, 1] | C(u, w) \le v\}$, for all $u, v \in [0, 1]$.

Definition 2.11. ([3, 32, 9]) A fuzzy implication $I: [0, 1]^2 \rightarrow [0, 1]$ satisfies:

- (**NP**) The left neutrality property $\Leftrightarrow \forall v \in [0, 1] : I(1, v) = v$.
- **(LOP)** The left ordering property $\Leftrightarrow \forall u, v \in [0, 1] : u \le v \Rightarrow I(u, v) = 1$.
- **(ROP)** The right ordering property $\Leftrightarrow \forall u, v \in [0, 1] : I(u, v) = 1 \Rightarrow u \leq v$.
- **(OP)** The ordering property $\Leftrightarrow \forall u, v \in [0, 1]: u \le v \Leftrightarrow I(u, v) = 1$.
- (**EP**) The exchange principle $\Leftrightarrow \forall u, v, w \in [0, 1] : I(u, I(v, w)) = I(v, I(u, w)).$
- (**IP**) The identity principle $\Leftrightarrow \forall u \in [0, 1]$: I(u, u) = 1.
- (**CB**) The consequent boundary $\Leftrightarrow \forall u, v \in [0, 1]: v \leq I(u, v)$.
- (SIB) The sub-iterative Boolean law $\Leftrightarrow \forall u, v \in [0, 1] : I(u, v) \leq I(u, I(u, v))$.
- **(IB)** The iterative Boolean law $\Leftrightarrow \forall u, v \in [0, 1]$: I(u, v) = I(u, I(u, v)).

3. Semi-overlap functions

3.1. Semi-overlap functions

In this part, we introduce the notion of semi-overlap functions and study some of their related algebraic properties.

Definition 3.1. A binary function $SO: [0,1]^2 \to [0,1]$ is called a semi-overlap function if the following conditions hold: for any $u, v \in [0,1]$,

- (S 1) S O(u, v) = S O(v, u);
- (S2) If uv = 0, then SO(u, v) = 0;
- (S3) If uv = 1, then SO(u, v) = 1;
- (S4) SO is increasing;
- (S5) SO is left-continuous.

Here, as shown in Figure 1, we discuss the relations between semi-overlap functions and other aggregation functions.

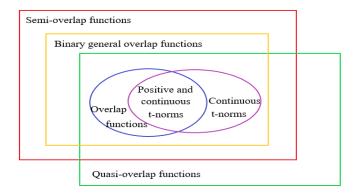


Figure 1: The included relation of different aggregation functions

- **Remark 3.2.** (1) Semi-overlap functions are bivariate general overlap functions which consider left continuity instead of continuity. So semi-overlap functions are generalization of bivariate general overlap functions.
- (2) The bivariate general overlap function is generalization of the overlap function. Hence, by (1), we can get that the semi-overlap function is also generalization of the overlap function.
- (3) There is no inclusive relations between semi-overlap functions and quasi-overlap functions. Since the latter does not require continuity, the former is left-continuous. On the other hand, quasi-overlap functions has no zero-divisor, but semi-overlap functions may have zero-divisor.

Then we will give some examples of semi-overlap functions.

Example 3.3. (1). For a fix $a \in (0, 1]$ and $u, v \in [0, 1]$, the function

$$SO(u,v) = \begin{cases} 0, & if \ u+v \le a; \\ u \wedge v, & if \ u+v > a. \end{cases}$$

is a semi-overlap function, but it is not a quasi-overlap function.

(2) The function, defined for any $u, v \in [0, 1]$ by

$$SO(u, v) = \begin{cases} 0, & if \ u + v \le 1; \\ uv, & otherwise. \end{cases}$$

is a semi-overlap function. However, it is not a binary general overlap function.

(3) The function $SO: [0,1]^2 \rightarrow [0,1]$ given by:

$$SO(u, v) = \min\{u, v\} \max\{u^2, v^2\}$$

is a non-associative semi-overlap function having 1 as neutral element.

(4) The function given by $SO_p(u, v) = u^p v^p$ with p > 0 and $p \ne 1$ is a semi-overlap function, but it is neither associative nor have 1 as neutral element.

Definition 3.4. A semi-overlap function $SO: [0,1]^2 \rightarrow [0,1]$ is called deflationary semi-overlap function if

 $(S6) \forall u \in [0,1]: SO(u,1) \leq u,$

and SO is called inflationary semi-overlap function if

 $(S7) \ \forall u \in [0,1]: SO(u,1) \ge u.$

Example 3.5. Let us consider the function given in Example 3.3(4). Then SO_p is deflationary when p > 1 and inflationary when 0 .

Theorem 3.6. The function $SO: [0,1]^2 \to [0,1]$ is a semi-overlap function if and only if there are two function $f,g: [0,1]^2 \to [0,1]$ such that

$$SO(u,v) = \frac{f(u,v)}{f(u,v)+g(u,v)},$$

and satisfies the following conditions:

- (1) $f(u, v) + g(u, v) \neq 0$ for all $u, v \in [0, 1]$.
- (2) f and g are commutative;
- (3) If uv = 0, then f(u, v) = 0;
- (4) If uv = 1, then g(u, v) = 0;
- (5) f is increasing and g is decreasing;
- (6) f is left-continuous and f + g is right-continuous;

PROOF. (\Rightarrow) Let SO be a semi-overlap function. Now, suppose that f(u, v) = SO(u, v) and g(u, v) = 1 - SO(u, v), then $f(u, v) + g(u, v) = 1 \neq 0$. Therefore, the function

$$SO(u, v) = \frac{f(u, v)}{f(u, v) + g(u, v)}$$

can be defined. And it is not hard to prove that the conditions (1) - (6) hold.

 (\Leftarrow) Suppose that $f, g : [0, 1]^2 \to [0, 1]$ satisfies the conditions (1) - (6), we will show that

$$SO(u, v) = \frac{f(u, v)}{f(u, v) + g(u, v)}$$

is a semi-overlap function. Obviously, it satisfies (S1), (S2) and (S3). Next, we will prove that it satisfies the conditions (S4) and (S5).

(S4) If $u_1 \le u_2$, since f is increasing and g is decreasing, then $f(u_1, v) \le f(u_2, v)$ and $g(u_2, v) \le g(u_1, v)$. Moreover, we can get

$$f(u_1, v)g(u_2, v) \le f(u_2, v)g(u_1, v)$$

and

$$f(u_1, v)f(u_2, v) + f(u_1, v)g(u_2, v) \le f(u_1, v)f(u_2, v) + f(u_2, v)g(u_1, v).$$

Hence

$$SO(u_{1}, v) = \frac{f(u_{1}, v)}{f(u_{1}, v) + g(u_{1}, v)}$$

$$\leq \frac{f(u_{2}, v)}{f(u_{2}, v) + g(u_{2}, v)}$$

$$= SO(u_{2}, v).$$

(S 5) Since f + g is right-continuous, we have $\frac{1}{f+g}$ is left-continuous. Then according to the left continuity of f, we can get $\frac{f}{f+g}$ is left-continuous.

Theorem 3.7. Let SO_1, SO_2, \dots, SO_m be semi-overlap functions and r_1, r_2, \dots, r_m be non-negative weights with $\sum_{j=1}^m r_j = 1$. Then $SO(u, v) = \sum_{j=1}^m r_j SO_j(u, v)$ is also a semi-overlap function.

PROOF. Conditions (S1) - (S4) are easy proved. Next, we prove that SO satisfies (S5). According to Definition 2.3, if SO is left-continuous, then for any $u \in [0, 1]$ and any $\{v_i | i \in I\} \subseteq [0, 1]$, it follows that:

$$SO(u, \sup\{v_i|i \in I\}) = \sup\{SO(u, v_i)|i \in I\}.$$

Hence, we have

$$SO(u, \sup_{i \in I} v_i) = \sum_{j=1}^{m} r_j SO_j(u, \sup_{i \in I} v_i)$$

$$= \sum_{j=1}^{m} r_j (\sup_{i \in I} SO_j(u, v_i))$$

$$= \sup_{i \in I} \sum_{j=1}^{m} r_j SO_j(u, v_i)$$

$$= \sup_{i \in I} SO(u, v_i).$$

3.2. Residual implications derived from semi-overlap functions

Let $SO: [0,1]^2 \to [0,1]$ be a semi-overlap functions. Then we defined $I_{SO}: [0,1]^2 \to [0,1]$ as follows:

$$I_{SO}(u, v) = \sup\{w \in [0, 1] | SO(u, w) \le v\}, \quad \text{for all } u, v \in [0, 1].$$

Lemma 3.8. If *SO* is an inflationary semi-overlap function, then it is a fuzzy conjunction that satisfies Eq.(2).

PROOF. Obviously, every semi-overlap function is a fuzzy conjunction. Now, if SO is inflationary, then by (S7), for any $v \in (0, 1]$, we can get $SO(1, v) \ge v > 0$.

From Theorem 2.9, Theorem 2.10 and Lemma 3.8, we can immediately draw the following conclusion.

Proposition 3.9. If SO is an inflationary semi-overlap function, then the function I_{SO} that induced by semi-overlap function is a residual implication. Moreover, SO and I_{SO} satisfies (**RP**).

In Table 1, we give some examples of semi-overlap functions and their residual implication.

Semi – overlap functions	Residual implications		
$(1) SO(u, v) = \min\{u^p, v^p\}$	$I_{SO}(u,v) = \begin{cases} 1, \\ \sqrt[q]{v}, \end{cases}$	$u^p \le v;$ $u^p > v.$	
(2) $SO(u, v) = \begin{cases} 0, & u + v \le a; a \in [0, 1] \\ min\{u, v\}, & u + v > a. \end{cases}$	$I_{SO}(u,v) = \begin{cases} 1, \\ max\{(a-u),v\}, \end{cases}$	$u \le v;$ $u > v.$	
(3) $SO(u, v) = \begin{cases} 0, & u + v \le 1; \\ uv, & otherwise. \end{cases}$	$I_{SO}(u,v) = \begin{cases} 1, \\ max\{(1-u), \frac{v}{u}\}, \end{cases}$	$u \le v$; otherwise.	
$(4) SO(u, v) = \begin{cases} 0, & u + v = 0; \\ \frac{2uv}{u+v}, & otherwise. \end{cases}$	$I_{SO}(u,v) = \begin{cases} 1, \\ max\{0, \frac{uv}{2u-v}\}, \end{cases}$	u = 0; otherwise.	
$(5) SO(u, v) = \max\{0, u + v - 1\}$	$I_{SO}(u,v) = \begin{cases} 1, \\ v - u + 1, \end{cases}$	$u \le v$; otherwise.	
(6) $SO(u, v) = \max\{\frac{u+v-1}{u+v-uv}, 0\}$	$I_{SO}(u,v) = \begin{cases} 1, \\ \frac{uv+1-u}{uv+1-v}, \end{cases}$	$u \le v$; otherwise.	
(7) SO(u, v) = uv	$I_{SO}(u,v) = \begin{cases} 1, \\ \frac{v}{u}, \end{cases}$	$u \le v;$ $u > v.$	
$(8) SO(u,v) = \frac{u+v}{2}$	$I_{SO}(u,v) = \begin{cases} 1, \\ 2v - u, \end{cases}$	$u \le 2v - 1;$ otherwise.	
$(9) SO(u, v) = \min\{u, v\}$	$I_{SO}(u,v) = \begin{cases} 1, \\ v, \end{cases}$	$u \le v$; otherwise.	

Table 1. Semi-overlap functions and their residual implications.

Proposition 3.10. Let $SO:[0,1]^2 \to [0,1]$ be a semi-overlap function and I_{SO} be a residual implication. Then it holds that:

- $(1) SO(u, v) = \min\{w \in [0, 1] | I_{SO}(u, w) \ge v\}.$
- (2) I_{SO} satisfies (**NP**) if and only if 1 is the neutral element of SO.

- (3) I_{SO} satisfies (**EP**) if and only if SO is associative.
- (4) I_{SO} satisfies (**IP**) if and only if SO satisfies (S6).
- (5) I_{SO} satisfies (**LOP**) if and only if SO satisfies (S6).
- (6) I_{SO} satisfies (**ROP**) if and only if SO satisfies (S7).
- (7) I_{SO} satisfies (**OP**) if and only if 1 is the neutral element of SO.
- (8) I_{SO} satisfies (**CB**) if and only if $SO(u, v) \le \min\{u, v\}$.
- (9) If I_{SO} satisfies (**CB**), then I_{SO} satisfies (**SIB**).
- (10) If $SO(u, v) = \min\{u, v\}$, then I_{SO} satisfies (**IB**).
- (11) If SO has 1 as neutral element, then I_{SO} satisfies (**CB**).

Proof. (1) Straightforward from Theorem 13 in [22].

(2) (\Rightarrow) Suppose that I_{SO} satisfies (**NP**), that is, for any $v \in [0, 1]$,

$$I_{SO}(1, \nu) = \max\{t \in [0, 1] | SO(1, t) \le \nu\} = \nu.$$
(3)

It follows that $SO(1, v) \le v$. Assume that there exists a $v_0 \in [0, 1]$ such that $SO(1, v_0) < v_0$. Then there also exists $w < v_0$ such that $SO(1, v_0) \le w$. According to (**RP**), we have $w < v_0 \le I_{SO}(1, w)$, which is contradiction to the Eq. (3). Hence SO(1, v) = v.

- (⇐) Suppose that for any $v \in [0, 1]$, SO(1, v) = v. Then we have $I_{SO}(1, v) = \max\{t \in [0, 1] | SO(1, t) \le v\} = \max\{t \in [0, 1] | t \le v\} = v$. Hence I_{SO} satisfies (**NP**).
 - (3) Let $u, v, w \in [0, 1]$.

$$(\Rightarrow) \ SO(u, SO(v, w)) = \min\{t \in [0, 1] | I_{SO}(u, t) \ge SO(v, w)\}$$
 (item (1))

$$= \min\{t \in [0, 1] | I_{SO}(w, I_{SO}(u, t)) \ge v\}$$
 (by (RP))

$$= \min\{t \in [0, 1] | I_{SO}(u, I_{SO}(w, t)) \ge v\}$$
 (by (EP))

$$= \min\{t \in [0, 1] | I_{SO}(w, t) \ge SO(u, v)\}$$
 (by (RP))

$$= SO(w, SO(u, v))$$
 (item (1))

$$= SO(SO(u, v), w)$$
 (by commutative).

$$(\Leftarrow) \ I_{SO}(u, I_{SO}(v, w)) = \max\{t \in [0, 1] | SO(u, t) \le I_{SO}(v, w)\}$$

$$= \max\{t \in [0, 1] | SO(v, SO(u, t)) \le w\} \quad (by \ (RP))$$

$$= \max\{t \in [0, 1] | SO(v, SO(v, t)) \le w\} \quad (by \ associative)$$

$$= \max\{t \in [0, 1] | SO(v, t) \le I_{SO}(u, w)\} \quad (by \ (RP))$$

$$= I_{SO}(v, I_{SO}(u, w)).$$

- (4) For each $u \in [0, 1]$, $I_{SO}(u, u) = \max\{t \in [0, 1] | SO(u, t) \le u\} = 1$ if and only if $SO(u, 1) \le u$.
- (5) (\Rightarrow) If $u \le v$, then $I_{SO}(u, v) = 1$. In particular, $I_{SO}(u, u) = 1$. Hence, by (4), we have SO satisfies (S6).
- (⇐) If $u \le v$, then by (S 6), we get $SO(u, 1) \le u \le v$. Thus, $I_{SO}(u, v) = \max\{w \in [0, 1] | SO(u, w) \le v\} = 1$.
- (6) (⇒) By item (1), we have $SO(u, 1) = \min\{w \in [0, 1] | I_{SO}(u, w) \ge 1\}$, then $I_{SO}(u, SO(u, 1)) = 1$. By (**ROP**), $I_{SO}(u, v) = 1 \Rightarrow u \le v$, hence, we can get $SO(u, 1) \ge u$.
 - (\Leftarrow) If $SO(u, 1) \ge u$ for each $u \in [0, 1]$, then

$$I_{SO}(u, v) = 1 \implies \max\{w \in [0, 1] | SO(u, w) \le v\} = 1$$

 $\implies SO(u, 1) \le v$
 $\implies u \le v.$

- (7) Obviously, it can be obtained from (5) and (6).
- (8)(⇒) Suppose that I_{SO} satisfies (**CB**). For any $u, v \in [0, 1]$, we have

$$v \le I_{SO}(u, v) \Rightarrow v \le \max\{w \in [0, 1] | SO(u, w) \le v\}$$

 $\Rightarrow v \in \max\{w \in [0, 1] | SO(u, w) \le v\}$
 $\Rightarrow SO(u, v) \le v.$

That is

$$v \le I_{SO}(u, v) \Rightarrow SO(u, v) \le v. \tag{4}$$

Analogously, we have

$$u \le I_{SO}(v, u) \Rightarrow u \le SO(v, u).$$
 (5)

Since SO is commutative, by Eqs.(4) and (5), we can conclude that

$$v \le I_{SO}(u, v) \Rightarrow SO(u, v) \le \min\{u, v\}.$$

 (\Leftarrow) For all $u, v \in [0, 1]$, it holds that

$$SO(u, v) \le \min\{u, v\} \implies SO(u, v) \le v$$

$$\Rightarrow v \in \{w \in [0, 1] | SO(u, w) \le v\}$$

$$\Rightarrow v \le \max\{w \in [0, 1] | SO(u, w) \le v\}$$

$$\Rightarrow v \le I_{SO}(u, v)$$

- (9) Straightforward from (I2).
- (10) For any $u, v \in [0, 1]$, if $SO(u, v) = \min\{u, v\}$, then it holds that

$$I_{SO}(u, v) = \max\{w \in [0, 1] | \min\{u, w\} \le v\} = \begin{cases} 1, & u \le v; \\ v, & u > v. \end{cases}$$

If $u \le v$, then $I_{SO}(u, I_{SO}(u, v)) = I_{SO}(u, 1) = 1 = I_{SO}(u, v)$. On the other hand, if u > v, then $I_{SO}(u, I_{SO}(u, v)) = I_{SO}(u, v)$.

(11) Assume that for any $v \in [0, 1]$, SO(1, v) = v. Since SO(0, v) is increasing, $SO(u, v) \le SO(1, v)$, that is $SO(u, v) \le v$. According to (**RP**), we have $v \le I_{SO}(u, v)$.

4. Quintuple implication principle

4.1. Quintuple implication principle of FMP

In this subsection, we will use the quintuple implication principle (QIP, for short), which induced by semi-overlap functions and their residual implications, to solve FMP problems. The FMP problems as follows:

rule
$$A \rightarrow B$$

input A^*
output B^* .

Definition 4.1. Let U and V be two non-empty sets, $A, A^* \in \mathcal{F}(U)$, $B \in \mathcal{F}(V)$. If B^* in FMP is the minimum fuzzy set of $\mathcal{F}(V)$ such that, for any $u \in U, v \in V$,

$$(A(u) \rightarrow B(v)) \rightarrow ((A^*(u) \rightarrow A(u)) \rightarrow (A^*(u) \rightarrow B^*(v)))$$

has the maximum value, i.e. 1, then B^* is called the FMP-solution of quintuple implication principle.

Theorem 4.2. Let the implication \rightarrow in FMP be a residual implication derived from the semi-overlap function \odot with 1 as neutral element. Then the FMP-solution B^* of quintuple implication principle is calculated as follows:

$$B^*(v) = \sup_{u \in U} \{A^*(u) \odot [(A^*(u) \rightarrow A(u)) \odot (A(u) \rightarrow B(v))]\},$$

for any $v \in V$.

PROOF. (1) First, we shall prove that, for each $u \in U$, $v \in V$,

$$(A(u) \to B(v)) \to [(A^*(u) \to A(u)) \to (A^*(u) \to B^*(v))] = 1.$$

In particular, by $B^*(v) = \sup_{u \in U} \{A^*(u) \odot [(A^*(u) \to A(u)) \odot (A(u) \to B(v))]\}$, we have

$$A^*(u) \odot [(A^*(u) \rightarrow A(u)) \odot (A(u) \rightarrow B(v))] \le B^*(v).$$

Further, by (**RP**), it follows that

$$(A^*(u) \to A(u)) \odot (A(u) \to B(v)) \le A^*(u) \to B^*(v).$$

Using (RP) again, we get

$$(A(u) \to B(v)) \le (A^*(u) \to A(u)) \to (A^*(u) \to B^*(v)).$$

Since the semi-overlap function has 1 as the neutral element, according to Proposition 3.10(7), we have

$$(A(u) \rightarrow B(v)) \rightarrow [(A^*(u) \rightarrow A(u)) \rightarrow (A^*(u) \rightarrow B^*(v))] = 1.$$

(2) Then, we prove that $B^*(v)$ is the minimum in $\mathcal{F}(V)$. Let $C^*(v) \in \mathcal{F}(V)$ such that

$$(A(u) \to B(v)) \to [(A^*(u) \to A(u)) \to (A^*(u) \to C^*(v))] = 1.$$

By Proposition 3.10(7), we have

$$A(u) \rightarrow B(v) \le (A^*(u) \rightarrow A(u)) \rightarrow (A^*(u) \rightarrow C^*(v)).$$

And by using (RP) twice, we can get

$$A^*(u) \odot [(A(u) \to B(v)) \odot (A^*(u) \to A(u))] \le C^*(v),$$

which implies that $C^*(v)$ is an upper bound of the set

$$\{A^*(u) \odot [(A^*(u) \rightarrow A(u)) \odot (A(u) \rightarrow B(v))] | u \in U, v \in V\}$$

that is, $B^*(v) \leq C^*(v)$. Therefore, B^* is the FMP-solution of quintuple implication principle.

Example 4.3. Let us consider the following rule:

rule: If u is A, then v is B

input: A^*

output: B^* .

where $A, A^* \in \mathcal{F}(U)$ with $U = \{u_1, u_2, u_3, u_4\}$ and $B, B^* \in \mathcal{F}(V)$ with $V = \{v_1, v_2, v_3, v_4\}$. These fuzzy sets are given by:

$$A = \{(u_1, 0), (u_2, 0.4), (u_3, 0.7), (u_4, 1)\},$$

$$B = \{(v_1, 0.2), (v_2, 0.5), (v_3, 0.9), (v_4, 1)\},$$

$$A^* = \{(u_1, 0.1), (u_2, 0.3), (u_3, 0.5), (u_4, 0.9)\}.$$

The semi-overlap function we choose and its corresponding residual implication are

$$u \odot v = min\{u, v\} = u \wedge v$$

and

$$u \to v = \begin{cases} 1, & if \ u \le v; \\ v, & if \ u > v. \end{cases}$$

Then we can calculate

(1)
$$A^*(u) \to A(u) = [0.1 \to 0 \ 0.3 \to 0.4 \ 0.5 \to 0.7 \ 0.9 \to 1]$$

= $[0 \ 1 \ 1 \ 1].$

$$(2) \quad A(u) \to B(v) = \begin{bmatrix} 0 \to 0.2 & 0 \to 0.5 & 0 \to 0.9 & 0 \to 1 \\ 0.4 \to 0.2 & 0.4 \to 0.5 & 0.4 \to 0.9 & 0.4 \to 1 \\ 0.7 \to 0.2 & 0.7 \to 0.5 & 0.7 \to 0.9 & 0.7 \to 1 \\ 1 \to 0.2 & 1 \to 0.5 & 1 \to 0.9 & 1 \to 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 1 & 1 & 1 \\ 0.2 & 0.5 & 1 & 1 \\ 0.2 & 0.5 & 0.9 & 1 \end{bmatrix}.$$

(3)
$$(A^*(u) \to A(u)) \odot (A(u) \to B(v)) = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \wedge \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.2 & 1 & 1 & 1 \\ 0.2 & 0.5 & 1 & 1 \\ 0.2 & 0.5 & 0.9 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 1 & 1 & 1 \\ 0.2 & 0.5 & 1 & 1 \\ 0.2 & 0.5 & 0.9 & 1 \end{bmatrix}.$$

Further, we can get

$$B^*(v) = \sup_{u \in U} \{A^*(u) \odot [(A^*(u) \to A(u)) \odot (A(u) \to B(v))]\}$$

$$= \sup_{u \in U} \left\{ [0.1 \ 0.3 \ 0.5 \ 0.9] \land \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 1 & 1 & 1 \\ 0.2 & 0.5 & 1 & 1 \\ 0.2 & 0.5 & 0.9 & 1 \end{bmatrix} \right\}$$

$$= \sup_{u \in U} \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.3 & 0.3 \\ 0.2 & 0.5 & 0.5 & 0.5 \\ 0.2 & 0.5 & 0.9 & 0.9 \end{bmatrix} \right\}$$

$$= [0.2 \ 0.5 \ 0.9 \ 0.9].$$

Hence, by Theorem 4.2, we have

$$B^*(v) = \{(v_1, 0.2), (v_2, 0.5), (v_3, 0.9), (v_4, 0.9)\}.$$

Next, we study the reducibility of QIP method of FMP problems. For FMP problems, reversibility means that if A^* equals A, we hope that B^* is equal to B.

Theorem 4.4. Let the implication \rightarrow in FMP be a residual implication derived from the semi-overlap function \odot with 1 as neutral element. If A is a normal fuzzy set, that is $\exists u_0 \in U$ such that $A(u_0) = 1$, then the FMP problem satisfies the reversibility of QIP method.

PROOF. According to Theorem 4.2, FMP-solution of QIP method is the following formula:

$$B^*(v) = \sup_{u \in U} \{A^*(u) \odot [(A^*(u) \rightarrow A(u)) \odot (A(u) \rightarrow B(v))]\}.$$

Suppose that $A^* = A$ is a normal fuzzy set, then we will prove that $B^* = B$. Since \odot has 1 as neutral element and $A(u) \to A(u) = 1$, it follows that

$$(A(u) \to A(u)) \odot (A(u) \to B(v)) \le A(u) \to B(v).$$

If $A^* = A$, then

$$(A^*(u) \to A(u)) \odot (A(u) \to B(v)) \le A^*(u) \to B(v).$$

By (**RP**), we can get

$$A^*(u) \odot ((A^*(u) \to A(u)) \odot (A(u) \to B(v))) \le B(v).$$

It means that *B* is an upper bound of the set

$${A^*(u) \odot (A^*(u) \rightarrow A(u)) \odot (A(u) \rightarrow B(v)) | u \in U, v \in V}.$$

Thus, $B^*(v) \leq B(v)$ for each $v \in V$.

On the other hand, if $A^* = A$ is a normal fuzzy set, that is, $\exists u_0 \in U$ such that $A^*(u_0) = A(u_0) = 1$, then

$$\begin{split} B^*(v) &= \sup_{u \in U} \{A^*(u) \odot ((A^*(u) \to A(u)) \odot (A(u) \to B(v)))\} \\ &\geq A^*(u_0) \odot ((A^*(u_0) \to A(u_0)) \odot (A(u_0) \to B(v))) \\ &= 1 \odot ((1 \to 1) \odot (1 \to B(v))) \\ &= B(v). \end{split}$$

That is $B(v) \leq B^*(v)$.

Example 4.5. Let us continue with Example 4.3. Now, if $A^*(u) = A(u)$, then

$$A^*(u) \to A(u) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}.$$

$$(A^*(u) \to A(u)) \odot (A(u) \to B(v)) = A(u) \to B(v).$$

Hence, we can get

$$B^*(v) = \sup_{u \in U} \{A^*(u) \odot ((A^*(u) \to A(u)) \odot (A(u) \to B(v)))\}$$

=
$$\sup_{u \in U} \{A^*(u) \odot (A(u) \to B(v))\}$$

= [0.2 0.5 0.9 1].

Therefore, $B^*(v) = B(v)$.

4.2. The quintuple implication principle of multiple-rules FMP problems

In this subsection, we will apply the QIP method to multiple-rules fuzzy reasoning. Because in many practical applications, most control systems are composed of multiple-rules.

A general multiple-rules model is as follows:

rule 1 If
$$u_1$$
 is A_{11} and \cdots and u_n is A_{1n} , then v is B_1 rule 2 If u_1 is A_{21} and \cdots and u_n is A_{2n} , then v is B_2

:
rule m If u_1 is A_{m1} and \cdots and u_n is A_{mn} , then v is B_m input u_1 is A_1^* and \cdots and u_n is A_n^* output v is v is

Model 1. General multiple-rules model

where A_{ij} , A_j^* are fuzzy sets on U_j , and B_i , B^* are fuzzy sets on V, $i = 1, 2, \dots, m$, $j = 1, 2, \dots, n$.

Since the variable u_1, u_2, \dots, u_n can be considered as a multi-dimensional variable u, suppose that

$$u = (u_1, u_2, \cdots, u_n) \in U = U_1 \times U_2 \times \cdots \times U_n$$
.

Every rules can be looked as a fuzzy relationship on U, that is

$$A_i(u) = A_{i1}(u_1) \odot A_{i2}(u_2) \odot \cdots \odot A_{in}(u_n)$$

Hence, the Model 1 can be simplified as Model 2:

rule 1 If
$$A_1(u)$$
, then $B_1(v)$
rule 2 If $A_2(u)$, then $B_2(v)$
:
rule m If $A_m(u)$, then $B_m(v)$
input $A^*(u)$
output $B^*(v)$.

Model 2

In fact, the variable u in Model 2 can be as either a multi-dimensional variable or a one-dimensional variable. In the following, suppose that u is a one-dimensional variable.

In fuzzy reasoning, the method often used is First Inference Then Aggregation(FITA). That is, first, the Model 2 is disassembled into the following FMP problems:

rule If
$$A_i(u)$$
, then $B_i(v)$
input $A^*(u)$
output $B_i^*(v)$.

where $i = 1, 2, \dots, n$. According to Theorem 4.2, the solution for every FMP problem is that

$$B_i^*(v) = \sup_{u \in U} \{A^*(u) \odot [(A^*(u) \to A_i(u)) \odot (A_i(u) \to B_i(v))]\}.$$

for any $v \in V$. Then we can aggregate $B_i^*(v)$, $i = 1, 2, \dots, n$, and get the solution of the Model 2:

$$B^{*}(v) = \bigvee_{i=1}^{m} B_{i}^{*}(v)$$

$$= \bigvee_{i=1}^{m} (\sup_{u \in U} \{A^{*}(u) \odot ((A^{*}(u) \to A_{i}(u)) \odot (A_{i}(u) \to B_{i}(v)))\}).$$

The following example shows that reversibility of QIP method in Theorem 4.4 is not applicable to multiple-rules FMP problems.

Example 4.6. Let U = V = [0,1], $A_1, A_2, A^* \in \mathcal{F}(U)$, $B_1, B_2, B^* \in \mathcal{F}(V)$. Then we consider the following fuzzy rules:

rule 1: If
$$A_1(u)$$
, then $B_1(v)$
rule 2: If $A_2(u)$, then $B_2(v)$

The correspond membership of the above fuzzy sets are given in the following:

$$A_1 = \{(u_1, 0.1), (u_2, 0.6), (u_3, 1)\},$$

$$B_1 = \{(v_1, 0.3), (v_2, 0.7), (v_3, 1)\},$$

$$A_2 = \{(u_1, 0.2), (u_2, 0.4), (u_3, 0.8)\},$$

$$B_2 = \{(v_1, 0.1), (v_2, 0.8), (v_3, 1)\}.$$

Suppose that $A^* = A_1$, we calculate the value of B^* . The semi-overlap function we choose and its corresponding residual implication are

$$u \odot v = min\{u, v\} = u \wedge v$$

and

$$u \to v = \begin{cases} 1, & if \ u \le v; \\ v, & if \ u > v. \end{cases}$$

According to FITA, we can do it in three steps:

$$(1) \ B_1^*(v) = \sup_{u \in U} \{A^*(u) \odot [(A^*(u) \to A_1(u)) \odot (A_1(u) \to B_1(v))]\} = [0.3 \ 0.7 \ 1];$$

(2)
$$B_2^*(v) = \sup_{u \in U} \{A^*(u) \odot ((A^*(u) \to A_2(u)) \odot (A_2(u) \to B_2(v)))\} = [0.1 \ 0.8 \ 0.8];$$

(3)
$$B^*(v) = B_1^*(v) \lor B_2^*(v) = [0.3 \ 0.8 \ 1].$$

Hence, $B^*(v) \ne B(v).$

In the following, we will study the reducibility of multiple-rules, which is based on similarity measure. Therefore, we introduce the definition of similarity measure.

Definition 4.7. ([5]) A function $S: \mathcal{F}(U) \times \mathcal{F}(U) \to [0,1]$ is called a similarity measure on $\mathcal{F}(U)$, if S has the following properties:

- (1) S(A, B) = S(B, A), for all $A, B \in \mathcal{F}(U)$;
- (2) $S(A, A^c) = 0$ if and only if A is crisp;
- (3) S(A, B) = 1 if and only if A = B;
- (4) For all $A, B, C, D \in \mathcal{F}(U)$, if $A \leq B \leq C \leq D$, then $S(A, B) \leq S(C, D)$;
- $(5) S(A, B) = S(A^c, B^c).$

Example 4.8. ([5]) Let $A, B \in \mathcal{F}(U)$. Then

$$S(A, B) = \frac{1}{n} \sum_{i=1}^{n} 1 - |A(u_i) - B(u_i)|.$$

is a similarity measure.

Then, the fuzzy reasoning algorithm based on similarity measure is as follows:

Algorithm 1

- Input: A set of rules R_i with $i \in \{1, \dots, m\}$ and a fact A_i^* with $j \in \{1, \dots, n\}$
- Output: B^* .

• According to
$$A_j^*, A_j^* \to A_{ij}$$
 and $A_{ij} \to B_i$, we can calculate that:
$$B_{ij}^* = \sup_{x \in X} \{A_j^* \odot ((A_j^* \to A_{ij}) \odot (A_{ij} \to B_i))\}.$$

• Making the union of these *m* results,

$$B_i^* = \cup B_{ij}^*$$

• Calculate the average similarity. Let S_{ij} be the similarity measure between A_{ij} and A_j^* , and the average similarity S_i is:

$$S_i = \frac{1}{n}(S_{i1} + S_{i2} + \dots + S_{in}).$$

• We give weight to rule
$$i$$
.
$$w_i = \begin{cases} \frac{1-S}{2}, & i \notin I \\ \frac{1+S}{2}, & i \in I. \end{cases}$$

where $S = max\{S_1, S_2, \dots, S_n\}$ and $I = \{i | S_i = S, 1 \le i \le n\}$.

Calculation reasoning results.

$$B^* = \cup w_i B_i^*.$$

Theorem 4.9. Let $A_{ij}(i = 1, 2, \dots, m; j = 1, 2, \dots, n)$ be a normal fuzzy set, that is exist a $u_{ij} \in U_j$ such that $A_{ij}(u_{ij}) = 1$. Then when $A_j^* = A_{i_0j}$, we can get $B^* = B_{i_0}$ according to the above algorithm.

PROOF. Suppose that for $i_0(1 \le i_0 \le m)$, $A_j^* = A_{i_0j}$, where $j = 1, 2, \dots, n$. By Theorem 4.4, we can get $B_{i_0j}^* = B_{i_0}$, hence $B_{i_0}^* = \bigcup B_{i_0j}^* = B_{i_0}$. If $A_j^* = A_{i_0j}$, we have $S_{i_0j}(A_{i_0j}, A_j^*) = 1$. Therefore, $S_{i_0} = \frac{1}{n}(S_{i_01} + S_{i_02} + \dots + S_{i_0n}) = 1$. By algorithm 1, $S = \max\{S_1, S_2, \dots, S_{i_0}, \dots, S_n\}$, we have $S = S_{i_0} = 1$, thus, $w_{i_0} = 1$, $w_i = 0$, where $i = 1, 2 \cdots, m$ and $i \ne i_0$. Hence $B^* = \bigcup w_i B_i^* = B_{i_0}$.

5. An application of semi-overlap functions and quintuple implication principle in classification problems

In this section, we apply the combination of semi-overlap functions and quintuple implication principle to classification problems. Firstly, we introduce fuzzy association rule-based classification method for high-dimensional problems (**FARC** – **HD**, for short). Then, based on semi-overlap functions and quintuple implication principle, we propose a new fuzzy classification algorithm which called **SO5I** – **FRC**. Finally, we apply the two methods to the classification problem respectively. And the experimental results show that the classification accuracy of SO5I-FRC algorithm is higher than that of FARC-HD algorithm.

5.1. Fuzzy rule-based classification systems

In the literature, there are multiple techniques used to solve classification problems. Among them, Fuzzy Rule-Based Classification Systems (FRBCSs, for short) are one of the most popular approaches, since they provide an interpretable model by means of the use of linguistic labels in their rules[19]. The main purpose of rule-based classification algorithm (fuzzy or clear rules) is to classify the samples that satisfied certain conditions in the data set. Here, we will introduce two fuzzy rule based classification algorithms.

• FARC - HD

J.Alcalá-Fdes([1]) propose a fuzzy association rule based classification method for highdimensional problems(FARC-HD) to obtain an accurate and compact fuzzy rule-based classifier with a low computational cost. In this paper, we only consider the classification problem with two variables. Hence, we will give the rule structure and algorithm with two variables in FARC-HD. The rule structure is as follows:

Rules
$$R_j$$
: If u_1 is A_{j1} and u_2 is A_{j2} ,
then Class is C_j with RW_j

where $R_j(j = 1, 2, \dots, m)$ is the label of the *j*-th rule, $u = (u_1, u_2)$ is a vector representing the example, A_{j1} and A_{j2} is a linguistic label modeled by a triangular membership function, C_j is the class label and RW_j is the rule weight computed using the certainty factor defined in [20].

The FARC-HD algorithm is as follows:

Algorithm 2

- Input: A set of rules R_i and rule weight RW_i and a sample $u^* = (u_1, u_2)$
- Output: C_i .

Algorithm 2

• Matching degree. Calculate the matching degree between the antecedent for all rules and the samples u^* .

$$\mu_{A_i}(u^*) = O(\mu_{A_{i1}}(u_1^*), \mu_{A_{i2}}(u_2^*)).$$

• Association degree. Calculate the association degree between the sample u^* and each rule in the rule base.

$$b_i(u^*) = \mu_{A_i}(u^*) \cdot RW_i$$

• Confidence degree. Calculate the confidence degree of each class

$$con f_i(u^*) = \sum b_i(u^*)$$

• Classification. Predict the class with the highest confidence.

Class=arg $max\{conf_i(u^*)\}$.

In algorithm 2, *O* represents binary overlap functions. In fact, in [12], based on FARC-HD algorithm, multi-class classification problems is studied by *n*-dimensional overlap functions, and good results are obtained. In this paper, we study the classification problem with two variables, hence we choose the binary overlap functions.

• SO5I - FRC

Based on semi-overlap functions and QIP method introduced in this paper, we propose a new fuzzy classification algorithm, which called SO5I-FRC, for short. The main idea of SO5I-FRC algorithm is as follows: we apply the semi-overlap functions and their residual implications to the QIP method and calculate the matching degree between an input sample and each rules in fuzzy rule set, then the class of the rule with the highest matching degree is the classification result of this sample. The difference between SO5I-FRC algorithm and FARC-HD algorithm is that in FARC-HD algorithm, we only need to calculate the matching degree between input A^* and the antecedent A_{ji} of the rule, however, in SO5I-FRC algorithm, we need to calculate the matching degree between A^* and the whole rule, including the matching degree of A_{ji} and B_{j} . The rule structure of SO5I-FRC algorithm is as follows: (we only consider the classification problem with two variables)

Rules
$$R_j$$
: If u_1 is A_{j1} and u_2 is A_{j2} ,
then C_j is B_j .

where $R_j(j = 1, 2, \dots, m)$ is the j-th rule of the fuzzy rule set; A_{j1} and A_{j2} are two onedimensional fuzzy sets of the antecedents of fuzzy rules; C_j is the classification label of the sample satisfying fuzzy sets A_{j1} and A_{j2} ; B_j is the degree to which C_j belongs to class 1 or class 2, and it is a single point fuzzy set.

The SO5I-FRC algorithm is as follows:

Algorithm 3

- Input: A set of rules R_j and a sample $u^* = (u_1, u_2)$
- Output: C_i .
- Calculating two-dimensional fuzzy sets A^* and A_j

$$A^*(u_1, u_2) = min\{A_1^*(u_1), A_2^*(u_2)\};$$

$$A_i(u_1, u_2) = min\{A_{i1}(u_1), A_{i2}(u_2)\}.$$

• Calculating the matching degree between the rules R_i and the antecedent A^*

$$B_{j}^{*}(v) = \sup_{u \in U} \{A^{*}(u) \odot ((A^{*}(u) \to A_{j}(u)) \odot (A_{j}(u) \to B_{j}(v)))\}$$

• The classification result of A^* is the class of the rule with the highest matching degree Class=arg $max\{B_i^*(v)\}$.

5.2. Experimental framework

• step 1. Selection of data sets

We will select the Banana data set in the KEEL data set repository(https://sci2s.ugr.es/keel/category.php?cat=cla), which is a binary classified data set with two numerical attributes, as shown in Table 2.

number	attribute 1	attribute 2	class
1	1.14	-0.114	1
2	-1.52	-1.15	2
3	-1.05	0.72	1
4	-0.916	0.397	2
5	-1.09	0.437	2
6	-0.584	0.0937	2
5298	1.57	-0.389	1
5299	-0.411	0.727	2
5300	-0.72	0.189	2

Table 2. Banana data set

The distribution of the two classes is shown in Figure 2. Here, the red is the data of the first class and the green is the data of the second class.

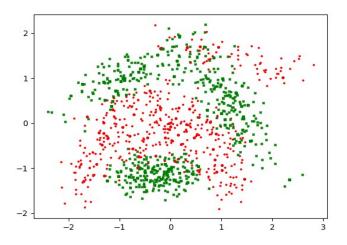


Figure 2: Classification distribution

• step 2. Establish accurate rule set

According to RIPPER algorithm, the rule set (I) of the Banana data set is obtained:

- (1) if $-2.41 \le u_1 \le -1.26$ and $-1.9 \le u_2 \le 0.52$, then class is 1;
- (2) if $-0.9 \le u_1 \le 0.6$ and $0.52 \le u_2 \le 0.8$, then class is 1;
- (3) if $-1.26 \le u_1 \le 0.9$ and $-0.65 \le u_2 \le 0.52$, then class is 1;
- (4) if $0.9 \le u_1 \le 1.23$ and $-0.65 \le u_2 \le -0.15$, then class is 1;
- (5) if $0.56 \le u_1 \le 1.6$ and $-1.9 \le u_2 \le -0.65$, then class is 1;
- (6) if $1.3 \le u_1 \le 2.81$ and $0.81 \le u_2 \le 2.2$, then class is 1;

otherwise, class is 2.

According to the actual classification result and data set characteristics, we subdivide "otherwise, class is 2." in the above rule set (I) into the following 7 rules:

- (7) if $-2.41 \le u_1 \le -0.9$ and $0.52 \le u_2 \le 2.2$, then class is 2;
- (8) if $-0.9 \le u_1 \le 0.6$ and $0.8 \le u_2 \le 2.2$, then class is 2;
- (9) if $0.6 \le u_1 \le 1.3$ and $0.52 \le u_2 \le 2.2$, then class is 2;
- (10) if $0.9 \le u_1 \le 1.23$ and $-0.15 \le u_2 \le 0.52$, then class is 2;
- (11) if $-1.26 \le u_1 \le 0.56$ and $-1.9 \le u_2 \le -0.65$, then class is 2;
- (12) if $1.23 \le u_1 \le 2.81$ and $-0.65 \le u_2 \le 0.81$, then class is 2;
- (13) if $1.6 \le u_1 \le 2.81$ and $-1.9 \le u_2 \le -0.65$, then class is 2.

Rule set (II) is composed of 6 rules in rule set (I) and 7 rules obtained by subdividing "otherwise, class is 2". Our experiment is based on the 13 rules in rule set (II), and applies SO5I-FRC algorithm and FARC-HD algorithm to carry out classification experiment, respectively.

• step 3. Fuzzification methods of major premise (rules) and minor premise

Fuzzification is to convert the determined real number input into fuzzy quantity, that is, fuzzy set and corresponding membership function. Here, we need to fuzzify each rules in rule set (II) to obtain the fuzzy rule set (III). When the condition of variable u_i of rules R_j in rule set (II) is $a \le u_i \le b$, the membership function in fuzzy rule set (III) is the trapezoidal membership function T(a,b) shown in Fig 3. Fuzzy set $A_{ji}(i=1,2)$ is shown in the following, where the domain of u_1 is [-2.41, 2.81] and the domain of u_2 is [-1.9, 2.2].

$$A_{ji}(u) = \begin{cases} 1, & if \ a+0.1 \le u_i \le b-0.1 \\ \frac{2(u-a+0.25)}{7}, & if \ a-0.25 \le u_i \le a+0.1 \\ \frac{-2(u-b-0.25)}{7}, & if \ b-0.1 \le u_i \le a+0.25 \\ 0, & otherwise \end{cases}$$

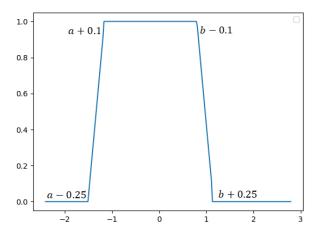


Figure 3: Trapezoidal membership function

The membership degree of the fuzzy rule consequent is the fuzzy set $B_j = \{a\}$, the value of a is the classification accuracy of all points in the area determined by variables u_1 and u_2 of R_j in rule set (II). That is

$$B_1 = \{0.891 | C_j = 1\}, B_2 = \{0.941 | C_j = 1\}, B_3 = \{0.896 | C_j = 1\},$$

$$B_4 = \{0.923 | C_j = 1\}, B_5 = \{0.958 | C_j = 1\}, B_6 = \{0.637 | C_j = 1\},$$

$$B_7 = \{0.996 | C_j = 2\}, B_8 = \{0.778 | C_j = 2\}, B_9 = \{0.795 | C_j = 2\},$$

$$B_{10} = \{0.968 | C_j = 2\}, B_{11} = \{0.794 | C_j = 2\}, B_{12} = \{0.987 | C_j = 2\},$$

$$B_{13} = \{0.983 | C_j = 2\}.$$

Next, we fuzzify the minor premise A_{ji}^* . The membership function expression of A_{ji}^* and its membership function graph(Fig.4) are as follows

$$A_{ji}^{*}(u) = \begin{cases} 4u - 4a + 1, & if \ a - 0.25 \le u_{i} \le a \\ -4u + 4a + 1, & if \ a \le u_{i} \le a + 0.25 \\ 0, & otherwise \end{cases}$$

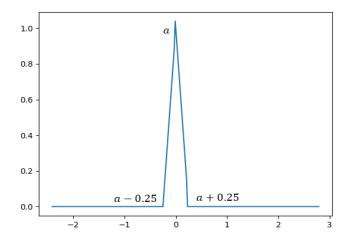


Figure 4: Triangular membership function

5.3. Experimental results and analysis

(1) In rule set (III), we use FARC-HD algorithm to classify banana data set. We select the following overlap functions for experiments and analyze the results.

$$O_a : u \odot_a v = min\{u, v\};$$

$$O_b : u \odot_b v = uv;$$

$$O_c : u \odot_c v = \frac{uv(u+v)}{2};$$

$$O_d: u \odot_d v = min\{\sqrt{u}, \sqrt{v}\};$$

$$O_e: u \odot_e v = min\{u^2, v^2\}.$$

In the following table, we will give the membership functions of variables u_1 and u_2 and the rule weight corresponding to each rules.

rules	membership function of u_1/u_2	rule weight
1	$Tra_{u_1}(-2.66, -2.31, -1.36, -1.01)$	1/6
•	$Tra_{u_2}(-2.15, -1.8, 0.42, 0.77)$	1,0
	$Tra_{u_1}(-1.15, -0.8, 0.5, 0.85)$	1/6
_	$Tra_{u_2}(0.27, 0.62, 0.7, 1.05)$	1,0
3	$Tra_{u_1}(-1.51, -1.16, 0.8, 1.15)$	1/6
5	$Tra_{u_2}(-0.9, -0.55, 0.42, 0.77)$	1,0
4	$Tra_{u_1}(0.65, 1.0, 1.13, 1.48)$	1/6
7	$Tra_{u_2}(-0.9, -0.55, -0.25, 0.1)$	170
	$Tra_{u_1}(0.31, 0.66, 1.5, 1.85)$	1/6
3	$Tra_{u_1}(0.51, 0.00, 1.5, 1.05)$ $Tra_{u_2}(-2.15, -1.8, -0.75, -0.4)$	170
6	$Tra_{u_1}(1.05, 1.4, 2.71, 3.06)$	1/6
O	$Tra_{u_2}(0.56, 0.91, 2.1, 2.45)$	170
7	$Tra_{u_1}(-2.66, -2.31, -1.0, -0.65)$	1/7
,	$Tra_{u_2}(0.27, 0.62, 2.1, 2.45)$	1,,
8	$Tra_{u_1}(-1.15, -0.8, 0.5, 0.85)$	1/7
Ü	$Tra_{u_2}(0.55, 0.9, 2.1, 2.45)$	1,,
9	$Tra_{u_1}(0.35, 0.7, 1.2, 1.55)$	1/7
	$Tra_{u_2}(0.27, 0.62, 2.1, 2.45)$	1,,
10	$Tra_{u_1}(0.65, 1.0, 1.13, 1.48)$	1/7
10	$Tra_{u_1}(-0.4, -0.05, 0.42, 0.77)$	1,,
11	$Tra_{u_1}(-1.51, -1.16, 0.46, 0.81)$	1/7
- 11	$Tra_{u_2}(-2.15, -1.8, -0.75, -0.4)$	1,,
12	$Tra_{u_1}(0.98, 1.33, 2.71, 3.06)$	1/7
12	$Tra_{u_2}(-0.9, -0.55, 0.71, 1.06)$	1,,
13	$Tra_{u_1}(1.35, 1.7, 2.71, 3.06)$	1/7
1.0	$Tra_{u_2}(-2.15, -1.8, -0.75, -0.4)$	1//
	$1.14u_2(-2.13, -1.0, -0.73, -0.4)$	

Table 3. Membership functions and rule weight

In the following, we will give the classification accuracy results. The Table 4 shows that the classification accuracy obtained by selecting the same overlap function for 13 rules, and the Table 5 shows the classification accuracy obtained by selecting different overlap functions for 13 rules.

experiments	overlap functions	accuracy
Exp 1	O_a	75.09%
Exp 2	O_b	74.25%
Exp 3	O_c	74.23%
Exp 4	O_d	72.94%
Exp 5	O_e	74.28%

Table 4. Classification accuracy results by selecting the same overlap function

From the Table 4, we can get that using FARC-HD algorithm and selecting the same overlap function for 13 rules, the classification accuracy of the five groups experiments are 75.09%, 74.25%, 74.23%, 72.94% and 74.28%, respectively, and the average accuracy is 74.158%.

rules	Exp 1	Exp 2	Exp 3	Exp 4	Exp 5
1	O_c	O_c	O_c	O_a	O_c
2	O_b	O_b	O_d	O_e	O_c
3	O_a	O_c	O_a	O_b	O_d
4	O_a	O_a	O_a	O_a	O_a
5	O_d	O_e	O_c	O_a	O_e
6	O_d	O_d	O_c	O_d	O_a
7	O_c	O_a	O_e	O_a	O_d
8	O_b	O_d	O_a	O_e	O_a
9	O_e	O_c	O_c	O_c	O_c
10	O_a	O_a	O_b	O_a	O_a
11	O_d	O_b	O_a	O_b	O_c
12	O_a	O_a	O_c	O_c	O_c
13	O_e	O_d	O_c	O_d	O_b
accuracy	76.92%	75.06%	75.11%	74.08%	74.92%

Table 5. Classification accuracy results by selecting the different overlap functions

From the Table 5, we can get that using FARC-HD algorithm and selecting the different overlap functions for 13 rules, the classification accuracy of the five groups experiments are 76.92%, 75.06%, 75.11%, 74.08% and 74.92%, respectively, and the average accuracy is 75.218%. It is easy to see that the average accuracy of selecting different overlap functions for 13 rules is higher than that of selecting the same overlap function.

(2) In rule set (III), we use SO5I-FRC algorithm to classify banana data set. When calculating the matching degree between A^* and rule R_j , the following four groups of semi-overlap functions with good experimental performance can be arbitrarily selected for 13 rules, with a total of C_{13}^4 choices. It should be noted that the implication and semi-overlap functions in each rules must be the same semi-overlap function and its corresponding implication.

SO₁:
$$u \odot_1 v = \begin{cases} 0, & if \ u + v \le 1 \\ uv, & otherwise \end{cases}$$

$$u \to_1 v = \begin{cases} 1, & if \ u \le v \\ max\{1 - u, \frac{v}{u}\}, & otherwise \end{cases}$$
SO₂: $u \odot_2 v = max\{\frac{u+v-1}{u+v-uv}, 0\}$

$$u \to_2 v = \begin{cases} 1, & if \ u \le v \\ \frac{uv+1-u}{uv+1-v}, & otherwise \end{cases}$$
SO₃: $u \odot_3 v = min\{u, v\}$

$$u \to_3 v = \begin{cases} 1, & if \ u \le v \\ v, & otherwise \end{cases}$$

$$SO_{4}: \quad u \odot_{4} v = \begin{cases} \frac{1 + min\{2u - 1, 2v - 1\}max\{(2u - 1)^{2}, (2v - 1)^{2}\}}{2}, & u, v \in (0.5, 1] \\ min\{u, v\}, & otherwise \end{cases}$$

$$u \rightarrow_{4} v = \begin{cases} min\{1, max\{\frac{\sqrt{2v - 1}}{2\sqrt{2u - 1}}, \frac{v}{2(2u - 1)^{2}}\} + \frac{1}{2}\}, & u \in (0.5, 1], \\ v \in [0.5, 1] \\ v, & v \in [0, 0.5), u > v \\ 1, & u \in [0, 0.5], u \leq v \end{cases}$$

In the following table, we will give the membership function of variables u_1 and u_2 in each rules.

rules	membership function of x_1/x_2	
Tuics	* -	
1	$Tra_{u_1}(-2.66, -2.31, -1.36, -1.01),$	$Tra_{u_2}(-2.15, -1.8, 0.42, 0.77)$
2	$Tra_{u_1}(-1.15, -0.8, 0.5, 0.85),$	$Tra_{u_2}(0.27, 0.62, 0.7, 1.05)$
3	$Tra_{u_1}(-1.51, -1.16, 0.8, 1.15),$	$Tra_{u_2}(-0.9, -0.55, 0.42, 0.77)$
4	$Tra_{u_1}(0.65, 1.0, 1.13, 1.48),$	$Tra_{u_2}(-0.9, -0.55, -0.25, 0.1)$
5	$Tra_{u_1}(0.31, 0.66, 1.5, 1.85),$	$Tra_{u_2}(-2.15, -1.8, -0.75, -0.4)$
6	$Tra_{u_1}(1.05, 1.4, 2.71, 3.06),$	$Tra_{u_2}(0.56, 0.91, 2.1, 2.45)$
7	$Tra_{u_1}(-2.66, -2.31, -1.0, -0.65),$	$Tra_{u_2}(0.27, 0.62, 2.1, 2.45)$
8	$Tra_{u_1}(-1.15, -0.8, 0.5, 0.85),$	$Tra_{u_2}(0.55, 0.9, 2.1, 2.45)$
9	$Tra_{u_1}(0.35, 0.7, 1.2, 1.55),$	$Tra_{u_2}(0.27, 0.62, 2.1, 2.45)$
10	$Tra_{u_1}(0.65, 1.0, 1.13, 1.48),$	$Tra_{u_2}(-0.4, -0.05, 0.42, 0.77)$
11	$Tra_{u_1}(-1.51, -1.16, 0.46, 0.81),$	$Tra_{u_2}(-2.15, -1.8, -0.75, -0.4)$
12	$Tra_{u_1}(0.98, 1.33, 2.71, 3.06),$	$Tra_{u_2}(-0.9, -0.55, 0.71, 1.06)$
13	$Tra_{u_1}(1.35, 1.7, 2.71, 3.06),$	$Tra_{u_2}(-2.15, -1.8, -0.75, -0.4)$

Table 6. Membership functions

The following table shows the results of five groups of experiments.

rules	Exp 1	Exp 2	Exp 3	Exp 4	Exp 5
1	SO_3	SO_3	SO_1	SO_3	SO_3
2	SO_4	SO_1	SO_2	SO_1	SO_4
3	SO_2	SO_1	SO_3	SO_3	SO_3
4	SO_3	SO_4	SO_4	SO_3	SO_2
5	SO_1	SO_3	SO_1	SO_1	SO_1
6	SO_2	SO_3	SO_2	SO_2	SO_1
7	SO_3	SO_4	SO_3	SO_4	SO_3
8	SO_4	SO_3	SO_4	SO_1	SO_4
9	SO_2	SO_2	SO_1	SO_4	SO_2
10	SO_3	SO_2	SO_2	SO_2	SO_2
11	SO_2	SO_1	SO_3	SO_1	SO_2
12	SO_2	SO_2	SO_4	SO_2	SO_1
13	SO_2	SO_2	SO_3	SO_4	SO_1
accuracy	86.58%	88.09%	88.09%	88.15%	86.58%

Table 7. Classification accuracy results

Using SO5I-FRC algorithm, we can get the classification accuracy of the five groups of experiments are 86.58%, 88.09%, 88.09%, 88.15% and 86.58% respectively, and the average accuracy is 87.50%.

The average accuracy of FARC-HD algorithm (using overlap functions but not quintuple implication principle) is 75.218%, and the average accuracy of SO5I-FRC algorithm (using semi-overlap functions and quintuple implication principle) proposed in this paper is 87.50%. The experimental results show that the average accuracy of SO5I-FRC algorithm is higher than that of FARC-HD algorithm. On the other hand, we can conclude that semi-overlap functions and QIP method has a wide range of application and certain advantages in classification problems.

6. Conclusions

In this paper, we have studied semi-overlap functions and its corresponding quintuple implication principle and proposed a new fuzzy classification algorithm, which called SO5I-FRC algorithm. By doing comparative experiments, we can conclude that the average accuracy of SO5I-FRC algorithm is higher than FARC-HD algorithm. In the future work, we will try to use SO5I-FRC algorithm in more fields to get some better conclusions.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (61976130) and by the Brazilian National Council for Scientific and Technological Development – CNPq (311429/2020-3).

References

- [1] J. Alcalá-Fdez, R. Alcalá, F. Herrera. A fuzzy association rule-based classification model for high-dimensional problems with genetic rule selection and lateral tuning, IEEE Transactions on Fuzzy Systems, 19, (2011), 857-872.
- [2] H. Bustince, J. Fernandez, R. Mesiar, J. Montero, R. Orduna, Overlap functions, Non-linear Analysis: Theory, Method and Applications, 72, (2010), 1488-1499.
- [3] B.C. Bedregal, G.P. Dimuro, H. Bustince, E. Barrenechea, New results on overlap and grouping functions, Information Sciences, 249, (2013), 148-170.
- [4] H. Bustince, M. Pagola, R. Mesiar, E. Hüllermeier, F. Herrera, Grouping, overlap, and generalized bientropic functions for fuzzy modeling of pairwise comparisons, IEEE Transactions on Fuzzy Systems, 20, (2012), 405-415.
- [5] H. Bustince, E. Barrenechea, M. Pagola, Relationship between restricted dissimilarity functions, restricted equivalence functions and normal En-functions: Image thresholding invariant, Pattern Recognition Letters, 29, (2008), 525-536.
- [6] M. Baczyński, On the applications of fuzzy implication functions, Soft Computing Applications, 195, (2013), 9-10.
- [7] Z. Chen, G. Chen, Building an associative classifier based on fuzzy association rules, International Journal of Computational Intelligence Systems, 1, (2008), 262-273.
- [8] L. De Miguel, D. Gómez, J.T. Rodríguez, J. Montero, H. Bustince, G.P. Dimuro, J.A. Sanz, General overlap functions, Fuzzy Sets and Systems, 372, (2019), 81-96.
- [9] G.P. Dimuro, B. Bedregal, On residual implications derived from overlap functions, Information Sciences, 312, (2015), 78-88.

- [10] M. Elkano, M. Galar, J. Sanz, H. Bustince, Fuzzy rule-based classification systems for multi-class problems using binary decomposition strategies: on the influence of n-dimensional overlap functions in the fuzzy reasoning method, Information Sciences, 332, (2016), 94-114.
- [11] M. Elkano, M. Galar, J.A. Sanz, A. Fernández, E. Barrenechea, F. Herrera, Enhancing multiclass classification in FARC-HD fuzzy classifier: on the synergy between n-dimensional overlap functions and decomposition strategies, IEEE Transactions on Fuzzy Systems, 23, (2015), 1562-1580.
- [12] M. Elkano, M. Galar, J. Sanz, H. Bustince, Study of n-dimensional overlap functions in Fuzzy Rule-Based Classification Systems,9th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT), 2015.
- [13] J.C. Fodor, T. Keresztfalvi, Nonstandard conjunctions and implications in fuzzy logic, International Journal of Approximate Reasoning, 12, (1995), 69-84.
- [14] D. Gómez, J.T. Rodríguez, J. Yáñez, J. Montero, A new modularity measure for fuzzy community detection problems based on overlap and grouping functions, International Journal of Approximate Reasoning, 74, (2016), 88-107.
- [15] D. Gómez, J.T. Rodríguez, J. Montero, H. Bustince, E. Barrenechea, n-Dimensional overlap functions, Fuzzy Sets and Systems, 287, (2016), 57-75.
- [16] Y.C. Hua, G.H. Tzeng, Elicitation of classification rules by fuzzy data mining, Engineering Applications of Artificial Intelligence, 16, (2003), 709-716.
- [17] Y. Hu, R. Chen, G. Tzeng, Finding fuzzy classification rules using data mining techniques, Pattern Recognition Letters, 24, (2003), 509-519.
- [18] Y.C. Hua, Determining membership functions and minimum fuzzy support in finding fuzzy association rules for classification problems, Knowledge-Based Systems, 19, (2006), 57-66.
- [19] H. Ishibuchi, T. Nakashima, M. Nii, Classification and Modeling with Linguistic Information Granules: Advanced Approaches to Linguistic Data Mining, Springer-Verlag, Berlin, 2004.
- [20] H. Ishibuchi, T. Yamamoto, T. Nakashima, Hybridization of fuzzy GBML approaches for pattern classification problems. IEEE Transactions on System, Man and Cybernetics B, 35, (2005), 359-365.
- [21] A. Jurio, H. Bustince, M. Pagola, A. Pradera, R. Yager, Some properties of overlap and grouping functions and their application to image thresholding, Fuzzy Sets and Systems, 229, (2013),69-90.
- [22] A. Król, Dependencies between fuzzy conjunctions and implications, in: Proceedings of EUSFLAT-LFA 2011, the 7th Conference of the European Society for Fuzzy Logic and Technology and "les rencontres francophones sur la Logique Floue et ses Applications", Atlantis Press, Amsterdam, (2011), 230-237.
- [23] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Springer, 2000.

- [24] M.X. Luo, X.L. Zhou, Interval-valued quintuple implication principle of fuzzy reasoning, International Journal of Approximate Reasoning, 84, (2017), 23-32.
- [25] T. Nakashima, G. Schaefer, Y. Yokota, A weighted fuzzy classifier and its application to image processing tasks, Fuzzy Sets and Systems, 158, (2007), 284-294.
- [26] F. Pach, A. Gyenesei, J. Abonyi, Compact fuzzy association rule-based classifier, Expert Systems with Applications, 34, (2008), 2406-2416.
- [27] R. Paiva, R. Santiago, B. Bedregal, E. Palmeira, Lattice-valued overlap and quasi-overlap functions, Information Sciences, 562, (2021), 180-199.
- [28] C. Tsang, S. Kwong, H. Wang, Genetic-fuzzy rule mining approach and evaluation of feature selection techniques for anomaly intrusion detection, Pattern Recognition, 40,(2007), 2373-2391.
- [29] G.J. Wang, The full implication triple I method of fuzzy reasoning, Sciences in China (Series E), 29, (1999), 43-53.
- [30] Y. Yi, E. Hullermeier, Learning complexity-bounded rule-based classifiers by combining association analysis and genetic algorithms, in Proceedings of 4th Conference of the European Society for Fuzzy Logic and Technology (EUSFLAT 2005), Barcelona, Spain, (2005), 47-52.
- [31] B.K. Zhou, G.Q. Xu, S.J. Li, The quintuple implication principle of fuzzy reasoning, Information Sciences, 297, (2015), 20-215.
- [32] H.J. Zhou, Characterizations of Fuzzy Implications Generated by Continuous Multiplicative Generators of T-Norms, IEEE Transactions on Fuzzy Systems, 29, (2020), 2988-3002.
- [33] L.A. Zadeh, Outline of a new approach to the analysis of complex systems and decision processes, IEEE Transactions on Systems Man Cybernetics, 3, (1973), 28-44.

Remark: The main results of this paper were introduced at a conference (The Seminar on Logics for New-Generation Artificial Intelligence, see: https://www.xixilogic.org/events/ngl2021/ or https://rwsk.zju.edu.cn/2021/1027/c2075a243 5645/page.htm) in October 2021, but it has not been published anywhere.