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Minimum implicational basis for \wedge -semidistributive lattices

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Received 13 January 2006; received in revised form 21 February 2006; accepted 21 April 2006

Communicated by L. Boasson

Abstract

For a \wedge -semidistributive lattice L , we study some particular implicational systems and show that the cardinality of a minimum implicational basis is polynomial in the size of join-irreducible elements of the lattice L . We also provide a polynomial time algorithm to compute a minimum implicational basis for L .

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Keywords: Algorithms; Lattice; Closure system; Minimum implicational basis

1. Introduction

This paper deals with the computation of a minimum implicational basis for a closure system. Computing a minimum implicational basis for a lattice given by its poset of irreducible elements is an important problem, which has applications to many areas of computer science, in particular to databases and AI [1,4,6,7,10]. For a survey on this problem and related areas, see [3].

The complexity of this problem remains open for general lattices. Recent progress on the status of this problem, and in particular solvability by limited non-determinism [5], suggests however that this problem is more likely to be expected tractable than intractable [4].

It has been already shown that this problem is tractable for the two classes of locally distributive lattice [2] and of modular lattices [14]. In this paper we

show by using a dependence relation in [11] that the class of \wedge -semidistributive lattices is another tractable case.

Consider a finite set U . A subset $C \subseteq 2^U$ is said to be a closure system if C is closed under set-intersection and containing the set U . An implication on U is an ordered pair (A, B) of subsets of U , denoted by $A \rightarrow B$. The set A is called the premise and the set B the conclusion of the implication $A \rightarrow B$. Let Σ be a set of implications on U . A subset $D \subseteq U$ is Σ -closed if for each implication $A \rightarrow B$ in Σ , $A \subseteq D$ implies $B \subseteq D$. The set of Σ -closed subsets of U , denoted by $C(\Sigma)$, is a closure system on U . Conversely, given a closure system C on U , a family Σ of implications on U is said an implicational basis for C if $C = C(\Sigma)$. An implicational basis is said minimum if it has a minimum number of implications.

In this paper, we study the latticial version of this problem. We view a lattice L as the closure system C_L on the set $J(L)$ of its join-irreducible elements. More precisely, put $J(a) = \{j \in J(L) : j \leq a\}$ for $a \in L$.

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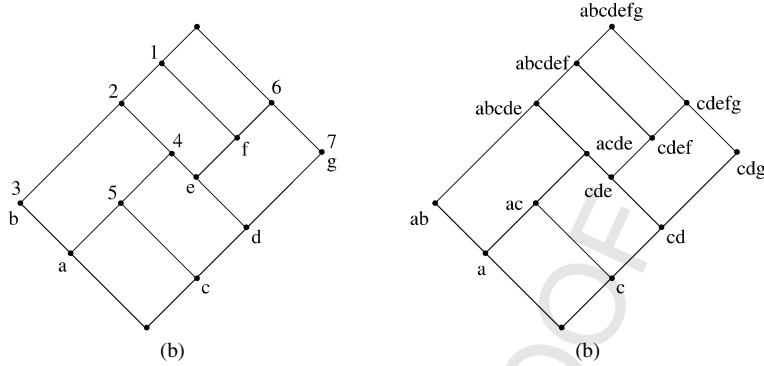


Fig. 1. (a) A lattice L where join-irreducible (resp. meet-irreducible) elements are labeled by letters (resp. numbers); (b) The closure system cl associated to L .

Then $\mathcal{C}_L = \{J(a) : a \in L\}$ is a closure system on $J(L)$ which, as a lattice ordered by inclusion, is isomorphic to L .

Fig. 1 gives an example of the closure system \mathcal{C}_L associated to a lattice L .

The closure system \mathcal{C}_L can be defined by the set of its meet-irreducible elements $\mathcal{M}(\mathcal{C}_L) = \{J(m) : m \in M(L)\}$, where $M(L)$ denotes the set of meet-irreducible elements of L . Each element of \mathcal{C}_L can be obtained as intersection of some elements of $\mathcal{M}(\mathcal{C}_L)$.

The problem we study is:

Problem: Minimum implicational basis

Instance: The set of meet-irreducible elements $\mathcal{M}(\mathcal{C}_L)$ of the closure system \mathcal{C}_L .

Question: Find a minimum basis Σ for \mathcal{C}_L .

This problem remains open for general lattices. Duquenne [2] has given a latticial version of this problem and shown that it is polynomial for upper locally distributive lattices or antimatroid. Recently, Wild [14] has proposed a polynomial time algorithm to compute an optimal¹ implicational basis for modular lattices. In the following, we study the case of \wedge -semidistributive lattices. For such lattices we show that the number of implications of a minimum implicational basis is at most $|J(L)|^2$ and give a polynomial time algorithm to compute such a basis.

2. Some properties of \wedge -semidistributive lattices

Let L be a finite lattice. We note \vee the join operation, \wedge the meet operation and \prec the cover relation of L . If j is a join-irreducible element of L , we use j_* to denote

the unique element covered by j . Dually, we use m^* to denote the unique element covering a meet-irreducible element m .

We will use the arrow relations introduced by Wille [15]: for $x, y \in L$, $x \downarrow y$ means that x is a minimal element of $\{z \in L : z \not\leq x\}$, $x \uparrow y$ means that y is a maximal element of $\{z \in L : z \not\geq y\}$ and $x \updownarrow y$ means that $x \uparrow y$ and $x \downarrow y$. Recall that $\uparrow, \downarrow, \updownarrow$ are relations defined on $J(L) \times M(L)$, where $J(L)$ is the set of join-irreducible elements and $M(L)$ the set of meet-irreducible elements of L .

In the following, we deal essentially with \wedge -semidistributive lattices. Let us recall that a lattice L is said \wedge -semidistributive if for all elements $x, y, z \in L$, $x \wedge y = x \wedge z$ implies $x \wedge y = x \wedge (y \vee z)$. A \wedge -semidistributive lattice is said semidistributive if for all elements x, y, z , $x \vee y = x \vee z$ implies $x \vee y = x \vee (y \wedge z)$. The following characterization of these lattices are well known (see, for example, [6]):

Property 1. A finite lattice L is \wedge -semidistributive if and only if for any $j \in J(L)$ there exists a unique $m \in M(L)$ such that $j \updownarrow m$.

For any \wedge -semidistributive lattice L and $j \in J(L)$, we denote by $m(j)$ the unique element $m \in M(L)$ such that $j \updownarrow m$.

We define the mapping $\gamma : J(L) \rightarrow 2^{M(L)}$ by $\gamma(j) = \{m \in M(L) : j \downarrow m\}$. This mapping was introduced in [12] to define colored posets, which provides a new representation for lattices, and specially for upper locally distributive lattices. Fig. 2 shows the γ mapping of the lattice of Fig. 1. Note that this lattice is semidistributive.

We consider one of the standard dependence relations defined on join-irreducible elements (assuming

¹ An implication is known as optimal if the sum of the cardinality of the premises and the conclusions of all the implications is minimal.

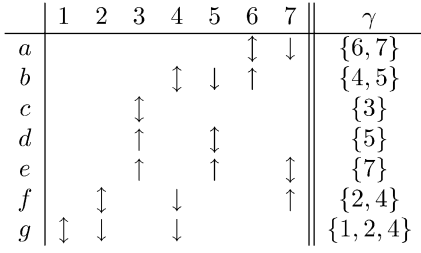


Fig. 2. The arrow relations and mapping γ of the lattice in Fig. 1.

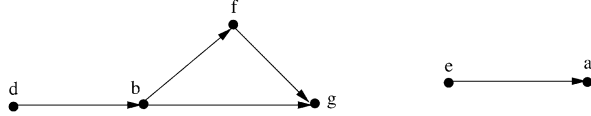


Fig. 3. The graph of the B relation for the lattice in Fig. 1.

that the lattice L is \wedge -semidistributive) as follows (see, for example, [8,11]):

Let $j, j' \in J(L)$.

Then $j B j'$ iff $j \neq j', j' \not\leq m(j), j'_* \leq m(j)$.

For an illustration of that definition, see Fig. 3.

There are relationships between the existence of cycles in the graph of the relation B and some classes of lattices. For example, Nation has shown that a \wedge -semidistributive lattice is semidistributive if and only if it contains no B -cycle of length 2 [11].

The following lemma gives a rewriting of the definition of the relation B using the mapping γ .

Lemma 1. Let L be a \wedge -semidistributive lattice, $j, j' \in J(L)$.

$j B j'$ iff $j \neq j'$ and $m(j) \in \gamma(j')$.

3. Minimum implicational basis for \wedge -semidistributive lattices

In this section, we give a polynomial time algorithm to compute a minimum implicational basis for a \wedge -semidistributive lattice.

We start with two technical lemmas on closed sets of a closure system \mathcal{C}_L . The first one is obvious since the elements of \mathcal{C}_L are order ideals of the induced poset by $J(L)$.

Lemma 2. Let $j, j' \in J(L)$ such that $j < j'$ and $X \in \mathcal{C}_L$. Then $j' \in X$ implies $j \in X$.

Consider now a \wedge -semidistributive lattice L and $j, j' \in J(L)$ such that $j B j'$. We denote by $P_{jj'}$ the set $J(j_*) \cup J(j')$.

Lemma 3. Let L be a \wedge -semidistributive lattice and $j, j' \in J(L)$ such that $j B j'$ and $X \in \mathcal{C}_L$. Then $P_{jj'} \subseteq X$ implies $j \in X$.

Proof. Let $x \in L$ such that $X = J(x)$ and $P_{jj'} \subseteq X$. Since $J(j_*) \subset X$ this implies that $j_* \vee j' \leq x$, and then it suffices to prove that $j \leq j_* \vee j'$.

Suppose that $j \not\leq j_* \vee j'$ and let $m' \in M(L)$ be a maximal element of $\{z \in L \mid z \not\leq j \text{ and } z \geq j_* \vee j'\}$. By definition of m' , we have $j \uparrow m'$. Moreover $j \downarrow m'$ since $j_* \leq m'$. Thus $j \downarrow m'$.

Consider now the meet-irreducible $m(j)$ associated with j . Then $j' \not\leq m(j)$ since $j B j'$. Thus since $j' \leq m'$, m' and $m(j)$ are two distinct elements such that $j \downarrow m'$ and $j \downarrow m(j)$. This contradicts the fact that L is \wedge -semidistributive. \square

We can now define a particular set of implications associated to a \wedge -semidistributive lattice L . Let $\Sigma_1 = \{j \rightarrow J(j)\}$, $\Sigma_2 = \{P_{jj'} \rightarrow j \mid j' \in J(L) \text{ and } j B j'\}$ and $\Sigma = \Sigma_1 \cup \Sigma_2$.

For example, the sets of implications Σ_1 and Σ_2 for the lattice in Fig. 1 are $\Sigma_1 = \{b \rightarrow ab, d \rightarrow cd, e \rightarrow cde, f \rightarrow cdef, g \rightarrow cdg\}$ and $\Sigma_2 = \{acd \rightarrow e, abc \rightarrow d, acdef \rightarrow b, acdg \rightarrow b, cdeg \rightarrow f\}$.

The following theorem shows that Σ is an implicational basis for \mathcal{C}_L .

Theorem 1. Let L be a \wedge -semidistributive lattice. Then the set of implications Σ is an implicational basis for \mathcal{C}_L .

Proof. We need to show that $\mathcal{C}_\Sigma = \mathcal{C}_L$.

Let $X \in \mathcal{C}_L$. By Lemma 2, X is Σ_1 -closed. By Lemma 3, X is Σ_2 -closed. Then X is Σ -closed and $\mathcal{C}_L \subseteq \mathcal{C}_\Sigma$.

Now let us show that $\mathcal{C}_\Sigma \subseteq \mathcal{C}_L$. Let $X \in \mathcal{C}_\Sigma$. Let $x_0 = \bigvee X$, i.e., the least closed set containing X . Clearly X is an ideal since it is Σ_1 -closed. Suppose that $X \notin \mathcal{C}_L$ and let j be a minimal element of $J(x_0) \setminus X$. Since $j \leq x_0$, we have $x_0 \not\leq m(j)$. Moreover $X \not\subseteq J(m(j))$, otherwise one would have $\bigvee X \leq m(j)$ and then $\bigvee X \neq x_0$. Thus there exists an element $j' \in X$ such that $m(j) \in \gamma(j')$ and therefore $P_{jj'} \rightarrow j \in \Sigma$ with $P_{jj'} \subseteq X$ and $j \notin X$. Then X is not Σ -closed, which concludes the proof. \square

Corollary 1. Let L be a \wedge -semidistributive lattice. Then there exists an implicational basis for \mathcal{C}_L with at most $|B| + |J(L)|$ implications, where $|B|$ is the number of arcs in the relation B .

Data: Let L be a \wedge -semidistributive lattice and $\mathcal{M}(C_L)$ the set of meet-irreducible elements of C_L .

Result: A minimum basis Σ of the closure system C_L .

begin

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 $\Sigma = \emptyset;$ 
for  $j \in J(L)$  do
   $\Sigma = \Sigma \cup \{j \rightarrow \varphi(j)\};$ 
  for  $j' \in J(L)$  do
     $P = (\varphi(j)) \setminus \{j\} \cup \varphi(j');$ 
     $\Sigma = \Sigma \cup \{P \rightarrow (P)\};$ 
 $\Sigma$  = a nonredundant cover of  $\Sigma;$ 
end

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Algorithm 1. Minimum-Basis($\mathcal{M}(C_L)$).

Clearly the set Σ of implications obtained as above is in general not minimum. For instance, for the set Σ associated to the lattice in Fig. 1, the implication $acd g \rightarrow b$ is redundant² and can be removed from Σ without changing $\mathcal{C}(\Sigma)$.

In the following we give a polynomial time algorithm to compute a minimum basis for a \wedge -semidistributive lattice.

3.1. Algorithm

This is based on Theorem 1 and the algorithm in [13]. Indeed, the algorithm in [13] computes a minimum basis (called there a minimum cover) from any given basis in polynomial time.

Let $\mathcal{M}(C_L)$ be the set of meet-irreducible elements. Consider the closure operator $\varphi: 2^J \rightarrow 2^J$, with for $X \subseteq J$, $\varphi(X) = \bigcap \{M \in \mathcal{M}(C_L) \mid X \subseteq M\}$. The images of the mapping φ are said closed sets, and they correspond to the elements of the closure system C_L .

Remark 1. We replaced $P \rightarrow j$ by $P \rightarrow \varphi(P)$ to guarantee the minimality after the calculation of a nonredundant cover of Σ .

Remark 2. Let us note that Algorithm 1 does not compute the same Σ as that of Theorem 1. This to avoid the computation of the relation B . But like the whole of the implications calculated by Algorithm 1 contains all implications of Theorem 1 (relative with the preceding remark), this guaranteed to us to have a cover of C_L .

Theorem 2. Let L be a \wedge -semidistributive lattice. Then Algorithm 1 computes a minimum implicational basis Σ

of C_L in $O(|J|^5 + |J|^3 |\mathcal{M}(C_L)|)$ time complexity. Moreover, the size of Σ is at most $|J(L)|^2$ implications.

Proof. Theorem 1 guarantees that Σ is a basis for the closure system C_L . Since the conclusions of all implications are closed by the mapping φ , the result in [13] guarantees that a not redundant basis is minimum.

Computing the closure of a set $X \subseteq J(L)$ by φ can be done in $O(|J(L)| |\mathcal{M}(C_L)|)$ time complexity. Thus the total time complexity for computing a basis is in $O(|J(L)|^3 |\mathcal{M}(C_L)|)$. Now computing a not redundant basis can be done in $O(|J(L)| |\Sigma|^2)$. Since Σ has at most $|J(L)|^2$ implications, we conclude that the time complexity of Algorithm 1 is in $O(|J(L)|^5 + |J(L)|^3 |\mathcal{M}(C_L)|)$. \square

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Acknowledgements

The authors are very grateful to the referees for very helpful comments.

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² An implication $A \rightarrow B$ in Σ is said redundant in Σ if it can be derived using Armstrong rules from $\Sigma \setminus \{A \rightarrow B\}$.

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