Large Alphabets and Incompressibility

Travis Gagie

Department of Computer Science University of Toronto

Abstract

We briefly survey some concepts related to empirical entropy — normal numbers, de Bruijn sequences and Markov processes — and investigate how well it approximates Kolmogorov complexity. Our results suggest ℓ th-order empirical entropy stops being a reasonable complexity metric for almost all strings of length m over alphabets of size n about when n^{ℓ} surpasses m.

Key words: Data compression; Kolmogorov complexity; Shannon's entropy; empirical entropy; normal numbers; de Bruijn sequences; threshold phenomena; self-information; Markov processes; relative entropy; birthday paradox

1 Introduction

For data compression, machine learning and cryptanalysis, we often want to know the Kolmogorov complexity K(S) [23,13,4,15] of a string S, that is, the minimum space needed to store S. It is formally defined as the length in bits of the shortest program that outputs S. Notice our choice of programming language does not affect this length by more than an additive constant, provided it is Turing-equivalent; for example, the length of the shortest such FORTRAN program exceeds the length of the shortest such LISP program by no more than the length of the shortest LISP-interpreter written in FORTRAN which does not depend on S. Unfortunately, a simple diagonalization shows Kolmogorov complexity is incomputable: Given a program \mathcal{A} for computing Kolmogorov complexity, we could write a program \mathcal{B} that searches until it finds and outputs a string S with $\mathcal{A}(S) = K(S)$ greater than \mathcal{B} 's length in bits, contradicting the definition of K(S). Thus, researchers substitute various other complexity metrics; in this paper we study one of the most popular empirical entropy.

Empirical entropy is rooted in information theory. Let X be a random variable that takes on one of n values according to $P = p_1, \ldots, p_n$. Shannon [20]

proposed that any function H(P) measuring our uncertainty about X should have three properties:

- (1) "*H* should be continuous in the p_i ."
- (2) "If all the p_i are equal, $p_i = \frac{1}{n}$, then H should be a monotonic increasing function of n."
- (3) "If a choice be broken down into two successive choices, the original H should be the weighted sum of the individual values of H."

He proved the only function with these properties is $H(P) = \sum_{i=1}^{n} p_i \log(1/p_i)$, which he called the *entropy* of P. The choice of the logarithm's base determines the unit; by convention, log means \log_2 and the units are bits.

Let ℓ be a non-negative integer and suppose $S = s_1 \cdots s_m$. The ℓ th-order empirical entropy of S (see, e.g., [16]) is our expected uncertainty about the random variable s_i given a context of length ℓ , as in the following experiment: i is chosen uniformly at random from $\{1, \ldots, m\}$; if $i \leq \ell$, then we are told s_i ; otherwise, we are told $s_{i-\ell} \cdots s_{i-1}$. Specifically,

$$H_{\ell}(S) = \begin{cases} \sum_{a \in S} \frac{\#_a(S)}{m} \log \frac{m}{\#_a(S)} & \text{if } \ell = 0, \\\\ \frac{1}{m} \sum_{|\alpha| = \ell} |S_{\alpha}| H_0(S_{\alpha}) & \text{if } \ell \ge 1. \end{cases}$$

In this paper, $a \in S$ means character a occurs in S; $\#_a(S)$ is the number of occurrences of a in S; and S_α is the string whose *i*th character is the one immediately following the *i*th occurrence of string α in S — the length of S_α is the number of occurrences of α in S, which we denote $\#_\alpha(S)$, unless α is a suffix of S, in which case it is 1 less. We assume $S_\alpha = S$ when α is empty. Notice $0 \leq H_{\ell+1}(S) \leq H_{\ell}(S) \leq \log |\{a : a \in S\}|$ for $\ell \geq 0$. For example, if Sis the string TORONTO, then

$$H_0(S) = \frac{1}{7}\log 7 + \frac{3}{7}\log \frac{7}{3} + \frac{1}{7}\log 7 + \frac{2}{7}\log \frac{7}{2} \approx 1.84 ,$$

$$H_1(S) = \frac{1}{7} \left(H_0(S_{\rm N}) + 2H_0(S_{\rm O}) + H_0(S_{\rm R}) + 2H_0(S_{\rm T}) \right)$$

$$= \frac{1}{7} \left(H_0({\rm T}) + 2H_0({\rm RN}) + H_0({\rm O}) + 2H_0({\rm OO}) \right)$$

$$= 2/7 \approx 0.29$$

and all higher-order empirical entropies of S are 0. This means if someone chooses a character uniformly at random from TORONTO and asks us to

guess it, then our uncertainty is about 1.84 bits. If they tell us the preceding character before we guess, then on average our uncertainty is about 0.29 bits; if they tell us the preceding two characters, then we are certain of the answer.

Empirical entropy has a surprising connection to number theory. Let $(x)_{n,m}$ denote the first m digits of the number x in base $n \ge 2$. Borel [2] called x normal in base n if, for $\alpha \in \{0, \ldots, n-1\}^*$, $\lim_{m\to\infty} \frac{\#_{\alpha}(x)_{n,m}}{m} = 1/n^{|\alpha|}$. For example, the Champernowne constant [5] and Copeland-Erdös constant [6], $0.123456789101112\ldots$ and $0.23571113171923\ldots$, are normal in base 10. Notice x being normal in base n is equivalent to $\lim_{m\to\infty} H_{\ell}((x)_{n,m}) = \log n$ for $\ell \ge 0$. Borel called x absolutely normal if it is normal in all bases. He proved almost all numbers are absolutely normal but Sierpinski [21] was the first to find an example, which is still not known to be computable. Turing [24] claimed there exist computable absolutely normal numbers' representations have finite Kolmogorov complexity yet look random if we consider only empirical entropy — regardless of base and order. Of course, we are sometimes fooled whatever computable complexity metric we consider.

Now consider de Bruijn sequences [7] from combinatorics. An *n*-ary linear de Bruijn sequence of order ℓ is a string over $\{0, \ldots, n-1\}$ containing every possible ℓ -tuple exactly once. For example, the binary linear de Bruijn sequences of order 3 are the 16 10-bit substrings of 00010111000101110 and its reverse: 0001011100, ..., 1000101110, 0111010001, ..., 0011101000. By definition, such strings have length $n^{\ell} + \ell - 1$ and ℓ th-order empirical entropy 0 (but $(\ell - 1)$ st-order empirical entropy $\frac{(n^{\ell}-1)\log n}{n^{\ell}+\ell-1}$). However, Rosenfeld [19] showed there are $(n!)^{n^{\ell-1}}$ of them. It follows that one randomly chosen has expected Kolmogorov complexity in $\Theta\left(\log(n!)^{n^{\ell-1}}\right) = \Theta(n^{\ell}\log n)$; whereas Borel's normal numbers can be much less complex than empirical entropy suggests, de Bruijn sequences can be much more complex.

Empirical entropy also has connections to algorithm design. For example, Munro and Spira [18] used 0th-order empirical entropy to analyze several sorting algorithms and Sleator and Tarjan [22] used it in the Static Optimality Theorem: Suppose we perform a sequence of m operations on a splay-tree, with s_i being the target of the *i*th operation; if $S = s_1 \cdots s_m$ includes every key in the tree, then we use $O((H_0(S) + 1)m)$ time. Of course, most of the algorithms analyzed in terms of empirical entropy are for data compression. Manzini's analysis [16] of the Burrows-Wheeler Transform [3] is particularly interesting. He proved an algorithm based on the Transform stores any string S of length m over an alphabet of size n in at most about $(8H_{\ell}(S) + 1/20)m + n^{\ell}(2n\log n + 9)$ bits, for all $\ell \geq 0$ simultaneously. Subsequent research by Ferragina, Manzini, Mäkinen and Navarro [8], for example, has shown that if $n^{\ell+1}\log m \in o(m\log n)$, then we can store an efficient index for S in $(H_{\ell}(S) + o(\log n))m$ bits. Notice we cannot lift the restriction on n and ℓ to $n^{\ell} \in O(m)$: If S is a randomly chosen n-ary linear de Bruijn sequence of order ℓ , then $m = n^{\ell} + \ell - 1$ and $H_{\ell}(S) = 0$, so $(cH_{\ell}(S) + o(\log n))m = o(n^{\ell} \log n)$ for any c, but $K(S) \in \Theta(n^{\ell} \log n)$ in the expected case.

In this paper we investigate further the relationship between the order ℓ , the alphabet size n and the string length m. Our results suggest ℓ th-order empirical entropy stops being a reasonable complexity metric for almost all strings about when n^{ℓ} surpasses m. For simplicity, we assume ℓ and n are given to us as (possibly constant) functions from m to the positive integers and consider $S \in \{1, \ldots, n\}^m$. In Section 2 we prove that, for any fixed $c \geq 1$ and $\epsilon > 0$, if $n^{\ell+1/c} \log n \in o(m)$ and m is sufficiently large, then $K(S) < (cH_{\ell}(S) + \epsilon)m$. We use a new upper bound for compressing probability distributions, which extends our results from [9] and may be of independent interest. In Section 3 we prove that if $\epsilon < 1/c$, ℓ is fixed, $n^{\ell+1/c-\epsilon} \in \Omega(m)$ and m is sufficiently large, then $K(S) > (cH_{\ell}(S) + \frac{\epsilon}{3} \log n)m$ with high probability for randomly chosen S. As a corollary we prove a nearly matching lower bound for compressing probability distributions.

It seems interesting that slightly changing the relationship between ℓ , n and m can change $(cH_{\ell}(S) + o(\log n))m$ from an upper bound on K(S) to an almost certain lower bound. Phenomena like this one, in which small shifts in parameters change a property asymptotically from very likely to very unlikely, are called *threshold phenomena*; they are common and well-studied in several disciplines (see, e.g., [12]) but we know of no others related to data compression. Although our proof of a threshold phenomenon requires ℓ to be fixed, we emphasize it holds for any constant coefficient $c \geq 1$ before $H_{\ell}(S)$ and any $o(\log n)$ second term in the formula.

2 Upper bounds

We first rephrase the definition of empirical entropy: For $\ell \geq 0$, the ℓ th-order empirical entropy of a string S is the minimum self-information per character of S emitted by an ℓ th-order Markov process. The *self-information* of an event with probability p is $\log(1/p)$. An ℓ th-order Markov process is a string of random variables in which each variable depends only on at most ℓ immediate predecessors (see, e.g., [20]); a process is said to emit the values of its variables. We use relative entropy [14], also called the Kullback-Leibler distance, to prove the two definitions equivalent. Let $P = p_1, \ldots, p_n$ and $Q = q_1, \ldots, q_n$ be probability distributions over $\{1, \ldots, n\}$; the *relative entropy* between Pand Q, $D(P||Q) = \sum_{i=1}^{n} p_i \log(p_i/q_i)$, is often used in information theory to measure how well Q approximates P. Although relative entropy is not a distance metric — it is not symmetric and does not obey the triangle inequality — it is 0 when P = Q and positive otherwise. **Theorem 1** For any string $S \in \{1, ..., n\}^m$ and $\ell \ge 0$, we have $H_{\ell}(S) = \frac{1}{m} \min \left\{ \log(1/\Pr[Q \text{ emits } S]) : Q \text{ is an } \ell \text{th-order Markov process} \right\}.$

PROOF. Consider the probability an ℓ th-order Markov process Q emits S. Assume, without loss of generality, that Q first emits $s_1 \cdots s_\ell$ with probability 1. For $\alpha \in \{1, \ldots, n\}^\ell$, let $P_\alpha = p_{\alpha,1}, \ldots, p_{\alpha,n}$ be the normalized distribution of the characters in S_α , so $H(P_\alpha) = H_0(S_\alpha)$; let $Q_\alpha = q_{\alpha,1}, \ldots, q_{\alpha,n}$, where $q_{\alpha,a}$ is the probability Q emits a immediately after an occurrence of α . Then

$$\log \frac{1}{\Pr[Q \text{ emits } S]}$$

$$= \log \prod_{i=\ell+1}^{m} \frac{1}{q_{s_{i-\ell}\cdots s_{i-1},s_i}}$$

$$= \sum_{i=\ell+1}^{m} \log \frac{1}{q_{s_{i-\ell}\cdots s_{i-1},s_i}}$$

$$= \sum_{|\alpha|=\ell} \sum_{a\in S_{\alpha}} \#_a(S_{\alpha}) \log \frac{1}{q_{\alpha,a}}$$

$$= \sum_{|\alpha|=\ell} |S_{\alpha}| \sum_{a\in S_{\alpha}} p_{\alpha,a} \left(\log \frac{p_{\alpha,a}}{q_{\alpha,a}} + \log \frac{1}{p_{\alpha,a}}\right)$$

$$= \sum_{|\alpha|=\ell} |S_{\alpha}| (D(P_{\alpha}||Q_{\alpha}) + H(P_{\alpha}))$$

$$\geq \sum_{|\alpha|=\ell} |S_{\alpha}| H(P_{\alpha})$$

$$= H_{\ell}(S)m ,$$

with equality throughout if $P_{\alpha} = Q_{\alpha}$ for $\alpha \in \{1, \ldots, n\}^{\ell}$. \Box

We now consider how compactly we can store probability distributions, Markov processes and, ultimately, strings.

Lemma 2 Fix $c \ge 1$ and $\epsilon > 0$ and let $P = p_1, \ldots, p_n$ be a probability distribution over $\{1, \ldots, n\}$. For some probability distribution Q with $D(P||Q) < (c-1)H(P) + \epsilon$, storing Q takes $O(n^{1/c} \log n)$ bits.

PROOF. Let $t \leq rn^{1/c}$ be the number of probabilities in P that are at least $\frac{1}{rn^{1/c}}$, where $r = \frac{2^{\epsilon/2}}{2^{\epsilon/2}-1}$. For each such p_i , we record i and $\lfloor p_i r^2 n \rfloor$. Since r depends only on ϵ , which is fixed, in total we use $O(n^{1/c} \log n)$ bits. This

information lets us later recover $Q = q_1, \ldots, q_n$, where

$$q_i = \begin{cases} \left(1 - \frac{1}{r}\right) \frac{\lfloor p_i r^2 n \rfloor}{\sum \left\{ \lfloor p_j r^2 n \rfloor : p_j \ge \frac{1}{r n^{1/c}} \right\}} & \text{if } p_i \ge \frac{1}{r n^{1/c}}, \\ \frac{1}{r(n-t)} & \text{otherwise.} \end{cases}$$

Suppose $p_i \ge \frac{1}{rn^{1/c}}$; then $p_i r^2 n \ge r$. Since $\sum \left\{ \lfloor p_j r^2 n \rfloor : p_j \ge \frac{1}{rn^{1/c}} \right\} \le r^2 n$,

$$p_i \log \frac{p_i}{q_i} \le p_i \log \left(\frac{r}{r-1} \cdot \frac{p_i r^2 n}{\lfloor p_i r^2 n \rfloor} \right) < 2p_i \log \frac{r}{r-1} = p_i \epsilon$$
.

Now suppose $p_i < \frac{1}{rn^{1/c}}$; then $p_i \log(1/p_i) > \frac{p_i}{c} \log n$. Thus,

$$p_i \log \frac{p_i}{q_i} < p_i \log \frac{r(n-t)}{rn^{1/c}} \le \frac{(c-1)p_i}{c} \log n < (c-1)p_i \log \frac{1}{p_i} \ .$$

Finally, since $p \log(1/p) \ge 0$ for $p \le 1$, we have

$$D(P||Q) < \sum \left\{ (c-1)p_i \log \frac{1}{p_i} : p_i < \frac{1}{rn^{1/c}} \right\} + \epsilon \le (c-1)H(P) + \epsilon .$$

Corollary 3 Fix $c \ge 1$ and $\epsilon > 0$ and consider a string $S \in \{1, \ldots, n\}^m$. For some ℓ th-order Markov process Q with $\log(1/\Pr[Q \text{ emits } S]) < (cH_{\ell}(S) + \epsilon)m$, storing Q takes $O(n^{\ell+1/c} \log n)$ bits.

PROOF. First we store $s_1 \cdots s_\ell$. For $\alpha \in \{1, \ldots, n\}^\ell$, let $P_\alpha = p_{\alpha,1}, \ldots, p_{\alpha,n}$ be the normalized distribution of characters in S_α and let $Q_\alpha = q_{\alpha,1}, \ldots, q_{\alpha,n}$ be the probability distribution with $D(P_\alpha || Q_\alpha) < (c-1)H(P_\alpha) + \epsilon$ obtained from applying Lemma 2 to c, ϵ and P_α . We store every Q_α , using a total of $O(n^{\ell+1/c} \log n)$ bits.

This information lets us later recover a Markov process Q that first emits $s_1 \cdots s_\ell$ and in which, for $\alpha \in \{1, \ldots, n\}^\ell$ and $a \in \{1, \ldots, n\}$, the probability a is emitted immediately after an occurrence of α is $q_{\alpha,a}$. As in the proof of Theorem 1, $\log(1/\Pr[Q \text{ emits } S]) = \sum_{|\alpha|=\ell} |S_{\alpha}| (D(P_{\alpha} || Q_{\alpha}) + H(P_{\alpha}))$, so

$$\log \frac{1}{\Pr[Q \text{ emits } S]} < \sum_{|\alpha|=\ell} |S_{\alpha}| (cH(P_{\alpha}) + \epsilon) \le (cH_{\ell}(S) + \epsilon)m .$$

We note that, given a string $S \in \{1, \ldots, n\}^m$, we can store an ℓ th-order Markov process Q with $\log(1/\Pr[Q \text{ emits } S]) = H_{\ell}(S)$ in $O\left(n^{\ell+1}\log\left(\frac{m}{n^{\ell+1}}+1\right)\right)$ bits, as a table containing $\#_a(S_{\alpha}) = \#_{\alpha a}(S) \leq m$ for $\alpha a \in \{1, \ldots, n\}^{\ell+1}$. Grossi, Gupta and Vitter [10] investigated the space needed for such a table; they also showed that, apart from the cost of storing the table, we can store S in $H_{\ell}(S)$ bits. However, because we do not see how to store the table in less space when there is a constant coefficient c > 1 before $H_{\ell}(S)$, we tolerate the ϵ term in Corollary 3 and the following theorem.

Theorem 4 Fix $c \ge 1$ and $\epsilon > 0$ and let ℓ and n be functions from m to the positive integers. Consider a string $S \in \{1, \ldots, n\}^m$. If $n^{\ell+1/c} \log n \in o(m)$ and m is sufficiently large, then $K(S) < (cH_{\ell}(S) + \epsilon)m$.

PROOF. By Corollary 3, since $n^{\ell+1/c} \log n \in o(m)$ and m is sufficiently large, we can store an ℓ th-order Markov process Q with $\log(1/\Pr[Q \text{ emits } S]) < (cH_{\ell}(S) + \epsilon/2)m$ in $\epsilon m/2 - 1$ bits. Shannon [20] showed how, given Q, we can store S in $\lceil \log(1/\Pr[Q \text{ emits } S]) \rceil$ bits. Thus, we can store Q and S together in fewer than $(cH_{\ell}(S) + \epsilon)m$ bits. \Box

3 Lower bounds

Consider the so-called *birthday paradox*: If we draw *m* times from $\{1, \ldots, n\}$, then the probability at least two of the numbers drawn are the same is about $1 - 1/e^{\frac{m(m-1)}{2n}}$. Thus, for $\ell \geq 1$, if $n^{1/2} \in \omega(m)$ and *S* is chosen randomly, then with high probability $H_{\ell}(S) = 0$ because no character appears more than once in *S*. (Notice also $H_0(S) \leq \log m \leq \log(n)/2$ for sufficiently large *m*.) Thus, we cannot lift the restriction on *n* and ℓ in Theorem 4 to $n^{1/2-\epsilon} \in O(m)$. We use a similar but more complicated argument to show we cannot even lift the restriction to $n^{\ell+1/c-\epsilon} \in O(m)$. Essentially, we use a Chernoff bound on the probability of there being any frequent ℓ -tuples in *S*. Since the probability of an ℓ -tuple occurring somewhere in *S* depends on whether it occurs in neighbouring positions, we apply the following intuitive lemma (proven in, e.g., [17]) before we apply the Chernoff bound.

Lemma 5 Let X_1, \ldots, X_m be binary random variables such that, for $1 \le i \le m$ and $b \in \{0,1\}^{i-1}$, $\Pr\left[X_i = 1 \mid X_1 \cdots X_{i-1} = b\right] \le p$. Let Y_1, \ldots, Y_m be independent binary random variables, each equal to 1 with probability p. For $0 \le q \le 1$,

$$\Pr\left[\sum_{j=1}^{m} X_j \ge qm\right] \le \Pr\left[\sum_{j=1}^{m} Y_j \ge qm\right]$$

Theorem 6 Fix $c \ge 1$, ϵ with $0 < \epsilon < 1/c$ and $\ell \ge 1$ and let n be a function from m to the positive integers. Choose a string $S \in \{1, \ldots, n\}^m$ uniformly at random. If $n^{\ell+1/c-\epsilon} \in \Omega(m)$ and m is sufficiently large, then $K(S) > \left(cH_{\ell}(S) + \frac{\epsilon}{3}\log n\right)m$ with high probability.

PROOF. Since there are n^m choices for S and only

$$\sum \left\{ 2^i : 0 \le i \le \lfloor (1 - \epsilon/3)m \log n \rfloor \right\} \le 2n^{(1 - \epsilon/3)m} - 1$$

binary strings of length at most $(1 - \epsilon/3)m \log n$, we have $K(S) \ge (1 - \epsilon/3)m \log n$ with probability greater than $1 - 2/n^{\epsilon m/3}$. Thus, we need only show $cH_{\ell}(S) < (1 - 2\epsilon/3) \log n$ with high probability. By definition,

$$\begin{aligned} H_{\ell}(S) \\ &\leq \max_{|\alpha|=\ell} \{H_0(S_{\alpha})\} \\ &\leq \max_{|\alpha|=\ell} \{\log |\{a : a \in S_{\alpha}\}|\} \\ &\leq \max_{|\alpha|=\ell} \left\{ \log \left(|\{a : a \in S_{\alpha}, a \notin \alpha\}| + \ell \right) \right\} . \end{aligned}$$

Notice $n \in \omega\left(m^{\frac{1}{\ell+1/c}}\right)$. We will show

$$\Pr\left[\left|\left\{a : a \in S_{\alpha}, a \notin \alpha\right\}\right| \ge n^{1/c - 2\epsilon/3} - \ell\right] \le \frac{1}{2^{n^{\epsilon/3} - \ell}}$$

for each $\alpha \in \{1, \ldots, n\}^{\ell}$, so

$$\Pr\left[\max_{|\alpha|=\ell}\{|\{a:a\in S_{\alpha},a\not\in\alpha\}|\}\geq n^{1/c-2\epsilon/3}-\ell\right]\leq \frac{n^{\ell}}{2^{n^{\epsilon/3}-\ell}}$$
$$\Pr\left[\max_{|\alpha|=\ell}\left\{\log\left(|\{a:a\in S_{\alpha},a\not\in\alpha\}|+\ell\right)\right\}\geq \left(\frac{1}{c}-\frac{2\epsilon}{3}\right)\log n\right]\leq \frac{n^{\ell}}{2^{n^{\epsilon/3}-\ell}}$$

and $cH_{\ell}(S) < (1 - 2\epsilon c/3) \log n \le (1 - 2\epsilon/3) \log n$ with high probability.

Consider $\alpha \in \{1, \ldots, n\}^{\ell}$. Let $X_1, \ldots, X_{m-\ell}$ be binary random variables, with $X_i = 1$ if $s_i \cdots s_{i+\ell-1} = \alpha$ and $s_{i+\ell} \notin \alpha$. Notice $|\{a : a \in S_{\alpha}, a \notin \alpha\}| + \ell \leq \sum_{i=1}^{m-\ell} X_i + \ell$. For $\ell + 1 \leq i \leq m - \ell$, by definition, X_i is independent of $X_1, \ldots, X_{i-\ell-1}$; if any of $X_{i-\ell}, \ldots, X_{i-1}$ are 1, then at least one of $s_i, \ldots, s_{i+\ell-1}$ is not in α , so $X_i = 0$; and

$$\Pr \left[X_i = 1 \mid X_{i-\ell} = \dots = X_{i-1} = 0 \right]$$

=
$$\frac{\Pr \left[X_i = 1 \text{ and } X_{i-\ell} = \dots = X_{i-1} = 0 \right]}{\Pr \left[X_{i-\ell} = \dots = X_{i-1} = 0 \right]}$$

=
$$\frac{\Pr \left[X_i = 1 \right]}{1 - \Pr \left[X_{i-\ell} = 1 \text{ or } \dots \text{ or } X_{i-1} = 1 \right]}$$

$$\leq \frac{\Pr \left[X_i = 1 \right]}{1 - \sum_{j=i-\ell}^{i-1} \Pr \left[X_j = 1 \right]}$$

$$\leq \frac{1/n^{\ell}}{1 - \ell/n^{\ell}}$$

=
$$\frac{1}{n^{\ell} - \ell} .$$

Let $Y_1, \ldots, Y_{m-\ell}$ be independent binary random variables, each equal to 1 with probability $p = \frac{1}{n^{\ell}-\ell}$, and let $q = \frac{n^{1/c-2\epsilon/3}-\ell}{m-\ell}$. If q > 1 the proof is finished, because $\Pr\left[\sum_{i=1}^{m-\ell} X_i \ge q(m-\ell)\right] = 0$; otherwise by Lemma 5,

$$\Pr\left[\sum_{i=1}^{m-\ell} X_i \ge q(m-\ell)\right] \le \Pr\left[\sum_{i=1}^{m-\ell} Y_i \ge q(m-\ell)\right]$$

and it remains for us to show

$$\Pr\left[\sum_{i=1}^{m-\ell} Y_i \ge q(m-\ell)\right] \le \frac{1}{2^{n^{\epsilon/3}-\ell}} \ .$$

Since ℓ is fixed and $n^{\ell+1/c-\epsilon} \in \Omega(m)$, we have $p(m-\ell) \in O(n^{1/c-\epsilon}) \subset o(q(m-\ell))$; thus, for sufficiently large $m, q(m-\ell) \geq 6p(m-\ell)$ and we can use the following simple Chernoff bound [11]:

$$\Pr\left[\sum_{i=1}^{m-\ell} Y_i \ge q(m-\ell)\right] \le \frac{1}{2^{q(m-\ell)}} = \frac{1}{2^{n^{1/c-2\epsilon/3}-\ell}} \ .$$

Finally, since $\epsilon < 1/c$,

$$\Pr\left[\sum_{i=1}^{m-\ell} Y_i \ge q(m-\ell)\right] \le \frac{1}{2^{n^{\epsilon/3}-\ell}} \ .$$

Corollary 7 Fix $c \ge 1$ and ϵ with $0 < \epsilon < 1/c$ and let P be a probability distribution over $\{1, \ldots, n\}$. In the worst case, for any probability distribution Q with $D(P||Q) \le (c-1)H(P) + o(\log n)$, storing Q takes $\omega(n^{1/c-\epsilon})$ bits.

PROOF. For the sake of a contradiction, assume there exists an algorithm \mathcal{A} that, given any probability distribution P over $\{1, \ldots, n\}$, stores a probability distribution Q with $D(P||Q) \leq (c-1)H(P) + o(\log n)$ in $O(n^{1/c-\epsilon})$ bits. Then a proof similar to that of Theorem 4, but substituting \mathcal{A} for Lemma 2, yields:

Fix $c \geq 1$ and ϵ with $0 < \epsilon < 1/c$ and let ℓ and n be functions from m to the positive integers. Consider a string $S \in \{1, \ldots, n\}^m$. If $n^{\ell+1/c-\epsilon} \in o(m)$, then $K(S) \leq (cH_{\ell}(S) + o(\log n))m$.

Suppose we fix c and ℓ , choose $\epsilon < 1/c$ and n such that $n^{\ell+1/c-\epsilon} \in o(m)$ but $n^{\ell+1/c-\epsilon/2} \in \Omega(m)$, and choose a string $S \in \{1, \ldots, n\}^m$ uniformly at random. The claim above gives $K(S) \leq (cH_\ell(S) + o(\log n))m$ but by Theorem 6, for sufficiently large m, $K(S) > (cH_\ell(S) + \frac{\epsilon}{6}\log n)m$ with high probability. \Box

4 Future work

Suppose we want to store a probability distribution P over a set of strings. We recently proved that, in theory, if the relative entropy is small between P and the probability distribution induced by a low-order Markov process Q, the we can store P accurately and efficiently by storing an approximation of Q. We hope experiments will show this technique to be practical.

Our proof of Theorem 6 is slightly complicated because if, for some ℓ -tuple α , a non-empty string is both a suffix and a prefix of α , then occurrences of α can overlap and any one occurrence increases the probability of others. In this paper we used the fact that if two occurrences of α overlap, the the first must be immediately followed by a character in α . We recently proved that, moreover, it must be immediately followed by one of $O(\log \ell)$ characters. We are now trying to use this result to prove a version of Theorem 6 that does not require ℓ to be fixed.

We are also trying another approach to generalize Theorem 6. Results about linear de Bruijn sequences are often proved by considering them as Eulerian tours on certain graphs, called de Bruijn graphs. In fact, any string can be considered as a walk on a de Bruijn graph; random strings correspond to random walks. Since de Bruijn graphs are good expanders, random walks on them have properties that may be useful in reasoning about random strings.

Acknowledgments

Many thanks to Giovanni Manzini and Charlie Rackoff, who supervised this research; Mark Braverman, Paolo Ferragina, Roberto Grossi and the anonymous reviewers, for helpful comments; and Alistair Moffat, for editorial patience.

References

- V. Becher and S. Figueira. An example of a computable absolutely normal number. *Theoretical Computer Science*, 270:947–958, 2002.
- [2] É. Borel. Les probabilités dénombrables et leur applications arithmétiques. Rendiconti del Circolo Matematico di Palermo, 27:247–271, 1909.
- [3] M. Burrows and D.J. Wheeler. A block-sorting lossless data compression algorithm. Technical Report 24, Digital Equipment Corporation, 1994.
- [4] G.J. Chaitin. On the length of programs for computing finite binary sequences: Statistical considerations. *Journal of the ACM*, 16:145–159, 1969.
- [5] D.G. Champernowne. The construction of decimals normal in the scale of 10. Journal of the London Mathematical Society, 8:254–260, 1933.
- [6] A.H. Copeland and P. Erdös. Note on normal numbers. Bulletin of the American Mathematical Society, 52:857–860, 1946.
- [7] N.G. de Bruijn. A combinatorial problem. Koninklijke Nederlandse Akademie van Wetenschappen, 49:758–764, 1946.
- [8] P. Ferragina, G. Manzini, V. Mäkinen, and G. Navarro. Compressed representations of sequences and full-text indexes. *ACM Transactions on Algorithms*, to appear.
- [9] T. Gagie. Compressing probability distributions. *Information Processing Letters*, 97:133–136, 2006.
- [10] R. Grossi, A. Gupta, and J.S. Vitter. An algorithmic framework for compression and text indexing. Submitted.
- [11] T. Hagerup and C. Rüb. A guided tour of Chernoff bounds. Information Processing Letters, 33:305–308, 1990.
- [12] G. Kalai and S. Safra. Threshold phenomena and influence. In A.G. Percus, G. Istrate, and C. Moore, editors, *Computational Complexity and Statistical Physics*. Oxford University Press, 2006.
- [13] A.N. Kolmogorov. Three approaches to the quantitative definition of information. Problems in Information Transmission, 1:1–7, 1965.

- [14] S. Kullback and R.A. Leibler. On information and sufficiency. Annals of Mathematical Statistics, 22:79–86, 1951.
- [15] M. Li and P. Vitányi. An Introduction to Kolmogorov Complexity and Its Applications. Springer-Verlag, 2nd edition, 1997.
- [16] G. Manzini. An analysis of the Burrows-Wheeler Transform. Journal of the ACM, 48:407–430, 2001.
- [17] R. Motwani and P. Raghavan. Randomized Algorithms. Cambridge University Press, 1995.
- [18] J.I. Munro and P.M. Spira. Sorting and searching in multisets. SIAM Journal on Computing, 5:1–8, 1976.
- [19] V.R. Rosenfeld. Enumerating De Bruijn sequences. MATCH Communications in Mathematical and in Computer Chemistry, 45:71–83, 2002.
- [20] C.E. Shannon. A mathematical theory of communication. Bell System Technical Journal, 27:379–423, 623–656, 1948.
- [21] M.W. Sierpinski. Démonstration élémentaire du théorème de M. Borel sur les nombres absolument normaux et détermination d'un tel nombre. Bulletin de la Société Mathématiques de France, 45:127–132, 1917.
- [22] D.D. Sleator and R.E. Tarjan. Self-adjusting binary search trees. Journal of the ACM, 32:652–686, 1985.
- [23] R.J. Solomonoff. A formal theory of inductive inference. Information and Control, 7:1–22, 224–254, 1964.
- [24] A.M. Turing. A note on normal numbers. In J.L. Britton, editor, Collected Works of A.M. Turing: Pure Mathematics, pages 117–119. North-Holland, 1992.