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Finding cycles in hierarchical hypercube networks $\stackrel{\mbox{\tiny{\sc black}}}{\longrightarrow}$

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ABSTRACT

The hierarchical hypercube network, which was proposed as an alternative to the hypercube, is suitable for building a large-scale multiprocessor system. A bipartite graph G = (V, E) is bipancyclic if it contains cycles of all even lengths ranging from 4 to |V|. In this paper, we show that the hierarchical hypercube network is bipancyclic.

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1. Introduction

In recent decades, many interconnection network topologies have been proposed in the literature (see [1,3,6]) for the purpose of connecting hundreds or thousands of processors together. Among them, the hypercube network possesses many attractive properties such as regularity, symmetry, logarithmic diameter, strong connectivity, recursive construction, partition ability, and relatively low link complexity [6].

The hierarchical hypercube network [8–10], which was proposed as an alternative to the hypercube network, is

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feasible to be implemented with thousands of or more processors, while retaining a good performance. It has a two-level structure; the hypercube networks are taken as basic modules and connected to form a larger hypercube network, where each basic module is regarded as one processor of the larger hypercube network. It bears the advantages of a hierarchical structure; thus, it has a smaller diameter, degree, link density and fanout than a comparable hypercube network. It also inherits some favorable properties, e.g., regularity, symmetry and logarithmic diameter, from the hypercube network [10].

Linear arrays and rings are two fundamental networks that are suitable for parallel and distributed computation. Many simple and efficient algorithms with low communication costs were designed on linear arrays and rings for solving a variety of algebraic and graph problems (see [6]). The hierarchical hypercube is known to be bipartite [10]. In this paper, we show that the hierarchical hypercube is bipancyclic. A bipartite graph G = (V, E) is *bipancyclic* if it contains cycles of all even lengths ranging from 4 to |V|. Consequently, previous algorithms on linear arrays and rings can be executed on the hierarchical hypercube network as well. In the next section, the topology of the hierarchical hypercube network is first reviewed.

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Fig. 1. Topology of a 6-HHC network.

2. Hierarchical hypercube networks

It is convenient to represent a network with a graph *G*, where each vertex (edge) of *G* uniquely represents a processor (link) of the network. By Q_k we denote a *k*-dimensional hypercube network, where $k \ge 1$. The vertex set of Q_k is $\{x_{k-1}x_{k-2}...x_0: x_i = 0 \text{ or } 1 \text{ for all } 0 \le i \le k-1\}$, and the edge set of Q_k is $\{(x_{k-1}x_{k-2}...x_0, x_{k-1}...x_{r+1}, x_rx_{r-1}...x_0): 0 \le r \le k-1\}$, where $\overline{x_r}$ is the complement of x_r . There are 2^k processors contained in Q_k . Throughout this paper, we use network and graph, processor and vertex, link and edge, interchangeably.

Let $n = 2^m + m$, where $m \ge 1$ is an integer. An *n*-dimensional hierarchical hypercube network (abbreviated to *n*-HHC network) can be obtained by replacing each vertex, say *P*, of Q_{2^m} with Q_m , where each vertex of Q_m is uniquely connected to an adjacent vertex of *P*. Each vertex of an *n*-HHC network can be identified with a two-tuple (S, P), where $S = s_{n-m-1}s_{n-m-2}\dots s_0$ is a binary sequence of length n - m telling which Q_m the vertex is located in and $P = p_{m-1}p_{m-2}\dots p_0$ is a binary sequence of length *m* giving the address of the vertex in the located Q_m .

For a binary sequence $X = x_{|X|-1}x_{|X|-2}...x_0$, we let $X^{(l)} = x_{|X|-1}...\overline{x_l}...x_0$ and dec(X) be the decimal value of X, where $0 \le l \le |X| - 1$ and |X| denotes the length of X. The following is a formal definition of an *n*-HHC network, in terms of graph theory.

Definition 1. The vertex set of an *n*-HHC network is $\{(S, P): S = s_{n-m-1}s_{n-m-2} \dots s_0 \text{ and } P = p_{m-1}p_{m-2} \dots p_0$ are two binary sequences of lengths n - m and m, respectively}, where $n = 2^m + m$ and $m \ge 1$. The vertex adjacency of an *n*-HHC network is defined as follows: (S, P) is adjacent to (1) $(S, P^{(l)})$ for all $0 \le l \le m - 1$ and (2) $(S^{(\text{dec}(P))}, P)$.

There are a total of $2^{2^m} \times 2^m = 2^{2^m+m} = 2^n$ vertices in an *n*-HHC network. Edges defined by (1) are referred to as *internal edges*, and those defined by (2) are referred to as *external edges*. Each internal edge is contained in Q_m , and each external edge connects two Q_m 's. Fig. 1 shows the topology of a 6-HHC network (i.e., m = 2), where vertices (0000, 01), (0000, 11) are connected by an internal edge and vertices (0000, 01), (0010, 01) are connected by an external edge.

An *n*-HHC network is regular of degree m + 1, and it has a diameter of 2^{m+1} (see [10]). Besides, it is vertex-symmetric. For each vertex *A* of an *n*-HHC network, we use A_S and A_P to denote its *S* part and *P* part, respectively, i.e., $A = (A_S, A_P)$. The following is a formal definition of Gray codes [4], which will be used in the next section.

Definition 2. An *m*-bit Gray code, denoted by G_m , defines an ordering among all 2^m *m*-bit binary sequences (codewords). Let $G_1 = (0, 1)$, and for m > 1, define $G_m = (0G_{m-1}, 1G_{m-1}^r)$, where G_{m-1}^r is the reverse of G_{m-1} and $0G_{m-1}$ $(1G_{m-1}^r)$ stands for prefixing each codeword of G_{m-1} (G_{m-1}^r) with 0 (1).

For example, we have $G_2 = (00, 01, 11, 10)$ and $G_3 = (000, 001, 011, 010, 110, 111, 101, 100)$. It is easy to see that every two adjacent codewords, including the first and the last, of G_m differ in exactly one bit position.

3. Cycle embedding

In this section, we embed cycles of all possible lengths into an *n*-HHC network. Since an *n*-HHC network is bipartite, only cycles of even lengths ranging from 4 to 2^{2^m+m} , can be embedded. For any two binary sequences $X = x_{|X|-1}x_{|X|-2}...x_0$ and $Y = y_{|Y|-1}y_{|Y|-2}...y_0$ of equal length (i.e., |X| = |Y|), we let $d_H(X, Y)$ denote the Hamming distance between X and Y, which is equal to the



Fig. 2. The ((0000, 00), (0000, 01))-augmented path.

number of different bits between *X* and *Y*. Throughout this section, we assume $n = 2^m + m$, where $m \ge 1$ is an integer. Each Q_m contained in an *n*-HHC network is referred to as an *embedded hypercube network* (or an *embedded Q_m*). The following lemma was shown in [2].

Lemma 1. (See [2].) Suppose that X and Y are two distinct vertices of Q_m and $d_H(X, Y) = d$. When m > 1, there are X-Y paths (i.e., paths from X to Y) in Q_m whose lengths are $d+2, d+4, \ldots, c$, where $c = 2^m - 1$ if d is odd and $c = 2^m - 2$ if d is even.

Suppose that *A* and *B* are two distinct vertices in an *n*-HHC network. An *A*–*B* path will contain internal edges interleaved with external edges provided it traverses more than one embedded Q_m . For example,

 $A = (0000000, 000)^{(e)} \rightarrow (0000001, 000)$

 $^{*} \rightarrow (0000001, 010)^{(e)} \rightarrow (00000101, 010)$

 $^{*} \rightarrow (00000101, 000)^{(e)} \rightarrow (00000100, 000)$

* \rightarrow (00000100, 010) ^(e) \rightarrow (00000000, 010) = B

shows an A-B path in an 11-HHC network that traverses four embedded Q_m 's, where ${}^{(e)} \rightarrow$ denotes an external edge and $^* \rightarrow$ denotes a shortest path in an embedded Q_3 . For the ease of discussion, we consider each path in an embedded Q_m a shortest path.

The sequence of the external edges contained in a path is referred to as an *external edge sequence* (abbreviated to *EES*) with respect to the path. For the example above, the EES can be represented as (((00000000, 000), (00000001, 000)), ((00000010, 010), (00000101, 010)), ((00000101, 000), (00000100, 000)), ((00000100, 010), (00000000, 010))). Further, according to the definition of external edges, the EES can be uniquely determined by the *P* parts of the external edges. Thus, its representation can be simplified to (000, 010, 000, 010). An EES corresponds to a path in an *n*-HHC network, while the shortest paths in the embedded Q_m 's are ignored.

Suppose that (A, B) is an internal edge in an *n*-HHC network, where $A_S = B_S$. Let us consider an A-B path whose EES is (A_P, B_P, A_P, B_P) . The path contains four external edges and three internal edges alternately. Since it can be uniquely determined, it is referred to as the (A, B)-augmented path. It passes through four embedded Q_m 's whose S parts are $I_0 = A_S = B_S$, $I_1 = I_0^{(dec(A_P))}$, $I_2 = I_1^{(dec(B_P))}$ and $I_3 = I_2^{(dec(A_P))}$, respectively. Notice that I_0, I_1, I_2 and I_3 only differ in the $(dec(A_P) + 1)$ th bit and the $(dec(B_P)) + 1$)th bit from the right. These four embedded Q_m 's are referred to as augmented Q_m 's. The three

internal edges contained in the (A, B)-augmented path are referred to as *augmented internal edges*.

For example, Fig. 2 shows the ((0000, 00), (0000, 01))augmented path in a 6-HHC network, where each circle represents a Q_2 and the associated four-bit sequence is its *S* part. The *S* parts of the four Q_2 's differ in the two rightmost bits. The three internal edges ((0001, 00), (0001, 01)), ((0011, 01), (0011, 00)) and ((0010, 00), (0010, 01)) are augmented internal edges.

A path (cycle) in a graph that contains every vertex exactly once is called a *Hamiltonian path* (*Hamiltonian cycle*). Let $G_{m-1}[i]$ denote the *i*th codeword in G_{m-1} and $G_{m-1}[i]0$ ($G_{m-1}[i]1$) stand for appending $G_{m-1}[i]$ with 0 (1), where $1 \le i \le 2^{m-1}$. The following lemma holds as a consequence of Gray codes.

Lemma 2. An embedded Q_m contains a Hamiltonian cycle as follows:

$$(I, G_{m-1}[1]0) \to (I, G_{m-1}[1]1) \to (I, G_{m-1}[2]1)$$

$$\to (I, G_{m-1}[2]0) \to \cdots$$

$$\to (I, G_{m-1}[2^{m-1} - 1]1)$$

$$\to (I, G_{m-1}[2^{m-1}]1) \to (I, G_{m-1}[2^{m-1}]0)$$

$$\to (I, G_{m-1}[1]0),$$

where I is the S part of the Q_m and \rightarrow denotes an internal edge.

The following theorem presents our main result.

Theorem 1. An *n*-HHC network contains cycles of all even lengths ranging from 4 to 2^n , where $n = 2^m + m$ for some integer m > 2.

Proof. We use C_l to denote a cycle of length l in an n-HHC network, where $4 \le l \le 2^n$. First of all, we construct $2^{m-1} + 1$ cycles whose lengths are $f(1), f(2), \ldots, f(2^{m-1}+1)$, respectively, where $f(i) = 4^{i-1} \times 2^m$ for all $1 \le i \le 2^{m-1} + 1$. These cycles are denoted by $C_{f(1)}, C_{f(2)}, \ldots, C_{f(2^{m-1}+1)}$.

 $C_{f(1)}$ is constructed in the Q_m with S part 0^{2^m} according to Lemma 2. $C_{f(2)}$ is obtained from $C_{f(1)}$ as follows. First, the edge $((0^{2^m}, G_{m-1}[1]0), (0^{2^m}, G_{m-1}[1]1))$ of $C_{f(1)}$ is replaced with the $((0^{2^m}, G_{m-1}[1]0), (0^{2^m}, G_{m-1}[1]1))$ -augmented path. Notice that the three augmented internal edges have the form $((*, G_{m-1}[1]0), (*, G_{m-1}[1]1))$ (we use * to denote the S part of a Q_m if it can be ignored). A Hamiltonian cycle is established in each augmented Q_m (exclusive of the Q_m with S part 0^{2^m}) according to Lemma 2, and then the three augmented internal edges are removed.

In general, for $2 \leq i \leq 2^{m-1}$, $C_{f(i+1)}$ can be obtained from $C_{f(i)}$ as follows. There are a total of $4^{i-1}Q_m$'s traversed by $C_{f(i)}$ and each of them has an edge $((*, G_{m-1}[i]0), (*, G_{m-1}[i]1))$ included in $C_{f(i)}$. First, each edge $((*, G_{m-1}[i]0), (*, G_{m-1}[i]1))$ is replaced with the $((*, G_{m-1}[i]0), (*, G_{m-1}[i]1))$ -augmented path (thus, passes through $4^i Q_m$'s). Then, in each new augmented Q_m , a Hamiltonian cycle is established according to Lemma 2

Table 1

Number of Q_m 's traversed by $C_{f(i)}$, length of $C_{f(i)}$, and edges (A, B) in $C_{f(i)}$ to be replaced with (A, B)-augmented paths

	Number of Q_m 's traversed	Length	(A, B)
1	4 ⁰	2^m	$\begin{array}{l} ((*,G_{m-1}[1]0),(*,G_{m-1}[1]1)) \\ ((*,G_{m-1}[2]0),(*,G_{m-1}[2]1)) \end{array}$
2	4 ¹	4×2^m	
: : i	\vdots 4 ⁱ⁻¹	$\frac{1}{2}$ $4^{i-1} \times 2^m$: ((*, $G_{m-1}[i]0$), (*, $G_{m-1}[i]1$))
:	$\frac{1}{4}$	$\frac{1}{2}$:
2 ^{m-1}		$4^{2^{m-1}} \times 2^m$	((*, $G_{m-1}[2^{m-1}]0$), (*, $G_{m-1}[2^{m-1}]1$)

and the three augmented internal edges are removed. Table 1 shows the number of Q_m 's traversed by $C_{f(i)}$, the length of $C_{f(i)}$, and the edges (A, B) in $C_{f(i)}$ to be replaced with (A, B)-augmented paths for constructing $C_{f(i+1)}$.

Cycles of other lengths can be obtained by the aid of Lemma 1, as elaborated below. By Lemma 1, cycles of lengths 4, 6, ..., $2^m - 2$ can be obtained in a Q_m . Cycles of lengths ranging from $2^m + 2$ to 4×2^m can be obtained, while constructing $C_{f(2)}$ from $C_{f(1)}$. Recall that the edge $((*, G_{m-1}[1]0), (*, G_{m-1}[1]1))$ of $C_{f(1)}$ was replaced with the $((*, G_{m-1}[1]0), (*, G_{m-1}[1]1))$ -augmented path (refer to Fig. 2 for the example of m = 2). By Lemma 1, each augmented internal edge can be expanded to paths of lengths $3, 5, \ldots, 2^m - 1$. Besides, paths of lengths $3, 5, \ldots, 2^m - 1$ can be also obtained between $(*, G_{m-1}[1]0)$ and $(*, G_{m-1}[1]1)$ in the Q_m that embeds $C_{f(1)}$. Consequently, the desired cycles can be obtained. In a similar way, for all $2 \le i \le 2^{m-1}$, cycles of lengths ranging from $4^{i-1} \times 2^m + 2$ to $4^i \times 2^m$ can be obtained by the aid of Lemma 1, while constructing $C_{f(i+1)}$ from $C_{f(i)}$.

Finally, it should be noted that the Q_m 's traversed by $C_{f(i)}$ are all distinct. Let $b_{2^m-1}b_{2^m-2}...b_0$ denote the *S* part of a Q_m , where $b_k \in \{0, 1\}$ for all $0 \le k \le 2^m - 1$, and S_i be the set of the *S* parts of those Q_m 's traversed by $C_{f(i)}$. We have $S_i = \{b_{2^m-1}b_{2^m-2}...b_0: b_{dec(G_{m-1}[r]1)} b_{dec(G_{m-1}[r]0)} \in \{00, 01, 11, 10\}$ for all $1 \le r \le i - 1$ and $b_j = 0$ for all $j \in \{0, 1, ..., 2^m - 1\}$ – $\{dec(G_{m-1}[r]0), dec(G_{m-1}[r]1): 1 \le r \le i - 1\}$. It is easy to see that $|S_i| = 4^{i-1}$. For example, $S_2 = (0^{2^m-2}00, 0^{2^m-2}01, 0^{2^m-2}11, 0^{2^m-2}10)$ and $S_3 = (0^{2^m-4}0000, 0^{2^m-4}0011, 0^{2^m-4}0110, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0110, 0^{2^m-4}0110, 0^{2^m-4}0110, 0^{2^m-4}0110, 0^{2^m-4}0110, 0^{2^m-4}0110, 0^{2^m-4}0100, 0^{2^m-4}010, 0^{2^m-4}010, 0^{2^m-4}010, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}0100, 0^{2^m-4}010,$

 $0^{2^m-4}1100, 0^{2^m-4}1101, 0^{2^m-4}1111, 0^{2^m-4}1110, 0^{2^m-4}1000, 0^{2^m-4}1001, 0^{2^m-4}1011, 0^{2^m-4}1010).$

4. Discussion and conclusion

In this paper, we showed that an *n*-dimensional hierarchical hypercube network is bipancyclic, where $n = 2^m + m$ for some integer m > 2. To say more concretely, we showed that it contains cycles of all even lengths ranging from 4 to 2^n . When m = 1, the hierarchical hypercube network is simply a cycle of length 8. When m = 2, the hierarchical hypercube network is composed of 16 Q₂'s, and it contains cycles of even lengths ranging from 4 to 64, exclusive of length 6.

Finally, some further research problems are suggested. The Hamiltonian-laceability of the hierarchical hypercube network is still open. More specifically, is the hierarchical hypercube network Hamiltonian-laceable or strongly Hamiltonian-laceable (see [5,7])? Besides, fault-tolerant embedding on the hierarchical hypercube network was not studied before. For example, how many link faults can the hierarchical hypercube network tolerate, while retaining a fault-free Hamiltonian cycle.

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