

On the approximability of minmax (regret) network optimization problems

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Abstract

In this paper the minmax (regret) versions of some basic polynomially solvable deterministic network problems are discussed. It is shown that if the number of scenarios is unbounded, then the problems under consideration are not approximable within $\log^{1-\epsilon} K$ for any $\epsilon > 0$ unless $\text{NP} \subseteq \text{DTIME}(n^{\text{poly log } n})$, where K is the number of scenarios.

Keywords: Combinatorial optimization; Approximation; Minmax; Minmax regret;

1 Introduction

We are given a network modeled by a directed or undirected graph $G = (V, E)$ with non-negative cost c_e associated with every edge $e \in E$. A set of feasible solutions Φ consists of some subsets of the edges of G . It may contain, for instance, all $s - t$ paths, spanning trees, $s - t$ cuts or matchings in G . In a classical *deterministic network problem* \mathcal{P} , i.e. the problem in which the costs c_e are precisely given, we wish to find a feasible solution $X \in \Phi$ that minimizes the total cost, namely the value of $\sum_{e \in X} c_e$. In this paper we assume that \mathcal{P} is polynomially solvable. A comprehensive review of various polynomially solvable deterministic network problems \mathcal{P} can be found in [1, 17].

In practice, the costs c_e in the objective may be uncertain. In *robust approach* [16] the uncertainty is modeled by specifying a set Γ that contains all possible realizations of the edge costs. Every particular realization $S = (c_e^S)_{e \in E}$ is called a *scenario* and the value of c_e^S denotes the cost of edge e under scenario S . There are two ways of describing the set Γ . In the *interval scenario case*, the value of every edge cost may fall within a given closed interval and Γ is the Cartesian product of all the uncertainty intervals. In the *discrete scenario case*, which is considered in this paper, the set of scenarios is defined by explicitly listing all scenarios. So, $\Gamma = \{S_1, \dots, S_K\}$ is finite and contains exactly K scenarios. The cost of solution $X \in \Phi$ under scenario $S \in \Gamma$ is $F(X, S) = \sum_{e \in X} c_e^S$. We will denote by $F^*(S) = \min_{X \in \Phi} F(X, S)$

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the cost of an optimal solution under S . In the *minmax* version of problem \mathcal{P} , we seek a solution that minimizes the worst case objective value over all scenarios, that is

$$\text{MINMAX } \mathcal{P} : \min_{X \in \Phi} \max_{S \in \Gamma} F(X, S).$$

In the *minmax regret* version of problem \mathcal{P} , we wish to find a solution that minimizes the *maximal regret*, that is

$$\text{MINMAX REGRET } \mathcal{P} : \min_{X \in \Phi} Z(X) = \min_{X \in \Phi} \max_{S \in \Gamma} \{F(X, S) - F^*(S)\}.$$

The motivation of the minmax (regret) approach and a deeper discussion on the two robust criteria can be found in [16]. Unfortunately, under the discrete scenario case, the minmax (regret) versions of basic network problems such as SHORTEST PATH, MINIMUM SPANNING TREE, MINIMUM ASSIGNMENT and MINIMUM S-T CUT turned out to be NP-hard even if Γ contains only 2 scenarios [2, 6, 16]. Furthermore, if the number of scenarios is unbounded (it is a part of the input), then MINMAX (REGRET) SHORTEST PATH is strongly NP-hard and not approximable within $(2 - \epsilon)$ and MINMAX (REGRET) MINIMUM SPANNING TREE is strongly NP-hard and not approximable within $(3/2 - \epsilon)$ for any $\epsilon > 0$ if $P \neq NP$ [3, 5]. For the interval scenario case, if \mathcal{P} is polynomially solvable, then MINMAX \mathcal{P} is polynomially solvable as well. On the other hand, the minmax regret versions of SHORTEST PATH, MINIMUM SPANNING TREE, MINIMUM ASSIGNMENT and MINIMUM S-T CUT are strongly NP-hard [2, 6, 8, 11, 19]. It is worth pointing out that there are some interesting differences between the discrete and interval scenario representations. In [10] a minmax regret problem has been described, which is polynomially solvable in the interval case and NP-hard for two explicitly given scenarios. On the other hand, the minmax regret linear programming problem is polynomially solvable in the discrete scenario case and becomes strongly NP-hard in the interval one [12].

Consider again the discrete scenario case. If problem \mathcal{P} is polynomially solvable and the edge costs under all scenarios are nonnegative, then both MINMAX \mathcal{P} and MINMAX REGRET \mathcal{P} are approximable within K [4]. A generic K -approximation algorithm proposed in [4] simply outputs an optimal solution to problem \mathcal{P} under costs $c_e = \frac{1}{K} \sum_{S \in \Gamma} c_e^S$ for all $e \in E$. In consequence, the problems are approximable within a constant if the number of scenarios K is assumed to be bounded (K is bounded by a constant). However, up to now the existence of an approximation algorithm with a constant performance ratio for the unbounded case has been an open question. In this paper we address this question and show that the minmax (regret) versions of SHORTEST PATH, MINIMUM ASSIGNMENT and MINIMUM S-T CUT are not approximable within $\log^{1-\epsilon} K$ for any $\epsilon > 0$ unless $NP \subseteq DTIME(n^{\text{poly log } n})$. Here and subsequently n denotes the length of the input. The last inclusion is widely believed to be untrue. We also show that MINMAX (REGRET) MINIMUM SPANNING TREE is not approximable within $(2 - \epsilon)$ for any $\epsilon > 0$ unless $P = NP$. Moreover, all the negative results remain true even for a class of graphs with a very simple structure. We can thus conclude that the discrete scenario representation of uncertainty leads to problems that are more complex to solve than the interval one. Recall that for the interval scenario case, if \mathcal{P} is polynomially solvable, then MINMAX REGRET \mathcal{P} is approximable within 2 [14].

2 The approximability of minmax (regret) network optimization problems

In this section, we present the main results of this paper. Namely, we give a negative answer to the question about the existence of approximation algorithms with a constant performance ratio for the minmax (regret) versions of SHORTEST PATH, MINIMUM ASSIGNMENT and MINIMUM S-T CUT, when the number of scenarios K is unbounded. We increase the gaps obtained in [3, 5] and prove that the problems of interest are hard to approximate within a ratio of $\log^{1-\epsilon} K$ for any $\epsilon > 0$. We first consider the minmax (regret) versions of SHORTEST PATH. In this problem set Φ consists of all paths between two distinguished nodes s and t of G .

Theorem 1. *The MINMAX (REGRET) SHORTEST PATH problem is not approximable within $\log^{1-\epsilon} K$ for any $\epsilon > 0$, unless $NP \subseteq DTIME(n^{\text{poly} \log n})$ even for edge series-parallel directed or undirected graphs.*

Proof. Consider the 3-SAT problem, in which we are given a set $\mathcal{U} = \{x_1, \dots, x_n\}$ of Boolean variables and a collection $\mathcal{C} = \{C_1, \dots, C_m\}$ of clauses, where every clause in \mathcal{C} has exactly three distinct literals. We ask if there is an assignment to \mathcal{U} that satisfies all clauses in \mathcal{C} . This problem is known to be strongly NP-complete [13].

Given an instance of 3-SAT, we construct the corresponding instance of MINMAX SHORTEST PATH as follows: we associate with each clause $C_i = (l_i^1 \vee l_i^2 \vee l_i^3)$ a digraph G_i composed of 5 nodes: $s_i, v_1^i, v_2^i, v_3^i, t_i$ and 6 arcs: the arcs $(s_i, v_1^i), (s_i, v_2^i), (s_i, v_3^i)$ correspond to literals in C_i (*literal arcs*), the arcs $(v_1^i, t_i), (v_2^i, t_i), (v_3^i, t_i)$ have costs equal to 0 under every scenario (*dummy arcs*); in order to construct digraph G , we connect all digraphs G_1, \dots, G_m by dummy arcs (t_i, s_{i+1}) for $i = 1, \dots, m-1$; we finish the construction of G by setting $s = s_1$ and $t = t_m$. We now form scenario set Γ as follows. For every pair of arcs of G , (s_i, v_j^i) and (s_q, v_r^q) , that correspond to contradictory literals l_i^j and l_q^r , i.e. $l_i^j = \sim l_q^r$, we create scenario S such that under this scenario the costs of the arcs (s_i, v_j^i) and (s_q, v_r^q) are set to 1 and the costs of all the remaining arcs are set to 0. An example of the reduction is shown in Figure 1. Notice that the resulting graph G has a simple series-parallel topology (see for instance [18] for a description of this class of graphs).

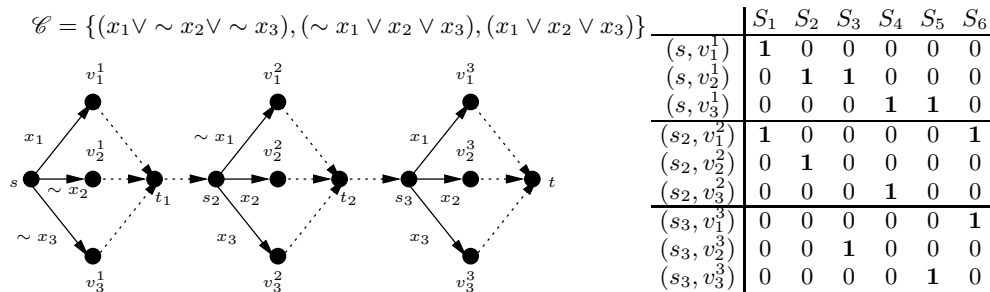


Figure 1: An example of the reduction. All the dummy arcs (the dotted arcs) have costs equal to 0 under all scenarios and they are not listed in the table.

It is easily verified that if the answer to 3-SAT is ‘Yes’, then there is a path P in G that does not use arcs corresponding to contradictory literals. So, $\max_{S \in \Gamma} F(P, S) \leq 1$. On the other hand, if the answer is ‘No’, then all paths in G must use at least two arcs corresponding

to contradictory literals and $\max_{S \in \Gamma} F(P, S) \geq 2$. This yields a gap of 2 and the MINMAX SHORTEST PATH problem is not approximable within $(2 - \epsilon)$ for any $\epsilon > 0$ unless P=NP.

We now show that the gap of 2 can be increased by applying an iterative construction that gradually increases the gap. A similar technique was applied to the problem of minimizing the number of unsatisfied linear equations [9] and to the problem of minimizing the number of nonzero variables in linear systems [7].

Let us transform the resulting graph $G = (V, E)$ into $G^{(1)}$ by replacing every arc in G by the whole graph G . We now associate scenario set $\Gamma^{(1)}$ with $G^{(1)}$ as follows. Initially, Γ has K scenarios. For every scenario $S \in \Gamma$ in graph G , we create K^2 scenarios so that two values of 1 in S are replaced by all pairs of scenarios $S_i \in \Gamma$ and $S_j \in \Gamma$, $i, j = 1, \dots, K$. In other words, two values of 1 in S are replaced by two matrices $\mathbf{S}_1^{(1)}$ and $\mathbf{S}_2^{(1)}$ of the size $|E| \times K^2$, respectively, where the columns of matrix

$$\begin{pmatrix} \mathbf{S}_1^{(1)} \\ \mathbf{S}_2^{(1)} \end{pmatrix} = \begin{pmatrix} S_1 & S_1 & \dots & S_1 & S_2 & S_2 & \dots & S_2 & \dots & S_K & S_K & \dots & S_K \\ S_1 & S_2 & \dots & S_K & S_1 & S_2 & \dots & S_K & \dots & S_1 & S_2 & \dots & S_K \end{pmatrix}, \quad S_i \in \Gamma, i = 1, \dots, K,$$

are the Cartesian product $\Gamma \times \Gamma$. Furthermore, every value of 0 in S is replaced by matrix $\mathbf{O}^{(1)}$ of the size $|E| \times K^2$ with all elements equal to zero. The resulting instance is graph $G^{(1)}$ with $|E|^2$ edges and K^3 scenarios. A sample construction of $\Gamma^{(1)}$ is shown in Figure 2. Now, if the answer to 3-SAT is ‘Yes’, then there is a path P in graph $G^{(1)}$ such that $\max_{S \in \Gamma^{(1)}} F(P, S) \leq 1$ and $\max_{S \in \Gamma^{(1)}} F(P, S) \geq 4$ otherwise. We thus get a gap of 4.

$G^{(1)}$	$\Gamma^{(1)}$					
$G_{(s, v_1^1)}$	$\mathbf{S}_1^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$
$G_{(s, v_2^1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_1^{(1)}$	$\mathbf{S}_1^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$
$G_{(s, v_3^1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_1^{(1)}$	$\mathbf{S}_1^{(1)}$	$\mathbf{O}^{(1)}$
$G_{(s_2, v_1^2)}$	$\mathbf{S}_2^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_1^{(1)}$
$G_{(s_2, v_2^2)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_2^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$
$G_{(s_2, v_3^2)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_2^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$
$G_{(s_3, v_1^3)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_2^{(1)}$
$G_{(s_3, v_2^3)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_2^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$
$G_{(s_3, v_3^3)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{O}^{(1)}$	$\mathbf{S}_2^{(1)}$	$\mathbf{O}^{(1)}$
$G^{(2)}$	$\Gamma^{(2)}$					
$G_{(s, v_1^1)}^{(1)}$	$\mathbf{S}_1^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$
$G_{(s, v_2^1)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_1^{(2)}$	$\mathbf{S}_1^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$
$G_{(s, v_3^1)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_1^{(2)}$	$\mathbf{S}_1^{(2)}$	$\mathbf{O}^{(2)}$
$G_{(s_2, v_1^2)}^{(1)}$	$\mathbf{S}_2^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_1^{(2)}$
$G_{(s_2, v_2^2)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_2^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$
$G_{(s_2, v_3^2)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_2^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$
$G_{(s_3, v_1^3)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_2^{(2)}$
$G_{(s_3, v_2^3)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_2^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$
$G_{(s_3, v_3^3)}^{(1)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{O}^{(2)}$	$\mathbf{S}_2^{(2)}$	$\mathbf{O}^{(2)}$

Figure 2: The construction of $\Gamma^{(1)}$ and $\Gamma^{(2)}$ for the sample problem shown in Figure 1. $G_{(s_i, v_j^i)}$ and $G_{(s_i, v_j^i)}^{(1)}$ are the graphs G and $G^{(1)}$, respectively, inserted in place of (s_i, v_j^i) in G . The graphs corresponding to the dummy arcs are not shown.

We can now repeat the construction to obtain an instance with a gap of 8. Namely, we construct $G^{(2)}$ by replacing every arc in graph G by the whole graph $G^{(1)}$. Then we form scenario set $\Gamma^{(2)}$ in the following way. For every scenario $S \in \Gamma$ in graph G , we create $(K^3)^2$ scenarios such that two values of 1 in S are replaced by two matrices $\mathbf{S}_1^{(2)}$ and $\mathbf{S}_2^{(2)}$ of the size $|E|^2 \times (K^3)^2$, respectively, where all columns of matrix

$$\begin{pmatrix} \mathbf{S}_1^{(2)} \\ \mathbf{S}_2^{(2)} \end{pmatrix} = \begin{pmatrix} S_1^{(1)} & S_1^{(1)} & \dots & S_1^{(1)} & S_2^{(1)} & S_2^{(1)} & \dots & S_2^{(1)} & \dots & S_{K^3}^{(1)} & S_{K^3}^{(1)} & \dots & S_{K^3}^{(1)} \\ S_1^{(1)} & S_2^{(1)} & \dots & S_{K^3}^{(1)} & S_1^{(1)} & S_2^{(1)} & \dots & S_{K^3}^{(1)} & \dots & S_1^{(1)} & S_2^{(1)} & \dots & S_{K^3}^{(1)} \end{pmatrix}, \quad S_i^{(1)} \in \Gamma^{(1)}, i = 1, \dots, K^3,$$

are the Cartesian product $\Gamma^{(1)} \times \Gamma^{(1)}$, and every value of 0 in S is replaced by matrix $\mathbf{O}^{(2)}$ of the size $|E|^2 \times (K^3)^2$ with all elements equal to zero (see Figure 2). By repeating the above construction t times, we get graph $G^{(t)}$ with $|E|^{t+1}$ edges together with scenario set $\Gamma^{(t)}$ containing $K^{2^{t+1}-1}$ scenarios that yield a total gap of 2^{t+1} . Let $t = \log \log^\beta n$ for some fixed $\beta > 0$, where n is the number of variables in the instance of 3-SAT. Now graph $G^{(t)}$ has $|E|^{\log \log^\beta n + 1}$ edges and $K^{2^{\log \log^\beta n} - 1}$ scenarios. Let $K' = K^{2^{\log \log^\beta n} - 1}$. Since $|E|$ and K are bounded by a polynomial in n , graph $G^{(t)}$ together with scenario set $\Gamma^{(t)}$ can be constructed in $O(n^{\text{poly log } n})$ time. The resulting instance of MINMAX SHORTEST PATH has a gap of $2^{\log \log^\beta n}$. Since $K' = K^{2^{\log \log^\beta n} - 1}$ and $K = O(n^c)$ for some constant c , we get $K' = 2^{\log K (2^{\log \log^\beta n} - 1)} = 2^{O(\log^{\beta+1} n)}$. So, $\log K' = O(\log^{\beta+1} n)$ and the gap is $2^{\log \log^\beta n} = O(\log^{\frac{\beta}{\beta+1}} K')$.

Assume, on the contrary, that a polynomial time algorithm approximates the MINMAX SHORTEST PATH problem within a factor $\log^{1-\epsilon} K$ for any $\epsilon > 1 - \frac{\beta}{\beta+1}$. Applying this algorithm to the resulting graph $G^{(t)}$ with scenario set $\Gamma^{(t)}$ containing K' scenarios, we could decide the 3-SAT problem in $O(n^{\text{poly log } n})$ time. But this would imply $\text{NP} \subseteq \text{DTIME}(n^{\text{poly log } n})$, a contradiction.

In order to prove the result for MINMAX REGRET SHORTEST PATH, we use exactly the same graph $G^{(t)}$ with scenario set $\Gamma^{(t)}$. A proof goes without any modifications. It follows from the fact that under every scenario $S \in \Gamma^{(t)}$ there is a path P^* in $G^{(t)}$ such that $F(P^*, S) = 0$, and thus $F^*(S) = 0$. In consequence the minmax and minmax regret criteria lead to optimal solutions with the same total costs in the resulting instances. Furthermore, if arc directions are ignored, then the results hold for the undirected graphs as well. \square

It is worth pointing out that Theorem 1 holds for the graphs having a very simple series-parallel structure. The class of series-parallel graphs is a subclass of various classes of graphs and its description can be found for instance in [18]. Recall also that in the interval scenario case the MINMAX REGRET SHORTEST PATH problem for edge series-parallel graphs admits a fully polynomial time approximation scheme [15].

In the MINIMUM ASSIGNMENT problem we assume that G is a bipartite graph and Φ consists of all perfect matchings in G . The following corollary is a consequence of Theorem 1:

Corollary 1. *The MINMAX (REGRET) MINIMUM ASSIGNMENT problem is not approximable within $\log^{1-\epsilon} K$ for any $\epsilon > 0$, unless $\text{NP} \subseteq \text{DTIME}(n^{\text{poly log } n})$.*

Proof. In [6] a cost preserving reduction from MINMAX (REGRET) SHORTEST PATH to MINMAX (REGRET) MINIMUM ASSIGNMENT has been proposed. Therefore, we have exactly the same inapproximability results for MINMAX (REGRET) MINIMUM ASSIGNMENT as for MINMAX (REGRET) SHORTEST PATH. \square

Recall that in the MINIMUM SPANNING TREE problem Φ consists of all spanning trees of G , that is all subsets of exactly $|V| - 1$ edges that form acyclic subgraphs of G . The following result is true:

Corollary 2. *The MINMAX (REGRET) MINIMUM SPANNING TREE problem is not approximable within $(2 - \epsilon)$ for any $\epsilon > 0$, unless $P = \text{NP}$ even for edge series-parallel graphs.*

Proof. It is enough to observe that an optimal minmax (regret) path in the first graph G from the proof of Theorem 1 can be transformed to an optimal minmax (regret) spanning tree and vice versa by adding or removing a number of dummy edges. Since the dummy edges have

costs equal to 0 under all scenarios, this transformation is cost preserving. We get a gap of 2 and the theorem follows. \square

Notice that Corollary 2 strengthens the results obtained in [3]. However, we are not able to show here that MINMAX (REGRET) MINIMUM SPANNING TREE is not approximable within a constant factor. The reduction, which is valid for the first graph G , is not true for the subsequent graphs $G^{(t)}$. In other words, it is not possible to transform a path in $G^{(t)}$ into a spanning tree by simply adding dummy edges. The question whether MINMAX (REGRET) MINIMUM SPANNING TREE is approximable within a constant remains open. It is also open for a more general class of minmax (regret) matroidal problems, where Φ consists of all bases of a given matroid [17].

Finally, let us consider the MINIMUM S-T CUT problem. In this problem we distinguish two nodes s and t in G and Φ consists of all $s - t$ -cuts in G , that is the subset of edges whose removal disconnects s and t .

Theorem 2. *The MINMAX (REGRET) S-T CUT problem is not approximable within $\log^{1-\epsilon} K$ for any $\epsilon > 0$, unless $NP \subseteq DTIME(n^{\text{poly log } n})$ even for edge series-parallel graphs.*

Proof. As in the proof of Theorem 1, we use a reduction from 3-SAT. For a given instance of 3-SAT, we construct the corresponding instance of MINMAX S-T CUT as follows: for each clause $C_i = (l_i^1 \vee l_i^2 \vee l_i^3)$ in \mathcal{C} , we create three edges of the form $\{s, v_1^i\}$, $\{v_1^i, v_2^i\}$ and $\{v_2^i, t\}$ that correspond to the literals in C_i . Observe that the resulting graph G is composed of exactly m disjoint $s - t$ paths and it has a series-parallel topology. For every pair of edges, that correspond to *contradictory* literals l_i^j and l_q^r , we form scenario S such that under this scenario the costs of edges that correspond to l_i^j and l_q^r are set to 1 and the costs of all the remaining edges are set to 0. An example of the reduction is shown in Figure 3.

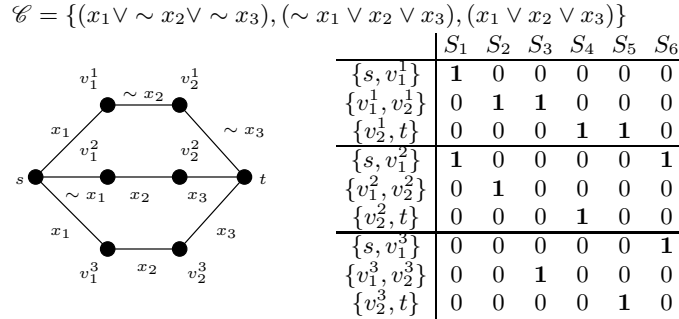


Figure 3: An example of the reduction.

It is easy to check that the answer to 3-SAT is ‘Yes’ if there is a cut C in G such that $\max_{S \in \Gamma} F(C, S) \leq 1$ and $\max_{S \in \Gamma} F(C, S) \geq 2$ otherwise. We thus get a gap of 2 and the MINMAX S-T CUT problem is not approximable within $(2 - \epsilon)$ for any $\epsilon > 0$ unless $P=NP$. The rest of the proof is the same as the one of Theorem 1. \square

3 Conclusions

In this paper, we have given a negative answer to the question about the existence of approximation algorithms with a constant performance ratio for the minmax and minmax regret

versions of SHORTEST PATH, MINIMUM ASSIGNMENT and MINIMUM S-T CUT under discrete scenario representation of uncertainty. Namely, we have shown that the considered problems are hard to approximate within a ratio of $\log^{1-\epsilon} K$ for any $\epsilon > 0$ if the number of scenarios K is unbounded. The question whether the performance ratio of K is the best possible for these problems remains open and it is the subject of future research. We have also strengthened the known results for the MINMAX (REGRET) MINIMUM SPANNING TREE problem. For this problem, however, it may be still possible to design an approximation algorithm with a constant performance ratio.

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