Fast Arithmetics Using Chinese Remaindering

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Abstract

In this paper, some issues concerning the Chinese remaindering representation are discussed. Some new converting methods, including an efficient probabilistic algorithm based on a recent result of von zur Gathen and Shparlinski [5], are described. An efficient refinement of the NC^1 division algorithm of Chiu, Davida and Litow [2] is given, where the number of moduli is reduced by a factor of $\log n$.

Keywords: Parallel algorithm; Chinese remaindering representation.

1 Introduction

For the fundamental arithmetic operations, it is often desirable to represent an integer as a vector of smaller integers. This can be done by selecting a set of pairwise coprime positive integers $m_1, m_2, \ldots, m_r$, and mapping an integer x to the vector of residues $(|x|_{m_1}, |x|_{m_2}, \cdots, |x|_{m_r})$, where $|x|_{m_i}$ denotes $x \pmod{m_i}$. This approach is called the *Chinese remaindering representation (CRR)*, as the *Chinese remainder theorem (CRT)* guarantees such mapping is meaningful. Using CRR, large calculations can be split as a series of smaller calculations that can be performed independently and in parallel. So, this approach has a significant role to play in applications such as cryptography and high precision scientific computation.

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It is well known that three basic arithmetic operations, addition, subtraction, and multiplication, can be performed in $O(\log n)$ time using $n^{O(1)}$ processors. These operations can also be done in the manner of log-space uniform. However, the parallel complexity of integer division is a subtle problem and has attracted a lot of attention. The first $O(\log n)$ time $n^{O(1)}$ sized circuit for integer division was exhibited by Beame, Cook and Hoover [1]. Recently, the log-depth, polynomial size, logspace-uniform circuit family for integer division (i.e., integer division is in logspace-uniform NC^1) was described by Chiu, Davida and Litow [2]. This settled a longstanding open problem and provided an optimal computation efficiency theoretically.

In this paper, we discuss some issues concerning the Chinese remaindering representation. The organization of the paper is as follows. Section 2 describes the Chinese remaindering system. Two methods for converting a vector to the corresponding integer are presented in this section. Section 3 focuses on the integer division using CRR. Under the framework of NC^1, an efficient refinement of the division algorithm of Chiu, Davida and Litow [2] is proposed.

2 Chinese Remainder Representation

Let $\mathcal{M} = \{m_1, m_2, \ldots, m_r\}$ be a set of pairwise coprime integers and $M = \prod_{i=1}^{r} m_i$. For a set of integers $x_1, x_2, \ldots, x_r$ with $0 \leq x_i < m_i$, the Chinese Remainder Theorem says that the system of congruence

$$\begin{cases} x \equiv x_1 \pmod{m_1} \\ x \equiv x_2 \pmod{m_2} \\ \cdots \\ x \equiv x_r \pmod{m_r} \end{cases}$$

has a unique solution $0 \leq x < M$. In fact, using the extended Euclidean algorithm, one finds integers $u_1, u_2, \cdots, u_r$ such that

$$\sum_{i=1}^{r} u_i \frac{M}{m_i} = 1,$$

and it is easy to verify that

$$x = \sum_{i=1}^{r} x_i u_i \frac{M}{m_i} \pmod{M} \tag{1}$$

gives the desired solution. It is remarked that one can also choose $u_i = (\frac{M}{m_i})^{-1} \pmod{m_i}$; and such choice of u_i will be used in the rest of our discussion.

The above system is called a Chinese remaindering representation (CRR) based on the set $\mathcal{M}$, and is denoted by CRR($\mathcal{M}$).

Now we present a method of finding u_i's which can be seen as an alternative to the Garner algorithm described in [7] (pages 290,293).

For each $j > 1$, m_j is coprime to $m_1 \cdots m_{j-1}$. Therefore, by the extended Euclidean algorithm, there exist integers α_j, β_j such that

$$\alpha_j m_j + \beta_j m_1 \cdots m_{j-1} = 1. \tag{2}$$

With these $r-1$ pairs of (α_i, β_i), the coefficients u_i can be computed as follows:

$$\begin{aligned}
u_1 &\leftarrow \alpha_2\alpha_3\cdots\alpha_r \pmod{m_1} \\
u_2 &\leftarrow \beta_2\alpha_3\cdots\alpha_r \pmod{m_2} \\
u_3 &\leftarrow \beta_3\alpha_4\cdots\alpha_r \pmod{m_3} \\
&\cdots \\
u_r &\leftarrow \beta_r \pmod{m_r}
\end{aligned}$$

The correctness of the above algorithm is based on the following identity:

$$\begin{aligned}
&(\alpha_2\cdots\alpha_r)m_2m_3\cdots m_r + (\beta_2\alpha_3\cdots\alpha_r)m_1m_3\cdots m_r + \\
&(\beta_3\alpha_4\cdots\alpha_r)m_1m_2m_4\cdots m_r + \cdots + \beta_r m_1m_2\cdots m_{r-1} = 1.
\end{aligned}$$

This identity can be verified using the standard mathematical induction: for $i > 2$, suppose that

$$\begin{aligned}
&(\alpha_2\cdots\alpha_{i-1})m_2m_3\cdots m_{i-1} + (\beta_2\alpha_3\cdots\alpha_{i-1})m_1m_3\cdots m_{i-1} + \\
&(\beta_3\alpha_4\cdots\alpha_{i-1})m_1m_2m_4\cdots m_{i-1} + \cdots + \beta_{i-1}m_1m_2\cdots m_{i-2} = 1.
\end{aligned}$$

Multiply both sides of the above by $\alpha_i m_i$, and apply the equation (1) for $j = i$, one gets

$$(\alpha_2 \cdots \alpha_i) m_2 m_3 \cdots m_i + (\beta_2 \alpha_3 \cdots \alpha_i) m_1 m_3 \cdots m_i + (\beta_3 \alpha_4 \cdots \alpha_i) m_1 m_2 m_4 \cdots m_i + \cdots + \beta_i m_1 m_2 \cdots m_{i-1} = 1.$$

It is remarked that in this process, we call the extended Euclidean algorithm $r-1$ times. For the method described in [7], $\frac{r(r-1)}{2}$ instances of extended Euclidean algorithm need to be invoked, for pairs (m_i, m_j) with $i < j$.

Next we present a probabilistic converting method for CRT. For positive integers N_1, N_2, let $a_1, a_2, \cdots, a_r$ be in $\{1, 2, \cdots, N_1\}$. Pick $2r$ uniformly distributed random integers $s_1, s_2, \cdots, s_r$ and $t_1, t_2, \cdots, t_r$ in $\{1, 2, \cdots, N_2\}$ and consider the linear forms

$$S = \sum_{i=1}^{r} a_i s_i, \ T = \sum_{i=1}^{r} a_i t_i.$$

It has been proved by Cooperman, Feisel, von zur Gathen and Havasin in [3] that with high probability

$$\gcd(a_1, a_2, \cdots, a_r) = \gcd(S, T). \tag{3}$$

This was improved recently by von zur Gathen and Shparlinski [5] and they gave the following strong result: with probability at least $\frac{6}{\pi^2} + o(1)$,

$$\gcd(a_1, a_2, \cdots, a_r) = \gcd(S, T),$$

provided that $\frac{N_2}{r + \ln N_1}$ is large enough.

This result can be used to produce a very efficient probabilistic algorithm for Chinese remaindering. Let us take $a_i = \frac{M}{m_i}$. We can find x such that

$$\begin{cases} x \equiv x_1 \pmod{m_1} \\ x \equiv x_2 \pmod{m_2} \\ \cdots \\ x \equiv x_r \pmod{m_r} \end{cases}$$

by the following steps:

1. Choose random linear forms S, T until

$$\gcd(S, T) = 1.$$

(The expected number for getting the desired pair of S, T is less than 2.)

2. Use extended Euclidean algorithm to get integers u, v such that

$$uS + vT = \sum_{i=1}^{r} (us_i + vt_i)\frac{M}{m_i} = 1.$$

3. The solution x is

$$x = \sum_{i=1}^{r} x_i(us_i + vt_i)\frac{M}{m_i} \pmod{M}.$$

Remark. It can be seen that in this routine, if the extended Euclidean algorithm is used to compute all *gcd*s, then the expected number of rounds to get u, v in step 2 is less than 2. In step 3, $us_i + vt_i$ can be replaced by $(us_i + vt_i) \pmod{m_i}$.

3 An Improved NC^1 Division Algorithm

In this section, we discuss the division algorithm of Chiu, Davida and Litow [2]. A careful analysis enables us to reduce the number of prime moduli by a factor of $\log n$.

Let α be a real number. A rational number α' is said to be an $n-$*bit under approximation to* α if

$$0 \leq \alpha - \alpha' \leq \frac{1}{2^n}.$$

The next result improves the lemma 3.2 of [2]:

Lemma 1 *Let* $\frac{1}{2} \leq \alpha < 1$ *and* $\beta = 1 - \alpha$. *If* $\frac{t_1}{A_1}, \frac{t_2}{A_2}, \ldots, \frac{t_{n+1}}{A_{n+1}}$ *are* $(n + 3)-$*bit underapproximations to* β, *then*

$$1 + \frac{t_1}{A_1} + \frac{t_1 t_2}{A_1 A_2} + \cdots + \prod_{i=1}^{n+1} \frac{t_i}{A_i}$$

is an $n-$*bit underapproximation to* $\frac{1}{\alpha}$.

Proof. Let

$$\eta = \min_{1 \le i \le n+1} \{\frac{t_i}{A_i}\}.$$

Note that $0 \le \beta \le \frac{1}{2}$ and $0 \le \beta - \eta \le \frac{1}{2^{n+3}}$, we see that

$$\begin{aligned}
\frac{1}{\alpha} - (1 + \frac{t_1}{A_1} + \frac{t_1 t_2}{A_1 A_2} + \cdots + \prod_{i=1}^{n+1} \frac{t_i}{A_i}) &\le \frac{1}{\alpha} - (1 + \eta + \eta^2 + \cdots + \eta^{n+1}) \\
&= \frac{1}{1-\beta} - \frac{1-\eta^{n+2}}{1-\eta} \\
&= \big(\frac{1}{1-\beta} - \frac{1}{1-\eta}\big) + \frac{\eta^{n+2}}{1-\eta} \\
&= \frac{\beta - \eta}{(1-\beta)(1-\eta)} + \frac{\eta^{n+2}}{1-\eta} \\
&\le \frac{\frac{1}{2^{n+3}}}{\frac{1}{2} \cdot \frac{1}{2}} + \frac{\frac{1}{2^{n+2}}}{\frac{1}{2}} \\
&= \frac{1}{2^n}.
\end{aligned}$$

In [2], the log-depth, polynomial size, logspace-uniform circuit family for integer division was constructed by Chiu, Davida and Litow. In other words, integer division is proved to be in logspace-uniform NC^1. This solves a longstanding open problem.

Notice that the original construction of the NC^1 circuit family for integer division needs $3n^2$ (actually $2n^2 + 5n$) primes numbers. The main purpose of this section is to refine the Chiu-Davida-Litow construction to achieve more efficiency. To be more specific, we shall show that $\frac{n^2}{\log n} + 3n$ primes will be sufficient.

Theorem 1 *The number of prime moduli of the Chiu-Davida-Litow NC^1 integer division algorithm can be reduced to* $\frac{n^2}{\log n} + 3n$.

Proof. The proof follows the similar line as in [2].

The goal is: given $x, y < 2^n$, compute the CRR of $\left\lfloor \frac{x}{y} \right\rfloor$.

Let $N = \left\lfloor \dfrac{n^2}{\log n} \right\rfloor + 3n$.

Suppose that x, y are represented in a CRR system with base $\{m_1, m_2, \ldots, m_n\}$ where m_i is the $(i+2)$th prime ($m_1 > 3$). This base is extended to

$$\{m_1, m_2, \ldots, m_n, m_{n+1}, \ldots, m_N\}.$$

A product D of the initial part of the base and some power of 2 will be constructed so that

$$\frac{1}{2} \leq \frac{y}{D} < 1.$$

According to [2], if $y = 2$, set $D = 2$. If $y > 2$, then take $j < n$ to be the number such that

$$m_1 m_2 \cdots m_j \leq y < m_1 m_2 \cdots m_j m_{j+1}.$$

Let k be the smallest positive integer such that $y < 2^k m_1 m_2 \cdots m_j$ (therefore $\dfrac{y}{2^k m_1 m_2 \cdots m_j} \geq \dfrac{1}{2}$), and set

$$D = 2^k m_1 m_2 \cdots m_j.$$

Let $r = \lfloor \dfrac{n}{\log n} \rfloor$. If $n \geq 2^6$, then $\dfrac{n - \log n - (\log n)^2}{\log n} > 3$. The fact that $m_{n+1} > 2n$ gives

$$\begin{aligned} (m_{n+1})^r &> (2n)^{\left\lfloor \frac{n}{\log n} \right\rfloor} \\ &\geq \left(2^{\log n + 1}\right)^{\frac{n}{\log n} - 1} \\ &= 2^{n + \frac{n - \log n - (\log n)^2}{\log n}} \\ &> 2^{n+3} \end{aligned} \tag{4}$$

Since $n + (n+1)r \leq N$, we can form the following products:

$$\begin{aligned} A_1 &= m_{n+1} m_{n+2} \cdots m_{n+r} \\ A_2 &= m_{n+r+1} m_{n+r+2} \cdots m_{n+2r} \\ &\cdots \\ A_{n+1} &= m_{n+nr+1} m_{n+nr+2} \cdots m_{n+(n+1)r}. \end{aligned}$$

We note that $A_i > 2^{n+3}$ for $i = 1, 2, \ldots, n+1$, by (4).

Next, choose

$$t_i = \lfloor \frac{(D-y)A_i}{D} \rfloor, \text{ for } i = 1, 2, \ldots, n+1.$$

Similar to [2], $\frac{t_i}{A_i}$ can be computed in NC^1. It is also routine to check that $\frac{t_i}{A_i}$ is an $(n+3)-$bit underapproximation to $\beta = \frac{D-y}{D}$.

Finally, by the lemma 1, we get an $n-$bit underapproximation to $\frac{1}{\alpha}$ where $\alpha = \frac{y}{D}$:

$$\gamma = 1 + \frac{t_1}{A_1} + \frac{t_1 t_2}{A_1 A_2} + \cdots + \frac{t_1 t_2 \cdots t_{n+1}}{A_1 A_2 \cdots A_{n+1}}.$$

Again, similar to [2], we have

$$\lfloor \frac{x}{y} \rfloor = \lfloor x\frac{\gamma}{D} \rfloor \text{ or } \lfloor \frac{x}{y} \rfloor = \lfloor x\frac{\gamma}{D} \rfloor + 1.$$

And all the computations are done in NC^1.

Remark. The Chebyshev bounds for primes can be used to get an inequality which is a bit sharper than the inequality (4), but there is no significant reduction on the number of prime moduli.

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