# Alphabetic Coding with Exponential Costs * 

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#### Abstract

An alphabetic binary tree formulation applies to problems in which an outcome needs to be determined via alphabetically ordered search prior to the termination of some window of opportunity. Rather than finding a decision tree minimizing $\sum_{i=1}^{n} w(i) l(i)$, this variant involves minimizing $\log _{a} \sum_{i=1}^{n} w(i) a^{l(i)}$ for a given $a \in(0,1)$. This note introduces a dynamic programming algorithm that finds the optimal solution in polynomial time and space, and shows that methods traditionally used to improve the speed of optimizations in related problems, such as the Hu-Tucker procedure, fail for this problem. This note thus also introduces two approximation algorithms which can find a suboptimal solution in linear time (for one) or $O(n \log n)$ time (for the other), with associated coding redundancy bounds.


Key words: Approximation algorithms; dynamic programming; information retrieval; Rényi entropy; tree searching

## 1 Introduction

Applications such as searching [8] and coding theory [6] make extensive use of binary trees. We denote the length (number of edges) of a path from the root to node $i \in\{1,2, \ldots, n\}$ of the tree as $l(i)$, and the weight (usually probability) of the leaf as $w(i)$. Given a set of weights, Huffman's algorithm [6] finds a tree minimizing cost function

$$
\begin{equation*}
\sum_{i=1}^{n} w(i) l(i) \tag{1}
\end{equation*}
$$

and Hu and Tucker's algorithm [5] finds an optimal alphabetic tree:

[^0]Definition 1 An alphabetic tree is a tree with leaves are in numerical order given inorder tree traversal (i.e., $1,2, \ldots, n$ from left to right).

Three papers independently considered the problem of minimizing

$$
\begin{equation*}
L_{a}(w, \boldsymbol{l}) \triangleq \log _{a} \sum_{i=1}^{n} w(i) a^{l(i)} \tag{2}
\end{equation*}
$$

for $a>1$ [5, p. 254] [10, p. 485] [7, p. 231] for unconstrained (Huffmanlike) minimization, the solution of which is very similar to that of Huffman's algorithm. One of these further noted that an algorithm similar to Hu and Tucker's solves the alphabetically constrained version of this problem [5], while another noted that the Huffman-like solution also solves the unconstrained (21) for $a<1$ [7], in which $\log _{a} x$ is monotonically decreasing and the objective's summation term is thus maximized.

A recent paper showed that the $a<1$ problem describes certain situations of single-shot decision-making [1]. Given a window of time corresponding to a memoryless random variable, if we wish to find the leaf of the binary tree through constant-time edge traversal, this is found in time with probability $a^{L_{a}(w, l)}$ - which we thus wish to minimize - for some known $a<1$. However, solving the alphabetic version of this problem remained unaddressed.

Here we present an $O\left(n^{3}\right)$ algorithm for minimizing (2) that is somewhat similar to Gilbert and Moore's method for (11) [4]. We then introduce counterexamples on attempts to minimize using faster methods, such the modification of Hu and Tucker's, which only succeeds for $a>1$. Finally we present approximation algorithms, related to those for the linear problem, which find suboptimal solutions in $O(n)$ and $O(n \log n)$, leading to simple bounds for both these solutions and the optimal ones.

## 2 Optimal Alphabetic Trees

Because the alphabetic tree imposes leaf order, each decision of which child to take, represented by a 0 (for left) or 1 (for right), is equivalent to a question of the form, "Is the output greater than or equal to $s$ ?" where $s$ is one of the possible symbols, a symbol we call the splitting point:

Definition 2 The splitting point of an internal node (or the corresponding subtree) is the smallest index among the leaves of the right subtree.

Definition 3 Each codeword $c(i)$ is the sequence of bits corresponding to the sequence of decisions (path) to arrive at leaf i. The overall set of codewords

- alphabetic code $C$ - fully describes the tree, as does length vector $\boldsymbol{l}$, the sequence of lengths $\{l(i)\}$.

The dynamic programming approach of Gilbert and Moore [4] is adapted to this problem (2):

Theorem 1 An algorithm finds the maximum tree weight $W_{j, k}$ (and corresponding optimum tree) for items $j$ through $k$ for each value of $k-j$ from 0 to $n-1$ (in order), by computing inductively

$$
\begin{equation*}
W_{j, k} \leftarrow a \max _{s \in\{j+1, j+2, \ldots, k\}}\left[W_{j, s-1}+W_{s, k}\right] \text { starting with } W_{j, j} \leftarrow w(j) \tag{3}
\end{equation*}
$$

for $1 \leq j<k \leq n$ in $O\left(n^{3}\right)$ time and $O\left(n^{2}\right)$ space.
Proof Recall first that maximizing $a^{L_{a}(w, l)}=\sum_{i} w(i) a^{l(i)}$ minimizes $L_{a}(w, \boldsymbol{l})$, which is why (3) is a maximization operation. One can see that $W_{1, n}=a^{L_{a}(w, l)}$ inductively by considering a (sub)tree's two subtrees as independent, rooted trees, one with summation $W_{1, s-1}=\sum_{i=1}^{s-1} w(i) a^{l(i)-1}$, the other with summation $W_{s, n}=\sum_{i=s}^{n} w(i) a^{l(i)-1}$. Then $W_{1, n}=a\left(W_{1, s-1}+W_{s, n}\right)$. Starting with $W_{j, j}=w(j)$, then, we see that these values can be built up accordingly (since the path length from a leaf to itself is $0, W_{j, j}=a^{0} w(j)$, and there is nothing numerically special about the final tree). Since all subtrees of an optimal tree are optimal - via a substitution argument, e.g., [8] - the maximization finds an optimal solution. This suggests the dynamic programming algorithm; similarly to [8], calculating all optimal subtrees of a size less than that of the (sub)tree in a current step, we can try all possible splitting points using optimal subtrees, yielding the optimal tree.
$O\left(n^{2}\right)$ items are stored - $O\left(n^{2}\right)$ weights for every possible range and the associated splitting points; these are used to recursively find the implied subtree - calculated by testing $O(n)$ splitting points for each internal node, thus the time and space complexity.

Knuth [8] reduced the algorithmic complexity of Gilbert and Moore's method for (11) by using the fact that the splitting point of an optimal tree of size $n$ must be between the splitting points of the two optimal subtrees of size $n-1$. With (2), this no longer holds. Consider $a=0.6$ with input weights $w=(8,1,9,6)$. The splitting point of $(8,1,9)$ is $s=3(w(s)=w(3)=9$, yielding subtrees with $(8,1)$ and $(9))$, and the splitting point of $(1,9,6)$ is $s=4(w(s)=6)$. However, the optimal splitting point of $(8,1,9,6)$ is $s=2$ $(w(s)=1)$.

Similarly, for (2) with $a>1$ [5], there is a procedure based on the Hu-Tucker algorithm for finding an optimal alphabetic solution. The Hu-Tucker algorithm begins with the input weights arranged as leaves in numerical order $(1,2, \ldots, n$ in a line). It then combines the two items $i$ and $j$ that, of all pairs of items without a leaf separating them, have a minimum weight sum, putting it in
the place of either node, both of which are now (ordered) children. In the original Hu-Tucker algorithm, this item is given weight $w(i)+w(j)$, whereas for (21) with $a>1$ it is given weight $a w(i)+a w(j)$. The algorithm then finds the minimum weighted pair among those pairs of distinct items (uncombined input leaves and combined items) without any uncombined leaf between them, placing the resulting node in the place of either original node. Continuing on, we obtain a tree that is not necessarily alphabetical, but which has the same lengths as an alphabetic tree which can be easily reconstructed, (optimally) solving the problem (for $a>1$ ).

However, consider again $a=0.6$, this time for weights ( $8,1,9,6,2$ ). The Hu-Tucker-like algorithm first combines 6 and 2 , then 8 and 1 , then the first combined node with 9 , and finally the remaining two nodes, resulting in a tree with lengths $\boldsymbol{l}^{\prime}=(2,2,2,3,3)$ and $L_{a}\left(w, \boldsymbol{l}^{\prime}\right) \approx-4.121$. However, a tree with lengths $\boldsymbol{l}^{\prime \prime}=(1,3,3,3,3)$, having $L_{a}\left(w, \boldsymbol{l}^{\prime \prime}\right) \approx-4.232$, shows that the Hu-Tucker-like solution is nonoptimal.

Result 1 Knuth's method for speeding up dynamic programming fails for $a<$ 1, as does using the Hu-Tucker-like method optimal for $a>1$.

## 3 Approximation Algorithms and Bounds

In this section, we add the assertion $\sum_{i=1}^{n} w(i)=1$ to our problem, which can be considered an optimization of (2) with constraints:

1. The Kraft inequality of binary trees, $\sum_{i=1}^{n} 2^{-l(i)} \leq 1$;
2. The integer constraint, $l(i) \in \mathbb{Z}$;
3. The alphabetic constraint.

The first and second of these are necessary and sufficient for the lengths to correspond to a binary tree. Relaxing the second and third allows for a numerical solution which can bound the performance of the optimal solution. The numerical solution, $l^{\dagger}$, shown by Campbell [2,3], results in the Shannon-like

$$
l^{s}(i) \triangleq\left\lceil l^{\dagger}\right\rceil=\left\lceil-\frac{1}{1+\log _{2} a} \log _{2} w(i)+\log _{2}\left(\sum_{j=1}^{n} w(j)^{\frac{1}{1+\log _{2} a}}\right)\right\rceil
$$

a valid (but not necessarily optimal) solution to the problem with only the alphabetic constraint relaxed, that is, the Huffman-like problem.

The approximation algorithm in Fig. 3 has a linear-time variant patterned after that in [11] - relying on $\boldsymbol{l}^{\mathrm{s}}$ - and a $O(n \log n)$-time variant patterned after [9] — instead using $\boldsymbol{l}^{\mathrm{h}}$, those lengths obtained from solving the optimal

## Procedure for Finding a Near-Optimal Code

1. Start with an optimal or near-optimal nonalphabetic code with length vector $\boldsymbol{l}^{\text {non }}$, either the Shannon-like $\boldsymbol{l}^{\mathrm{S}}$ or the Huffman-like $\boldsymbol{l}^{\mathrm{h}}$.
2. Find the set of all minimal points: $i$ such that $1<i<n$, $l^{\text {non }}(i)<$ $l^{\text {non }}(i-1)$, and $l^{\text {non }}(i)<l^{\text {non }}(i+1)$; or $i \in[j, j+k]$ minimizing $w(i)$ for $l^{\text {non }}(j-1)>l^{\text {non }}(j)=l^{\text {non }}(j+1)=\cdots=l^{\text {non }}(j+k)<l^{\text {non }}(j+k+1)$.
3. Assign a preliminary alphabetic code with lengths $l^{\text {pre }}(i)=l^{\text {non }}(i)+1$ for all minimal points and $l^{\text {pre }}(i)=l^{\text {non }}(i)$ for all other items. The first codeword is $l^{\text {pre }}(1)$ zeros, and each additional codeword $c(i)$ is obtained by either truncating $c(i-1)$ to $l^{\text {pre }}(i)$ bits and adding 1 to the integer that the binary codeword represents (if $l^{\text {pre }}(i) \leq l^{\text {pre }}(i-1)$ ) or by adding 1 to the integer/codeword $c(i-1)$ and appending $l^{\text {pre }}(i)-l^{\text {pre }}(i-1)$ zeros (if $l^{\text {pre }}(i)>l^{\text {pre }}(i-1)$ ), defining the binary tree.
4. Go through the code tree (with, e.g., a depth-first search), and remove any redundant nodes. Any node with only one child can replace the child by its grandchild or grandchildren. At the end of this process, an alphabetic code with $\sum_{i=1}^{n} 2^{-l(i)}=1$ is obtained.
code tree for the Huffman-like problem.
Every step after the first takes linear time with linear space, thus the overall complexity of the algorithms. Step 3 is the method by which Nakatsu showed that any nonalphabetic code can be made into an alphabetic code with similar lengths [9]. (The use of weights as a tie breaker and the nonlinearity of the problem do not change the validity of the algorithm.) Step 4 is the method by which Yeung showed that any alphabetic code can be made into another alphabetic code with $\sum_{i=1}^{n} 2^{-l(i)}=1$ without lengthening any codewords [11]. Thus this is a hybrid and extension of these two approaches.

For $w=(8 / 26,1 / 26,9 / 26,6 / 26,2 / 26)$ with $a=0.6$, applying the Shannonlike version of this algorithm, we find that $\boldsymbol{l}^{\mathrm{s}}=(2,13,1,4,10)$, preliminary codeword lengths are $\boldsymbol{l}_{\mathrm{s}}^{\text {pre }}=(2,13,2,4,10)$, and the preliminary code tree is as follows:

$$
C=(00,0100000000000,10,1100,1101000000)
$$

The italicized bits are redundant, and therefore so are the corresponding nodes in the code tree. They are thus removed in Step 4, which means the final tree has lengths $(2,2,2,3,3)$. The probability of success is $a^{L_{a}(w, l)} \approx$ $0.316\left(L_{a}(w, \boldsymbol{l}) \approx 0.851\right)$, close to the optimal probability of about 0.334 $\left(L_{a}\left(w, \boldsymbol{l}^{*}\right) \approx 0.843\right)$. Using the Huffman-like approximation algorithm yields $\boldsymbol{l}^{\mathrm{h}}=(2,4,1,3,4)$, a preliminary tree of lengths $\boldsymbol{l}_{\mathrm{h}}^{\text {pre }}=(2,4,2,3,4)$, and an output tree with lengths $(2,2,2,3,3)$, which are identical to the above. The same probability mass function with $a=0.7$ yields an optimal tree in the Huffmanlike version. For $a \in(0.5,1)$, coding bounds follow from these approaches:

Theorem 2 Let

- $L_{a}^{\bar{a}}(w)$ be the minimized (2) for the alphabetic problem,
- $L_{a}^{\mathrm{h}}(w)$ be that obtained using the $\boldsymbol{l}^{\mathrm{h}}$-based approximation algorithm,
- $L_{a}^{\tilde{\mathrm{s}}}(w)$ be that obtained using the $\boldsymbol{l}^{\mathbf{s}}$-based approximation algorithm,
- $L_{a}^{\mathrm{non}}(w)=L_{a}\left(w, \boldsymbol{l}^{\mathrm{non}}\right), L_{a}^{\mathrm{s}}(w)=L_{a}\left(w, \boldsymbol{l}^{\mathrm{s}}\right)$, and $L_{a}^{\mathrm{h}}(w)=L_{a}\left(w, \boldsymbol{l}^{\mathrm{h}}\right)($ using those l values from Fig. (3).

Then

$$
\begin{align*}
& H_{\alpha}(w) \leq L_{a}^{\mathrm{h}}(w) \leq L_{a}^{\overline{\mathrm{a}}}(w) \leq L_{a}^{\tilde{\mathrm{h}}}(w)<1+L_{a}^{\mathrm{h}}(w)<2+H_{\alpha}(w)  \tag{4}\\
& H_{\alpha}(w) \leq L_{a}^{\mathrm{h}}(w) \leq L_{a}^{\overline{\mathrm{a}}}(w) \leq L_{a}^{\tilde{\mathrm{s}}}(w)<1+L_{a}^{\mathrm{s}}(w)<2+H_{\alpha}(w) \tag{5}
\end{align*}
$$

where $H_{\alpha}(w)$ is the Rényi entropy for $\alpha=\left(1+\log _{2} a\right)^{-1}$ :

$$
H_{\alpha}(w)=\frac{1}{1-\alpha} \log _{2} \sum_{i=1}^{n} w(i)^{\alpha}=\left(\log _{a} 2 a\right)\left(\log _{2} \sum_{i=1}^{n} w(i)^{\frac{1}{1+\log _{2} a}}\right)
$$

Proof This is a corollary of Campbell's Shannon-like bounds for $a>0.5-$ $H_{\alpha}(w) \leq L_{a}^{\mathrm{h}}(w) \leq L_{a}^{\mathrm{s}}(w)<1+H_{\alpha}(w)$ - along with the facts that (a) the two approximation algorithm lengths corresponding to items 1 and $n$ are no greater than those in $\boldsymbol{l}^{\text {non }}$ and (b) no other length exceeds the corresponding length in $\boldsymbol{l}^{\mathrm{non}}$ by 1 or more. This results in $L_{a}^{\tilde{\mathrm{h}}}(w)<1+L_{a}^{\mathrm{h}}(w)$ and $L_{a}^{\tilde{\mathrm{s}}}(w)<1+L_{a}^{\mathrm{s}}(w)$ due to (2), and, since no alphabetic tree is better than the optimal alphabetic tree and no alphabetic tree is better than the optimal Huffman-like tree, we arrive at (4) and (5).

The lower limit to $L_{a}^{\bar{a}}(w)$ is satisfied by $(2,2)$, while the upper limit is approached by $(\epsilon, 1-2 \epsilon, \epsilon)$, which approaches entropy 0 and penalty 2 .

Both these algorithms and the bounds due to analogous inequalities apply to $a>1$ and to the traditional alphabetic problem $\left(a \rightarrow 1\right.$, where $H_{1}$ is Shannon entropy [3]). For the traditional problem, due to Step 4, the Huffmanbased approximation version of the above algorithm is a strict improvement on Yeung's Huffman-based approximation [11].

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