## Finding bipartite subgraphs efficiently

Dhruv Mubayi<sup>\*</sup> and György Turán<sup>†</sup>

#### Abstract

Polynomial algorithms are given for the following two problems:

- given a graph with n vertices and m edges, where  $m \ge 3n^{3/2}$ , find a complete balanced bipartite subgraph with parts about  $\frac{\ln n}{\ln(n^2/m)}$ ,
- given a graph with n vertices, find a decomposition of its edges into complete balanced bipartite graphs having altogether  $O(n^2/\ln n)$  vertices.

Previous proofs of the existence of such objects, due to Kővári-Sós-Turán [10], Chung-Erdős-Spencer [5], Bublitz [4] and Tuza [13] were non-constructive.

## 1 Introduction

Determining the minimal number of edges in a bipartite graph which guarantees the existence of a complete balanced bipartite subgraph  $K_{q,q}$  is known as the Zarankiewicz problem (see, e.g., Bollobás [3]). It was shown by Kővári, Sós and Turán [10] that every bipartite graph with n vertices in both sides and  $c_q n^{2-1/q}$  edges contains a  $K_{q,q}$ . The same bound (with different constant  $c_q$ ) holds for general n-vertex graphs. The argument from [10] also shows that n-vertex graphs of constant density, i.e., graphs with  $\epsilon n^2$  edges, contain a complete bipartite graph with parts of size at least  $c_{\epsilon} \ln n$ . The proofs of all these results are based on counting, and thus are *non-constructive*.

<sup>\*</sup>Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, IL 60607; email: mubayi@math.uic.edu; research supported in part by NSF grant DMS 0653946

<sup>&</sup>lt;sup>†</sup>Department of Mathematics, Statistics, and Computer Science, University of Illinois at Chicago, IL 60607 and Research Group on Artificial Intelligence, Hungarian Academy of Sciences, University of Szeged; email: gyt@uic.edu

We consider the question whether such subgraphs can be found by *efficient*, i.e., polynomial time, algorithms. This question has been considered recently by Kirchner [9], who gave an efficient algorithm to find a complete balanced bipartite subgraph with parts of size  $\Omega(\sqrt{\ln n})$  in graphs of constant density. We improve this result by giving an efficient algorithm which finds a complete balanced bipartite subgraph with parts of size  $\Omega(\ln n)$ , i.e. of the optimal order of magnitude, in graphs of constant density. Our algorithm gives subgraphs of similar size as the counting argument in other ranges as well <sup>1</sup>.

Finding a largest balanced complete bipartite subgraph is an important optimization problem, which is known to be NP-hard, and even hard to approximate (see, e.g., Feige and Kogan [6]). We would like to emphasize that we are *not* trying to give an approximation algorithm for this problem. Our objective is to give an efficient algorithm which finds a balanced complete bipartite subgraph of size close to the largest size that is guaranteed to exist knowing *only* the number of edges in the graph. Thus, even in a dense graph, we are finding a subgraph of logarithmic size only. Results of this type are given, for example, in Alon *et al.* [1].

The counting argument of [10] has several applications to other combinatorial problems. It seems to be an interesting question whether the algorithmic version of the counting argument leads to further algorithmic results in these applications. As a case in point, we consider the question of decomposing, or partitioning, the edge set of a graph into complete bipartite graphs. The motivation to look for such algorithms comes from an application in approximation algorithms [2].

Every *n*-vertex graph can be decomposed into at most n-1 stars, and Graham and Pollak [7] showed that n-1 complete bipartite graphs are necessary for the *n*-vertex complete graph. Instead of minimizing the number of complete bipartite graphs in a decomposition, one can also try to minimize the complexity of decompositions, measured by the sum of the number of vertices of the complete bipartite graphs used in the decomposition. This measure of complexity was suggested by Tarján [12] in the context of circuit complexity. For recent connections to circuit complexity see Jukna [8].

It was shown by Chung, Erdős and Spencer [5], and by Bublitz [4], that there is always a decomposition of complexity  $O(n^2/\ln n)$ , and this order of magnitude is best possible. Similar results were obtained by Tuza [13] for decomposing bipartite graphs. These results are obtained by repeatedly applying the counting argument to show the existence of a

<sup>&</sup>lt;sup>1</sup>Note that the problem becomes meaningless in the sense studied here for fewer than  $n^{3/2}$  edges, as such graphs do not always contain even  $K_{2,2}$  subgraphs.

large complete bipartite graph and removing its edges. Thus the decomposition results obtained in [4, 5, 13] are also non-constructive. As a direct application of our algorithm for finding bipartite subgraphs, we obtain efficient algorithms to find decompositions of complexity  $O(n^2/\ln n)$ .

## 2 Complete balanced bipartite subgraphs

Searching for a  $K_{q,q}$  by checking all subgraphs of that size would give an algorithm with superpolynomial running time if q is, say, logarithmic in the number of vertices. A polynomial algorithm could be given by restricting the search space to a polynomial size set of candidate subgraphs. One possibility for that would be to find a bipartite subgraph (R, S) with the following properties:

- it is dense enough for the known results to guarantee the *existence* of a  $K_{q,q}$ , and
- the number of q-element subsets of R is only *polynomial*.

If such an (R, S) can be found efficiently then a required  $K_{q,q}$  is obtained by checking the common neighborhood of all q-element subsets of R. It turns out that this approach indeed works if one chooses R to be the right number of vertices with maximal degree and S to be the remaining vertices. Thus, we consider the following algorithm, where q(n,m)and r(n,m) are functions to be determined.

#### Algorithm **FIND-BIPARTITE**

input: G = (V, E) with |V| = n and |E| = m

 $q := q(n,m), \ r := r(n,m),$ 

R := r vertices having highest degree

for all subsets  $C \subseteq R$  with |C| = q do

 $D := \bigcap \{ N(v) - R : v \in C \}$ 

if  $|D| \ge q$  then D' := the first q elements of D, return (C, D')

We now show that with the appropriate choice of q(n, m) and r(n, m) the algorithm works.

Theorem 1 Let

$$q := \left\lfloor \frac{\ln(n/2)}{\ln(2en^2/m)} \right\rfloor, \quad r := \left\lfloor \frac{qn^2}{m} \right\rfloor.$$

If n is sufficiently large and  $m \ge 3n^{3/2}$  then Algorithm **FIND-BIPARTITE** returns a  $K_{q,q}$  (with  $q \ge 2$  as long as  $m > 8n^{3/2}$ ). The running time of the algorithm is polynomial in n.

**Remark.** Note that our algorithm finds a  $K_{q,q}$  in an *n*-vertex graph with  $m = c_q n^{2-1/q}$  edges as long as  $c_q$  is large. This is optimal for q = 2, 3 as there exist *n*-vertex graphs with  $c'_q n^{2-1/q}$  edges and no  $K_{q,q}$ , and if certain conjectures in extremal graph theory are true, then it is also optimal for q > 3.

**Proof.** After selecting i < r vertices, the number of edges incident to these vertices is less than rn. Hence in the subgraph induced by the remaining vertices there is a vertex of degree at least 2(m-rn)/n. Thus if R is the set of r highest degree vertices in G then

$$\sum_{v \in R} \deg_G(v) \ge \frac{2r(m-rn)}{n}.$$

Hence the bipartite graph H with parts R, V - R and edge set comprising those edges of G with one endpoint in R and the other in V - R has at least  $2rm/n - 3r^2$  edges.

We will now argue that  $rm/n \ge 3r^2$ . Indeed,  $rm/n \ge 3r^2$  is equivalent to  $r \le m/3n$ . Now  $r \le qn^2/m$  so it is enough to show that  $qn^2/m \le m/3n$  or equivalently, that  $3qn^3 \le m^2$ . Using the definition of q, we see that  $3qn^3 \le m^2$  follows from

$$m^2 \ln(2en^2/m) \ge 3n^3 \ln(n/2).$$

Suppose first that  $3n^{3/2} \le m \le 3n^{3/2}\sqrt{\ln n}$ . Then

$$m^{2} \ln\left(\frac{2en^{2}}{m}\right) \ge 9n^{3} \ln\left(\frac{2en^{2}}{3n^{3/2}\sqrt{\ln n}}\right) > 9n^{3} \ln\left(\sqrt{\frac{n}{\ln n}}\right) > 4n^{3} \ln n > 3n^{3} \ln(n/2).$$

On the other hand, if  $m \ge 3n^{3/2}\sqrt{\ln n}$ , then using  $m < n^2/2$  we have

$$m^2 \ln(2en^2/m) \ge 9n^3 \ln n \ln(2en^2/m) > 9n^3 \ln n \ln(4e) > 3n^3 \ln(n/2).$$

We conclude that H has at least  $2rm/n - 3r^2 \ge rm/n$  edges.

For the correctness of the algorithm it is sufficient to show that H contains a copy of  $K_{q,q}$ . This follows by the counting argument referred to in the introduction, which is included here for completeness. Let s denote the number of stars with centers in V - R and qleaves. Then

$$s = \sum_{v \in V-R} \binom{\deg_H(v)}{q} \geq \frac{n}{2} \binom{rm/n^2}{q},$$

using the convexity of the function which is  $\binom{x}{q}$  if  $x \ge q-1$  and 0 otherwise, and using  $r \le n/2$  which follows by the lower bound on m. If the latter quantity is greater than  $(q-1)\binom{r}{q}$  then there is a q-subset of R which is the leaf set for at least q distinct stars, and this gives a copy of  $K_{q,q}$ . Observe that the definition of q implies that  $n/2 \ge (2en^2/m)^q$  and this is equivalent to

$$\frac{n}{2} \left(\frac{rm}{n^2 q}\right)^q \ge \left(\frac{2er}{q}\right)^q.$$

Now the inequality above and standard estimates of the binomial coefficients give

$$\frac{n}{2}\binom{rm/n^2}{q} > \frac{n}{2}\left(\frac{rm}{n^2q}\right)^q \ge \left(\frac{2er}{q}\right)^q \ge q\left(\frac{re}{q}\right)^q > (q-1)\binom{r}{q},$$

Thus H indeed contains a  $K_{q,q}$ .

In order to show that the running time of the algorithm is polynomial, note first that, assuming an adjacency matrix representation, the set R can be found in  $O(n^2)$  steps. For a given q-subset of R, the common neighbors can be found in O(nq) steps. All q-subsets can be listed in  $O(\binom{r}{q})$  steps (see, e.g. [11]). Thus the algorithm requires time

$$O\left(n^2 + \binom{r}{q}nq\right).$$

The number of iterations is at most

$$\binom{r}{q} \le \left(\frac{re}{q}\right)^q \le e^q \left(\frac{n^2}{m}\right)^q = e^q e^{q \ln(n^2/m)}$$

Now  $m < n^2/2$  implies that

$$e^q \le e^{\ln n / \ln 4e} = n^{1/\ln 4e} < n^{0.4195}$$

and  $q < \ln n / \ln (n^2/m)$  implies that

$$e^{q\ln(n^2/m)} < e^{\ln n} = n.$$

Therefore the running time of the algorithm is  $O(n^{2.42})$ .  $\Box$ 

# 3 Decomposition into balanced complete bipartite subgraphs

Given a graph G = (V, E), we consider complete bipartite subgraphs  $G_i = (A_i, B_i, E_i), i = 1, \ldots, t$  such that the edges sets  $E_i$  form a partition of E. The *complexity* of such a decomposition is measured by the total number of vertices, i.e., by

$$\sum_{i=1}^{t} |A_i| + |B_i|$$

We find a decomposition of complexity  $O(n^2/\ln n)$ . The decomposition contains balanced bipartite graphs, thus  $|A_i| = |B_i|$  holds as well. The algorithm uses Algorithm **FIND-BIPARTITE BIPARTITE** in a straightforward manner. As stated, Algorithm **FIND-BIPARTITE** is guaranteed to work only if  $n \ge n_0$  for some  $n_0$ . As we are only interested in proving an asymptotic result, let us assume that graphs on fewer vertices are handled by some brute-force method.

#### Algorithm FIND-DECOMPOSITION

Given an *n*-vertex input graph G = (V, E), if  $n < n_0$ , use a brute-force method to find an optimal decomposition of G. Else, use Algorithm **FIND-BIPARTITE** repeatedly to find a complete balanced bipartite subgraph and delete it from the current graph, as long as there are more than  $n^2/\ln n$  edges. After that, form a separate bipartite graph from each remaining edge.

**Theorem 2** For every n-vertex graph G, Algorithm **FIND-DECOMPOSITION** finds a decomposition of G into balanced complete bipartite graphs, having complexity

$$O\left(\frac{n^2}{\ln n}\right).$$

The running time of the algorithm is polynomial in n.

**Proof.** As the size of the subgraphs produced by Algorithm **FIND-BIPARTITE** is of the same order of magnitude as guaranteed by the existence theorems, the theorem follows as in [4, 5, 13]. For completeness, we give the argument, following [13].

Let the subgraphs produced by the calls of Algorithm **FIND-BIPARTITE** be  $G_i = (A_i, B_i)$  with  $|A_i| = |B_i| = q_i$ , where i = 1, ..., t for some t. We need to show that

$$\sum_{i} q_i = O\left(\frac{n^2}{\ln n}\right). \tag{1}$$

Let us divide the iterations of the algorithm into *phases*. The  $\ell$ 'th phase consists of those iterations where the number of edges in the input graph of Algorithm **FIND-BIPARTITE** is more than  $n^2/(\ell + 1)$  and at most  $n^2/\ell$ . Dividing up the term  $q_i$  in (1) between the  $q_i^2$  edges of  $G_i$ , each edge gets a weight of  $1/q_i$ . We have to upper bound the sum of the weights assigned to the edges.

It follows from the definition of  $q_i$  in Theorem 1 that graphs formed in the  $\ell$ 'th phase have  $q_i = \Theta(\ln n / \ln \ell)$ . Thus edges, which get their weight in the  $\ell$ 'th phase, get a weight of  $\Theta(\ln \ell / \ln n)$ . The number of edges getting their weight in the  $\ell$ 'th phase is  $\Theta((\frac{1}{\ell} - \frac{1}{\ell+1})n^2) = \Theta(n^2/\ell^2)$ . Hence the total weight assigned to the edges is at most of the order of magnitude

$$\sum_{\ell=1}^{\infty} \frac{\ln \ell}{\ln n} \cdot \frac{n^2}{\ell^2} = \Theta\left(\frac{n^2}{\ln n}\right),$$

as  $\sum \frac{\ln \ell}{\ell^2}$  is convergent. The polynomiality of the running time follows directly from the polynomial running time of Algorithm **FIND-BIPARTITE**.  $\Box$ 

## 4 Subgraphs and decompositions of bipartite graphs

In this section we formulate the result analogous to Theorem 2 for bipartite graphs G = (A, B, E) having parts A and B, with |A| = a, |B| = b and |E| = m. We assume w.l.o.g. that  $a \ge b$ .

The algorithms and their analysis are straightforward modifications of those for general graphs. Algorithm **FIND-BIPARTITE-IN-BIPARTITE**, a modified version of **FIND-BIPARTITE**, uses functions q(a, b, m) and r(a, b, m). It constructs R as the set of r highest degree vertices in B, and checks the common neighborhood of all q element subsets of R. Algorithm **FIND-DECOMPOSITION-IN-BIPARTITE**, a modified version of **FIND-DECOMPOSITION**, uses this modified algorithm while the number of edges is greater than  $ab/\ln(a + b)$ .

**Theorem 3** Let G be a bipartite graph with sides of size a and b. Algorithm **FIND**-**DECOMPOSITION-IN-BIPARTITE** finds a decomposition of G into balanced com-

plete bipartite graphs, having complexity

$$O\left(\frac{ab}{\ln(a+b)}\right).$$

The running time of the algorithm is polynomial in a + b.

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