# Variants of Constrained Longest Common Subsequence

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#### Abstract

In this work, we consider a variant of the classical Longest Common Subsequence problem called Doubly-Constrained Longest Common Subsequence (DC-LCS). Given two strings  $s_1$  and  $s_2$  over an alphabet  $\Sigma$ , a set  $C_s$  of strings, and a function  $C_o: \Sigma \to N$ , the DC-LCS problem consists in finding the longest subsequence s of  $s_1$ and  $s_2$  such that s is a supersequence of all the strings in  $C_s$  and such that the number of occurrences in s of each symbol  $\sigma \in \Sigma$  is upper bounded by  $C_o(\sigma)$ . The DC-LCS problem provides a clear mathematical formulation of a sequence comparison problem in Computational Biology and generalizes two other constrained variants of the LCS problem: the Constrained LCS and the Repetition-Free LCS. We present two results for the DC-LCS problem. First, we illustrate a fixed-parameter algorithm where the parameter is the length of the solution. Secondly, we prove a parameterized hardness result for the Constrained LCS problem when the parameter is the number of the constraint strings  $(|C_s|)$  and the size of the alphabet  $\Sigma$ . This hardness result also implies the parameterized hardness of the DC-LCS problem (with the same parameters) and its NP-hardness when the size of the alphabet is constant.

#### 1 Introduction

The problem of computing the longest common subsequence (LCS) of two sequences is a fundamental problem in stringology and in the whole field of

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algorithms, as it couples a wide range of applications with a simple mathematical formulation. Applications of variants of LCS range from Computational Biology to data compression, syntactic pattern recognition and file comparison (for instance it is used in the Unix *diff* command).

A few basic definitions are in order. Given two sequences s and t over a finite alphabet  $\Sigma$ , s is a *subsequence* of t if s can be obtained from t by removing some (possibly zero) characters. When s is a subsequence of t, then t is a *supersequence* of s. Given two sequences  $s_1$  and  $s_2$ , the longest common subsequence problem asks for a longest possible sequence t that is a subsequence of both  $s_1$  and  $s_2$ .

The problem of computing the longest common subsequence of two sequences has been deeply investigated and polynomial time algorithms are well-known for the problem [11]. It is possible to generalize the LCS problem to a set of sequences: in such case the result is a sequence that is a subsequence of all input sequences. The problem is NP-hard even on binary alphabet [10] and it is not approximable within factor  $O(n^{1-\varepsilon})$ , for any constant  $\varepsilon > 0$ , on arbitrary alphabet [9].

Computational Biology is a field where several variants of the LCS problem have been introduced for various purposes. For instance researchers defined some similarity measures between genome sequences based on constrained forms of the LCS problem. More precisely, it has been studied an LCS-like problem that deals with two types of symbols (mandatory and optional symbols) to model the differences in the number of occurrences allowed for each gene [4, 2]. An illustrative example is the definition of repetition-free longest common subsequence [2] where, given two sequences  $s_1$  and  $s_2$ , a repetition-free common subsequence is a subsequence of both  $s_1, s_2$  that contains at most one occurrence of each symbol. Such a model can be useful in the genome rearrangement analysis, in particular when dealing with the exemplar model. In such framework we want to compute an exemplar sequence, that is a sequence that contains only one representative (called the exemplar) for each family of duplicated genes inside a genome. In biological terms, the exemplar gene may correspond to the original copy of the gene, from which all other copies have been originated.

A different variant of LCS that has been introduced to compare biological sequences is called Constrained Longest Common Subsequence [15]. More precisely, such variant of LCS can be useful when comparing two biological sequences that have a known substructure in common [15]. Given two sequences  $s_1$ ,  $s_2$ , and a constraint sequence  $s_c$ , we look for a longest common subsequence s of  $s_1$ ,  $s_2$ , such that  $s_c$  is a subsequence of s. The constrained LCS problem admits polynomial-time algorithms [15, 3, 5] but it becomes NP-hard when generalized to a set of input sequences or to a set of constraint sequences [8].

In this paper we introduce a new problem, called *Doubly-Constrained* Longest Common Subsequence and denoted as DC-LCS, that extends both the repetition-free longest common subsequence problem and the constrained longest common subsequence problem. More precisely, given two input sequences  $s_1$ ,  $s_2$ , the DC-LCS problem asks for the longest common subsequence s that satisfies two constraints: (i) the number of occurrences of each symbol  $\sigma$  is upper bounded by a quantity  $C_o(\sigma)$ , and (ii) s is a supersequence of the strings of a specified constraint set. First, we design a fixed-parameter algorithm [7] when the parameter is the length of the solution. Then we give a parameterized hardness result for the Constraint Longest Common Subsequence, when the number of constraint sequences and the size of the alphabet are considered as parameters. This result implies the same parameterized hardness result of DC-LCS.

## 2 Basic Definitions

Let  $s_1, s_2$  be two strings over an alphabet  $\Sigma$ . Given a string s, we denote by s[i] the symbol at position i in string s, and by  $s[i \dots j]$ , the substring of s starting at position i and ending at position j. A string constraint  $C_S$ consists of a set of strings, while an occurrence constraint  $C_o$  is a function  $C_o: \Sigma \to \mathbb{N}$ , assigning an upper bound on the number of occurrences of each symbol in  $\Sigma$ . First, consider the following variant of the LCS problem.

**Problem 1.** CONSTRAINED LONGEST COMMON SUBSEQUENCE (C-LCS) **Input:** two strings  $s_1$  and  $s_2$ , a string constraint  $C_s$ .

**Output:** a longest common subsequence s of  $s_1$  and  $s_2$ , so that each string in  $C_s$  is a subsequence of s.

The problem admits a polynomial time algorithm when  $C_s$  consists of a single string [15, 3, 5], while it is NP-hard when  $C_s$  consists of an arbitrary number of strings [8]. In the latter case, notice that C-LCS cannot be approximated, since a feasible solution for the C-LCS problem must be a supersequence of all the strings in the constraint  $C_s$  and computing if such a feasible solution exists is NP-complete [8].

**Problem 2.** REPETITION-FREE LONGEST COMMON SUBSEQUENCE (RF-LCS) **Input:** two strings  $s_1$  and  $s_2$ .

**Output:** a longest common subsequence s of  $s_1$  and  $s_2$ , so that s contains at most one occurrence of each symbol  $\sigma \in \Sigma$ .

The problem is APX-hard even when each symbol occurs at most twice in each of the input strings  $s_1$  and  $s_2$  [2]. A positive note is that allowing at most k occurrences of each symbol in each of  $s_1$  and  $s_2$  results in a  $\frac{1}{k}$ approximation algorithm [2].

We can introduce an even more general version of both the C-LCS and RF-LCS problem, called *Doubly-Constrained Longest Common Subsequence* (DC-LCS) problem.

**Problem 3.** DOUBLY-CONSTRAINED LONGEST COMMON SUBSEQUENCE (DC-LCS)

**Input:** two strings  $s_1$  and  $s_2$ , a string constraint  $C_s$ , and an occurrence constraint  $C_o$ .

**Output:** a longest common subsequence s of  $s_1$  and  $s_2$ , so that each string in  $C_s$  is a subsequence of s and s contains at most  $C_o(\sigma)$  occurrences of each symbol  $\sigma \in \Sigma$ .

It is easy to see that C-LCS problem is the restriction of the DC-LCS problem when  $C_o(\sigma) = |s_1| + |s_2|$  for each  $\sigma \in \Sigma$ . At the same time, the RF-LCS problem is the restriction of the C-LCS problem when  $C_s = \emptyset$  and  $C_o(\sigma) = 1$  for each  $\sigma \in \Sigma$ . Therefore the DC-LCS problem is APX-hard, since it inherits all hardness properties of C-LCS and RF-LCS.

## 3 A Fixed-Parameter Algorithm for DC-LCS

Initially we present a fixed-parameter algorithm for the DC-LCS problem when  $|C_s| \leq 1$  (hence the result holds also for the RF-LCS problem), where the parameter is the size of a solution of DC-LCS. Later on, we will extend the algorithm to a generic set  $C_s$ .

The algorithm is based on the color coding technique [1]. We recall the basic definition of perfect family of hash functions [14]. Given a set S, a family F of hash functions from S to  $\{1, 2, \ldots, k\}$  is called *perfect* if for any  $S' \subseteq S$  of size k, there exists an injective hash function  $f \in F$  from S' to the set of labels  $\{1, 2, \ldots, k\}$ .

Since  $|C_s| \leq 1$ , we denote by  $s_c$  the only sequence in  $C_s$ . Let k be the size of a solution for DC-LCS, and recall that a solution contains at most  $C_o(\sigma)$  occurrences of each symbol  $\sigma \in \Sigma$ . Notice that, since s is a subsequence of both  $s_1$  and  $s_2$ , and by the definition of  $C_o$ , the number of occurrences of each symbol  $\sigma \in \Sigma$  in a solution s is also (upper) bounded by the number of occurrences of  $\sigma$  in each  $s_1$  and  $s_2$  (i.e.  $occ(\sigma, s) \leq$  $\min\{C_o(\sigma), occ(\sigma, s_1), occ(\sigma, s_2)\}$ ). Let  $C'_o$  be a function from  $\Sigma$  to  $\mathbb{N}$  defined as  $C'_o(\sigma) := \min\{C_o(\sigma), occ(\sigma, s_1), occ(\sigma, s_1), occ(\sigma, s_2)\}$ .

Given  $C'_o$  and the sequences  $s_1$  and  $s_2$ , we construct a set  $\Sigma$  that contains the pairs  $(\sigma, i)$  for each  $\sigma \in \Sigma$  and  $i \in \{1, \ldots, C'_o(\sigma)\}$ . For example, if  $s_1 = aaaa \, bbb \, cc \, d$ ,  $s_2 = dd \, c \, bbbb \, aaaa$ , and  $C_o(a) = C_o(b) = C_o(c) = C_o(d) = 3$ , then the set  $\widetilde{\Sigma}$  is equal to  $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3), (c, 1), (d, 1)\}$ .

Consider now a perfect family F of hash functions from  $\Sigma$  to the set  $\{1, 2, \ldots, k\}$ . We can associate a function  $l: \Sigma \to 2^{\{1, 2, \ldots, k\}}$  with each  $f \in F$ , where  $l(\sigma) = \{f(\sigma, i) : (\sigma, i) \in \widetilde{\Sigma}\}$ . Let s be a solution of the DC-LCS problem of length at most k, and let L be a subset of  $\{1, \ldots, k\}$ . Then s is an *L*-colorful solution w.r.t. a hash function  $f \in F$  (and its associated function l) if and only if there exists a function  $l_1: \Sigma \to 2^{\{1, 2, \ldots, k\}}$  which satisfies the following conditions:

- (i)  $\forall \sigma \in \Sigma, l_1(\sigma) \subseteq l(\sigma) \cap L$ ,
- (ii)  $\forall \sigma \in \Sigma$ ,  $|l_1(\sigma)|$  is equal to the number of occurrences of  $\sigma$  in s,
- (iii)  $\forall \sigma_1, \sigma_2 \in \Sigma, \ l_1(\sigma_1) \cap l_1(\sigma_2) = \varnothing.$

Intuitively, an *L*-colorful solution *s* is a sequence such that it is possible to associate distinct elements (labels) of the set *L* with all the characters of *s* by using the function *l*. Notice that the length of an *L*-colorful solution *s* is equal to the number of labels that *s* uses, and each symbol  $\sigma$  does not occur more than  $C'_o(\sigma)$  times in *s*.

The basic idea of our algorithm is to verify if there exists an *L*-colorful solution that uses all labels in *L* or, equivalently, if the length of an optimal *L*-colorful solution is |L|. Such task is fulfilled via a dynamic programming recurrence. Since *F* is a perfect family of hash functions, for each feasible solution *s* of length *k*, there exists a hash function  $f \in F$  such that *s* is  $\{1, \ldots, k\}$ -colorful w.r.t. *f*. Therefore, by computing the recurrence for all hash functions of *F*, we are guaranteed to find a solution of length *k*, if such a solution exists.

Given a hash function f, we define V[i, j, h, L] which takes value 1 if and only if there exists an L-colorful common subsequence s of  $s_1[1...i]$ and  $s_2[1...j]$ , such that s is a supersequence of  $s_c[1...h]$  and s has length equal to |L| (or, equivalently, s uses all labels in L). Notice that the actual supersequence can be computed by a standard backtracking technique. Theorem 3.1 states that V[i, j, h, L] can be computed by the following dynamic programming recurrence which is an extension of the standard equation for the Longest Common Subsequence (LCS) problem [6].

$$V[i, j, h, L] = \max \begin{cases} V[i - 1, j, h, L] \\ V[i, j - 1, h, L] \\ V[i - 1, j - 1, h, L \setminus \{\lambda\}] & \text{if } s_1[i] = s_2[j] \text{ and} \\ \lambda \in L \cap l(s_1[i]) \\ V[i - 1, j - 1, h - 1, L \setminus \{\lambda\}] & \text{if } s_1[i] = s_2[j] = s_c[h] \text{ and} \\ \lambda \in L \cap l(s_1[i]) \end{cases}$$
(1)

The boundary conditions are V[0, j, h, L] = 0 and V[i, 0, h, L] = 0 if  $L \neq \emptyset$ , while  $V[i, j, 0, \emptyset] = 1$ , and  $V[i, j, h, \emptyset] = 0$  when h > 0. Moreover, notice that, as a consequence of the recurrence's definition, we have V[i, j, h, L] = 0 for all h > |L|. A feasible solution of length k is  $\{1, \ldots, k\}$ colorful w.r.t. f if and only if  $V[|s_1|, |s_2|, |s_c|, \{1, \ldots, k\}] = 1$ . In this case, a standard backtracking search can reconstruct the actual solution.

**Theorem 3.1.** Let  $f \in F$  be a hash function mapping injectively the solution s to the set of labels  $\{1, \ldots, k\}$ . Then Equation (1) is correct.

*Proof.* We will prove the theorem by induction, that is we will prove the correctness of the value in  $V[i_a, j_a, h_a, L_a]$  by assuming that of  $V[i_b, j_b, h_b, L_b]$  when  $i_b \leq i_a$ ,  $j_b \leq i_a$ ,  $h_b \leq h_a$ ,  $L_b \subseteq L_a$ , and at least one inequality is strict.

Let s be an optimal  $L_a$ -colorful solution for the sequences  $s_1[1, \ldots, i_a]$ ,  $s_2[1, \ldots, j_a]$ ,  $s_c[1, \ldots, h_a]$ , and let  $\beta$  be the last symbol of s, that is  $s = t\beta$ , where t is the prefix of s consisting of all but the last character.

If  $\alpha \neq \beta$  then, just as for the recurrences of the standard LCS problem [6], the theorem holds.

Therefore we can assume now that  $\alpha = \beta$ . Since *s* is  $L_a$ -colorful, then there exists a mapping  $l_1$  satisfying the definition of *L*-colorfulness. By condition (ii),  $|l_1(\beta)|$  is equal to the number of occurrences of  $\beta$  in *s*. Let *z* be the label which is image through *f* of the last character of *s*. Then there exists an  $L \setminus \{z\}$ -colorful solution *t* of  $s_1[1, \ldots, i_a - 1], s_2[1, \ldots, j_a - 1],$  $s_c[1, \ldots, j_a]$  (if *t* is a supersequence of  $s_c[1, \ldots, j_a]$ ) or of  $s_1[1, \ldots, i_a - 1],$  $s_2[1, \ldots, j_a - 1], s_c[1, \ldots, j_a - 1]$ , hence completing the proof.

If f is a hash function that does not map injectively the solution s of length k to the set of labels  $\{1, \ldots, k\}$  then, by definition of hash function, there is a label  $z \in \{1, \ldots, k\}$  that is not in the image through fof any character of s. The latter observation also implies that z is not in the image through l of any symbol, therefore for each set L including z, the last two cases of our recurrence equation cannot apply, which implies that  $V[i, j, h, \{1, \ldots, k\}] = 0$  for all values of i, j, h, hence establishing the correctness of our algorithm.

It is immediate to notice that the total number of entries of the matrix  $V[\cdot, \cdot, \cdot, \cdot]$  is  $|s_1||s_2||s_c|2^k$ . Furthermore notice that computing each entry requires at most O(k) time, as case 1 and case 2 of the recurrence require constant time, while case 2 and case 4 require at most O(k) time, since  $|L| \leq k$ . Since there exists a perfect family of hash functions whose size is  $O(\log |\tilde{\Sigma}|)2^{O(k)}$  and that can be computed in  $O(|\tilde{\Sigma}|\log |\tilde{\Sigma}|)2^{O(k)}$  time [1], and  $|\tilde{\Sigma}| \leq |s_1$ , the algorithm has an overall  $O(|s_1|\log |s_1|)2^{O(k)} + O(|s_1||s_2||s_c|k2^k)$  time complexity.

The algorithm actually computes a longest supersequence of  $s_c$  that is a feasible solution of the problem. Assume now that  $C_s$  is a generic occurrence set, and let x be an optimal solution of a generic instance of the DC-LCS problem of size k. It is immediate to notice that, by removing from x all symbols that are not also in one of the sequence of  $C_s$ , we obtain a common supersequence  $x_1$  of  $C_s$  that is a subsequence of x. Moreover, as x has size k,  $x_1$  contains at most k characters (where k is the length of an optimal solution).

Notice that the alphabet consisting of the symbols appearing in at least one sequence of  $C_s$  contains at most k symbols, for otherwise all supersequences of  $C_s$  would be longer than k. Consequently there are at most  $k^k$ such supersequences. Our algorithm for a generic  $C_s$  enumerates all such supersequences  $s_c$ , and applies the algorithm for  $|C_s| = 1$  on the new set of constraint sequences made only of  $s_c$ , returning the longest feasible solution computed.

The overall time complexity is clearly  $O\left(k^k(|s_1|\log|s_1|)2^{O(k)}+|s_1||s_2||s_c|k2^k)\right)$ .

# 4 W[1]-hardness of C-LCS

In this section we prove that computing if there exists a feasible solution of C-LCS is not only NP-complete, but it is also W[1]-hard when the parameter is the number of string in  $C_s$  and the alphabet  $\Sigma$  (see [7] for an exposition on the consequences of W[1]-hardness).

We reduce the Shortest Common Supersequence (SCS) problem parameterized by the number of input strings and the size of alphabet  $\Sigma$ , which is known to be W[1]-hard [12]. Let  $R = \{r_1, \ldots, r_k\}$  be a set of sequences over alphabet  $\Sigma$ , hence R is a generic instance of the SCS problem. In what follows we denote by l the size of a solution of the SCS problem.

The input of the C-LCS consists of two sequences  $s_1$ ,  $s_2$ , and a string constraint  $C_s$ . Let # be a delimiter symbol not in  $\Sigma$ . Moreover, given a sequence  $r_i = y_1 y_2 \cdots y_z$  over alphabet  $\Sigma$ , let  $c(r_i)$  be the sequence  $y_1 \# y_2 \# \cdots \# y_z \#$ . Pose  $C_s = \{\#^l\} \cup \{c(r_i) : r_i \in R\}$ , let w be a sequence over  $\Sigma$  such that wcontains exactly one occurrence of each symbol in  $\Sigma$ , and let rev(w) be the reversal of w. Finally, let  $s_1 = (w\#)^l$  and  $s_2 = (rev(w)\#)^l$ . In the following we call each occurrence of w or of rev(w) a *block*.

Let t be any supersequence of  $\#^l$  that is also a common subsequence of  $s_1$  and  $s_2$ . Since in each of those sequences there are  $l \ \#s$ , then also t must contain  $l \ \#s$ , which in turn implies that, by construction of w, at most one symbol of each block can be in t. Therefore t contains at most 2lsymbols. At the same time, let p be a generic sequence no longer than 2l, ending with a # and such that no two symbols from  $\Sigma$  appear consecutively in p. Since each symbol of  $\Sigma$  occurs exactly once in w, it is immediate to notice that p is a common subsequence of  $s_1$  and  $s_2$ . Consequently, the set of all supersequences of  $\#^l$  that are also common subsequences of  $s_1$  and  $s_2$ is equal to the set of sequences q with length not larger than 2l and such that (i) q contains exactly  $l \ \#s$ , (ii) q ends with a #, and (iii) taken two consecutive symbols from q, at least one of those symbols is equal to #.

An immediate consequence is that there exists a feasible solution of length 2l of the instance of C-LCS made of the set  $C_s$  and the two sequences  $s_1$  and  $s_2$  iff there exists a supersequence of length 2l of the set R of sequences.

The reduction described is an FPT-reduction [7]. Finally, notice that the W[1]-hardness of C-LCS with parameters  $|C_s|$  and  $|\Sigma|$  implies the W[1]hardness of DC-LCS with parameters  $|C_s|$  and  $|\Sigma|$  since C-LCS is a restriction of the DC-LCS problem. Moreover, notice that the same reduction can be applied starting from the SCS problem over binary alphabet, implying that the DC-LCS problem is NP-hard over a fixed ternary alphabet, as the SCS problem is NP-hard over a binary alphabet [13].

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