

On the super connectivity of Kronecker products of graphs*

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Abstract

Let G_1 and G_2 be two graphs. The *Kronecker product* $G_1 \times G_2$ has vertex set $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and edge set $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$. A graph G is *super connected*, or simply *super- κ* , if every minimum separating set is the neighbors of a vertex of G , that is, every separating set isolates a vertex. In this paper we show that for an arbitrary graph G with $\kappa(G) = \delta(G)$ and K_n ($n \geq 3$) a complete graph on n vertices, $G \times K_n$ is super- κ , where $\kappa(G)$ and $\delta(G)$ are the connectivity and the minimum degree of G , respectively.

Keywords: Super connectivity; Kronecker product

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1 Introduction and terminology

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow Bondy [5] for terminology and definitions.

Let $G = (V(G), E(G))$ be a graph. For two vertices $u, v \in V(G)$, u and v are *neighbors* if u and v is adjacent. The set of vertices adjacent to the vertex v is called the *neighborhood* of v and denoted by $N(v)$, i.e., $N(v) = \{u \mid uv \in E(G)\}$. The *degree* of v is equal to $|N(v)|$, denoted by $d_G(v)$ or simply $d(v)$. The number $\delta(G) = \min\{d_G(v) \mid v \in V(G)\}$ is the *minimum degree* of G . For a subset $S \subseteq V(G)$, the neighborhood of S is $N(S) = \bigcup_{v \in S} N(v)$. The subgraph induced by S is denoted by $G[S]$, and let $d_S(v)$ denote the number of vertices in S that are adjacent to the vertex v . As usual, K_n denotes the complete graph on n vertices and C_n is the cycle on n vertices.

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A set $S \subset V$ is a *separating set* of a connected graph G , if either $G - S$ is disconnected or reduces to the trivial graph K_1 . The *connectivity* of G , denoted by $\kappa(G)$, is the minimum cardinality of a separating set of G . In particular, $\kappa(K_n) = n - 1$ and $\kappa(G) = 0$ if and only if G is disconnected or a K_1 . Clearly, $\kappa(G) \leq \delta(G)$. A graph G with minimum degree $\delta(G)$ is *maximally connected* if $\kappa(G) = \delta(G)$.

The notion of super-connectedness proposed in [2, 3, 4] aims at pushing the analysis of the connectivity properties of graphs beyond the standard connectivity. A graph G is *super connected*, or simply *super- κ* , if every minimum separating set is the neighbors of a vertex of G , that is, every separating set isolates a vertex. Observe that a super-connected graph G is necessarily maximally connected, i.e., $\kappa(G) = \delta(G)$, but the converse is not true. It is easily to see from the cycle graph C_n ($n \geq 6$).

The Kronecker product, together with the Cartesian, the strong, and the lexicographic product, is one of the four standard graph products [12]. The Kronecker product of two graphs G_1 and G_2 is defined as $V(G_1 \times G_2) = V(G_1) \times V(G_2)$ and $E(G_1 \times G_2) = \{(u_1, v_1)(u_2, v_2) : u_1u_2 \in E(G_1) \text{ and } v_1v_2 \in E(G_2)\}$ (see, [6, 21]). The Kronecker product has been introduced and studied from several points of view and is known under many different names, for instance as the direct product, cardinal product, categorical product, tensor product and cross product. Moreover, it is universal in the sense that every graph is an induced subgraph of a suitable direct product of complete graphs [18]. The Kronecker product of graphs has been extensively investigated concerning graph colorings, graph recognition and decomposition, graph embedding, matching theory, stability and domination theory in graphs (see, for example, [1, 7, 16]). The properties on the structure of Kronecker product of graphs can be found in [10, 11, 15, 19]. One has known that it has many interesting applications, for instance it can be used in automata theory [9], complex networks [14] and modeling concurrency in multiprocessor systems [13].

Miller [17] and Weichsel [21] investigated the connectedness of Kronecker product of two connected graphs. Recently, the connectivity of Cartesian products and strong products of two connected graphs have been studied, and in all cases the explicit formulae have been obtained in terms of the graph invariants of the factor graphs (see, [8, 19, 22] for more details). The connectivity of Kronecker products of graphs seems to be more complex than that with the Cartesian or strong products. Guji and Vumar [10] presented the connectivity of Kronecker product of a bipartite graph and a complete graph and they proposed to investigate the connectivity of Kronecker product of a nontrivial graph and a complete graph. Recently, Wang and Xue [20] settled the problem and they obtained the following result.

Theorem 1 ([20]). *For any graph G , $\kappa(G \times K_n) = \min\{n\kappa(G), (n - 1)\delta(G)\}$ for $n \geq 3$.*

Recently, Guo et al. [11] studied the super connectivity of Kronecker product of a bipartite graph and a complete graph and they proved the following result.

Theorem 2 ([11]). *If G is a bipartite graph with $\kappa(G) = \delta(G)$, then $G \times K_n$ ($n \geq 3$) is super- κ .*

In this paper, motivated by the above results, we further investigate the super connectivity of Kronecker product of an arbitrary graph and a complete graph K_n ($n \geq 3$). Our main result is as follows.

Theorem 3. *For an arbitrary graph G with $\kappa(G) = \delta(G)$, $G \times K_n$ ($n \geq 3$) is super- κ .*

2 Preliminaries

In this section we give some properties on Kronecker product of graphs, and we will use them in the proof of our main result. We first give two known results.

Observation 1 ([6]). *Let $H = G_1 \times G_2 = (V(H), E(H))$. Then*

- (1) $|V(H)| = |V(G_1)| \cdot |V(G_2)|$,
- (2) $|E(H)| = 2|E(G_1)| \cdot |E(G_2)|$,
- (3) *for every $(u, v) \in V(H)$, $d_H((u, v)) = d_{G_1}(u) \cdot d_{G_2}(v)$.*

By Observation 1, we have $\delta(G \times K_n) = (n - 1)\delta(G)$ for any graph G .

Lemma 1 ([21]). *Let G_1 and G_2 be connected graph. The graph $H = G_1 \times G_2$ is connected if and only if G_1 or G_2 contains an odd cycle.*

When considering the Kronecker product of a graph G and K_n ($n \geq 3$), we shall always let $V(G) = \{u_1, u_2, \dots, u_m\}$, $V(K_n) = \{v_1, v_2, \dots, v_n\}$ and set $S_i = \{u_i\} \times V(K_n)$, for $i = 1, 2, \dots, m$. Then S_i is an independent set in $G \times K_n$, and $V(G \times K_n)$ has a partition $V(G \times K_n) = S_1 \cup S_2 \cup \dots \cup S_m$.

Let $S \subseteq V(G \times K_n)$ satisfy the following three conditions:

- (1). $|S| = (n - 1)\delta(G)$, and
- (2). $S'_i := S_i - S \neq \emptyset$, for $i = 1, 2, \dots, m$, and
- (3). $G \times K_n - S$ has no isolated vertex.

Associated with G , S and S'_i , we define the following new graph G^* as described in [20].

(i). $V(G^*) = \{S'_1, S'_2, \dots, S'_m\}$, and

(ii). $E(G^*) = \{S'_i S'_j : E(S'_i, S'_j) \neq \emptyset\}$, where $E(S'_i, S'_j)$ denotes the collections of all edges in $G \times K_n - S$ with one end in S'_i and the other in S'_j .

Next we give two lemmas about the connectedness of G^* and the structure of S'_i , which play a key role in the proof of our main result. In the next proofs, we always assume $\kappa(G) = \delta(G) > 0$.

In [20] proved that G^* is connected if G is connected and $|S| < (n-1)\delta(G)$. In fact, when $|S| = (n-1)\delta(G)$ here, we still get the result by using the same method in [20].

Lemma 2. *If G is connected, then G^* is connected.*

Lemma 3. *Let G be nonbipartite graph with $\kappa(G) = \delta(G)$. Then for any vertex of G^* , S'_i , as a subset of $V(G \times K_n)$, it is contained in the vertex set of some component of $G \times K_n - S$.*

Proof. It suffices to prove the lemma for $i = 1$. We consider the cardinality of the set S'_1 .

If $|S'_1| = 1$, then the assertion holds trivially. So we may assume that $|S'| \geq 2$. We consider the following two cases:

Case 1: $|S'_1| \geq 3$. Suppose to the contrary that S'_1 is not contained in any component of $G \times K_n - S$. Then there must exist a component, say C , such that $0 < |S'_1 \cap V(C)| \leq |S'_1|/2 < |S'_1| - 1$. Let $(u_1, v_p) \in S'_1 \cap V(C)$. By the conditions (1) and (3) of the definition of S , $G \times K_n - S$ has at least one vertex, say (u_j, v_q) ($j \in \{2, \dots, m\}$), such that (u_j, v_q) and (u_1, v_p) are neighbors. Clearly, $(u_j, v_q) \in V(C)$, and $S'_1 - \{(u_1, v_p)\} \subseteq V(C)$ since every vertex in $S'_1 - \{(u_1, v_p)\}$ is adjacent to (u_j, v_q) . It follows $|S'_1 \cap V(C)| \geq |S'_1| - 1$, a contradiction.

Case 2: $|S'_1| = 2$. Let $Z^* = \{S'_j : j \in \{1, 2, \dots, m\}, |S'_j| = 1\}$ and C^* be a component of $G^* - Z^*$ which containing S'_1 . Let $|C^*| = r$, without loss of generality, we may assume $V(C^*) = \{S'_1, S'_2, \dots, S'_r\}$.

Since each $S'_j \in V(C^*)$ contains at least two elements, any edge $S'_j S'_k$ in C^* implies every vertex in S'_k has at least one neighbor in S'_j in $G \times K_n - S$. Therefore, if there is a vertex S'_j in C^* contains in the vertex set of some component C of $G \times K_n - S$, then every S'_k with $S'_k S'_j \in E(C^*)$ is contained in $V(C)$ as each $|S'_j| > 1$. It follows from the connectedness of C^* that $\bigcup_{i=1}^r S'_i \subseteq V(C)$ and hence $S'_1 \subseteq V(C)$.

By Case 1, we may assume each $S'_j \in V(C^*)$ contains exactly two elements. Let $S'_j = \{u_j\} \times F_j$, $j = 1, 2, \dots, r$, and $F_j \subseteq V(K_n)$.

Suppose that there exists an edge $S'_j S'_k$ in C^* with $F_j \neq F_k$. It is easily to see that $S'_j \cup S'_k$ induces a connected subgraph of $G \times K_n$. This implies that S'_j and S'_k are contained in the

same component, say C , of $G \times K_n - S$. As mentioned above, we have $S'_1 \subseteq \bigcup_{i=1}^r S'_i \subseteq V(C)$, the lemma follows.

Otherwise, by the connectedness of C^* , we have $F_1 = F_2 = \dots = F_r$. Notice that C^* and $G[\bigcup_{i=1}^r S'_i]$ are isomorphic to $G[u_1, u_2, \dots, u_r]$ and $G[u_1, u_2, \dots, u_r] \times K_2$, respectively. If we can prove that $G[u_1, u_2, \dots, u_r] \times K_2$ is connected in $G \times K_n - S$, then the lemma follows.

Note that G is connected. If $G[u_1, u_2, \dots, u_r]$ contains an odd cycle, then the assertion follows by Lemma 1.

Suppose that $G[u_1, u_2, \dots, u_r]$ does not contain an odd cycle, that is, $G[u_1, u_2, \dots, u_r]$ is bipartite. This implies that $\delta(G[u_1, u_2, \dots, u_r]) \leq r/2$. Let $j \in \{1, 2, \dots, r\}$ such that $d_{G[u_1, u_2, \dots, u_r]}(u_j) = \delta(G[u_1, u_2, \dots, u_r])$.

Let u_k be a neighbor of u_j in G . Then either $S'_k \in Z^*$ or S'_k is adjacent to S'_j in C^* . Thus,

$$\delta(G) \leq d_G(u_j) \leq d_{C^*}(S'_j) + |Z^*| = d_{G[u_1, u_2, \dots, u_r]}(u_j) + |Z^*| \leq r/2 + |Z^*|. \quad (1)$$

Therefore, we have

$$(n-1)\delta(G) = |S| \geq (n-2)r + (n-1)|Z^*| \geq (n-1)\left(\frac{r}{2} + |Z^*|\right) \geq (n-1)\delta(G). \quad (2)$$

This means that the equations holds in (1) and (2). Hence, $n = 3$, $\delta(G) = d_G(u_j) = r/2 + |Z^*|$ and $d_{G[u_1, u_2, \dots, u_r]}(u_j) = \delta(G[u_1, u_2, \dots, u_r]) = r/2$. Since $G[u_1, u_2, \dots, u_r]$ is bipartite, it is $r/2$ -regular, so each vertex u_j ($1 \leq j \leq r$) has the same degree $r/2 + |Z^*|$ in G . This implies that each S'_j ($1 \leq j \leq r$) is adjacent to all the vertices of Z^* in G^* .

We claim that $Z^* \neq \emptyset$.

Indeed, if $Z^* = \emptyset$, then $C^* = G^*$, so $G[u_1, u_2, \dots, u_r] = G[u_1, u_2, \dots, u_m] = G$, which contradicts our assumption that G is a nonbipartite graph. Clearly, $G[u_1, u_2, \dots, u_r] \times K_2$ can be connected by the vertices of Z^* , as desired. \square

By the definition, the following lemma is straight.

Lemma 4. *Let $m = |G| \leq 2$ and u_i be any vertex of G . Then*

- (1). $\delta(G - u_i) \geq \delta(G) - 1$, and
- (2). $\kappa(G - u_i) \geq \kappa(G) - 1$.

3 Proof of the main result

Now we are ready to give the proof of Theorem 3. By Theorem 2, we only need to show that the assertion in Theorem 3 is true for a nonbipartite graph G . Therefore, we always assume that G is nonbipartite in our proof below.

Proof of Theorem 3. If G is disconnected, i.e., $\kappa(G) = \delta(G) = 0$, then $G \times K_n$ is disconnected, and the assertion holds. So we may assume that G is connected and $\delta(G) \geq 1$.

If we can show that for every subset S of $G \times K_n$ with $|S| = (n-1)\delta(G)$, either $G \times K_n - S$ is connected or $G \times K_n - S$ has an isolated vertex, then we are done. Our proof is by contradiction. Suppose that $G \times K_n$ is not super- κ . Then there is a separating set S with $|S| = (n-1)\delta(G)$ such that $G \times K_n - S$ is not connected but has no isolated vertex. If we can show that $G \times K_n - S$ is connected, then we shall arrive at a contradiction and the assertion holds.

We will distinguish two possibilities as follows.

Case 1: If S satisfies condition (2), i.e., $S'_i := S_i - S \neq \emptyset$, for $i = 1, 2, \dots, m$. It follows from Lemma 2 and Lemma 3 that $G \times K_n - S$ is connected.

Case 2: S does not satisfies condition (2). Then there exists an S_i contained in S . So $S - S_i \subseteq V((G - u_i) \times K_n)$ and

$$\begin{aligned} |S - S_i| &= |S| - n = (n-1)\delta(G) - n \\ &< \min\{n\kappa(G) - n, (n-1)\delta(G) - (n-1)\} \\ &\leq \min\{n\kappa(G - u_i), (n-1)\delta(G - u_i)\}, \end{aligned}$$

the last inequality above follows from Lemma 4.

By Theorem 1, we have $\kappa((G - u_i) \times K_n) = \min\{n\kappa(G - u_i), (n-1)\delta(G - u_i)\}$. Hence, $(G - u_i) \times K_n - (S - S_i)$ is connected. Note that $(G - u_i) \times K_n - (S - S_i) = G \times K_n - S$. Hence $G \times K_n - S$ is connected.

In all cases, we show that $G \times K_n - S$ is connected, this contradicts our assumption on S . So the assertion follows.

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