# Approximate pattern matching with $k$-mismatches in packed text ${ }^{\star}$ 

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#### Abstract

Given strings $P$ of length $m$ and $T$ of length $n$ over an alphabet of size $\sigma$, the string matching with $k$-mismatches problem is to find the positions of all the substrings in $T$ that are at Hamming distance at most $k$ from $P$. If $T$ can be read only one character at the time the best known bounds are $O(n \sqrt{k \log k})$ and $O(n+n \sqrt{k / w} \log k)$ in the word-RAM model with word length $w$. In the RAM models (including $A C^{0}$ and word-RAM) it is possible to read up to $\lfloor w / \log \sigma\rfloor$ characters in constant time if the characters of $T$ are encoded using $\lceil\log \sigma\rceil$ bits. The only solution for $k$-mismatches in packed text works in $O\left((n \log \sigma / \log n)\lceil m \log (k+\log n / \log \sigma) / w\rceil+n^{\varepsilon}\right)$ time, for any $\varepsilon>0$. We present an algorithm that runs in time $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor}(1+\right.$ $\log \min (k, \sigma) \log m / \log \sigma))$ in the $A C^{0}$ model if $m=O(w / \log \sigma)$ and $T$ is given packed. We also describe a simpler variant that runs in time $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor} \log \min (m, \log w / \log \sigma)\right)$ in the word-RAM model. The algorithms improve the existing bound for $w=\Omega\left(\log ^{1+\epsilon} n\right)$, for any $\epsilon>0$. Based on the introduced technique, we present algorithms for several other approximate matching problems.


## 1 Introduction

The string matching problem consists in reporting all the occurrences of a pattern $P$ of length $m$ in a text $T$ of length $n$, both strings over a common alphabet. The occurrences may be exact or approximate according to a specified matching model. For most matching problems, all the characters from the text and the pattern need to be read at least once in the worst case; hence, if they are read one at a time, the worst-case lower bound is $\Omega(n)$. Interestingly, for some standard problems (e.g., exact pattern matching) it is possible to achieve a sublinear search time, for short patterns, even in the worst case, if the word-RAM computational model is assumed and the text is packed. In a packed encoding,

[^0]the characters of a string are stored adjacently in memory and each character is encoded using $\log \sigma$ bits $^{4}$, where $\sigma$ is the alphabet size. A single machine word, of size $w \geq \log n$ bits, thus contains up to $\alpha=\lfloor w / \log \sigma\rfloor$ characters. While the word size of current architectures is 64 (which, for example, permits up to 32 DNA symbols to be encoded in a word), there are also vector instruction sets where the word size is larger, such as SSE and AVX (128 and 256 bits) or the Intel Xeon Phi coprocessor (512 bits).

For this setting and the exact string matching problem, several sublinear-time algorithms have been given in recent years $[13,7,5,6,8]$.

In this paper we study the string matching with $k$-mismatches problem in the packed scenario. This problem is to find the positions of all the substrings in $T$ that are at Hamming distance at most $k$ from $P$, i.e., that match $P$ with at most $k$ mismatches. For this problem, the best known bounds in the worstcase are $O(n \sqrt{k \log k})$ time for the algorithm by Amir et al. [2] and $O(n+$ $n \sqrt{k / w} \log k)$ time for its implementation based on word-level parallelism [14]. One classical result in the word-RAM model that is also practical is Shift-Add [4]. The best worst-case bound of this algorithm, based on the Matryoshka counters technique [15], is $O(n\lceil m / w\rceil)$.

In [13] Fredriksson presented a Shift-Add variant, based on the superalphabet technique, that works in $O((n \log \sigma / \log n)\lceil m \log (k+\log n / \log \sigma) / w\rceil+$ $\left.n^{\varepsilon}\right)$ time, for any $\varepsilon>0$. To our knowledge, this is the only solution for the $k$ mismatches problem that works on packed text and that achieves sublinear time complexity when $m$ and $k$ are sufficiently small.

In this work, we present an algorithm for the $k$-mismatches problem that runs in time $O\left(\frac{n}{\lfloor w /(m \log \sigma)}(1+\log \min (k, \sigma) \log m / \log \sigma)\right)$ in the $A C^{0}$ model for $m \leq \alpha$ if $T$ is given packed. We also describe a simpler variant that runs in time $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor} \log \min (m, \log w / \log \sigma)\right)$ in the word-RAM model. In particular, it achieves sublinear worst-case time when $m \log \sigma \log \min (m, \log w / \log \sigma)=$ $o(w)$. Note that for $w=\Theta(\log n)$ Fredriksson's solution is better, but our algorithm dominates if $w=\Omega\left(\log ^{1+\epsilon} n\right)$, for any $\epsilon>0$, or, more precisely, if $w=\omega(\log n \log \log w)$.

## 2 Basic notions and definitions

Let $\Sigma=\{0,1, \ldots, \sigma-1\}$ denote an integer alphabet and $\Sigma^{m}$ the set of all possible sequences of length $m$ over $\Sigma . S[i], i \geq 0$, denotes the $(i+1)$-th character of string $S$, and $S[i \ldots j]$ its substring between the $(i+1)$-st and the $(j+1)$-st characters (inclusive).

The $k$-mismatches problem consists in, given a pattern (string) $P$ of length $m$ and a text (string) $T$ of length $n$, reporting all the positions $0 \leq j \leq n-m$ such that $|\{0 \leq h<m: T[j+h] \neq P[h]\}| \leq k$, i.e., such that the Hamming distance between $P$ and the substring $T[j \ldots j+m-1]$ is at most $k$.

[^1]The word-RAM model is assumed, with machine word size $w \geq \log n$. We use some bitwise operations following the standard notation as in C language: $\&, \mid$, $\wedge, \sim, \ll, \gg$ for and, or, xor, not, left shift and right shift, respectively.

## 3 Operations on words

We define a $(f)$-word as a machine word logically divided into $\lfloor w / f\rfloor$ fields of $f$ bits. Given a $(f)$-word $W$, we denote with $W[i]$ its $i$-th field, for $i=1, \ldots,\lfloor w / f\rfloor$. The most significant bit in a field is called the top bit. A field where only the top bit is set is thus equal to $2^{f-1}$. We also define, for a given field size $f$, the mask $V_{f}$ where $V_{f}[i]=2^{f-1}$, for $i=1, \ldots,\lfloor w / f\rfloor$. We define the following primitives on words:
find non-zero fields $(\operatorname{fnf}(A, f))$ : given a $(f)$-word $A$, return a $(f)$-word $A^{\prime}$ such that $A^{\prime}[i]$ is equal to $2^{f-1}$ if $A[i] \neq 0$, and to 0 otherwise, for $i=1, \ldots,\lfloor w / f\rfloor$.
How to implement this primitive in $O(1)$ time was presented in [8, Sect. 4], but we give a simpler method, in three simple steps and in constant time.

1. $W \leftarrow A \& \sim V_{f}$
2. $X \leftarrow V_{f}-W$
3. $A^{\prime} \leftarrow(\sim X \mid A) \& V_{f}$

The first two steps generate a $(\log \sigma)$-word $X$ such that $X[i]=2^{\log \sigma-1}$ if $A[i]$ with the top-bit masked is zero, and $<2^{\log \sigma-1}$ otherwise. It is then not hard to see that $A[i]$ is non-zero iff either $A[i] \geq 2^{\log \sigma-1}$ or $(\sim X)[i] \geq 2^{\log \sigma-1}$, from which follows the correctness of the last step.
sideways addition $(\mathrm{sa}(A))$ : given a word $A$, return the number of bits set in $A$.
This primitive is a well-known bitwise operation, also known as popcount. The folklore method [19] to compute it in the word-RAM model has $O(\log \log w)$ time complexity.
interleaved blockwise sideways addition (ibsa $(A, f, b)$ ): Given a $(f)$-word $A$, such that $f$ divides $\log \sigma$ and only the top bit of each field may be set, and a power of two $b$, return a $(f)$-word $A^{\prime}$ such that $A^{\prime}[i]$ is equal to

$$
\sum_{j=0}^{\min (b,\lceil i /(\log \sigma / f)\rceil)-1} X[i-j(\log \sigma / f)]
$$

where $X=A \gg(f-1)$, i.e., the $i$-th field contains the number of bits set in the sequence of $\min (b,\lceil i /(\log \sigma / f)\rceil)$ fields in $A$ spaced by $\log \sigma$ bits ending at $i$.
This operation is a variant of the parallel prefix-sum operation described in [16] and can be implemented in $O(\log b)$ steps, where the $j$-th step computes

$$
A_{j}^{\prime}= \begin{cases}A_{j-1}^{\prime}+\left(A_{j-1}^{\prime} \ll\left(2^{j-1} \times \log \sigma\right)\right) & \text { if } j>0 \\ A \gg(f-1) & \text { otherwise }\end{cases}
$$

Since $f \geq \log (b+1)$ does not necessarily hold, the top bits of all the fields are masked out before each addition and restored afterwards. In this way, if a sum is $\geq 2^{f-1}$, its encoded value is $\geq 2^{f-1}$ but the exact value is undetermined.
blockwise sideways addition (bsa $(A, f, b)$ ): Given a $(f)$-word $A$, such that only the top bit of each field may be set, and a power of two $b$, return a $(b f)$-word $A^{\prime}$ such that $A^{\prime}[i]$ is equal to

$$
\sum_{j=0}^{b-1} X[i b-j]
$$

where $X=A \gg(f-1)$, i.e., the $i$-th field contains the number of bits set in the block of $b$ fields in $A$ ending at $i b$.
This operation can be implemented in time $O(\log \min (b, \log w / f))$ in word-RAM and in time $O(\log b)$ in $A C^{0}$ using the following method. We assume that the word size $w$ is a power of two. Let $r$ be the smallest power of two greater than or equal to $\log (w+1) / f$. The first step consists in computing a word logically divided into fields of $\min (b, r) f$ bits, such that each field contains the number of bits set in the corresponding $\min (b, r)$ fields in the original word. This widening operation can be performed in $\log \min (b, r)=O(\log \min (b, \log w / f))$ steps using simple bitwise operations and $\log \min (b, r)$ masks.

Since both $r$ and $b$ are a power of two, each block of $b f$ bits spans an integral number of fields of $\min (b, r) f$ bits. Observe that there can be at most $w$ bits set in a word, so $r f$ bits are enough to encode the total number of bits. If $b \leq r$, then since $b$ is a power of two after the last widening step we have a word divided into fields of $b f$ bits, each one containing the desired number of ones. Otherwise, if $b>r$, we compute the prefix sum of the sequence of numbers given by the fields, i.e., we store into each field the sum of the previous fields including itself. In word-RAM we do this by performing a multiplication (which is $O(1)$ ), with the mask $0^{r f-1} 1 \ldots 0^{r f-1} 1$. Instead, in $A C^{0}$ we use again the parallel prefix-sum algorithm described in [16], which is $O(\log b)$. It is not hard to see that, after this operation, the number of bits in a block is equal to the last field of the block minus the last field of the previous block. This operation can be implemented in parallel for all the blocks with a shift and a subtraction. Finally, to obtain the desired output word we reset to zero all the fields but the last of each block, using an and with the mask $1^{r f} 0^{(b-r) f} \ldots 1^{r f} 0^{(b-r) f}$, and shift the word to the right by $(b-r) f$ bits.

The pseudocode of bsa in word-RAM is the following:

$$
\begin{aligned}
& X \leftarrow A \gg(f-1) \\
& l \leftarrow f \\
& \text { for } i \leftarrow 1 \text { to } \log \min (b, r) \text { do } \\
& \quad H \leftarrow X \gg l \\
& \quad X \leftarrow\left(X \& 0^{l} 1^{l} \ldots 0^{l} 1^{l}\right)+\left(H \& 0^{l} 1^{l} \ldots 0^{l} 1^{l}\right) \\
& \quad l \leftarrow l \times 2 \\
& \text { if } \min (b, r)=r \text { then } \\
& \quad X \leftarrow X \times 0^{r f-1} 1 \ldots 0^{r f-1} 1
\end{aligned}
$$

9. $\quad X \leftarrow X-(X \ll b f)$
10. $A^{\prime} \leftarrow\left(X \& 1^{\left.r f 0^{(b-r) f} \ldots 1^{r f} 0^{(b-r) f}\right) \gg(b-r) f}\right.$
parallel minima (maxima) [18] (pmin (pmax) $(A, B, f)$ ): Given two $(f)$-words $A$ and $B$, return a $(f)$-word $W$ such that $W[i]$ is equal to $2^{f-1}$ if $A[i] \leq B[i]$ $(A[i] \geq B[i])$, and to 0 otherwise, for $i=1, \ldots,\lfloor w / f\rfloor$. pvmin (pvmax) is similar, but $W[i]$ is equal to $\min (A[i], B[i])(\max (A[i], B[i]))$.
These operations can be implemented in constant time, as demonstrated by the following code (pmin):
11. $T_{A} \leftarrow A \& V_{f}$
12. $T_{B} \leftarrow B \& V_{f}$
13. $A^{\prime} \leftarrow A \& \sim V_{f}$
14. $A^{\prime \prime} \leftarrow\left(B \mid V_{f}\right)-A^{\prime}$
15. $H_{1} \leftarrow \sim T_{A} \& T_{B}$
16. $H_{2} \leftarrow A^{\prime \prime} \&\left(T_{A} \wedge T_{B}{ }^{\wedge} V_{f}\right)$
17. $W \leftarrow\left(H_{1} \mid H_{2}\right) \& V_{f}$

All the given bounds do not include the time to compute the used masks, if any.

## 4 The algorithm

We start the presentation with a simple idea, which is then extended and modified in some ways. Consider two $(\log \sigma)$-words $A$ and $B$, each containing a packed string of length $m \leq \alpha$ in its $m \log \sigma$ least significant bits (i.e., each field of $\log \sigma$ bits encodes a character). The higher bits in both words, if any, are all 0s We perform the xor operation of $A$ and $B$ and the number of non-zero fields in the result is exactly the Hamming distance between the two strings. To count the number of such fields, we first convert, using the fnf operation, each non-zero field into a field with only the top-bit set, and then count the number of bits set using the sa operation. The procedure to compute the Hamming distance of $A$ and $B$ can thus be implemented in time $O(\log \log w)$ with the following operations:

1. $X \leftarrow A^{\wedge} B$
2. $A^{\prime} \leftarrow \mathrm{fnf}(X, \log \sigma)$
3. return sa $\left(A^{\prime}\right)$

For arbitrary $m$, observe that the packed encoding of a string of length $m$ requires $\lceil m \log \sigma / w\rceil$ words, and the Hamming distance between two such strings can be computed by running the above procedure for each word and summing the outputs.

Using this method, we can obtain an algorithm for the string matching with $k$-mismatches problem that runs in $O(n\lceil m \log \sigma / w\rceil \log \log w)$ time for any $m$. Note that the resulting algorithm is also practical and compares favorably with
the classical Shift-Add [4] for small alphabets and large $k$, although it is less flexible (no support for classes of characters). It is also worth noting that recent processors include a POPCNT instruction to compute the sideways addition of a word, so the $\log \log w$ term disappears in practice.

We now show how to apply the described ideas in an (improved) algorithm for the $k$-mismatches problem on packed text for short patterns. In the following, we shall assume $m \leq \alpha$. Our method exploits a general technique [17] to increase the parallelism in string matching algorithms based on word-level parallelism. We present a solution in the $A C^{0}$ model and a simpler variant in the word-RAM model. We start with the word-RAM algorithm. Let $\bar{m}$ be the smallest power of two greater than or equal to $m$ and let $\ell=\lfloor w /(\bar{m} \log \sigma)\rfloor$. We first preprocess the pattern $P$ to create a word $A$ with $\ell$ copies of $P$ of length $\bar{m} \log \sigma$ starting from the least significant bit. The last $\bar{m}-m$ fields of each copy are set to zero. We perform this padding because the bsa and ibsa operations which we shall use require the size of the blocks to be a power of two. Let $B_{i}$ be the word containing the packed encoding of the substrings $T[j+s \bar{m} \ldots j+s \bar{m}+m-1]$, for $s=0, \ldots, \ell-1$, where $j=\ell\lfloor i / m\rfloor m+i \bmod m$, with $\bar{m}-m$ zero (padding) fields every $m$ fields (i.e., at the end of each substring). For example, if $\sigma=4, m=3$, $T=0110111110110110$ and $w=16$, then we have $B_{0}=011011 \# 101101 \#$ and $B_{1}=101111 \# 110110 \#$, where \# denotes a padding field. Note that because of this partitioning we do not process all the text substrings in linear order. The word $B_{i}$ can be computed in constant time by extracting the substring $T[j \ldots j+\ell \bar{m}-1]$ from the packed text and clearing the padding fields with a mask. Our search algorithm performs the following main steps, for each $0 \leq i<$ $n / \ell$ :

1. $X \leftarrow A^{\wedge} B_{i}$
2. $A^{\prime} \leftarrow \operatorname{fnf}(X, \log \sigma)$
3. $M \leftarrow \operatorname{pmin}\left(\mathrm{bsa}\left(A^{\prime}, \log \sigma, \bar{m}\right), K, f\right)$
4. report $(M)$
where $f=\bar{m} \log \sigma$ and $K$ is a $(f)$-word with a copy of the integer $k$ in each field. At each iteration, our algorithm processes $\ell$ substrings of $T$ in parallel using the technique to compute the Hamming distance of two words described before. First, we perform the xor and fnf operations to identify the mismatches for the $\ell$ substrings encoded in $B_{i}$. Then, we use the bsa operation to count the number of mismatches for each substring, i.e., we compute a (f)-word such that each field of $\bar{m} \log \sigma$ bits contains the number of bits set (mismatches) in the corresponding block of $\bar{m}$ fields of $A^{\prime}$. Observe that in this setting bsa has $O(\log \min (m, \log w / \log \sigma))$ time complexity. Then, to find all the occurrences with at most $k$ mismatches we use the pmin operation with the word $K$ to identify the blocks with a bit count less than or equal to $k$. Finally, to iterate over all the occurrences we use the well-known bitwise operation that computes the position of the highest bit set in a word. Observe that this operation is in $A C^{0}$ and takes constant time [3]. Hence, our algorithm has $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor} \log \min (m, \log w / \log \sigma)\right)$ time complexity, and it obtains the $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor}\right)$ bound, corresponding to no overhead for the bitwise operations,

| $A \leftarrow 011001 \underline{0001100100}$ |  |  |  |  |
| :---: | :---: | :---: | :---: | :---: |
| $B$ | $\leftarrow$ | $011011 \underline{00}$ | 101101 | $\underline{00}$ |
| $X \leftarrow A{ }^{\wedge} B_{0}$ |  | $000010 \underline{\underline{00}}$ | 110100 |  |
| $A^{\prime} \leftarrow \operatorname{fnf}(X, 2)$ |  |  |  |  |
| $A \leftarrow 000010 \underline{00110100 \underline{00}}$ |  |  |  |  |
|  | $\leftarrow$ | $101010 \underline{10}$ | 101010 |  |
| $W \leftarrow A \& \sim V$ | = | $000000 \underline{00}$ | 010100 | $\underline{00}$ |
| $X \leftarrow V-W$ |  | $101010 \underline{10}$ | 010110 | $\underline{10}$ |
| $A^{\prime} \leftarrow(\sim X \mid A) \& V=000010 \underline{00101000 \underline{00}}$ |  |  |  |  |
| $X \leftarrow \mathrm{bsa}\left(A^{\prime}, 2,4\right)$ |  | 00000001 | 0000001 |  |
|  | $\leftarrow$ | 00000001 | 0000000 |  |
| $M \leftarrow \mathrm{pmin}(X, K, 4$ | $=$ | 10000000 | 0000000 |  |

Fig. 1. Example of the algorithm and of the fnf operation for $\sigma=4, P=011001$, $T=0110111110110110, w=16$ and $k=1$. The word $A$ encodes two pattern copies while the word $B_{0}$ encodes the text substrings $T[0 \ldots 2]$ and $T[4 \ldots 6]$. Padding fields are underlined. The pattern matches the first and second substring with 1 and 2 mismatches, respectively. Since $k=1$, only the first field is nonzero in the pmin output word.
for $\log \sigma=\Omega(\log w)$ or constant $m$. An example of the algorithm is depicted in Figure 1.

We now present the algorithm in the $A C^{0}$ model. Let $\bar{k}$ be the smallest power of two greater than $k$. We distinguish two cases: if $\log \sigma<\log \bar{k}+1$ we simply run the word-RAM solution. In $A C^{0}$ bsa has $O(\log m)$ time complexity and so the algorithm runs in $O\left(\frac{n}{[w /(m \log \sigma)]} \log m\right)$ time. Otherwise, the algorithm performs the following main steps, for each $0 \leq i<n / \ell$ :
$X \leftarrow A^{\wedge} B_{i}$
2. $A^{\prime} \leftarrow \mathrm{fnf}(X, \log \sigma)$
3. $H \leftarrow(H \ll f) \mid A^{\prime}$
4. if $i>0$ and $i \bmod \lfloor\log \sigma / f\rfloor=0$
5. $\quad M \leftarrow \operatorname{pmin}(\mathrm{ibsa}(H, f, \bar{m}), K, f)$
6. $\quad \operatorname{report}(M)$
7. $H \leftarrow 0$
where in this case $f=\log \bar{k}+1$ and $H$ is a word initialized to 0 . The main difference in this algorithm is that we report the occurrences every $\lfloor\log \sigma / f\rfloor$ iterations, so as to reduce the overhead due to counting the number of mismatches when $\log k=o(\log \sigma)$. To this end, we compact the fields in the word $\operatorname{fnf}\left(A^{\wedge}\right.$ $\left.B_{i}, \log \sigma\right)$ into fields of size $f$ in the word $H$. If $i>0$ and $i \bmod \lfloor\log \sigma / f\rfloor=0$, i.e., every $\ell\lfloor\log \sigma / f\rfloor$ processed substrings, we report the occurrences as follows. First, observe that the word $H$ contains $\ell \bar{m}\lfloor\log \sigma / f\rfloor$ fields of $f$ bits, encoding the mismatches for the substrings of $T$ of length $m$ corresponding to the words $B_{i-j}$, for $j=0, \ldots,\lfloor\log \sigma / f\rfloor-1$. More precisely, the $l$-th sequence of $\operatorname{fnf}\left(A^{\wedge} B_{i-j}, \log \sigma\right)$ spans the fields $s, s+\lfloor\log \sigma / f\rfloor, \ldots, s+\lfloor\log \sigma / f\rfloor(\bar{m}-1)$, where $s=j+(l-1) \bar{m}\lfloor\log \sigma / f\rfloor$, for $l=1, \ldots, \ell$. Using a suitable algorithm, i.e, the ibsa operation, we compute a word such that the last field of each sequence has value equal to the number of bits set (mismatches) in all the fields of the sequence if the number of mismatches is less than $\bar{k}$ and to a value $\geq \bar{k}$ otherwise.

Then, we proceed as in the word-RAM algorithm. We assumed for simplicity that $f$ divides $\log \sigma$ so that $H$ is a $(f)$-word. In general, we have $\log \sigma \bmod f$ unused bits every $\lfloor\log \sigma / f\rfloor$ fields in $H$. The algorithm works correctly also in this case, by suitably honoring this layout in $K$, ibsa and pmin. The time complexity of this algorithm is $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor}(1+\log \min (k, \sigma) \log m / \log \sigma)\right)$. It obtains the $O\left(\frac{n}{[w /(m \log \sigma)\rfloor}\right)$ bound if $\log \min (k, \sigma) \log m=O(\log \sigma)$.

Finally, we give a variant useful for two extreme cases: either $k$ or $m-k$ is very small. More precisely, it is competitive when $k=o(\log \min (k, \sigma) \log m / \log \sigma)$ or $m-k=o(\log \min (k, \sigma) \log m / \log \sigma)$. It uses only $A C^{0}$ instructions. In this variant, first presented for the case of small $k$, we compute $A$ and $B_{i}$ using $\bar{m}=$ $m+1$. Each block in $A$ and $B_{i}$ has thus one padding field and $p=(m+1) \log \sigma$ associated bits. The most significant bit of the padding field is a sentinel that will signal that there are more than $k$ mismatches, as will be shown shortly. The idea is to parallelize the well-known sideways addition implementation in which the least significant bit set is cleared in a loop ${ }^{5}$. To this end, we perform the following procedure:

1. $X \leftarrow A^{\wedge} B_{i}$
2. $A^{\prime} \leftarrow \mathrm{fnf}(X, \log \sigma)$
3. for $i \leftarrow 1$ to $k+1$
4. $A^{\prime} \leftarrow A^{\prime} \mid V_{p}$
5. $\quad A^{\prime} \leftarrow A^{\prime} \&\left(A^{\prime}-\left(V_{p} \gg(p-1)\right)\right)$
6. $M \leftarrow\left(A^{\prime} \& V_{p}\right)^{\wedge} V_{p}$
7. $\operatorname{report}(M)$

At each iteration of the loop we add the value $2^{p-1}$ (corresponding to the sentinel bit) to each block in $A^{\prime}$ and clear the least significant bit set. In this way, after $k+1$ iterations, the sentinel bit of any block is set iff the number of mismatches is at least $k+1$. We then replace the value of each block with $2^{p-1}$ if the sentinel bit is not set and with 0 otherwise. The complexity of the described operation is $O(k)$. The time complexity of this algorithm is $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor} k\right)$. A twin solution handles the case of small $m-k$. The idea is to find the blocks where the number of matching symbols is at least $m-k$, which basically consists in using the same method on the bitwise complement of the top bits of $A^{\prime}$.

## 5 Applications

The presented technique can be used for several other string matching problems. We show how to adapt it for particular models in the following subsections.

### 5.1 Matching with $k$-mismatches and wildcards

Assume that the integer alphabet $\Sigma$, of size $\sigma$, contains a wildcard symbol $\phi$, i.e., a special symbol that matches any other symbol of the alphabet. We consider the

[^2]$k$-mismatches problem with wildcards [11], which consists in reporting all the positions $j$ such that $|\{0 \leq h<m: T[j+h] \neq P[h] \wedge T[j+h] \neq \phi \wedge P[h] \neq \phi\}| \leq$ $k$. Let $A$ and $B_{i}$ be defined as in Sect. 4. The idea is to modify our algorithm so as to reset to zero all the fields $j \operatorname{in} \operatorname{fnf}\left(A^{\wedge} B_{i}, \log \sigma\right)$ such that $A[j]=\phi$ or $B_{i}[j]=\phi$, since there can be no mismatch in a position where either a pattern or text wildcard occurs.

In the preprocessing we create two $(\log \sigma)$-words $W_{P}$ and $H_{T}$. A field $W_{P}[i]$ in $W_{P}$ is equal to 0 if $i>\ell \bar{m}$ or $A[i]=\phi$, to $2^{\log \sigma-1}$ otherwise. A field $H_{T}[i]$ in $H_{T}$ is equal to $\phi$ if $i \leq \ell \bar{m}$, to 0 otherwise.

At each iteration $i$ of the searching phase, we compute the word $W_{T}=$ $\operatorname{fnf}\left(B_{i} \wedge H_{T}, \log \sigma\right)$. Analogously to $W_{P}$, a field $W_{T}[j]$ in $W_{T}$ is equal to 0 if $i>\ell \bar{m}$ or $B_{i}[j]=\phi$, to $2^{\log \sigma-1}$ otherwise.

Then, we and the result of operation 2 of the algorithm with $W_{P} \& W_{T}$ (i.e., $\left.A^{\prime} \leftarrow A^{\prime} \&\left(W_{P} \& W_{T}\right)\right)$. The rest of the procedure is unchanged. The overall time complexity is also unchanged.

## $5.2 \delta$-matching with $k$-mismatches and $(\delta, \gamma)$-matching

We consider the problem of $\delta$-matching $[9,10]$ with $k$-mismatches. In this problem we want to report, given an integer $\delta$, all the positions $j$ such that $|\{0 \leq i<m:|T[j+i]-P[i]|>\delta\}| \leq k$. In the exact case, i.e., when $k=0$, this is equivalent to matching under the $L_{\infty}$ distance [10]. In $\delta$-matching any two characters $t$ and $p$ are defined to match iff $|t-p| \leq \delta$. Note that in the algorithm to be presented in this section we can allow $\delta_{i}$ to be different for each pattern position $i$, while usually in $\delta$-matching the allowed error $\delta$ is the same for each character. This also yields an alternate solution to matching with wildcards in the pattern, by simply using $\delta_{i}=\sigma-1$ for pattern positions $i$ corresponding to wildcards, and $\delta_{i}=0$ elsewhere.

The idea is to compute the absolute difference $\left|A[j]-B_{i}[j]\right|$ for each field $j$ using pvmax and pvmin, i.e., we compute a $(\log \sigma)$-word $X$ such that $X[j]=\max \left(A[j], B_{i}[j]\right)-\min \left(A[j], B_{i}[j]\right)$. Then we replace each difference with $2^{\log \sigma-1}$ if it is greater than $\delta_{j} \bmod m$ and with 0 otherwise, using pmin and a xor operation. In this way we can count the number of symbols that do not $\delta$-match using ibsa or bsa. In the preprocessing phase we compute $D^{\prime}[j]=\delta_{j}$ and construct its packed representation $D$, a $(\log \sigma)$-word holding $\ell$ copies of $D^{\prime}$. Let $W$ be a $(\log \sigma)$-word such that $W[i]$ is equal to $2^{\log \sigma-1}$ if $i \leq \ell m$, to 0 otherwise. Then at iteration $i$ of the searching phase we compute

1. $X \leftarrow \operatorname{pvmax}\left(A, B_{i}, \log \sigma\right)-\operatorname{pvmin}\left(A, B_{i}, \log \sigma\right)$
2. $A^{\prime} \leftarrow \operatorname{pmin}(X, D, \log \sigma)^{\wedge} W$

The result is that each field of $A^{\prime}$ is equal to $2^{\log \sigma-1}$ iff the corresponding pattern and text characters do not $\delta$-match. The rest of the algorithm is as before, only the steps $1-2$ of either of the main algorithms (for $A C^{0}$ and word-RAM models) are replaced with the two steps above. The time complexities remain the same.

If we are interested in the (more conventional) exact $\delta$-matching variant (i.e. assume that $k=0$ ), we can improve the time to $O\left(\frac{n}{[w /(m / \log \sigma)]}\right)$ using the following algorithm:

1. $X \leftarrow \operatorname{pvmax}\left(A, B_{i}, \log \sigma\right)-\operatorname{pvmin}\left(A, B_{i}, \log \sigma\right)$
2. $A^{\prime} \leftarrow \operatorname{pmin}(X, D, \log \sigma)^{\wedge} W$
3. $M \leftarrow \operatorname{fnf}\left(A^{\prime}, m \log \sigma\right)^{\wedge} V_{m} \log \sigma$
4. $\operatorname{report}(M)$

The fnf operation interprets the word $A^{\prime}$ as a $(m \log \sigma)$-word, and returns a ( $m \log \sigma$ )-word where the $i$-th field is equal to $2^{m \log \sigma-1}$ if at least one character of the $i$-th pattern copy did not $\delta$-match, and to 0 otherwise. The xor operation then inverts the fields' values, so that a field is equal to $2^{m \log \sigma-1}$ if all the characters of the corresponding pattern copy did $\delta$-match.

We note that a close relative to $\delta$-matching is less-than matching, where characters $p$ and $t$ match if $p \leq t$. This model has applications in other pattern matching problems, see e.g. [1]. The less-than matching problem can be easily solved with our methods in the same way as $\delta$-matching, that is, the first two lines are simply replaced with $A^{\prime} \leftarrow \operatorname{pmin}\left(A, B_{i}, \log \sigma\right)^{\wedge} W$. It is also possible to solve $\delta$-matching, with the same time complexity, by combining less-than and greater-than matching.

The $\gamma$-matching problem consists in, given an integer $\gamma$, finding all the positions $j$ such that $\sum_{0 \leq i<m}|T[j+i]-P[i]| \leq \gamma$. If both $\delta$ and $\gamma$ conditions must hold, we speak of $(\delta, \gamma)$-matching. There are many algorithms devoted to this model, see e.g. [12]. In order to solve $\gamma$-matching we need to sum the fields of each pattern copy in $X$ and compare each value against $\gamma$ (note that we need to check also the $\delta$ condition). If we do not use field compaction and deferred reporting of occurrences, this is easiest to do in the same way as the first widening phase of bsa operation, i.e. by simply shifting and adding the fields in parallel in $O(\log m)$ time, giving $O\left(\frac{n}{[w /(m \log \sigma)\rfloor} \log m\right)$ total time in $A C^{0}$. This result can be improved in both $A C^{0}$ and word-RAM models. We first define one more operation:
interleave two words (interleave $(A, B, Z, f)$ ): given three $(f)$-words $A, B$ and $Z$, return a $(f)$-word $W$ such that $W[i]$ is equal to $A[i]$ if $Z[i]=0$, and to $B[i]$ otherwise. This can be implemented in $O(1)$ time as follows:

1. $Z \leftarrow \mathrm{fnf}(Z, f)$
2. $Z \leftarrow(Z-(Z \gg(f-1))) \mid Z$
3. $W \leftarrow(A \& \sim Z) \mid(B \& Z)$

The idea is to first find all the pattern copies with at least one difference greater than $\delta$, by computing the $(\bar{m} \log \sigma)$-word $Z=\operatorname{fnf}\left(A^{\prime}, \bar{m} \log \sigma\right)$, as in the algorithm described before for $\delta$-matching. Then, we prevent all these pattern copies from $\gamma$-matching by replacing all the corresponding differences in $X$ with $\delta$. The effect is that the sum of the fields for a pattern copy with at least one difference greater than $\delta$ is equal to $m \delta$ instead of the exact sum. This works because in
$\gamma$-matching it always holds that $m \delta>\gamma$, as otherwise the $\gamma$ condition does not prune anything. To this end, using $Z$ and interleave, we interleave the words $X$ and $D$, interpreted as $(m \log \sigma)$-words, into the $(\log \sigma)$-word $X^{\prime}$. In this way, a field $X^{\prime}[i]$ is equal to $X[i]$ if no difference is greater than $\delta$ for the corresponding pattern copy (i.e., if $Z[\lfloor i / \bar{m}\rfloor]=0$ ), and to $D[i]$ otherwise.

In word-RAM we then use bsa and pmin to accumulate the absolute differences in $X^{\prime}$ and compare the sums against the threshold value $\gamma$ in parallel. Note that we need to adjust bsa to use as $r$ the smallest power of two greater than or equal to $\log (w \delta+1) / \log \sigma$, so as to not cause overflows. The algorithm is

1. $X \leftarrow \operatorname{pvmax}\left(A, B_{i}, \log \sigma\right)-\operatorname{pvmin}\left(A, B_{i}, \log \sigma\right)$
2. $A^{\prime} \leftarrow \operatorname{pmin}(X, D, \log \sigma)^{\wedge} W$
3. $Z \leftarrow \operatorname{fnf}\left(A^{\prime}, \bar{m} \log \sigma\right)$
4. $X^{\prime} \leftarrow$ interleave $(X, D, Z, \bar{m} \log \sigma)$
5. $M \leftarrow \operatorname{pmin}\left(\mathrm{bsa}\left(X^{\prime}, \log \sigma, \bar{m}\right), G, f\right)$
6. $\operatorname{report}(M)$
where $f=\bar{m} \log \sigma$ and $G$ is a $(f)$-word containing a copy of the integer $\gamma$ in each field. The time complexity is $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor} \log \min (m, \log (w \delta) / \log \sigma)\right)$ which again obtains $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor}\right)$ time for $\log \sigma=\Omega(\log (w \delta))$ or constant $m$.

Consider now the $A C^{0}$ model. We use an approach similar to the one used in the $k$-mismatches case, the only difference is that we need more bits to represent the accumulated sums. That is, we replace $\bar{k}$ with $\bar{\gamma}=2^{\lceil\log (\gamma+1)\rceil}$ (the smallest power of two greater than $\gamma$ ), and use $f=\log \bar{\gamma}+1$. The ibsa operation then works correctly, i.e. accumulates the absolute differences without overflowing the sums, provided that all characters $\delta$-match, as otherwise the corresponding absolute difference may be $\sigma-1$, which in turn can be larger than $\bar{\gamma}$. This holds because no difference is greater than $\delta$ in $X^{\prime}$. The complete pseudocode follows.

1. $X \leftarrow \operatorname{pvmax}\left(A, B_{i}, \log \sigma\right)-\operatorname{pvmin}\left(A, B_{i}, \log \sigma\right)$
2. $A^{\prime} \leftarrow \operatorname{pmin}(X, D, \log \sigma)^{\wedge} W$
3. $Z \leftarrow \mathrm{fnf}\left(A^{\prime}, \bar{m} \log \sigma\right)$
4. $X^{\prime} \leftarrow$ interleave $(X, D, Z, \bar{m} \log \sigma)$
5. $H \leftarrow(H \ll f) \mid X^{\prime}$
6. if $i>0$ and $i \bmod \lfloor\log \sigma / f\rfloor=0$
$M \leftarrow \operatorname{pmin}(\mathrm{ibsa}(H, f, \bar{m}), G, f)$
$\operatorname{report}(M)$
$H \leftarrow 0$
The total time is $O\left(\frac{n}{[w /(m \log \sigma)]}(1+\log m \log \gamma / \log \sigma)\right)$, This becomes $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor}\right)$ for $\log m \log \gamma=O(\log \sigma)$.

We can also combine the two models, $\delta$-matching with $k$-mismatches and $(\delta, \gamma)$-matching, to obtain $(\delta, k, \gamma)$-matching. In this model we limit the number of characters not $\delta$-matching by $k$, and the accumulated sum of the absolute differences by $\gamma$. The basic idea is to compute two match vectors, $M_{\delta}$ and $M_{\gamma}$, take their bitwise and as $M \leftarrow M_{\delta} \& M_{\gamma}$ and then report the occurrences with
respect to $M$. The vector $M_{\delta}$ can be computed as was already shown. To compute $M_{\gamma}$ we just skip the interleave operation to take the absolute differences raw without saturating them with $\delta$. In $A C^{0}$ we can use the basic algorithm to compute $M_{\gamma}$ in $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor} \log m\right)$ time, which dominates the total time. However, since the sums need more bits now, there is a significant overhead in the case of the word-RAM algorithm and of the improved $A C^{0}$ algorithm. In particular, in the case of the word-RAM algorithm, the time complexity becomes $O\left(\frac{n}{[w /(m \log \sigma)\rfloor} \log \min (m, \log (w \sigma) / \log \sigma)\right)$, i.e., slightly worse. Instead, in the case of the improved $A C^{0}$ algorithm, the overhead makes the algorithm useless. However, we can still manage to obtain $O\left(\frac{n}{\lfloor w /(m \log \sigma)\rfloor}(1+\log m \log \gamma / \log \sigma)\right)$ time by modifying the model so that we accumulate the absolute differences only on $\delta$-matching character positions. This can be easily done with the tools already presented, namely using the interleave operation to set non- $\delta$-matching character positions to 0 in word $X^{\prime}$.

## 6 Conclusion

We presented a novel technique for approximate pattern matching with $k$ mismatches when the text is given in packed form. Assuming the pattern is short enough, it is possible to achieve a sublinear search time, if several pattern copies are matched against different text substrings at the same time. We described variants of our simple method in the $A C^{0}$ and word-RAM models and also considered the case when the number $k$ of allowed errors is small. Moreover, we showed how to adapt our algorithms to other matching models, including approximate matching with wildcard (don't-care) symbols, $\delta$-matching with $k$ mismatches and $(\delta, \gamma)$-matching.

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[^0]:    * This paper is an extended version of the article that appeared in Information Processing Letters 113(19-21):693-697 (2013), http://dx.doi.org/10.1016/j.ipl.2013.07.002

[^1]:    ${ }^{4}$ Throughout the paper, all logarithms are in base 2 . W.l.o.g. we also assume that $\sigma$ is a power of two.

[^2]:    ${ }^{5}$ http://graphics.stanford.edu/~seander/bithacks.html\#
    CountBitsSetKernighan

