

# Strong chromatic index of sparse graphs

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## Abstract

A coloring of the edges of a graph  $G$  is *strong* if each color class is an induced matching of  $G$ . The *strong chromatic index* of  $G$ , denoted by  $\chi'_s(G)$ , is the least number of colors in a strong edge coloring of  $G$ . In this note we prove that  $\chi'_s(G) \leq (4k - 1)\Delta(G) - k(2k + 1) + 1$  for every  $k$ -degenerate graph  $G$ . This confirms the strong version of conjecture stated recently by Chang and Narayanan [3]. Our approach allows also to improve the upper bound from [3] for chordless graphs. We get that  $\chi'_s(G) \leq 4\Delta - 3$  for any chordless graph  $G$ . Both bounds remain valid for the list version of the strong edge coloring of these graphs.

## 1 Introduction

A *strong coloring* of a graph  $G$  is an edge coloring in which every color class is an *induced matching*, that is, any two vertices that belong to distinct edges of the same color are not adjacent. The *strong chromatic index* of  $G$ , denoted by  $\chi'_s(G)$ , is the minimum number of colors in a strong edge coloring of  $G$ . This notion was introduced by Erdős and Nešetřil around 1985, who were perhaps inspired by a beautiful construction, due to Ruzsa and Szemerédi [8], of dense graphs decomposable into large induced matchings.

A basic conjecture formulated by Erdős and Nešetřil states that  $\chi'_s(G) \leq \frac{5}{4}\Delta^2$  for every graph  $G$  with maximum degree  $\Delta$ . If true, this bound is sharp, as is seen on the graph obtained from the cycle  $C_5$  by blowing-up each vertex to an independent set of size  $\frac{\Delta}{2}$ . For the general case, the trivial upper bound on  $\chi'_s(G)$  is  $2\Delta^2 - 2\Delta + 1$ , as is seen easily by applying the greedy algorithm for any ordering of the edges. Using the probabilistic method, Molloy and Reed [7] improved the trivial bound to  $(2 - \epsilon)\Delta^2$  for  $\Delta$  sufficiently large, where  $\epsilon$  is a small constant around  $\frac{1}{50}$ . They claim that this result can be slightly improved, but increasing  $c$  above  $\frac{1}{10}$  requires essentially new ideas. Another intriguing conjecture is due to Faudree, Gyárfás, Schelp, and Tuza [4]. It states that  $\chi'_s(G) \leq \Delta^2$  when  $G$  is a bipartite

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graph of maximum degree  $\Delta$ . Its stronger version formulated by Brualdi and Quinn [2] states that  $\chi'_s(G) \leq \Delta_1 \Delta_2$  if  $G$  is a bipartite graph and  $\Delta_i$  is the maximum degree of a vertex in the  $i$ -th partition class.

In this note we focus on graphs with bounded degeneracy. Recall that a graph  $G$  is  $k$ -degenerate if every subgraph of  $G$  contains a vertex of degree at most  $k$ . Chang and Narayanan [3] proved recently that  $\chi'_s(G) \leq 10\Delta - 10$  for a 2-degenerate graph  $G$ . They also conjectured that for any  $k$ -degenerate graph  $G$  we have a linear bound  $\chi'_s(G) \leq ck^2\Delta$ , where  $c$  is an absolute constant. We confirm this conjecture in a stronger form, by proving that inequality

$$\chi'_s(G) \leq (4k - 1)\Delta - k(2k + 1) + 1$$

holds for every  $k$ -degenerate graph  $G$ . Our proof is algorithmic: we produce appropriate ordering of the edges of a graph  $G$ , and then apply the greedy coloring algorithm. By the same technique, using a structural lemma from [3], we get an improved bound in a special case of *chordless* graphs (graphs in which every cycle is induced). These graphs form a proper subclass of 2-degenerate graphs, and contain graphs with cycle lengths divisible by four. It was conjectured in [4] that for such graphs the strong chromatic index is linear in  $\Delta$ . We prove that every chordless graph  $G$  of maximum degree  $\Delta$  satisfies  $\chi'_s(G) \leq 4\Delta - 3$ .

## 2 The results

In our proof we will use the following simple lemma concerning local structure of  $k$ -degenerate graphs. We say that a vertex  $v$  is *nice* in a graph  $G$  if  $v$  has at least one neighbor of degree at most  $k$ , and at most  $k$  neighbors of degree strictly greater than  $k$ .

**Lemma 1 ([3, Lemma 3])** *Every  $k$ -degenerate graph  $G$  with at least one edge contains a nice vertex.*

**Proof.** Let  $X \subseteq V(G)$  be the set of all vertices of degree at most  $k$  in  $G$ . If there is an edge  $e$  between any two vertices from  $X$ , then each of the end points of  $e$  is nice. Otherwise, when  $X$  is an independent set, consider the subgraphs  $H$  induced by the set  $V(G) \setminus X$  of all the vertices whose degree in  $G$  is strictly greater than  $k$ . Notice that  $V(G) \setminus X$  is nonempty because  $G$  has at least one edge. Let  $v \in V(G) \setminus X$  be any vertex whose degree in the subgraph  $H$  is at most  $k$  (which must exist, as  $H$  is also  $k$ -degenerate). We claim that  $v$  is nice in  $G$ . Indeed, it has at most  $k$  neighbors in  $V(G) \setminus X$ , and since it does not belong to  $X$ , it must have at least one neighbor in  $X$ . ■

Our plan for the proof of the main result is simple: we will construct an ordering of the edges of a  $k$ -degenerate graph  $G$  such that for any edge  $e$ , the number of edges that are within distance one from  $e$  and come after  $e$  in the order, is appropriately bounded. The desired coloring is then obtained by coloring edges greedily in the reverse order. The ordering will be constructed in a number of steps. Initially all edges are white. In each step we will grey out a bunch of edges and add them to the ordering (at the end). During the whole procedure we will keep under control the number of white edges incident to grey edges, thereby obtaining a desired result.

**Theorem 2** *Every  $k$ -degenerate graph  $G$  of maximum degree  $\Delta$  satisfies*

$$\chi'_s(G) \leq (4k - 1)\Delta - k(2k + 1) + 1.$$

**Proof.** We will iteratively construct an (ordered) list of edges of  $G$ , starting with  $L_0$  being an empty list. We will find an increasing chain of lists  $L_1, L_2, \dots$  such that  $L_i$  is obtained from  $L_{i-1}$  by appending a (nonzero) number of new edges. For some  $s$ , the last list  $L_s$  will contain all edges of  $G$ , and it will be the desired ordering of the set  $E(G)$ .

Suppose that  $L_i$ , with  $i > 0$ , is defined, and let  $X_i \subset E(G)$  be a set of edges that appear in  $L_i$ . We may imagine the edges from  $X_i$  to be greyed out, while the rest of edges being still white. Let  $H_i = (V(G), E(G) \setminus X_i)$ , and let  $v_i$  be a nice vertex in  $H_i$ . By Lemma 1, such a vertex must exist. Now, let  $Y_i$  be the set of white edges incident with  $v_i$  and a vertex of degree at most  $k$ , that is,

$$Y_i = \{v_i w \in E(H_i) : \deg_{H_i}(w) \leq k\}.$$

Take  $L_{i+1}$  to be the list obtained from  $L_i$  by appending edges from  $Y_i$  in any order. Clearly,  $Y_i$  is nonempty, so for some  $s$ , the list  $L_s$  will contain all edges of  $G$ . In order to prove the desired property of  $L_s$  we will keep the following invariant: the number of white edges incident to each end point of a grey edge is at most  $k$ . More formally, we shall prove the following claim.

**Claim 3** *For every  $i = 0, 1, \dots, s$ , and every vertex  $v \in V(G)$ , if  $v$  is incident to at least one edge from  $X_i$ , then it is incident to at most  $k$  edges outside  $X_i$ .*

To prove the claim we use induction on  $i$ . For  $i = 0$  we have  $X_i = \emptyset$ , so there is nothing to prove. Now, suppose that the invariant holds for some  $i$  (where  $0 \leq i < s$ ), and consider some vertex  $v$  incident with at least one edge from  $X_{i+1}$ . Note that if  $v$  is incident to an edge from  $X_i$ , then the statement follows from induction hypothesis, as  $X_i \subset X_{i+1}$ . In the remaining case,  $v$  is incident to an edge from  $Y_i$ . So, either  $v \neq v_i$  and  $\deg_{H_i}(v) \leq k$  by definition of  $H_i$ , or  $v = v_i$  and, as a nice vertex in  $H_i$ , it is incident to at most  $k$  edges outside  $X_i \cup Y_i$ . This finishes proof of the Claim.

Now, for any edge  $e \in E(G)$ , we will count the number of edges that are within distance one from  $e$  and appear on  $L_s$  later than  $e$ . Let  $i$  be the unique number for which  $e \in Y_i$ , and let  $e = v_i w$ , with  $v_i$  being nice in  $H_i$ . We will estimate the number of white edges (outside  $X_i$ ) that are within distance one from  $e$ .

We will say that a vertex  $v$  *sees* an edge  $e$  if  $v$  is incident to  $e$ , or it is a neighbor of a vertex incident to  $e$ . First we count white edges that are seen from vertex  $v_i$  (except  $e$  itself). Let  $a$  be the number of grey edges incident to  $v_i$ . By the Claim, there are at most  $ak$  white edges in total, incident to the end points of such edges (other than  $v_i$ ). Next, let  $b$  denote the number of neighbors of  $v_i$  in  $H_i$  whose white degree is at most  $k$ . So, we have another  $bk$  white edges seen from  $v_i$ . Finally, let  $c$  be the number of neighbors of  $v_i$  in  $H_i$  whose white degree is strictly larger than  $k$ . Thus we have at most  $ak + bk + c\Delta$  white edges in total that can be seen from  $v_i$ . But  $a + b + c \leq \Delta - 1$  and  $c \leq k$  (since  $v_i$  is nice in  $H_i$ ). So, we get that

$$ak + bk + c\Delta = (a + b + c)k + c(\Delta - k) \leq (\Delta - 1)k + k(\Delta - k).$$

Now, we count similarly the number of white edges seen from the vertex  $w$ . Let  $p$  and  $q$  be the number of white and grey edges (different from  $e$ ) incident to  $w$ , respectively. Arguing as before we get at most  $p\Delta + qk$  white edges seen from  $w$ . Since  $\deg_{H_i}(w) \leq k$ , we have  $p \leq k - 1$ . Also,  $p + q \leq \Delta - 1$ . Hence

$$p\Delta + qk = p(\Delta - k) + (p + q)k \leq (k - 1)(\Delta - k) + (\Delta - 1)k.$$

Putting these two estimates together, we obtain that the total number of white edges seen from the end points of  $e$  is at most

$$(4k - 1)\Delta - k(2k + 1).$$

To finish the proof of the theorem, notice only that we need one more color to guarantee success of the greedy coloring procedure with respect to the linear ordering of the edges we have produced. ■

As every chordless graph  $G$  is 2-degenerate, our theorem implies that  $\chi'_s(G) \leq 7\Delta - 9$ . However, one may improve it by using the following lemma from [3].

**Lemma 4 ([3, Lemma 7])** *Every chordless graph  $G$  contains a nice vertex with at most one neighbor of degree strictly greater than two.*

Indeed, the lemma implies that in the process of constructing lists  $L_i$  we can keep the stronger invariant, namely, that the number of white edges incident to each end point of a grey edge is at most one. By a similar counting one can obtain the following theorem (a detailed proof is omitted).

**Theorem 5** *Every chordless graph  $G$  of maximum degree  $\Delta$  satisfies*

$$\chi'_s(G) \leq 4\Delta - 3.$$

An important subclass of  $k$ -degenerate graphs are graphs of treewidth at most  $k$ . For these graphs a much better bound for the strong chromatic index can be obtained using a different argument. The following result was (implicitly) proved in [5], and we repeat its nice proof here for completeness.

**Theorem 6** *Let  $\mathcal{G}$  be a proper minor-closed class of graphs. Let  $r = r(\mathcal{G})$  be the maximum chromatic number of a member of  $\mathcal{G}$ . Then every graph  $G \in \mathcal{G}$  of maximum degree  $\Delta$  satisfies*

$$\chi'_s(G) \leq r(\Delta + 1).$$

**Proof.** Suppose we have  $\Delta + 1$  basic colors, each having  $r$  different shades. Start with the usual proper edge coloring of  $G$  using at most  $\Delta + 1$  basic colors. Let  $M$  be a fixed color class (which is a usual matching in  $G$ ). Consider a graph  $G_M$  whose vertex set is  $M$ , with two edges  $e, f \in M$  adjacent whenever at least one pair of their corresponding end points is joined by an edge in  $G$ . In other words,  $G_M$  is obtained by taking a subgraph of  $G$  induced

by the end points of all edges in  $M$ , and then their contraction. Clearly,  $G_M$  belongs to  $\mathcal{G}$ , so it has a proper vertex coloring using at most  $r$  colors. It follows that we may color the edges of  $M$  with  $r$  shades of the basic color of  $M$  so that each shade forms now an induced matching in  $G$ . Repeating this operation for every matching of the initial edge coloring of  $G$  we get the desired strong coloring using at most  $r(\Delta + 1)$  colors. ■

Since graphs of treewidth at most  $k$  are  $(k + 1)$ -colorable we get the following result.

**Corollary 7** *Every graph of treewidth at most  $k$  satisfies*

$$\chi'_s(G) \leq (k + 1)(\Delta + 1).$$

Notice that all upper bounds obtained by greedy colorings remain valid for the natural list version of the strong chromatic index.

### 3 Final remarks

Our main theorem asserts, that the strong chromatic index of graphs of bounded degeneracy is at most linear in their maximum degree, which is best possible up to a multiplicative constant. This covers all natural classes of graphs that have bounded *average* degree, such as graphs of bounded treewidth, graphs of bounded arboricity, planar graphs (where natural means closed under taking subgraphs). On the other hand, when average degree is a function  $d(n)$  of the number of vertices in a graph, our result implies a bound on  $\chi'_s(G)$  that is not better than  $cd(n)^2$  for some constant  $c$ .

It is natural to wonder whether our bound is optimal in some sense, in terms of average degree. Namely, does there exist a family of graphs of average degree  $d(n)$ , greater than a constant, that have strong chromatic index bounded by a linear function of  $d(n)$ ? Note that it is in fact a question concerning a nontrivial lower bound on  $\chi'_s(G)$  that depends only on the average degree. As we suspect that such a bound may not exist, we conjecture the following.

**Conjecture 8** *There exists a function  $d(n)$  that goes to infinity such that for any constant  $c$  there exists a graph on  $n$  vertices of average degree  $d(n)$ , that have strong chromatic index at most  $cd(n)$ .*

Surprisingly, a slightly weaker assertion holds even for very dense graphs. Alon, Moitra, and Sudakov [1] construct graphs of average degree  $n - o(n)$  that have strong chromatic index of order  $O(n^{1+o(1)})$ . This implies that an analog of Conjecture 8, where we replace bound on strong chromatic index with  $d(n)^{1+\epsilon}$ , is true even for  $d(n) = n - o(n)$ . However, their results do not imply anything for graphs of strong chromatic index at most  $cd(n)$ .

Knowing that there are graphs of average degree  $d(n)$  and chromatic index at most  $d(n)^{1+\epsilon}$ , a natural continuation of our research is to find a characterization of those graphs that would allow us to strengthen the upper bound given by Theorem 2 and, in a wider perspective, reveal some new properties of this intriguing chromatic parameter.

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