

COMPLEXITY OF A DISJOINT MATCHING PROBLEM ON BIPARTITE GRAPHS

GREGORY J. PULEO

ABSTRACT. We consider the following question: given an (X, Y) -bigraph G and a set $S \subseteq X$, does G contain two disjoint matchings M_1 and M_2 such that M_1 saturates X and M_2 saturates S ? When $|S| \geq |X| - 1$, this question is solvable by finding an appropriate factor of the graph. In contrast, we show that when S is allowed to be an arbitrary subset of X , the problem is NP-hard.

1. INTRODUCTION

A *matching* in a graph G is a set of pairwise disjoint edges. A matching *covers* a vertex $v \in V(G)$ if v lies in some edge of the matching, and a matching *saturates* a set $S \subseteq V(G)$ if it covers every vertex of S .

An (X, Y) -bigraph is a bipartite graph with partite sets X and Y . The fundamental result of matching theory is Hall's Theorem [5], which states that an (X, Y) -bigraph contains a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$. While Hall's Theorem does not immediately suggest an efficient algorithm for finding a maximum matching, such algorithms have been discovered and are well-known [1, 6].

A natural way to extend Hall's Theorem is to ask for necessary and sufficient conditions under which *multiple* disjoint matchings can be found. This approach was taken by Lebensold, who obtained the following generalization of Hall's Theorem.

Theorem 1.1 (Lebensold [9]). *An (X, Y) -bigraph has k disjoint matchings, each saturating X , if and only if*

$$(1) \quad \sum_{y \in Y} \min\{k, |N(y) \cap S|\} \geq k|S|$$

for all $S \subseteq X$.

When $k = 1$, the left side of (1) is just $|N(S)|$, so Theorem 1.1 contains Hall's Theorem as a special case. As observed by Brualdi, Theorem 1.1 is equivalent to a theorem of Fulkerson [3] about disjoint permutations of 0, 1-matrices. Theorem 1.1 is also a special case of Lovasz's (g, f) -factor theorem [10]. Like Hall's Theorem, Theorem 1.1 does not immediately suggest an efficient algorithm, but efficient algorithms exist for solving the (g, f) -factor problem [4], and these algorithms can be applied to find the desired k disjoint matchings. We discuss the algorithmic aspects further in Section 4.

A different extension was considered by Frieze [2], who considered the following problem:

Disjoint Matchings (DM)

Input: Two (X, Y) -bigraphs G_1, G_2 on the same vertex set.

Question: Are there matchings $M_1 \subseteq G_1$, $M_2 \subseteq G_2$ such that $M_1 \cap M_2 = \emptyset$ and each M_i saturates X ?

When $G_1 = G_2$, this problem is just the $k = 2$ case of the problem considered by Lebensold, and is therefore polynomially solvable. On the other hand, Frieze proved that the Disjoint Matchings problem is NP-hard in general.

In this paper, we consider the following disjoint-matching problem, which can be naturally viewed as a restricted case of the Disjoint Matchings problem:

Single-Graph Disjoint Matchings (SDM)

Input: An (X, Y) -bigraph G and a vertex set $S \subseteq X$.

Question: Are there matchings $M_1, M_2 \subseteq G$ such that $M_1 \cap M_2 = \emptyset$, M_1 saturates X , and M_2 saturates S ?

We call such a pair (M_1, M_2) an S -pair. When $S = X$, this problem is also equivalent to the $k = 2$ case of Lebensold's problem. The problem SDM is similar to a problem considered by Kamalian and Mkrtchyan [7], who proved that the following problem is NP-hard:

Residual Matching

Input: An (X, Y) -bigraph G and a nonnegative integer k .

Question: Are there matchings $M_1, M_2 \subseteq G$ such that $M_1 \cap M_2 = \emptyset$, M_1 is a maximum matching, and $|M_2| \geq k$?

When G has a perfect matching, we can think of the Residual Matching problem as asking whether there is *some* $S \subseteq X$ with $|S| = k$ such that G has an S -pair. In contrast, the SDM problem asks whether some *particular* S admits an S -pair. Since k is part of the input to the Residual Matching problem, it is *a priori* possible that SDM could be polynomially solvable while the Residual Matching problem is NP-hard, since one might need to check exponentially many candidate sets S .

In Section 2, we give a quick reduction from SDM to DM, justifying the view of SDM as a special case of DM, and in Section 3 we show that SDM is NP-hard, thereby strengthening Frieze's result. In Section 4 we show that SDM is polynomially solvable under the additional restriction $|S| \geq |X| - 1$.

2. REDUCING SDM TO DM

In this section, we show that any instance of SDM with $|S| < |X| - 1$ reduces naturally to an instance of DM. Since SDM-instances with $|S| \geq |X| - 1$ are polynomially solvable, as we show in Section 4, this justifies the claim that SDM is a special case of DM.

Theorem 2.1. *Let G be an (X, Y) -bigraph and let $S \subseteq V(G)$ with $|S| < |X| - 1$. Construct graphs G_1, G_2 as follows:*

$$\begin{aligned} V(G_1) &= V(G_2) = V(G), \\ E(G_1) &= E(G), \\ E(G_2) &= E(G) \cup \{xy : x \in X - S, y \in Y\}. \end{aligned}$$

The graph G has an S -pair if and only if there are disjoint matchings M_1, M_2 contained in G_1, G_2 respectively, each saturating X .

Proof. If $|Y| < |X|$, then it is clear that G has no S -pair and that G_1, G_2 do not have perfect matchings, so assume that $|Y| \geq |X|$.

First suppose that M_1, M_2 are disjoint matchings contained in G_1, G_2 respectively, each saturating X . Let $M'_1 = M_1$ and let $M'_2 = \{e \in M_2: e \cap X \subseteq S\}$. It is clear that (M'_1, M'_2) is an S -pair.

Now suppose that we are given an S -pair (M'_1, M'_2) . In order to obtain the matchings M_1, M_2 in G_1, G_2 as needed, we need to enlarge M'_2 so that it saturates all of X , rather than only saturating S . Let $Y' = \{y \in Y: y \notin V(M'_2)\}$, and let $H = G_2[(X - S) \cup Y'] - M'_1$.

We claim that H has a matching that saturates $X - S$, and prove this by verifying Hall's Condition. Let any $X_0 \subseteq X - S$ be given. If $|X_0| = 1$, say $X_0 = \{x_0\}$, then $N_H(X_0)$ contains all of Y' except possibly the mate of x_0 in M'_1 . Hence

$$|N_H(X_0)| \geq |Y'| - 1 = |Y| - |S| - 1 \geq |X| - |S| - 1 \geq 1 = |X_0|,$$

as desired. On the other hand, if $|X_0| \geq 2$, then $N_H(X_0)$ contains all of Y' , so that

$$|N_H(X_0)| = |Y'| = |Y| - |S| \geq |X| - |S| \geq |X_0|.$$

Hence Hall's Condition holds for H . Now let M be a perfect matching in H , let $M_1 = M'_1$, and let $M_2 = M'_2 \cup M$. By construction, M_2 is a matching in G_2 that saturates X . It is clear that $M_1 \cap M_2 = \emptyset$, since the edges in M'_1 were omitted from H . Hence M_1 and M_2 are as desired. \square

3. FINDING TWO MATCHINGS IS NP-HARD

Given an instance (G, S) of SDM, we call a pair of matchings (M_1, M_2) satisfying the desired condition an S -pair. When G' is a subgraph of G and $S' = S \cap V(G')$, we say that an S -pair (M_1, M_2) contains an S' -pair (M'_1, M'_2) if $M'_1 \subseteq M_1$ and $M'_2 \subseteq M_2$.

We prove that SDM is NP-hard via a reduction from 3SAT. Let c_1, \dots, c_s be the clauses and $\theta_1, \dots, \theta_t$ be the variables of an arbitrary 3SAT instance. We define a graph G as follows.

For each variable θ_i , let H_i be a copy of the cycle C_{4s} , with vertices $v_{i,1}, \dots, v_{i,4s}$ written in order. Define

$$\begin{aligned} X_i &= \{v_{i,j}: j \text{ is even}\}, \\ S_i &= \{v_{i,j}: j \equiv 2 \pmod{4}\}. \end{aligned}$$

Since H_i is an even cycle, it has exactly two perfect matchings, one containing the edge $v_{i,1}v_{i,2}$ and the other containing the edge $v_{i,2}v_{i,3}$. In an S_i -pair (M_1, M_2) for H_i , we have $v_{i,1}v_{i,2} \in M_1$ if and only if $v_{i,2}v_{i,3} \in M_2$, and the same argument holds for the other vertices of S_i . Thus, H_i has only two possible S_i -pairs, illustrated in Figure 1. We call these pairs the *true pair* and *false pair* for H_i .

In the full graph G , we will not add any new edges incident to the vertices of X_i , so it will still be the case that any S -pair in the full graph induces either the true pair or the false pair in H_i . We use these pairs to encode the truth values of the corresponding 3SAT-variables.

For each clause c_k , let L_k be a copy of K_2 , with vertices w_k, z_k . Let $G = (\bigcup_j H_j) \cup (\bigcup_k L_k)$. Add edges to G as follows: if the variable θ_i appears positively in the clause c_k , add an edge from w_k to $v_{i,4k-3}$, and if the variable θ_i appears negatively in the clause c_k , add an edge from w_k to $v_{i,4k-1}$.

Let $X = \bigcup_j (X_j \cup \{w_j\})$, and let $Y = V(G) - X$. Observe that (X, Y) is a bipartition of $V(G)$. Let $S = (\bigcup S_j) \cup \bigcup \{w_j\}$.

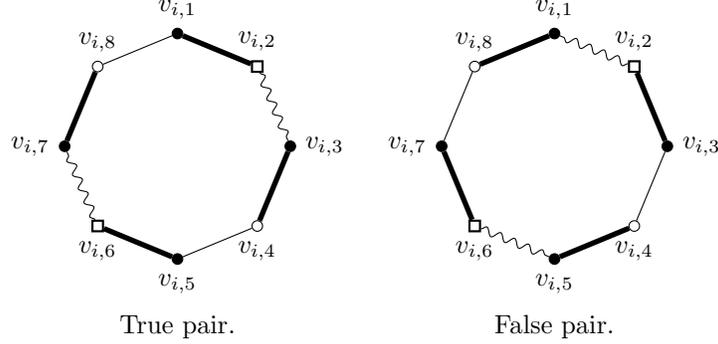


FIGURE 1. True and false pairs for H_i in the case $s = 2$. White vertices lie in X_i ; black vertices lie in Y_i ; square vertices lie in S_i . Thick lines denote edges in M_1 , wavy lines denote edges in M_2 .

Lemma 3.1. *G has an S -pair if and only if the given 3SAT instance is satisfiable.*

Proof. Let (M_1, M_2) be an S -pair. We show that the 3SAT instance is satisfiable.

For any variable θ_i , the vertices of $X \cap H_i$ have neighborhoods contained in H_i . Hence, (M_1, M_2) contains an S_i -pair, and in particular contains either the true pair or the false pair for H_i . Construct an assignment by setting each variable θ_i to be true if (M_1, M_2) contains the true pair for H_i and false otherwise. We claim that this is a satisfying assignment.

Consider any clause c_k . Since M_1 is a perfect matching and w_k is the only neighbor of z_k , we have $w_k z_k \in M_1$. Since $w_k \in S$, some edge $w_k v_{i,4k-3}$ or $w_k v_{i,4k-1}$ lies in M_2 .

If $w_k v_{i,4k-1} \in M_2$, then $v_{i,4k-2} v_{i,4k-2} \notin M_2$, so the given S -pair contains the false pair for H_i . Since $w_k v_{i,4k-1} \in E(G)$, the clause c_k contains a negative instance of θ_i , so the constructed assignment satisfies the clause c_k . On the other hand, if $w_k v_{i,4k-3} \in M_2$, then the given S -pair contains the true pair for H_i and θ_i appears positively in w_k , so we again see that w_k is satisfied.

Conversely, suppose that the 3SAT problem has a satisfying assignment. Consider the pair of matchings (M_1, M_2) in G obtained as follows. For each variable i , add the true pair for each H_i where θ_i is true and the false pair for each H_i where θ_i is false. For each clause c_k , add the edge $w_k z_k$ to M_1 . Choose some variable θ_i that satisfies the clause c_k . If θ_i is true, add the edge $w_k v_{i,4k-3}$ to M_2 , and otherwise add $w_k v_{i,4k-1}$ to M_2 . It is straightforward to check that this is an S -pair for G . \square

Corollary 3.2. *SDM is NP-hard.*

Viewing SDM as a special case of DM as in Section 2, we obtain the following NP-hardness result for DM.

Corollary 3.3. *DM is NP-hard, even when restricted to instances for which $E(G_1) \subseteq E(G_2)$.*

4. AN ALGORITHM FOR THE CASE $|S| \geq |X| - 1$

In this section, we provide a polynomial-time algorithm for solving SDM in the special case $|S| \geq |X| - 1$. Our algorithm requires the notion of a (g, f) -factor as well as the notion of *edge coloring*.

Definition 4.1. If G is a graph and g and f are functions from $V(G)$ into the nonnegative integers, a (g, f) -factor is a subgraph $H \subseteq G$ such that $g(v) \leq d_H(v) \leq f(v)$ for all $v \in V(G)$.

Lovasz [10] gave a Hall-like condition for a graph to have a g, f -factor, and polynomial-time algorithms are known for determining whether such a factor exists (for example, [4]). In the bipartite case we are considering here, the problem of determining whether such a factor exists can also be reduced to a feasible-flow problem.

Definition 4.2. For a nonnegative integer k , a k -edge coloring of a graph G is a function $f : E(G) \rightarrow \{1, \dots, k\}$ such that $f(e_1) \neq f(e_2)$ whenever e_1, e_2 are distinct edges sharing an endpoint. The *edge-chromatic number* of G , written $\chi'(G)$, is the smallest integer k such that G has a k -edge-coloring.

Theorem 4.3 (König’s line-coloring theorem [8]). *If G is a bipartite graph, then $\chi'(G) = \Delta(G)$, where $\Delta(G)$ is the maximum degree of G .*

Theorem 4.4. *There is a polynomial-time algorithm to solve SDM restricted to instances for which $|S| \geq |X| - 1$.*

Proof. To avoid triviality, assume that $|X| > 1$. Define functions f and g as follows.

$$f(v) = \begin{cases} 1, & \text{if } v \in X - S, \\ 2, & \text{otherwise} \end{cases}$$

$$g(v) = \begin{cases} f(v), & \text{if } v \in X, \\ 0, & \text{otherwise.} \end{cases}$$

We can check in polynomial time whether G has a (g, f) -factor. On the other hand, any (g, f) -factor H has maximum degree 2, and thus satisfies $\chi'(H) = 2$, by König’s line-coloring theorem. Since $d_H(v) = 2$ for all $v \in S$, any 2-edge-coloring of H uses colors $\{1, 2\}$ at each vertex of S . Furthermore, if $X - S \neq \emptyset$, then by switching colors if necessary, we can assume that the vertex in $X - S$ has only 1 as an incident color. Taking M_1 and M_2 to consist of the edges of color 1 and 2 respectively, we see that (M_1, M_2) is an S -pair in G . Conversely, if (M'_1, M'_2) is any S -pair in G , then $M'_1 \cup M'_2$ is a (g, f) -factor.

Hence, G has a (g, f) -factor if and only if G has an S -pair, so checking for such a factor solves the problem in polynomial time. \square

For any fixed k , the problem SDM is polynomial-time solvable on instances with $|S| \leq k$: we can iterate over the $O(|Y|^k)$ possible choices for M_2 , and for each possible choice, check whether $G - M_2$ has a perfect matching M_1 . Since the reduction in Section 3 produces SDM instances in which $|X - S|$ is arbitrarily large, Theorem 4.4 suggests that SDM might also be polynomially solvable when $|S|$ is bounded less strongly from below. However, the trick of using (g, f) -factors is no longer sufficient by itself to solve the problem when $k > 1$.

Question 4.5. For fixed $k > 1$, is there a polynomial-time algorithm to solve SDM on instances with $|S| \geq |X| - k$?

REFERENCES

1. H. Alt, N. Blum, K. Mehlhorn, and M. Paul, *Computing a maximum cardinality matching in a bipartite graph in time $O(n^{1.5}\sqrt{m/\log n})$* , Inform. Process. Lett. **37** (1991), no. 4, 237–240. MR 1095712 (91m:68141)
2. A. M. Frieze, *Complexity of a 3-dimensional assignment problem*, European J. Oper. Res. **13** (1983), no. 2, 161–164. MR 708379 (84i:68064)
3. D. R. Fulkerson, *The maximum number of disjoint permutations contained in a matrix of zeros and ones*, Canad. J. Math. **16** (1964), 729–735. MR 0168583 (29 #5843)
4. Harold N. Gabow, *An efficient reduction technique for degree-constrained subgraph and bidirected network flow problems*, Proceedings of the Fifteenth Annual ACM Symposium on Theory of Computing (New York, NY, USA), STOC '83, ACM, 1983, pp. 448–456.
5. Philip Hall, *On representatives of subsets*, J. London Math. Soc **10** (1935), no. 1, 26–30.
6. John E. Hopcroft and Richard M. Karp, *An $n^{5/2}$ algorithm for maximum matchings in bipartite graphs*, SIAM J. Comput. **2** (1973), 225–231. MR 0337699 (49 #2468)
7. R. R. Kamalian and V. V. Mkrtchyan, *On complexity of special maximum matchings constructing*, Discrete Math. **308** (2008), no. 10, 1792–1800. MR 2394447 (2009d:05197)
8. Dénes König, *Graphok és alkalmazásuk a determinánsok és a halmazok elméletére*, Matematikai és Természettudományi Ertesito **34** (1916), 104–119.
9. Kenneth Lebensold, *Disjoint matchings of graphs*, J. Combinatorial Theory Ser. B **22** (1977), no. 3, 207–210. MR 0450138 (56 #8435)
10. László Lovász, *Subgraphs with prescribed valencies*, J. Combinatorial Theory **8** (1970), 391–416. MR 0265201 (42 #113)