# The equidistribution of some length three vincular patterns on $S_{n}(132)$ 

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#### Abstract

In 2012 Bóna showed the rather surprising fact that the cumulative number of occurrences of the classical patterns 231 and 213 are the same on the set of permutations avoiding 132, beside the pattern based statistics 231 and 213 do not have the same distribution on this set. Here we show that if it is required for the symbols playing the role of 1 and 3 in the occurrences of 231 and 213 to be adjacent, then the obtained statistics are equidistributed on the set of 132 -avoiding permutations. Actually, expressed in terms of vincular patterns, we prove the following more general results: the statistics based on the patterns $2 \underline{31}, 2 \underline{13}$ and $\underline{21} 3$, together with other statistics, have the same joint distribution on $S_{n}(132)$, and so do the patterns $\underline{231}$ and $3 \underline{12}$; and up to trivial transformations, these statistics are the only based on length three proper (not classical nor adjacent) vincular patterns which are equidistributed on a set of permutations avoiding a classical length three pattern.


## 1 Introduction

In [2] Barnabei, Bonetti and Silimbani showed the equidistribution of some length three consecutive patterns involvement statistics on the set of permutations avoiding the classical pattern 312 (or equivalently, 132). And in [3] Bóna showed the surprising fact that the total number of occurrences of the patterns 231 and 213 is the same on the set of 132 -avoiding permutations, beside the pattern based statistics 231 and 213 are not equidistributed on this set. In [5], Homberger, generalizing Bóna's result, gave the total number of occurrences of each classical length three pattern on the set of 123 -avoiding permutations, and showed that the total number of occurrences of the pattern 231 is the same in the set of 123 - and 132 -avoiding permutations, despite the pattern statistic 231 has different distribution on the two sets. Motivated by these, Burnstein and Elizalde gave in [4, in a much more general context, the total number of occurrences of any vincular pattern of length three on 231 -avoiding (or equivalently, 132 -avoiding) permutations.

In this paper we show that, on the set of 132 -avoiding permutations, the vincular pattern based statistics $2 \underline{31}, \underline{213}$ and $\underline{21} 3$ are equidistributed, and so are $\underline{231}$ and $3 \underline{12}$; and numerical evidences show that, up to trivial transformations, these patterns are the only length three proper (not classical nor adjacent) vincular patterns equidistributed on a set of permutations avoiding a classical length three pattern.

It is worth to mention that, on the set of unrestricted permutations, the statistics $\underline{231}$ and $3 \underline{12}$ are trivially equidistributed, and so are $2 \underline{31}$ and $2 \underline{13}$ (which is all but obvious on 132-avoiding permutations), and this last distribution is different from that of $\underline{213}$.

More precisely, we show bijectively the equidistribution on 132-avoiding permutations of the multistatistics

- $(2 \underline{3} \underline{1}, 2 \underline{13}$, rlmin, rImax $)$ and ( $2 \underline{13}, 2 \underline{31}$, rlmax, rlmin $)$,
- $(2 \underline{31}$, des $)$ and $(2 \underline{13}$, des $)$,
- ( $\underline{213} 3$, des, 12 J$)$ and ( $2 \underline{13}$, des, 12$\rfloor$ ),
- $(\underline{23} 1,3 \underline{12}$, des $)$ and ( $3 \underline{12}, \underline{231}$, des $)$,
where rlmax, rlmin and des are respectively, the number of right-to-left maxima, right-to-left minima and descents. The corresponding bijections (the last of them being straightforward) are presented in Subsection 3.


## 2 Notations and definitions

A permutation of length $n$ is a bijection from the set $\{1,2, \ldots, n\}$ to itself and we write permutations in one-line notation, that is, as words $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$, where $\pi_{i}$ is the image of $i$ under $\pi$. We let $S_{n}$ denote the set of permutations of length $n \geq 0$, and $S=\cup_{n \geq 0} S_{n}$.

## Vincular patterns

Let $\sigma \in S_{k}$ and $\pi=\pi_{1} \pi_{2} \ldots \pi_{n} \in S_{n}, k \leq n$, be two permutations. One says that $\sigma$ occurs as a (classical) pattern in $\pi$ if there is a sequence $1 \leq i_{1}<i_{2}<\cdots<i_{k} \leq n$ such that $\pi_{i_{1}} \pi_{i_{2}} \cdots \pi_{i_{k}}$ is order-isomorphic with $\pi$. For example, 231 occurs as a pattern in 13452, and the three occurrences of it are 342,352 and 452.

Vincular patterns were introduced in [1 and they were extensively studied since then (see Chapter 7 in [6] for a comprehensive description of results on these patterns). Vincular patterns generalize classical patterns and they are defined as follows:

- Any pair of two adjacent letters may now be underlined, which means that the corresponding letters in the permutation must be adjacent. (The original notation for vincular patterns uses dashes: the absence of a dash between two letters of a pattern means that these letters are adjacent in the permutation.) For example, the pattern $2 \underline{1} \underline{3}$ occurs in the permutation 425163 four times, namely, as the subsequences 425, 416, 216 and 516. Note that, the subsequences 426 and 213 are not occurrences of the pattern because their last two letters are not adjacent in the permutation.
- If a pattern begins (resp., ends) with a hook then its occurrence is required to begin (resp., end) with the leftmost (resp., rightmost) letter in the permutation. (In the original notation the role of hooks was played by square brackets.) For example, there are two occurrences of the pattern $2 \underline{213}$ in the permutation 425163 , which are the subsequences 425 and 416.


## Statistics

A statistic on a set of permutations is simply a function from the set to $\mathbb{N}$, and a multistatistic is a tuple of statistics. A classical example of statistic on $S_{n}$ is the descent number

$$
\operatorname{des} \pi=\operatorname{card}\left\{i: 1 \leq i<n, \pi_{i}>\pi_{i+1}\right\},
$$

for example des $45312=2$.
In a permutation $\pi=\pi_{1} \pi_{2} \ldots \pi_{n}$, $\pi_{i}$ is a right-to-left maximum if $\pi_{i}>\pi_{j}$ for all $j>i$; and the number of right-to-left maxima of $\pi$ is denoted by rlmax $\pi$. Similarly, $\pi_{i}$ is a right-to-left minimum if $\pi_{i}<\pi_{j}$ for all $j>i$; and the number of right-to-left minima of $\pi$ is denoted by rlmin $\pi$. Both, rImax and rlmin are statistics on $S_{n}$.

For a set of permutations $S$, two statistics ST and $\mathrm{ST}^{\prime}$ have the same distribution (or are equidistributed) on $S$ if, for any $k$,

$$
\operatorname{card}\{\pi \in S: \mathrm{st} \pi=k\}=\operatorname{card}\left\{\pi \in S: \mathrm{sT}^{\prime} \pi=k\right\},
$$

and the multistatistics $\left(\mathrm{ST}_{1}, \mathrm{ST}_{2}, \ldots, \mathrm{ST}_{p}\right)$ and $\left(\mathrm{ST}_{1}^{\prime}, \mathrm{ST}_{2}^{\prime}, \ldots, \mathrm{ST}_{p}^{\prime}\right)$ have the same distribution if, for any $p$-tuple $k=\left(k_{1}, k_{2}, \ldots, k_{p}\right)$,

$$
\operatorname{card}\left\{\pi \in S:\left(\mathrm{sT}_{1}, \mathrm{ST}_{2}, \ldots, \mathrm{sT}_{p}\right) \pi=k\right\}=\operatorname{card}\left\{\pi \in S:\left(\mathrm{sT}_{1}^{\prime}, \mathrm{ST}_{2}^{\prime}, \ldots, \mathrm{ST}_{p}^{\prime}\right) \pi=k\right\} .
$$

For a permutation $\pi$ and a (vincular) patterns $\sigma$ we denote by $(\sigma) \pi$ the number of occurrences of this pattern in $\pi$, and $(\sigma)$ becomes a permutation statistic. For example, (21) $\pi$ is des $\pi$; (21) $\pi$ is the inversion number of $\pi$; and (12」) $\pi$ is the last value of $\pi$ minus one. Similarly, for a set of (vincular) patterns $\{\sigma, \tau, \ldots\}$, we denote by $(\sigma+\tau+\cdots) \pi$ the number of occurrences of these patterns in $\pi$.

## Sum decomposition

For a permutation $\pi,|\pi|$ denotes its length (and so, $\pi \in S_{|\pi|}$ ), and for two permutations $\alpha$ and $\beta$

- the skew sum of $\alpha$ and $\beta$, denoted $\alpha \ominus \beta$, is the permutation $\pi$ of length $|\alpha|+|\beta|$ with

$$
\pi_{i}=\left\{\begin{array}{cll}
\alpha_{i}+|\beta| & \text { if } & 1 \leq i \leq|\alpha|, \\
\beta_{i-|\alpha|} & \text { if } & |\alpha|+1 \leq i \leq|\alpha|+|\beta|,
\end{array}\right.
$$

and

- the direct sum of $\alpha$ and $\beta$, denoted $\alpha \oplus \beta$, is the permutation $\pi$ of length $|\alpha|+|\beta|$ with

$$
\pi_{i}=\left\{\begin{array}{cll}
\alpha_{i} & \text { if } & 1 \leq i \leq|\alpha|, \\
\beta_{i-|\alpha|}+|\alpha| & \text { if } & |\alpha|+1 \leq i \leq|\alpha|+|\beta| .
\end{array}\right.
$$

It is easy to check the following.
Fact 1. For two permutations $\alpha$ and $\beta$, $\operatorname{des} \alpha \oplus \beta=\operatorname{des} \alpha+\operatorname{des} \beta$ and, when $\alpha$ is not empty, $\operatorname{des} \alpha \ominus \beta=\operatorname{des} \alpha+\operatorname{des} \beta+1$.

The following characterization of 132 -avoiding permutations is folklore.


Figure 1: The decomposition: (1) $\pi=(\alpha \oplus 1) \ominus \beta$, and (2) $\pi=\alpha \ominus(\beta \oplus 1)$ of $\pi \in S_{n}(132)$, $n \geq 1$.

Fact 2. For a non-empty permutation $\pi \in S_{n}, n \geq 1$, the following are equivalent:

- $\pi$ avoids 132 ;
- $\pi$ can uniquely be written as $(\alpha \oplus 1) \ominus \beta$, where $\alpha$ and $\beta$ are (possibly empty) 132-avoiding permutations;
- $\pi$ can uniquely be written as $\alpha \ominus(\beta \oplus 1)$, where $\alpha$ and $\beta$ are (possibly empty) 132-avoiding permutations.

See Figure 1 .

## Permutation symmetries

For a permutation $\pi \in S_{n}$, the reverse and complement of $\pi$, denoted $\pi^{r}$ and $\pi^{c}$ respectively, are defined as:

- $\pi_{i}^{r}=\pi_{n-i+1}$,
- $\pi_{i}^{c}=n-\pi_{i}+1$,
and both operations can naturally be extended to vincular patters; for instance, the reverse of $\underline{231}$ is $1 \underline{32}$, and the complement of $\underline{231}$ is $\underline{213}$. These operations preserve pattern containment, in the sense that, if the (vincular) patter $\sigma$ is contained in the permutation $\pi$, then $\sigma^{r}$ is contained in $\pi^{r}$, and $\sigma^{c}$ is contained in $\pi^{c}$.

The inverse of $\pi$, denoted $\pi^{-1}$, is defined as:

- $\pi_{\pi_{i}}^{-1}=i$,
but, unlike the reverse and complement, it can not be extended to vincular patters: in general, the inverse of a vincular pattern is a bivincular pattern (see for example [6, p. 13] for its formal definition), we will not consider here.


## 3 Main results

3.1 Equidistribution of ( $2 \underline{31}, 2 \underline{13}$, rlmax, rlmin) and ( $2 \underline{13}, 2 \underline{3}$, rlmin, rlmax) on $S_{n}(132)$ : bijection $\phi$
We define a mapping $\phi$ on $S_{n}(132)$ and we will see that it is an involution, that is, a bijection from $S_{n}(132)$ into itself, which is its own inverse; and Theorem $\square$ below shows the desired equidistribution.

The mapping $\phi$ is recursively defined as：if $\pi$ is the empty permutation（that is，$n=0$ ），then $\phi(\pi)=\pi$ ；and if $\pi \in S_{n}(132), n \geq 1$ ，with $\pi=\alpha \ominus(\beta \oplus 1)$ for some 132 －voiding permutations $\alpha$ and $\beta$ ，then

$$
\phi(\pi)=\phi(\beta) \ominus(\phi(\alpha) \oplus 1) .
$$

See Figure 2 for this definition．
Note that $\phi$ is in some sense similar with the inversion ${ }^{-1}$ ，but is fundamentally different from it．Indeed，the inversion when restricted to $S_{n}(132)$ satisfies：$(\alpha \ominus(\beta \oplus 1))^{-1}=\left(\beta^{-1} \oplus 1\right) \ominus \alpha^{-1}$ ， see Figure 5


Figure 2：The recursive definition of $\phi(\pi)$ ．
By induction on $n$ ，from the definition of $\phi$ it follows that if $\pi \in S_{n}(132)$ ，then $\phi(\pi) \in S_{n}(132)$ and $\phi(\phi(\pi))=\pi$ ，and in particular $\phi$ is a bijection on $S_{n}(132)$ to itself．

In the proof of the next theorem we will need the following result．
Proposition 1．For $\pi \in S_{n}(132)$ we have（12」）$\left.\pi=(21\rfloor\right) \phi(\pi)$ and（21」）$\left.\pi=(12\rfloor\right) \phi(\pi)$ ．
Proof．If $\pi=\alpha \ominus(\beta \oplus 1)$ is a non－empty 132－avoiding permutation，then we have
－$(21\rfloor) \phi(\pi)=|\phi(\beta)|=|\beta|=(12\rfloor) \pi$ ；and
－（12」）$\phi(\pi)=|\phi(\alpha)|=|\alpha|=(21\rfloor) \pi$ ．

Theorem 1．If $\pi \in S_{n}(132)$ ，then

$$
(2 \underline{13}, 2 \underline{3}, \text { rlmin }, \text { rlmax }) \phi(\pi)=(2 \underline{31}, 2 \underline{13}, \text { rlmax }, \text { rlmin }) \pi .
$$

Proof．By induction on $n$ ．Trivially，the statement holds for $n=0$ ，and let suppose that it is true for any $n<m$ ，and consider $\pi=\alpha \ominus(\beta \oplus 1) \in S_{m}(132)$ for some 132－avoiding permutations $\alpha$ and $\beta$ ．

First we prove that rlmax $\pi=r \operatorname{lmin} \phi(\pi)$ ：

$$
\begin{aligned}
\mathrm{rlmin} \phi(\pi) & =1+\mathrm{r} \min \phi(\alpha) \\
& =1+\mathrm{rlmax} \alpha \quad \text { (by the induction hypothesis) } \\
& =\mathrm{r} \max \pi .
\end{aligned}
$$

An occurrence of $2 \underline{13}$ in $\phi(\pi)$ can be found either in $\phi(\alpha)$ ，or in $\phi(\beta)$ ，or has the form abc with $a b$ an occurrence of 21 J in $\phi(\alpha)$ and $c$ the last symbol of $\phi(\pi)$ ．Thus

$$
\begin{array}{rlr}
(2 \underline{13}) \phi(\pi) & =(2 \underline{13}) \phi(\alpha)+(2 \underline{13}) \phi(\beta)+(21\rfloor) \phi(\alpha) \quad \\
& =(2 \underline{31}) \alpha+(2 \underline{1}) \beta+(12\rfloor) \alpha \quad \text { (by the induction hypothesis } \\
& \text { and Proposition (1) })
\end{array}
$$

And an occurrence of $2 \underline{31}$ in $\pi$ can be found either in $\alpha$, or in $\beta$, or has the form $a b c$ with $a b$ an occurrence of 12」 in $\alpha$ and $c$ the first symbol of $\beta$ if it is not empty, and the last symbol of $\pi$ when $\beta$ is empty. Thus

$$
(2 \underline{31}) \pi=(2 \underline{31}) \alpha+(2 \underline{31}) \beta+(12 \jmath) \alpha,
$$

and finally $(2 \underline{13}) \phi(\pi)=(2 \underline{31}) \pi$.
Moreover, since $\phi$ is an involution, it follows that $(2 \underline{31}) \phi(\pi)=(2 \underline{13}) \pi$ and $\operatorname{rlmin} \pi=$ rlmax $\phi(\pi)$.

The three-statistics $(2 \underline{31}, 2 \underline{13}$, des) and ( $2 \underline{13}, 2 \underline{31}$, des) do not have the same distribution on $S_{n}(132)$; however, Theorem 2 below says that separetely (231) and (213), together with des, have the same joint distribution.

### 3.2 Equidistribution of ( $2 \underline{31}$, des) and (213, des) on $S_{n}(132)$ : bijection $\psi$

Now, we define the mapping $\psi: S_{n}(132) \rightarrow S_{n}$ by $\psi(1)=1$ if $n=1$, and for $n \geq 2, \psi(\pi)$ is defined recursively below, according to three cases: $\pi_{n}^{-1}=1,2 \leq \pi_{n}^{-1} \leq n-1$, and $\pi_{n}^{-1}=n$; see Figure 3 ,

Let $\pi \in S_{n}(132), n \geq 2$.

1. If $\pi$ has the form $1 \ominus \alpha$ (or equivalently, $\pi_{n}^{-1}=1$ ), then $\psi(\pi)$ is simply $1 \ominus \psi(\alpha)$.
2. If $\pi$ has the form $(\alpha \oplus 1) \ominus \beta$ for some non-empty permutations $\alpha$ and $\beta$ (or equivalently, $\left.2 \leq \pi_{n}^{-1} \leq n-1\right)$, then $\psi(\pi)$ is obtained by:
(a) considering the (possibly empty) permutations $\gamma$ and $\delta$ with $\psi(\beta)=\gamma \ominus(\delta \oplus 1)$, and (b) defining $\psi(\pi)$ as $((\psi(\alpha) \ominus(\delta \oplus 1)) \oplus 1) \ominus \gamma$.
(Note that $\alpha, \beta, \gamma$ and $\delta$ are 132-avoiding permutations.)
3. If $\pi$ has the form $\alpha \oplus 1$ for some non-empty permutation $\alpha$ (or equivalently, $\pi_{n}^{-1}=n$ ), then $\psi(\pi)$ is obtained by:
(a) considering the (possibly empty) permutations $\gamma$ and $\delta$ with $\psi(\alpha)=(\gamma \oplus 1) \ominus \delta$, and
(b) defining $\psi(\pi)$ as $((\gamma \oplus 1) \oplus 1) \ominus \delta$.
(Note that $\alpha, \gamma$ and $\delta$ are 132-avoiding permutations.)
From the above definition of $\psi$ it is easy to check the following.
Proposition 2. Let $\pi \in S_{n}(132)$ and $\sigma=\psi(\pi)$.
4. $\pi_{n}^{-1}=1$ iff $\sigma_{n}^{-1}=1$,
5. $2 \leq \pi_{n}^{-1} \leq n-1$ iff $\sigma_{n}^{-1}>1$ and $\sigma_{n}^{-1}-\sigma_{n-1}^{-1}>1$,
6. $\pi_{n}^{-1}=n$ iff $\sigma_{n}^{-1}>1$ and $\sigma_{n}^{-1}-\sigma_{n-1}^{-1}=1$.

Theorem 2. The mapping $\psi$ is a bijection on $S_{n}(132)$, and for any $\pi \in S_{n}(132)$, we have

$$
(2 \underline{13}, \text { des }) \psi(\pi)=(2 \underline{31}, \text { des }) \pi
$$



Figure 3: The three cases occurring in the definition of $\psi$ : (1) $\pi_{n}^{-1}=1$, (2) $2 \leq \pi_{n}^{-1} \leq n-1$, and (3) $\pi_{n}^{-1}=n$.

Proof. Let $\pi \in S_{n}(132)$. By the construction of $\psi$ and iteratively applying Fact 2 we have that $\psi(\pi) \in S_{n}(132)$. And by induction on $n$, from Proposition 2 it follows that $\psi$ is injective and thus bijective; and from Fact 1 it follows that des $\pi=\operatorname{des} \sigma$.

Now we show by induction on $n$ that $(2 \underline{213}) \psi(\pi)=(2 \underline{31}) \pi$, for any $\pi \in S_{n}(132), n \geq 1$.
Clearly, for $n=1$, (213) $\psi(\pi)=(2 \underline{31}) \pi$, and let suppose that $(2 \underline{13}) \psi(\pi)=(2 \underline{31}) \pi$ for any $\pi \in S_{n}$ and $n<m$, and we will prove it for $\pi \in S_{m}$.

1. If $\pi_{m}^{-1}=1$, then $\pi=1 \ominus \alpha$ for some $\alpha \in S_{m-1}(132)$, and by definition, $\psi(\pi)=1 \ominus$ $\psi(\alpha)$. By the induction hypothesis we have (213) $\psi(\alpha)=(2 \underline{31}) \alpha$, and thus (213) $\psi(\pi)=$ $(2 \underline{13}) \psi(\alpha)=(2 \underline{31}) \alpha=(2 \underline{31}) \pi$.
2. If $1 \leq \pi_{m}^{-1} \leq m-1$, let $\alpha, \beta, \gamma$ and $\delta$ be the permutations appearing in the second case of the definition of $\psi$, and we have

$$
\begin{array}{rlr}
(2 \underline{13}) \psi(\pi) & =(2 \underline{13}) \psi(\alpha)+|\alpha|+(2 \underline{13}) \delta \oplus 1+(2 \underline{13}) \gamma & \\
& =(2 \underline{13}) \psi(\alpha)+|\alpha|+(2 \underline{13}) \psi(\beta) \quad \\
& =(2 \underline{31}) \alpha+|\alpha|+(2 \underline{3}) \beta \\
& =(2 \underline{31}) \pi . & \\
\text { (by the induction hypothesis) }
\end{array}
$$

3. If $\pi_{m}^{-1}=m$, let $\alpha, \gamma$ and $\delta$ be the permutations appearing in the third case of the definition of $\psi$. Again, $(2 \underline{13}) \psi(\alpha)=(2 \underline{31}) \alpha$, and

$$
\begin{aligned}
(2 \underline{13}) \psi(\pi) & =(2 \underline{13}) \gamma \oplus 1+(2 \underline{13}) \delta \\
& =(2 \underline{13}) \psi(\alpha) \\
& =(2 \underline{31}) \alpha \\
& =(2 \underline{31}) \pi .
\end{aligned}
$$

3.3 Equidistribution of ( $\underline{21} 3$, des, 12$\rfloor$ ) and ( $2 \underline{13}$, des, 12$\rfloor$ ) on $S_{n}(132)$ : bijection $\mu$

Based on the previously defined bijection $\psi$ we give a mapping $\mu$ on $S_{n}(132)$ and show that it is a bijection on $S_{n}(132)$, and Theorem 3 proves the desired equidistribution.

Expressing in two different ways the major index of a permutation, Lemma 2 in [7] (see also Corollary 14 in [8]) shows that for any permutation $\pi$ (not necessarily in $S_{n}(132)$ ) we have

$$
(2 \underline{13}+21\rfloor) \pi=(2 \underline{31}+\underline{21}) \pi .
$$

Actually, (21) $\pi$ is equal to des $\pi$, and the above relation becomes

$$
\begin{equation*}
(2 \underline{13}+21\rfloor) \pi=(2 \underline{3} \underline{1}+\mathrm{des}) \pi, \tag{1}
\end{equation*}
$$

and the next lemma follows.
Lemma 1. For any permutation $\pi$ we have

$$
(2 \underline{13} \underline{3}) \pi \oplus 1=(2 \underline{31}+\mathrm{des}) \pi .
$$

Proof. From relation (1) we have

$$
(2 \underline{13}+21\rfloor) \pi \oplus 1=(2 \underline{31}+\mathrm{des}) \pi \oplus 1,
$$

and the statement holds by considering that (21」) $\pi \oplus 1=0$, $\operatorname{des} \pi \oplus 1=\operatorname{des} \pi$, and (231) $\pi \oplus 1=$ (231) $\pi$.

In the proof of the following theorem we will use the next easy to understand fact.
Fact 3. For any permutation $\pi$, we have $(\underline{21} 3) \pi \oplus 1=(\underline{21} 3+$ des $) \pi$.
The mapping $\mu$ on $S_{n}(132)$ is recursively defined as: if $\pi$ is the empty permutation, then $\mu(\pi)=\pi$; and if $\pi \in S_{n}(132), n \geq 1$, with $\pi=\alpha \ominus(\beta \oplus 1)$ for some 132 -voiding permutations $\alpha$ and $\beta$, then

$$
\mu(\pi)=\mu(\alpha) \ominus(\mu(\psi(\beta)) \oplus 1)
$$

where $\psi$ is the bijection define in Subsection 3.2. See Figure 4 for this recursive construction.


Figure 4: The recursive definition of $\mu(\pi)$.
Since $\psi$ is a bijection on $S_{n}(132), n \geq 0$, it follows that $\mu(\pi)$ avoids 132 whenever $\pi$ does so, and thus $\mu(\pi) \in S_{n}(132)$ for any $\pi \in S_{n}(132)$. With the notations above, it is clear that (12ן) $\mu(\pi)=(12\rfloor) \pi=|\beta|$ and considering again the bijectivity of $\psi$, by induction on $n$ it follows that $\mu$ is injective, and thus bijective.
Theorem 3. If $\pi \in S_{n}(132)$, then

$$
(\underline{21} 3, \text { des, } 12\rfloor) \mu(\pi)=(2 \underline{13}, \text { des, } 12\rfloor) \pi .
$$

Proof. Clearly, (12」) $\mu(\pi)=(12\rfloor) \pi$, and the remaining of the proof is by induction on $n$. Obviously, the statement holds for $n=0$, and let suppose that it is true for any $n<m$, and consider $\pi=\alpha \ominus(\beta \oplus 1) \in S_{m}(132)$ for some 132-avoiding permutations $\alpha$ and $\beta$.

The bijection $\mu$ preserves des statistic. Indeed, (using the Iverson bracket notation) considering $[|\alpha| \neq 0]$ equal to 0 (resp. 1) if $\alpha$ is empty (resp. not empty) we have

$$
\begin{array}{rlr}
\operatorname{des} \mu(\pi) & =\operatorname{des} \mu(\alpha) \ominus(\mu(\psi(\beta)) \oplus 1) & \\
& =\operatorname{des} \mu(\alpha)+\operatorname{des} \mu(\psi(\beta))+[|\alpha| \neq 0] & \\
& =\operatorname{des} \alpha+\operatorname{des} \psi(\beta)+[|\alpha| \neq 0] & \text { (by the induction hypothesis) } \\
& =\operatorname{des} \alpha+\operatorname{des} \beta+[|\alpha| \neq 0] & \text { (since } \psi \text { preserves des) } \\
& =\operatorname{des} \alpha \ominus(\beta \oplus 1) & \\
& =\operatorname{des} \pi . &
\end{array}
$$

Finally, we show that $(\underline{213}) \mu(\pi)=(2 \underline{13}) \pi$.

$$
\begin{aligned}
& (\underline{213}) \mu(\pi)=(\underline{213}) \mu(\alpha) \ominus(\mu(\psi(\beta)) \oplus 1) \\
& =(\underline{21} 3) \mu(\alpha)+(\underline{21} 3) \mu(\psi(\beta)) \oplus 1 \\
& =(\underline{21} 3) \mu(\alpha)+(\underline{21} 3) \mu(\psi(\beta))+\operatorname{des} \mu(\psi(\beta)) \quad \text { (by Fact 3) } \\
& =(2 \underline{13}) \alpha+(2 \underline{13}) \psi(\beta)+\operatorname{des} \beta \quad \text { (by the induction hypothesis } \\
& =(2 \underline{13}) \alpha+(2 \underline{31}) \beta+\operatorname{des} \beta \\
& =(2 \underline{13}) \alpha+(2 \underline{13}) \beta \oplus 1 \\
& =(2 \underline{13}) \alpha \ominus(\beta \oplus 1) \\
& =(2 \underline{13}) \pi \text {. }
\end{aligned}
$$

### 3.4 Equidistribution of ( $\underline{231}, 3 \underline{12}$, des) and (312, 231, des)

It is easy to see that the inverse of a permutation (defined at the end of Section (2) satisfies: if $\pi=\alpha \ominus(\beta \oplus 1)$, then $\pi^{-1}=\left(\beta^{-1} \oplus 1\right) \ominus \alpha^{-1}$, see Figure 5.

As mentioned at the end of Section 2, the inverse of a vincular pattern is not longer a vincular pattern, however we have the following.
Proposition 3. For any $\pi \in S_{n}(132)$, we have

$$
(\underline{23} 1,3 \underline{12}, \text { des }) \pi^{-1}=(3 \underline{12}, \underline{2} \underline{3} 1, \text { des }) \pi .
$$

Proof. Trivially, the statement holds for $n=0$, and let suppose that it is true for any $n<m$, and consider $\pi=\alpha \ominus(\beta \oplus 1) \in S_{m}(132)$ for some 132-avoiding permutations $\alpha$ and $\beta$.

If $\alpha$ is empty, then $\operatorname{des} \pi^{-1}=\operatorname{des} \beta^{-1}$ and $\operatorname{des} \pi=\operatorname{des} \beta$; otherwise, $\operatorname{des} \pi^{-1}=\operatorname{des} \beta^{-1}+$ $\operatorname{des} \alpha^{-1}+1$ and $\operatorname{des} \pi=\operatorname{des} \alpha+\operatorname{des} \beta+1$. In both cases, by the induction hypothesis it follows that des $\pi^{-1}=\operatorname{des} \pi$.

An occurrence of $\underline{231}$ in $\pi^{-1}$ can be found either in $\beta^{-1}$, or in $\alpha^{-1}$, or when $\beta$ is not empty, has the form $a b c$ with $a$ the last symbol of $\beta^{-1}, b=m$ (the largest symbol of $\pi$ ) and $c$ a symbol of $\alpha^{-1}$. Similarly, an occurrence of $3 \underline{12}$ in $\pi$ can be found either in $\alpha$, or in $\beta$, or when $\beta$ is not empty, has the form $a b c$ with $a$ a symbol of $\alpha, b$ the last symbol of $\beta$ and $c$ the last symbol of $\pi$.

Thus, when $\beta$ is not empty

$$
(\underline{23} 1) \pi^{-1}=(\underline{231}) \beta^{-1}+(\underline{23} 1) \alpha^{-1}+\left|\alpha^{-1}\right|
$$

and

$$
(3 \underline{12}) \pi=(3 \underline{12}) \alpha+(3 \underline{12}) \beta+|\alpha|,
$$

and by the induction hypothesis it follows that (231) $\pi^{-1}=(3 \underline{12}) \pi$. And when $\beta$ is empty

$$
(\underline{23} 1) \pi^{-1}=(\underline{23} 1) \alpha^{-1}
$$

and

$$
(3 \underline{12}) \pi=(3 \underline{12}) \alpha,
$$

and again $(\underline{231}) \pi^{-1}=(3 \underline{12}) \pi$.
Since ${ }^{-1}$ is an involution on $S_{n}(132)$, it follows that $(\underline{231}) \pi^{-1}=(3 \underline{12}) \pi$, and the statement holds.


Figure 5: The recursive construction of $\pi^{-1}$, for $\pi \in S(132)$.

## 4 Conclusions

We showed bijectively the joint equidistribution on the set $S_{n}(132)$ of 132 -voiding permutations of some length three vincular patterns together with other statistics. In particular, for the sets of vincular patterns $\{2 \underline{31}, 2 \underline{13}, \underline{213\}}$ and $\{\underline{231}, 3 \underline{12\}}$, we showed that the patterns within each set are equidistributed on $S_{n}(132)$. By applying permutation symmetries, other similar results can be derived. For instance, from the equidistribution of $\underline{213}$ and $2 \underline{13}$ on $S_{n}(132)$ (belonging to the first set, see Subsection 3.3) it follows, by applying

- the reverse operation, the equidistribution of $3 \underline{12}$ and $\underline{312}$ on $S_{n}(231)$,
- the complement operation, the equidistribution of $\underline{23} 1$ and $2 \underline{3}$ on $S_{n}(312)$, and
- the complement and the reverse operations (in any order), the equidistribution of $1 \underline{32}$ and 132 on $S_{n}(213)$.

Moreover, computer experiments show that, up to these two symmetries, the patterns in $\{2 \underline{2} \underline{1}, 2 \underline{13}, \underline{21} 3\}$ and those in $\{\underline{231}, 3 \underline{12}\}$ are the only length three proper (not classical nor adjacent) vincular patterns which are equidistributed on a set of permutations avoiding a classical length three pattern.

## References

[1] E. Babson, E. Steingrímsson, Generalized permutation petterns and a classification of Mahonian statistics, Sém. Lothar. Combin. (electronic), 44 (2000).
[2] M. Barnabei, F. Bonetti, M. Silimbani, The joint distribution of consecutive patterns and descents in permutations avoiding 3-1-2. Eur. J. Comb. 31(5 )(2010), 1360-1371.
[3] M. Bóna, Surprising symmetries in objects counted by Catalan numbers. Electr. J. Comb. 19(1) (2012), P62.
[4] A. Burstein, S. Elizalde, Total occurrence statistics on restricted permutations, ArXiv 2013, http://arxiv.org/pdf/1305.3177v1.pdf.
[5] C. Homberger, Expected patterns in permutation classes, Electr. J. Comb. 19(3) (2012), P43.
[6] S. Kitaev, Patterns in permutations and words, Springer-Verlag, 2011.
[7] V. Vajnovszki, Lehmer code transforms and Mahonian statistics on permutations, Discrete Mathematics, 313 (2013), 581-589.
[8] V. Vajnovszki, A new Euler-Mahonian constructive bijection, Discrete Applied Mathematics, 159 (2011), 1453-1459.

