# A Note on Multiparty Communication Complexity and the Hales-Jewett Theorem 

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#### Abstract

For integers $n$ and $k$, the density Hales-Jewett number $c_{n, k}$ is defined as the maximal size of a subset of $[k]^{n}$ that contains no combinatorial line. We show that for $k \geq 3$ the density Hales-Jewett number $c_{n, k}$ is equal to the maximal size of a cylinder intersection in the problem Part $_{n, k}$ of testing whether $k$ subsets of $[n]$ form a partition. It follows that the communication complexity, in the Number On the Forehead (NOF) model, of $\operatorname{Part}_{n, k}$, is equal to the minimal size of a partition of $[k]^{n}$ into subsets that do not contain a combinatorial line. Thus, the bound in [7 on Part ${ }_{n, k}$ using the Hales-Jewett theorem is in fact tight, and the density Hales-Jewett number can be thought of as a quantity in communication complexity. This gives a new angle to this well studied quantity.

As a simple application we prove a lower bound on $c_{n, k}$, similar to the lower bound in [19] which is roughly $c_{n, k} / k^{n} \geq \exp \left(-O(\log n)^{\left.1 /\left[\log _{2} k\right\rceil\right]}\right)$. This lower bound follows from a protocol for Part $_{n, k}$. It is interesting to better understand the communication complexity of Part $_{n, k}$ as this will also lead to the better understanding of the HalesJewett number. The main purpose of this note is to motivate this study.


## 1 Introduction

For any integers $n \geq 1$ and $k \geq 1$, consider the set $[k]^{n}$, of words of length $n$ over the alphabet $[k]$. Define a combinatorial line in $[k]^{n}$ as a subset of $k$ distinct words such that we can place these words in a $k \times n$ table so that all columns in this table belong to the set $\{(x, x, \ldots, x): x \in[k]\} \cup\{(1,2, \ldots, k)\}$. The density Hales-Jewett number $c_{n, k}$ is defined to be the maximal cardinality of a subset of $[k]^{n}$ which does not contain a combinatorial line.

Clearly, $c_{n, k} \leq k^{n}$, and a deep theorem of Furstenberg and Katznelson (11) 12 says that $c_{n, k}$ is asymptotically smaller than $k^{n}$ :
Theorem 1 (Density Hales-Jewett theorem) For every positive integer $k$ and every real number $\delta>0$ there exists a positive integer $\operatorname{DHJ}(k, \delta)$ such that if $n \geq$ $D H J(k, \delta)$ then any subset of $[k]^{n}$ of cardinality at least $\delta k^{n}$ contains a combinatorial line.

The above theorem is a density version of the Hales-Jewett theorem:
Theorem 2 (Hales-Jewett theorem) For every pair of positive integers $k$ and $r$ there exists a positive number $H J(k, r)$ such that for every $n \geq H J(k, r)$ and every $r$-coloring of the set $[k]^{n}$ there is a monochromatic combinatorial line.

Note that the density Hales-Jewett theorem implies the Hales-Jewett theorem but not the other way around. The density Hales-Jewett theorem is a fundamental result of Ramsey theory. It implies several well known results, such as van der Waerden's theorem [23], Szemerédi's theorem on arithmetic progressions of arbitrary length [22] and its multidimensional version [10].

The proof of Furstenberg and Katznelson used ergodic-theory and gave no explicit bound on $c_{n, k}$. Recently, additional proofs of this theorem were found [18, 2, 8]. The proof of [18] is the first combinatorial proof of the density Hales-Jewett theorem, and also provides effective bounds for $c_{n, k}$. In a second paper [19] in this project, several values of $c_{n, 3}$ are computed for small values of $n$. Using ideas from recent work [9, 13, 17] on the construction of Behrend [4] and Rankin [21], they also prove the following asymptotic bound on $c_{n, k}$. Let $r_{k}(n)$ be the maximal size of a subset of $[n]$ without an arithmetic progression of length $k$, then:
Theorem 3 ([19]) For each $k \geq 3$, there is an absolute constant $C>0$ such that

$$
c_{n, k} \geq C k^{n}\left(\frac{r_{k}(\sqrt{n})}{\sqrt{n}}\right)^{k-1}=k^{n} \exp \left(-O(\log n)^{1 /\left\lceil\log _{2} k\right\rceil}\right)
$$

We show analogues of the Hales-Jewett theorem, the density Hales-Jewett theorem, the above lower bound and other related quantities, in the communication complexity framework. The model used is the Number On the Forehead (NOF) model [6]. In this model $k$ players compute together a boolean function $f: X_{1} \times \cdots \times X_{k} \rightarrow\{0,1\}$. The input, $\left(x_{1}, x_{2}, \ldots, x_{k}\right) \in X_{1} \times \cdots \times X_{k}$, is presented to the players in such a way that the $i$-th player sees the entire input except $x_{i}$. A protocol is comprised of rounds, in each of which every player writes one bit ( 0 or 1 ) on a board that is visible to all players. The choice of the written bit may depend on the player's input and on all bits previously written by himself and others on the board. The protocol ends when all players know $f\left(x_{1}, x_{2}, \ldots, x_{k}\right)$. The cost of a protocol is the number of bits written on the board, for the worst input. The deterministic communication complexity of $f, D(f)$, is the cost of the best protocol for $f$.

Two key definitions in the number on the forehead model are a cylinder and a cylinder intersection. We say that $C \subseteq X_{1} \times \cdots \times X_{k}$ is a cylinder in the $i$-th coordinate if membership in $C$ does not depend on the $i$-th coordinate. Namely, for every $y, y^{\prime}$ and $x_{1}, x_{2}, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{k}$ there holds $\left(x_{1}, x_{2}, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{k}\right) \in C$ iff $\left(x_{1}, x_{2}, \ldots, x_{i-1}, y^{\prime}, x_{i+1}, \ldots, x_{k}\right) \in C$. A cylinder intersection is a set $C$ of the form $C=\cap_{i=1}^{k} C_{i}$ where $C_{i}$ is a cylinder in the $i$-th coordinate.

Every $c$-bit communication protocol for a function $f$ partitions the input space into at most $2^{c}$ cylinder intersections that are monochromatic with respect to $f$ (see 15 for more details). Thus, one way to relax $D(f)$ is to view it as a coloring problem. Denote by $\alpha(f)$ the largest size of a 1-monochromatic cylinder intersection with respect to $f$, and by $\chi(f)$ the least number of monochromatic cylinder intersections that form a partition of $f^{-1}(1)$. Obviously, $D(f) \geq \log \chi(f)$, and as we shall see, for special families of functions this bound is nearly tight, including the function Part $_{n, k}$ that we are interested in. Also observe that $\chi(f) \geq\left|f^{-1}(1)\right| / \alpha(f)$.

The function $\operatorname{Part}_{n, k}:\left(2^{[n]}\right)^{k} \rightarrow\{0,1\}$ is defined as follows, $\operatorname{Part}_{n, k}\left(S_{1}, \ldots, S_{k}\right)=1$ if and only if $\left(S_{1}, \ldots, S_{k}\right)$ is a partition of $[n]$. In [7] the Hales-Jewett theorem was used to prove that $D\left(\right.$ Part $\left._{n, k}\right) \geq \omega(1)$. We observe that in fact the Hales-Jewett theorem is equivalent to this statement. This follows from the following strong relation Part $_{n, k}$ has with the Hales-Jewett number.

Theorem 4 For every $k \geq 3$ and $n \geq 1$ there holds:

1. $c_{n, k}=\alpha\left(\right.$ Part $\left._{n, k}\right)$, and
2. $\chi\left(\right.$ Part $\left._{n, k}\right)$ is equal to the minimal number of colors required to color $[k]^{n}$ so that there is no monochromatic combinatorial line.

Theorem 4 entails an alternative characterization of the Hales-Jewett theorem and its density version:

Theorem 5 (Hales-Jewett theorem) For every fixed $k \geq 3$, one has $D\left(\right.$ Part $\left._{n, k}\right)=$ $\omega(1)$.

Theorem 6 (Density Hales-Jewett theorem) For every $k \geq 3$ there holds

$$
\lim _{n \rightarrow \infty} \alpha\left(\text { Part }_{n, k}\right) / k^{n}=0
$$

Given the central role the Hales-Jewett theorem plays in Ramsey Theory, and the intricacy of its proof, it would be very nice to find a proof of the Hales-Jewett theorem in the framework of communication complexity.

The relation between $c_{n, k}$ and communication complexity also suggests a way to prove a lower bound on $c_{n, k}$ : prove an efficient communication protocol for Part $_{n, k}$. We show indeed that $D\left(\operatorname{Part}_{n, k}\right) \leq O(\log n)^{1 /\left[\log _{2} k\right\rceil}$, and thus the lower bound follows. We prove the relationship between $c_{n, k}$ and $\alpha\left(\right.$ Part $\left._{n, k}\right)$ in Section2, and the lower bound on $c_{n, k}$ is proved in Section 3. Lastly, Section 4 contains a discussion on Fujimura sets, mentioned in [19], and their communication complexity analogues.

## 2 A communication complexity version of Hales-Jewett

We start with the definition of a star: A star is a subset of $X_{1} \times \cdots \times X_{k}$ of the form

$$
\left\{\left(x_{1}^{\prime}, x_{2}, \ldots, x_{k}\right),\left(x_{1}, x_{2}^{\prime}, \ldots, x_{k}\right), \ldots,\left(x_{1}, x_{2}, \ldots, x_{k}^{\prime}\right)\right\}
$$

where $x_{i} \neq x_{i}^{\prime}$ for each $i$. We refer to $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$ as the star's center. Cylinder intersections can be easily characterized in terms of stars.

Lemma 7 ([15]) A subset $C \subseteq X_{1} \times \cdots \times X_{k}$ is a cylinder intersection if and only if for every star that is contained in $C$, its center also belongs to $C$.

The function Part $_{n, k}$ has the property that for every $S_{1}, \ldots, S_{k-1} \in 2^{[n]}$ there is at most one set $S \subset[n]$ such that $\operatorname{Part}_{n, k}\left(S_{1}, \ldots, S_{k-1}, S\right)=1$. We call such a function a weak graph function, as opposed to a graph function [3] where there is always exactly one such $S$.

Graph functions have some particularly convenient properties, one of which is that 1-monochromatic cylinder intersections are characterized simply by the existence of stars, as proved in [16. The same proof also works for weak graph functions and gives:

Lemma 8 ([16]) Let $f: X_{1} \times \cdots \times X_{k} \rightarrow\{0,1\}$ be a weak graph function and $C \subseteq$ $f^{-1}(1)$. The set $C$ is a (1-monochromatic) cylinder intersection with respect to $f$ if and only if it does not contain a star.

Proof If $C$ does not contain a star, then $C$ is a cylinder intersection by Lemma 7 On the other hand, if $C$ contains a star whose center is $\left(x_{1}, x_{2}, \ldots, x_{k}\right)$, then by definition of a weak graph function $f\left(x_{1}, x_{2}, \ldots, x_{k}\right)=0$. Thus, $C$ does not contain the center of star, and therefore $C$ is not a cylinder intersection (again using Lemma 7).

Proof [of Theorem4] As in [7], define a bijection $\psi$ from $\operatorname{Part}_{n, k}^{-1}(1)$ to [k] ${ }^{n}$. A $k$-tuple $\left(S_{1}, \ldots, S_{k}\right)$ is mapped to $\left(j_{1}, \ldots, j_{n}\right) \in[k]^{n}$ where $j_{i}$ is the index of the set $S_{j_{i}}$ that contains $i$. Since $S_{1}, \ldots, S_{k}$ form a partition of $[n]$ this map is a bijection.

Now consider a 1-monochromatic star $\left(S_{1}^{\prime}, \ldots, S_{k}\right), \ldots,\left(S_{1}, \ldots, S_{k}^{\prime}\right)$ with respect to Part $_{n, k}$. Since this star is 1-monochromatic, it implies that in each of the families $\left(S_{1}^{\prime}, \ldots, S_{k}\right), \ldots,\left(S_{1}, \ldots, S_{k}^{\prime}\right)$, all subsets are pairwise disjoint. As a result, because $k \geq 3$, we get that the subsets $\left(S_{1}, \ldots, S_{k}\right)$ are also pairwise disjoint. This determines $S_{j}^{\prime}$ uniquely: $S_{j}^{\prime}=S_{j} \cup\left([n] \backslash\left(\cup_{i=1}^{k} S_{i}\right)\right)$, for every $j=1, \ldots, k$. Therefore, if we consider $\psi\left(S_{1}^{\prime}, \ldots, S_{k}\right), \ldots, \psi\left(S_{1}, \ldots, S_{k}^{\prime}\right)$ and place them in a $k \times n$ table, then the columns of this table all belong to $\{(x, x, \ldots, x): x \in[k]\} \cup\{(1,2, \ldots, k)\}$. The $i$-th column of this table is in $\{(x, x, \ldots, x): x \in[k]\}$ if $i \in S_{j}$ for some $j \in[k]$ and otherwise the $i$-th column is equal to $(1,2, \ldots, k)$. Thus the stars in $\operatorname{Part}_{n, k}^{-1}(1)$ are mapped to combinatorial lines in $[k]^{n}$.

On the other hand, consider a combinatorial line in $[k]^{n}$ given by a $k \times n$ matrix $L$. Let $L_{1}, \ldots, L_{k}$ be the rows of $L$, then it is not hard to check that similarly to the above, $\psi^{-1}\left(L_{1}\right), \ldots, \psi^{-1}\left(L_{k}\right)$ form a 1-monochromatic star with respect to $\operatorname{Part}_{n, k}$. The center of this star is $\left(S_{1}, \ldots, S_{k}\right)$ where $S_{j}$ contains all indices of columns that are equal to $(j, j, \ldots, j)$.

Hence the stars in $\operatorname{Part}_{n, k}^{-1}(1)$ are in one-to-one correspondence with combinatorial lines in $[k]^{n}$. It follows that $c_{n, k}=\alpha\left(\right.$ Part $\left._{n, k}\right)$, and that $\chi\left(\operatorname{Part}_{n, k}\right)$ is equal to the minimal number of colors required to color $[k]^{n}$ so that there is no monochromatic combinatorial line.

It is left to show the equivalence between Theorem 5 and the Hales-Jewett theorem, and Theorem 6 with its density version. The latter equivalence follows immediately from part 1 of Theorem 4. The equivalence of Theorem 5 to the Hales-Jewett theorem follows from part 2 of Theorem 4, and the following theorem:

Theorem 9 ([16]) For every weak graph function $f: X_{1} \times \cdots \times X_{k} \rightarrow\{0,1\}$, there holds

$$
\log \chi(f) \leq D(f) \leq\lceil\log \chi(f)\rceil+k
$$

Theorem 9 was proved in [16] for graph functions. The same proof with a minor change works for weak graph functions. For completeness we add the proof.
Proof The lower bound is standard and holds for every function, thus it is only required to prove the upper bound. Fix a $\chi(f)$-coloring of $f^{-1}(1)$ where every color class is starfree. On input $x_{1}, x_{2}, \ldots, x_{k-1}, y$, the last player first checks and announces whether there is a value $y^{\prime}$ such that $f\left(x_{1}, x_{2}, \ldots, x_{k-1}, y^{\prime}\right)=1$, using 1 bit. If $y^{\prime}$ exists, the last player then computes and publishes the color $b$ of $\left(x_{1}, x_{2}, \ldots, x_{k-1}, y^{\prime}\right)$. If $y^{\prime}$ does not exist, the protocol ends with value 0 . Note that since $f$ is a weak graph function, if $y^{\prime}$ exists then it is unique.

Then, for each $i=1, \ldots, k-1$, player $P_{i}$ checks whether there is a value $x_{i}^{\prime}$ such that $f\left(x_{1}, x_{2}, \ldots, x_{i-1}, x_{i}^{\prime}, x_{i+1}, . ., y\right)=1$ and $\left(x_{1}, x_{2}, \ldots, x_{i-1}, x_{i}^{\prime}, x_{i+1}, . ., y\right)$ is colored $b$. He writes 1 on the board if such an $x_{i}^{\prime}$ exists and writes 0 otherwise. The protocol's value is 1 if and only if all players wrote 1 on the board.

The total number of bits communicated in this protocol is $\lceil\log \chi(f)\rceil+k$. We turn to prove that the protocol is correct. When $f\left(x_{1}, x_{2}, \ldots, x_{k-1}, y\right)=1$, the protocol clearly outputs 1. Now suppose that it outputs 1 , even though $f\left(x_{1}, x_{2}, \ldots, x_{k-1}, y\right)=0$. This means that there is a choice of $x_{1}^{\prime}, x_{2}^{\prime}, \ldots, y^{\prime}$ for which

$$
f\left(x_{1}^{\prime}, x_{2}, \ldots, x_{k-1}, y\right)=f\left(x_{1}, x_{2}^{\prime}, \ldots, x_{k-1}, y\right)=\ldots=f\left(x_{1}, x_{2}, \ldots, x_{k-1}, y^{\prime}\right)=1
$$

and all points are in the same color set. But then this color set in $f^{-1}(1)$ cannot constitute a star-free set, a contradiction.

## 3 A lower bound on $c_{n, k}$

In this section we prove the following lower bound, similar to that of [19]:
Theorem 10 For each $k \geq 3$, there is an absolute constant $C>0$ such that

$$
c_{n, k} \geq C k^{n} \frac{r_{k}(k n)}{k n \log k n}=k^{n} \exp \left(-O(\log n)^{1 /\left\lceil\log _{2} k\right\rceil}\right) .
$$

We first give an efficient protocol for Part $_{n, k}$, and then explain how it implies Theorem 10
Lemma 11 For every fixed $k \geq 3$ it holds that

$$
D\left(\text { Part }_{n, k}\right) \leq O\left(\log \frac{k n \log k n}{r_{k}(k n)}\right)=O(\log n)^{1 /\left\lceil\log _{2} k\right\rceil}
$$

Proof The protocol uses a known reduction to the Exactly-n function, see e.g. [5, 7]. Define Exactly En,k $\left.^{( } x_{1}, \ldots, x_{k}\right)=1$ if and only if $\sum_{i=1}^{k} x_{i}=n$, where $\left(x_{1}, \ldots, x_{k}\right)$ are non-negative integers. The reduction is simple, given an instance $\left(S_{1}, \ldots, S_{k}\right)$ to be computed, the players do the following:

1. The $k$-th player checks whether $S_{1}, \ldots, S_{k-1}$ are pairwise disjoint. If they are not pairwise disjoint then the protocol ends with rejection.
2. The first player checks whether $S_{2}, \ldots, S_{k-1}$ are each disjoint from $S_{k}$. If this is not the case then the protocol ends with rejection.
3. The second player checks whether $S_{1} \cap S_{k}=\emptyset$ and rejects if not.
4. The players use a protocol for Exactly E $_{n, k}$ to determine whether $\sum_{i=1}^{k}\left|S_{i}\right|=n$. The protocol accepts if and only if equality holds, and the sum is exactly $n$.

The first three steps of the above protocol require three bits of communication, and the last part uses a protocol for Exactly E $_{n, k}$. Chandra, Furst and Lipton [6] gave a surprising protocol for Exactly $y_{n, k}$ with at most $O\left(\log \frac{k n \log k n}{r_{k}(k n)}\right)$ bits of communication. It was later observed by Beigel, Gasarch and Glenn [5] that when plugging in the bounds on $r_{k}(n)$ given by the construction of Rankin [21] one gets $O\left(\log \frac{k n \log k n}{r_{k}(k n)}\right)=$ $O(\log n)^{1 /\left\lceil\log _{2} k\right\rceil}$.

Proof [of Theorem 10] As mentioned before $\log \chi(f) \leq D(f)$ holds for every function f , combined with Lemma 11 this gives

$$
\chi\left(\text { Part }_{n, k}\right) \leq \exp \left(D\left(\text { Part }_{n, k}\right)\right) \leq O\left(\frac{k n \log k n}{r_{k}(k n)}\right)=\exp \left(O(\log n)^{1 /\left\lceil\log _{2} k\right\rceil}\right)
$$

Since $\alpha(f) \geq\left|f^{-1}(1)\right| / \chi(f)$ holds also for every $f$ and $\mid$ Part $_{n, k}^{-1}(1) \mid=k^{n}$, we get

$$
\alpha\left(\text { Part }_{n, k}\right) \geq \frac{k^{n}}{\chi\left(\text { Part }_{n, k}\right)} \geq \Omega\left(k^{n} \frac{r_{k}(k n)}{k n \log k n}\right)=k^{n} \exp \left(-O(\log n)^{1 /\left\lceil\log _{2} k\right\rceil}\right) .
$$

The lower bound on $c_{n, k}$ now follows from part 1 of Theorem 4

## 4 Fujimura sets

The following definitions are from [19]. Let $\Delta_{n, k}$ denote the set of $k$-tuples $\left(a_{1}, \ldots, a_{k}\right) \in$ $\mathbb{N}^{k}$ such that $\sum_{i=1}^{k} a_{i}=n$. Define a simplex to be a set of $k$ points in $\Delta_{n, k}$ of the form $\left(a_{1}+r, a_{2}, \ldots, a_{k}\right),\left(a_{1}, a_{2}+r, \ldots, a_{k}\right), \ldots,\left(a_{1}, a_{2}, \ldots, a_{k}+r\right)$ for some $0<r \leq n$. Define a Fujimura set to be a subset $B \subset \Delta_{n, k}$ that contains no simplices.

Theorem 10 actually proves a lower bound on the maximal size of a Fujimura set in $\Delta_{n, k}$, similarly to the proof in [19]. In fact

- $\Delta_{n, k}=\left(\text { Exactly }_{n, k}\right)^{-1}(1)$.
- A simplex in $\Delta_{n, k}$ is equivalent to a star.
- $\alpha\left(\right.$ Exactly $\left._{n, k}\right)$ is equal to the maximal size of a Fujimura set in $\Delta_{n, k}$, which is denoted by $c_{n, k}^{\mu}$ in [19].
The proof of Theorem 10 gives essentially a lower bound for $\alpha\left(\right.$ Exactly $\left._{n, k}\right)$ via an efficient protocol for Exactly- $n$, and the lower bound for $c_{n, k}$ is implied from the fact that $D\left(\right.$ Part $\left._{n, k}\right) \leq D\left(\right.$ Exactly $\left._{n, k}\right)+3$. It is an interesting question whether this bound is tight, or is it the case that $D\left(\right.$ Part $\left._{n, k}\right)$ can be significantly smaller than $D$ (Exactly ${ }_{n, k}$ ). This is equivalent to asking whether lower bounds on the density Hales-Jewett number via bounds on the maximal size of a Fujimura set can be tight, or close to tight. In this respect it is interesting to note the following characterization of the communication complexity of Exactly $y_{n, k}$ in the language of Part $_{n, k}$.

Let $m \geq n$ be natural numbers, define the function Part $_{m, k, n}:\left(2^{[m]}\right)^{k} \rightarrow\{0,1\}$ as follows, $\operatorname{Part}_{m, k, n}\left(S_{1}, \ldots, S_{k}\right)=1$ if and only if $\left(S_{1}, \ldots, S_{k}\right)$ are pairwise disjoint and $\left|S_{1} \cup S_{2} \cup \ldots \cup S_{k}\right|=n$. Clearly, $\operatorname{Part}_{n, k}=\operatorname{Part}_{n, k, n}$ and as we observe in the next theorem $D\left(\right.$ Exactly $\left._{n, k}\right)$ is also equivalent to the complexity of some function in this family.

We call a map $g:[n]^{k} \rightarrow\left(2^{[m]}\right)^{k}$ sum preserving if the following two properties hold for every $\left(a_{1}, \ldots, a_{k}\right) \in([n])^{k}:$ (i) $\left|g\left(a_{1}, \ldots, a_{k}\right)_{i}\right|=a_{i}$, (ii) $g\left(a_{1}, \ldots, a_{k}\right)$ are pairwise disjoint whenever $\sum_{i=1}^{k} a_{i}=n$.
Theorem 12 Let $n, k$ and $m$ be natural numbers, if there exists a sum preserving map $g:[n]^{k} \rightarrow\left(2^{[m]}\right)^{k}$. Then

$$
D\left(\text { Exactly }_{n, k}\right) \leq D\left(\text { Part }_{m, k, n}\right) \leq D\left(\text { Exactly }_{n, k}\right)+3
$$

Proof Similarly to the case $m=n$, given an instance $\left(S_{1}, \ldots, S_{k}\right)$ to be computed, the protocol is:

1. The players check whether $S_{1}, \ldots, S_{k}$ are pairwise disjoint, using three bits of communication. If they are not pairwise disjoint then the protocol ends with a rejection.
 The protocol accepts if and only if equality holds, and the sum is equal to $n$.
It follows that $D\left(\right.$ Part $\left._{m, k, n}\right) \leq D\left(\right.$ Exactly $\left._{n, k}\right)+3$. Note that Part $_{m_{1}, k, n} \leq$ Part $_{m_{2}, k, n}$ whenever $m_{1} \leq m_{2}$. Therefore the minimal value of $D\left(\right.$ Part $\left._{m, k, n}\right)$ is achieved when $m=n$, i.e. for Part $_{n, k}$.

On the other direction, to get a protocol for $\operatorname{Exactly}_{n, k}$ the players decide before hand on a sum preserving map $g:[n]^{k} \rightarrow\left(2^{[m]}\right)^{k}$. Then, given an instance $\left(a_{1}, \ldots, a_{k}\right)$ to Exactly $y_{n, k}$ the players solve the instance $g\left(a_{1}, \ldots, a_{k}\right)$ using an optimal protocol for Part $_{m, k, n}$. Since $g$ is sum preserving this reduction always gives the correct answer. We conclude that $D\left(\right.$ Exactl $\left._{n, k}\right) \leq D\left(\right.$ Part $\left._{m, k, n}\right)$.

Therefore, the question of separating $D\left(\right.$ Part $\left._{n, k}\right)$ from $D\left(\right.$ Exactly $\left._{n, k}\right)$ is actually a question of separating $D\left(\right.$ Part $\left._{m, k, n}\right)$ for different values of $m$. Notice that for $m=k n$ there already exists a sum preserving map $g:[n]^{k} \rightarrow\left(2^{[m]}\right)^{k}$, simply take $g\left(a_{1}, \ldots, a_{k}\right)=$ $\left(\left\{1, \ldots, a_{1}\right\},\left\{n+1, \ldots, n+a_{2}\right\}, \ldots,\left\{(k-1) n+1, \ldots,(k-1) n+a_{k}\right\}\right)$.

Thus the ranges of $m$ we are interested in are $m \leq k n$. This can be improved to $m \leq\left\lceil\frac{k n}{2}\right\rceil$, by pairing adjacent entries and considering the map $g\left(a_{1}, \ldots, a_{k}\right)=$ $\left(\left\{1, \ldots, a_{1}\right\},\left\{n-1, \ldots, n-a_{2}\right\},\left\{n+1, \ldots, n+a_{3}\right\},\left\{2 n-1, \ldots, 2 n-a_{4}\right\}, \ldots\right)$. The sets in this case might intersect, but if they do it implies that the sum $\sum a_{i}$ is greater than $n$ and the value 0 is correct. For $k=3$ this gives $m \leq 2 n$ and the question is to separate $D\left(\right.$ Part $\left._{n, k, n}\right)$ from $D\left(\right.$ Part $\left._{2 n, k, n}\right)$.

## 5 Discussion and open problems

The relation between the NOF model of communication complexity and Ramsey theory and related areas of mathematics was evident already in the initial paper of Chandra, Furst and Lipton [6]. Since then, the breadth and profoundness of this relation is better understood, see e.g. [20, 5, 7, 1, 16]. This note offers another strong bridge, showing that the Hales-Jewett theorem, a pillar of Ramsey theory, and related questions, are naturally formulated in this model of communication complexity.

We already know that the NOF model is rich enough to formulate many interesting questions in the theory of computer science, e.g. proving lower bounds on the size of $A C C^{0}$ circuits [14]. The new relations that we find give yet another proof to the richness and significance of this model, not only in computer science. For these relations to bear fruit though, it is not enough to describe the problems in communication complexity language, we need to also develop the tools to handle them in this setting. The main purpose of this note was to further motivate this study. Some interesting open questions in the context of the problems described here, are:

1. Find a protocol for Part $_{n, k}$ that does not rely on the construction of Behrend (4) and Rankin [21].
2. Find more efficient protocols for Part $_{n, k}$, and thus improve the lower bound on the density Hales-Jewett number. For $k>3$ we believe that the protocol described here is not optimal.
3. Prove a lower bound for the communication complexity of Part $_{n, k}$ using communication complexity tools, e.g. via a reduction. Currently the tools of communication complexity do not seem to even give $D_{k}\left(\right.$ Part $\left._{n, k}\right) \rightarrow \infty$.
4. Determine the relation between $D_{k}\left(\right.$ Part $\left._{n, k}\right)$ and $D_{k}\left(\right.$ Exactly $\left._{n, k}\right)$. From the point of view of communication complexity it makes sense to believe that these two are closely related, since determining whether pairwise disjoint sets $S_{1}, \ldots, S_{k}$ form a partition of $[n]$ essentially amounts to verifying that $\left|S_{1}\right|+\left|S_{2}\right|+\cdots+\left|S_{k}\right|=n$. It is therefore reasonable to make the following conjecture (see Section 4 for further discussion):

Conjecture $13 D_{k}\left(\right.$ Part $\left._{n, k}\right)=\Theta\left(D_{k}\left(\right.\right.$ Exactly $\left.\left._{n, k}\right)\right)$.
If the above conjecture is true, it in particular gives a strong proof of the HalesJewett theorem as well as deep insight into the relation between the Hales-Jewett theorem and multidimensional Szemerédi theorems.

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