# On shuffle products, acyclic automata and piecewise-testable languages 

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#### Abstract

We show that the shuffle $L \amalg F$ of a piecewise-testable language $L$ and a finite language $F$ is piecewise-testable. The proof relies on a classic but little-used automata-theoretic characterization of piecewise-testable languages. We also discuss some mild generalizations of the main result, and provide bounds on the piecewise complexity of $L Ш F$.


## 1 Introduction

Piecewise-testable languages, introduced in [Sim72, Sim75], are an important variety of simple dot-depth one, hence star-free, regular languages. As such they are closed under boolean operations, left and right derivatives, and inverse morphisms.

We prove in this paper that the shuffle product $L \amalg F$ of $L$ with some finite language $F$ is piecewise-testable when $L$ is.

Some motivations. The question was raised by our investigations of $\mathrm{FO}\left(A^{*}, \preccurlyeq\right)$, the first-order "logic of subwords", and its decidable two-variable fragment [KS16, HSZ17]. Let us use $u \preccurlyeq v$ to denote that $u$ is a (scattered) subword, or a subsequence, of $v$. For example, simon $\preccurlyeq$ stimulation while ordering $\nprec$ wordprocessing. Given a formula $\psi(x)$ with one free variable, e.g.,

$$
\begin{equation*}
\mathrm{ab} \preccurlyeq x \wedge \mathrm{bc} \preccurlyeq x \wedge \mathrm{ac} \npreceq x, \tag{x}
\end{equation*}
$$

we write $\operatorname{Sol}(\psi)$ for its set of solutions. In this example, $\operatorname{Sol}(\psi)$ is the set of all words that have $\mathrm{ab}, \mathrm{bc}$, but not ac , among their subwords. If we assume that the alphabet under consideration is $A=\{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, then $\operatorname{Sol}(\psi)$ is the language described via $\mathrm{c}^{*} \mathrm{~b}^{+} \mathrm{c}(\mathrm{b}+\mathrm{c})^{*} \mathrm{a}^{+} \mathrm{b}(\mathrm{a}+\mathrm{b})^{*}$, a simple regular expression. It is shown in [KS16, HSZ17] how to compute such solutions automatically. Let us extend the framework with the predicate $\preccurlyeq 1$, defined via

$$
u \preccurlyeq_{1} v \Longleftrightarrow u \preccurlyeq v \wedge|u|=|v|-1
$$

where $|u|$ is the length of $u$, so that $\preccurlyeq$ is the reflexive transitive closure of $\preccurlyeq 1$. Now an $\operatorname{FO}^{2}\left(A^{*}, \preccurlyeq, \preccurlyeq_{1}\right)$ formula of the form

$$
\begin{equation*}
\exists y: y \preccurlyeq{ }_{1} x \wedge \psi(y) \tag{x}
\end{equation*}
$$

has $\operatorname{Sol}(\phi)=\operatorname{Sol}(\psi) \amalg A$ as set of solutions. This is because $L \amalg A$ is the union of all $u \amalg a$ for $u \in L$ and $a \in A$, and $u \amalg a$ is the set of all words that can be obtained by inserting the letter $a \in A$ somewhere in $u$. Such equalities provide an effective quantifier-elimination procedure for (a fragment of) the logic. Extending the complexity analysis from [KS16] requires proving that $\operatorname{Sol}(\phi)$ is piecewise-testable when $\operatorname{Sol}(\psi)$ is. This will be a consequence of the main result in this paper.

Through the mirror automaton. It took us some time to find a simple proof that $L \amalg A$ is piecewise-testable when $L$ is. In particular, starting from any of the well-known characterizations of piecewise-testable languages (see Definition 2.1 below) did not take us very far. Neither could we use the approach developed for star-free languages - see [CR12, Coro. 3.3]- since piecewise-testable languages are not closed under bounded shuffle. We eventually found a simple proof based on a classic but little-used characterization: a regular language $L$ is piecewisetestable if, and only if, $L$ and its mirror image $L \leftarrow$ are $\mathcal{R}$-trivial, that is, iff the minimal DFAs for $L$ and for $L \leftarrow$ are both acyclic. This characterization is not explicitly mentioned in the main references on piecewise-testable languages, be they classic (e.g., [SS83]) or recent (e.g., [MT17]). As far as we know, it was first given explicitly by Brzozowski [Brz76]. Beyond that, we only saw it in [STV02, KP12] (and derived works).

Outline of the paper. In section 2 we recall the necessary notions on automata, languages, piecewise-testability, etc., state our main result and discuss extensions. In Section 3 we prove the main technical result: the class of $\mathcal{R}$-trivial regular languages is closed under interpolation products with finite languages. The proof is by inspecting the (nondeterministic) shuffle automaton and checking that the standard determinization procedure yields an acyclic automaton. In Section 4 we provide bounds on the piecewise complexity of some shuffle languages. In the conclusion, we list some questions raised by this work.

## 2 Basics

Finite automata. We consider languages over a fixed finite alphabet $A=$ $\{a, b, \ldots\}$ and finite automata (NFAs) of the form $\mathcal{A}=(Q, A, \cdot, I, F)$ where "." denotes the transition function. For $p \in Q$ and $a \in A, p \cdot a$ is a subset of $Q$. The transition function is extended to sets of states $S \subseteq Q$ via $S \cdot a=\bigcup_{p \in S} p \cdot a$ and to words by $S \cdot \epsilon=S$ and $S \cdot(a u)=(S \cdot a) \cdot u$. We often write $p \xrightarrow{u} q$ rather than $q \in(p \cdot u)$. The language recognized by $\mathcal{A}$ is $L(\mathcal{A}) \stackrel{\text { def }}{=}\left\{u \in A^{*} \mid(I \cdot u) \cap F \neq \emptyset\right\}$.
$\mathcal{A}$ is deterministic (is a DFA) if $|I| \leq 1$ and $|p \cdot a| \leq 1$ for all $p$ and $a$. It is complete if $|I| \geq 1$ and $|p \cdot a| \geq 1$ for all $p$ and $a$.

The transition function induces a quasi-ordering on the states of $\mathcal{A}: p \leq_{\mathcal{A}} q$ if there is a word $u$ such that $p \xrightarrow{u} q$, i.e., when $q$ can be reached from $p$ in the directed graph underlying $\mathcal{A}$. The quasi-ordering is a partial ordering if $\mathcal{A}$ is acyclic, i.e., $p \xrightarrow{u} q \xrightarrow{v} p$ implies $p=q$; or in other words, when the only loops in $\mathcal{A}$ are self-loops. It is well known that the $\mathcal{R}$-trivial languages are exactly the languages accepted by (deterministic) acyclic automata [BF80]. Regarding self-loops, we say that $p$ is $a$-stable when $p \cdot a=\{p\}$, and that it is $B$-stable,
where $B \subseteq A$ is some subalphabet, if it is $a$-stable for each $a \in B$.
Subwords and piecewise-testable languages. We write $u \preccurlyeq v$ when $u$ is a (scattered) subword of $v$, i.e., can be obtained from $v$ by removing some of its letters (possibly none, possibly all). A word $u=a_{1} a_{2} \cdots a_{n}$ generates a principal filter in $\left(A^{*}, \preccurlyeq\right)$. This is the language $L_{u}=\{v \mid u \preccurlyeq v\}$, also denoted by the regular expression $A^{*} a_{1} A^{*} a_{2} \ldots A^{*} a_{n} A^{*}$. The example in the introduction has $\operatorname{Sol}(\psi)=L_{\mathrm{ab}} \cap L_{\mathrm{bc}} \cap\left(A^{*} \backslash L_{\mathrm{ac}}\right)$.

For $k \in \mathbb{N}$, we write $u \sim_{k} v$ when $u$ and $v$ have the same subwords of length at most $k$ [Sim72]. This equivalence is called Simon's congruence since $u \sim_{k} v$ implies $x u y \sim_{k} x v y$ for all $x, y \in A^{*}$. Furthermore, $\sim_{k}$ partitions $A^{*}$ in a finite number of equivalence classes.

Definition 2.1 (Piecewise-testable languages). A language $L \subseteq A^{*}$ is piecewisetestable if it satisfies one of the equivalent following properties: ${ }^{1}$

- L is a finite boolean combination of principal filters,
- L is a union $\left[u_{1}\right]_{k} \cup \cdots \cup\left[u_{\ell}\right]_{k}$ of $\sim_{k}$-classes for some $k \in \mathbb{N}$,
- L can be defined by a $\mathcal{B} \Sigma_{1}$-formula in the first-order logic over words [DGK08],
- the syntactic monoid of $L$ is finite and $\mathcal{J}$-trivial (Simon's theorem) [Sim72],
- the minimal automaton for $L$ is finite, acyclic, and satisfies the UMS property [Sim75, Ste85],
- the minimal automaton for $L$ is finite, acyclic, and locally confluent [KP13].

The piecewise-testable languages over some $A$ form a variety and we mentioned the associated closure properties in our introduction. Note that piecewisetestable languages are not closed under alphabetic morphisms, concatenations, or star-closures.

Shuffling languages. In this note we focus on the shuffle product of words and languages, and more generally on their parameterized infiltration product. When $C \subseteq A$ is a subalphabet and $u, v$ are two words, we let $u \uparrow_{C} v$ denote the language of all words that are obtained by shuffling $u$ and $v$ with possible sharing of letters from $C$. This is better defined via a notation for extracting subwords: for a word $u=a_{1} a_{2} \cdots a_{n}$ of length $n$ and a subset $K=\left\{i_{1}, \ldots, i_{r}\right\} \subseteq\{1, \ldots, n\}$ of positions in $u$ where $i_{1}<i_{2}<\cdots<i_{r}$, we write $u_{K}$ for the subword $a_{i_{1}} a_{i_{2}} \cdots a_{i_{r}}$ of $u$. Then we let

$$
x \in u \uparrow_{C} v \Longleftrightarrow\left\{\begin{array}{l}
\exists K, K^{\prime}: K \cup K^{\prime}=\{1,2, \ldots,|x|\}, \\
x_{K}=u, x_{K^{\prime}}=v, \text { and } x_{K \cap K^{\prime}} \in C^{*} .
\end{array}\right.
$$

The operation is lifted from words to languages in the standard way via $L \uparrow_{C}$ $L^{\prime}=\bigcup_{u \in L} \bigcup_{v \in L^{\prime}} u \uparrow_{C} v$. This generalizes shuffle products and the interpolation products $L \uparrow L^{\prime}$ from [PS83, SS83] since $L Ш L^{\prime}=L \uparrow_{\emptyset} L^{\prime}$ and $L \uparrow L^{\prime}=L \uparrow_{A} L^{\prime}$. Note that $L \uparrow_{C} L^{\prime} \subseteq L \uparrow_{C^{\prime}} L^{\prime}$ when $C \subseteq C^{\prime}$. Also note that $L \uparrow_{C} L^{\prime}=$ $L \amalg L^{\prime}$ when $L$ or $L^{\prime}$ is subword-closed. A shuffle ideal is any language of the

[^0]form $L Ш A^{*}$. It is well-known that shuffle ideals are finite unions of principal filters [Hai69, Héa02] hence they are piecewise-testable.

Theorem 2.2 (Main result). If $L$ is regular and $X$-trivial (where $X$ can be $\mathcal{R}$, $\mathcal{L}$, or $\mathcal{J}$ ) then $L \uparrow_{C} L^{\prime}$ is regular and $X$-trivial when $L^{\prime}$ is finite, or cofinite, or is a shuffle ideal.

Let us first note that, since $A$ is finite, Theorem 2.2 answers the question about $L \amalg A$ raised in our introduction. A proof of the Theorem is given in the next section after a few observations that we now make.

Let us mention a few directions in which our main result cannot be extended:

- The shuffle of two piecewise-testable languages is star-free [CR12, Theorem 4.4] but is not always piecewise-testable: for example $a^{*} \amalg a b^{*}$, being $a(a+b)^{*}$, is not piecewise-testable while $a^{*}$ and $a b^{*}$ are.
- The concatenation $L \cdot F$ of a piecewise-testable $L$ and a finite $F$ is not always piecewise-testable: $(a+b)^{*}$ is piecewise-testable but $(a+b)^{*} a$ is not. Note that $L \cdot F$ is included in $L \amalg F$ that we claim is piecewise-testable.
- The scattered residual $L \xrightarrow{-} u$ of a piecewise-testable $L$ by some word $u$ is not always piecewise-testable. For example $a c(a+b)^{*} \rightarrow c=a(a+b)^{*}$. (Recall that $w \longrightarrow u$ is the set of all words $v$ such that $w \in u Ш v$, obtained by removing the subword $v$ somewhere along $w[\operatorname{Kar} 94]$.)

Finally, there are some (admittedly degenerate) situations that are not covered by Theorem 2.2 and where the shuffle of two piecewise-testable languages is piecewise-testable.

Proposition 2.3. If $L_{1}, \ldots, L_{m} \subseteq A^{*}$ are piecewise-testable then $L_{1} Ш \cdots 山 L_{m}$ is piecewise-testable in any of the following cases:

- the $L_{i}$ 's are all complements of shuffle ideals, i.e., they are subword-closed;
- their subalphabets are pairwise disjoint.

The first claim is easy to see since the shuffle of subword-closed languages is subword-closed, and the second claim ${ }^{2}$ is a consequence of the following Lemma.

Lemma 2.4 (See also [ÉS98, Lemma 6]). Let $\mathfrak{F}$ be a family of languages over A that is closed under intersections and inverse morphisms. If $L_{1}, L_{2} \in \mathfrak{F}$ use disjoint subalphabets, then $L_{1} \amalg L_{2}$ is in $\mathfrak{F}$ too.

Proof. Write $e_{B}: A^{*} \rightarrow A^{*}$ for the erasing morphism that replaces all letters from some subalphabet $B$ with $\epsilon$ and leaves other letters unchanged. Assuming $L_{1} \subseteq A_{1}^{*}$ and $L_{2} \subseteq A_{2}^{*}$, with furthermore $A_{1} \cap A_{2}=\emptyset$, one has

$$
L_{1} \amalg L_{2}=\left(L_{1} \amalg A_{2}^{*}\right) \cap\left(L_{2} \amalg A_{1}^{*}\right)=e_{A_{2}}^{-1}\left(L_{1}\right) \cap e_{A_{1}}^{-1}\left(L_{2}\right)
$$

The last equality shows that $L_{1} \amalg L_{2}$ is in $\mathfrak{F}$.

[^1]
## 3 Shuffling acyclic automata

In this section we first prove Proposition 3.1 by inspecting the shuffling of automata.

Proposition 3.1. If $L \subseteq A^{*}$ is regular and $\mathcal{R}$-trivial then $L \uparrow_{C} w$ is too, for any $w \in A^{*}$ and $C \subseteq A$.

Let $\mathcal{A}=(Q, A, \cdot, i, F)$ be an acyclic complete deterministic automaton for $L$, and let $w=z_{1} \cdots z_{m} \in A^{*}$ be the word under consideration. When building the shuffle automaton for $L \uparrow_{C} w$, it is more convenient to consider the smallest automaton for $w$, deterministic but not complete. Formally, we let $\mathcal{B}=\left(Q^{\prime}, A, \circ, i^{\prime}, F^{\prime}\right)$ given by $Q^{\prime}=Q \times\{0,1, \ldots, m\}, i^{\prime}=(i, 0), F^{\prime}=F \times\{m\}$, and a transition table given by

$$
\begin{equation*}
(p, k) \circ a=\{(p \cdot a, k), \overbrace{(p, k+1)}^{\text {if } a=z_{k+1}}, \overbrace{(p \cdot a, k+1)}^{\text {if furthermore } a \in C}\} . \tag{1}
\end{equation*}
$$

This is a standard construction: $\mathcal{B}$ is nondeterministic in general, and it is easy to see that it accepts exactly $L \uparrow_{C} w$.

Observe that $\mathcal{B}$ too is acyclic: by Eq. (1), for any transition $(p, k) \xrightarrow{a}(q, \ell)$ one has $p \leq_{\mathcal{A}} q$ and $k \leq \ell$ and this extends to any path $(p, k) \xrightarrow{u}(q, \ell)$ by transitivity. Thus $\leq_{\mathcal{B}}$ is included in the Cartesian product of two partial orderings.

From $\mathcal{B}=\left(Q^{\prime}, A, \circ, i, F^{\prime}\right)$ we derive a powerset automaton $\mathcal{P}=(\boldsymbol{Q}, A, \bullet, \boldsymbol{i}, \boldsymbol{F})$ in the standard way, i.e., $\boldsymbol{Q}=2^{Q^{\prime}}=\left\{S \mid S \subseteq Q^{\prime}\right\}, \boldsymbol{i}=\left\{i^{\prime}\right\}, \boldsymbol{F}=\{S \in$ $\left.\boldsymbol{Q} \mid S \cap F^{\prime} \neq \emptyset\right\}$ and $S \bullet a=\{S \circ a\}$. It is well known that $\mathcal{P}$ is deterministic, complete, and accepts exactly the language accepted by $\mathcal{B}$, i.e., $L \uparrow_{C} w$.
Lemma 3.2. $\mathcal{P}$ is acyclic.
Proof. Let $S_{0} \xrightarrow{a_{1}} S_{1} \xrightarrow{a_{2}} S_{2} \ldots \xrightarrow{a_{n}} S_{n}=S_{0}$ be a non-empty cycle in $\mathcal{P}$ and write $S=\bigcup_{i=0}^{n} S_{i}$ and $B=\left\{a_{1}, \ldots, a_{n}\right\}$ for the set of states (resp., set of letters) appearing along the cycle.

We first claim that for any $(p, k) \in S_{n}, p$ is $B$-stable in $\mathcal{A}$, which mean that $p \cdot a_{i}=p$ for $i=1, \ldots, n$. We prove this by induction on $\leq_{\mathcal{B}}$ : so consider an arbitrary $(p, k) \in S_{n}$ and assume that $p^{\prime}$ is $B$-stable whenever there is some $\left(p^{\prime}, k^{\prime}\right) \in S_{n}$ with $\left(p^{\prime}, k^{\prime}\right)<_{\mathcal{B}}(p, k)$. Since $S_{0} \xrightarrow{a_{1}} S_{1} \cdots \xrightarrow{a_{n}} S_{n}$ and $(p, k) \in S_{n}, \mathcal{B}$ has a sequence of transitions

$$
\left(p_{0}, \ell_{0}\right) \xrightarrow{a_{1}}\left(p_{1}, \ell_{1}\right) \xrightarrow{a_{2}}\left(p_{2}, \ell_{2}\right) \cdots \xrightarrow{a_{n}}\left(p_{n}, \ell_{n}\right)=(p, k)
$$

with $\left(p_{i}, \ell_{i}\right) \in S_{i}$ for all $i=1, \ldots, n$. Thus $p_{0} \leq_{\mathcal{A}} p_{1} \cdots \leq_{\mathcal{A}} p_{n}=p$ and $\ell_{0} \leq \ell_{1} \cdots \leq \ell_{n}=k$. If $p_{0} \neq p$, then $p_{0}=p_{1}=\ldots=p_{i-1} \neq p_{i} \leq \mathcal{A}_{\mathcal{A}} p_{n}$ for some $i$. Given $\left(p_{i-1}, \ell_{i-1}\right) \xrightarrow{a_{i}}\left(p_{i}, \ell_{i}\right)$ and $p_{i-1} \neq p_{i}$, Eq. (1) requires that $p_{i-1} \cdot a_{i}=p_{i}$ in $\mathcal{A}$, hence $p_{i-1}$ is not $B$-stable, but this contradicts the induction hypothesis since $p_{i-1}=p_{0},\left(p_{0}, \ell_{0}\right)$ belongs to $S_{n}$, and $\left(p_{0}, \ell_{0}\right)<_{\mathcal{B}}(p, k)$. Thus $p_{0}=p_{1}=\cdots=p_{n}=p$. If $\ell_{0}<\ell_{n}$, the induction hypothesis applies and states that $p_{0}$ is $B$-stable. If $\ell_{0}=\ell_{1}=\cdots=\ell_{n}$, then Eq. (1) requires that $p_{i-1} \cdot a_{i}=p_{i}$ for all $i<n$, which proves the claim.

Since we can change the origin of the cycle, we conclude that $p$ is $B$-stable in $\mathcal{A}$ for any $(p, k)$ in $S$, not just in $S_{n}$. If $p$ is $B$-stable, then $(p, k) \circ a_{i} \ni(p, k)$ by Eq. (1). Thus $S_{i-1} \bullet a_{i} \supseteq S_{i-1}$ for all $i=1, \ldots, n$. This entails $S_{0} \subseteq S_{1} \subseteq$
$\cdots \subseteq S_{n}=S_{0}$ and then $S_{0}=S_{1}=\ldots=S_{n}$. We have proved that all cycles in $\mathcal{P}$ are self-loops, hence $\mathcal{P}$ is acyclic as claimed.

This entails that $L \uparrow_{C} w$, the language recognized by $\mathcal{P}$, is $\mathcal{R}$-trivial and concludes the proof of Proposition 3.1.


Figure 1: NFA for $a^{*} \amalg b^{*} a$ and associated powerset DFA.

Remark 3.3. Lemma 3.2 needs a proof because determinizing an acyclic NFA does not always yield an acyclic DFA. ${ }^{3}$ For example, the NFA obtained by shuffling DFAs for $a^{*}$ and for $b^{*} a$ is acyclic (see left of Fig. 1). However, its powerset automaton and the minimal DFA are not (see right of the figure). Indeed, $a^{*} \amalg b^{*} a=(a+b)^{*} a$ is not $\mathcal{R}$-trivial.

With Proposition 3.1 it is easy to prove our main result.
Proof of Theorem 2.2. We first assume that $L$ is $\mathcal{R}$-trivial and consider several cases for $L^{\prime}$ :

- If $L^{\prime}$ is finite, we use distributivity of shuffle over unions: $L \uparrow_{C} L^{\prime}$ is $\mathcal{R}$-trivial since it is a finite union $\bigcup_{w \in L} L \uparrow_{C} w$ of $\mathcal{R}$-trivial languages.
- If $L^{\prime}$ is a shuffle ideal, i.e., if $L^{\prime}=L^{\prime} ш A^{*}=L^{\prime} \uparrow_{C} A^{*}$, then $L \uparrow_{C} L^{\prime}$ is a shuffle ideal too in view of

$$
L \uparrow_{C} L^{\prime}=L \uparrow_{C}\left(L^{\prime} \uparrow_{C} A^{*}\right)=\left(L \uparrow_{C} L^{\prime}\right) \uparrow_{C} A^{*}
$$

Recall now that shuffle ideals are always $\mathcal{R}$-trivial.

- If $L^{\prime}$ is cofinite, it is the union of a finite language and a shuffle ideal, so this case reduces to the previous two cases by distributing shuffle over union.

Once the result is proved for $X=\mathcal{R}$, it extends to $X=\mathcal{L}$ by mirroring since $L$ is $\mathcal{L}$-trivial if, and only if, its mirror $L^{\leftarrow}$ is $\mathcal{R}$-trivial, and since $\left(L \uparrow_{C} L^{\prime}\right)^{\leftarrow}=$ $L^{\leftarrow} \uparrow_{C} L^{\prime \leftarrow}$.

Finally, it extends to $X=\mathcal{J}$ since a finite monoid is $\mathcal{J}$-trivial if, and only if, it is both $\mathcal{R}$ - and $\mathcal{L}$-trivial.

Remark 3.4. Masopust and Thomazo extended the UMS criterion to nondeterministic automata. They showed that $L$ is piecewise-testable if it is recognized by a complete acyclic NFA with the UMS property [MT17, Thm. 25]. The NFA that one obtains by shuffling minimal DFAs for $L$ and $w$ is indeed acyclic and complete. However it does not satisfy the UMS property in general (already with $\left.a^{*} Ш a\right)$ so this additional characterization of piecewise-testable language does not directly entail our main result.

[^2]
## 4 The question of piecewise complexity

We write $h_{A}(L)$ for the piecewise complexity of $L$, defined as the smallest $k$ such that $L$ is $k$-PT, i.e., can be written as a union $L=\left[u_{1}\right]_{k} \cup \cdots \cup\left[u_{r}\right]_{k}$ of $\sim_{k}$-classes over $A^{*}$. We let $h_{A}(L)=\infty$ when $L$ is not piecewise-testable. For notational convenience, we usually write $h(L)$ when the alphabet is understood ${ }^{4}$ and write $h(u)$ for $h(\{u\})$ when $L=\{u\}$ is a singleton.

It was argued in [KS16] that $h(L)$ is an important, robust and useful, descriptive complexity measure for PT languages. In this light, a natural question is to provide upper-bounds on $h\left(L Ш L^{\prime}\right)$ as a function of $h(L)$ and $h\left(L^{\prime}\right)$. Computing or bounding $h(L)$ has received little attention until [KS16], and the available toolset for these questions is still primitive. In this section we provide some preliminary answers for $L \amalg L^{\prime}$ and slightly enrich the available toolset.

Before looking at simpler situations, let us note that, in general, the piecewisecomplexity of $L Ш w$ can be much higher than $h(L)$ and $h(w)$.

Proposition 4.1 (Complexity blowup). One cannot bound $h(L ш w)$ with a polynomial of $h(L)+h(w)$, even if we require $h(L)=0$. (NB: this statement assumes unbounded alphabets.)

Proof. Pick some $\lambda \in \mathbb{N}$ and let $U_{n}$ be a word over a $n$-letter alphabet $A_{n}=$ $\left\{a_{1}, \ldots, a_{n}\right\}$, given by $U_{0}=\epsilon$ and $U_{i+1}=\left(U_{i} a_{i+1}\right)^{\lambda} U_{i}$. It is known that $h\left(U_{n}\right)=$ $n \lambda+1$ [KS16, Prop. 3.1]. On the other hand $h\left(A_{n}^{*} \amalg U_{n}\right)=h\left(L_{U_{n}}\right)=\left|U_{n}\right|=$ $(\lambda+1)^{n}-1$ since, for any word $u, h\left(L_{u}\right)=|u|[$ KS16, Prop. 4.1].

### 4.1 Simple shuffles

Proposition 4.2. Assume that $L_{1}$ and $L_{2}$ are two non-empty piecewise-testable languages on disjoint alphabets. Then $h\left(L_{1} \amalg L_{2}\right)=\max \left(h\left(L_{1}\right), h\left(L_{2}\right)\right)$.

Proof. Since $k$-PT languages form a variety [Thé81, Lemma 2.3], Lemma 2.4 applies and yields $h\left(L_{1} \amalg L_{2}\right) \leq \max \left(h\left(L_{1}\right), h\left(L_{2}\right)\right)$.

To see that $h\left(L_{1} \amalg L_{2}\right) \geq h\left(L_{1}\right)$, we write $k=h\left(L_{1} \amalg L_{2}\right)$ and show that $L_{1}$ and $L_{2}$ are closed under $\sim_{k}$ : Pick any word $u \in L_{1}$ and any $u^{\prime} \in A_{1}^{*}$ with $u \sim_{k} u^{\prime}$. Since $L_{2}$ is not empty, there is some $v \in L_{2}$ and we obtain $u v \in L_{1} \amalg L_{2}$, and also $u^{\prime} v \in L_{1} \amalg L_{2}$ since $u v \sim_{k} u^{\prime} v$. Necessarily $u^{\prime} \in L_{1}$ since $L_{1}$ and $L_{2}$ have disjoint alphabets. Hence $L_{1}$ is closed under $\sim_{k}$, i.e., $h\left(L_{1}\right) \leq k$. The same reasoning applies to $L_{2}$.

Proposition 4.3. Assume that $L_{u}$ and $L_{v}$ are two principal filters. Then $h\left(L_{u} Ш L_{v}\right) \leq h\left(L_{u}\right)+h\left(L_{v}\right)$.

Proof. Recall that $h\left(L_{u}\right)=|u|$ as noted above. We then observe that $L_{u} \amalg L_{v}=$ $\bigcup_{w \in u ш v} L_{w}$ and that $|w|=|u|+|v|$ for all $w \in u ш v$.

The upper bound in Proposition 4.3 can be reached, an easy example being $h\left(L_{a^{n}}\right.$ Ш $\left.L_{a^{m}}\right)=h\left(L_{a^{n+m}}\right)=n+m$. The inequality can also be strict, as exemplified by Proposition 4.2.

[^3]
### 4.2 Shuffling finitely many words

Finite languages are piecewise-testable and closed under shuffle products. Their piecewise complexity reduces to the case of individual words in view of the following (from [KS16]):

$$
\begin{equation*}
h(F)=\max _{u \in F} h(u) \quad \text { when } F \text { is finite. } \tag{2}
\end{equation*}
$$

Lemma 4.4. $h\left(u_{1} \amalg u_{2} Ш \cdots ш u_{m}\right) \leq 1+\max _{a \in A}\left(\left|u_{1}\right|_{a}+\cdots+\left|u_{m}\right|_{a}\right)$.
Proof. Assume $A=\left\{a_{1}, \ldots, a_{n}\right\}$ and define $\ell_{1}, \ell_{2}, \ldots, \ell_{n}$ via $\ell_{j}=\left|u_{1}\right|_{a_{j}}+\cdots+$ $\left|u_{m}\right|_{a_{j}}$. From

$$
u_{1} \amalg \cdots ш u_{m} \subseteq a_{1}^{\ell_{1}} \amalg \cdots ш a_{n}^{\ell_{n}},
$$

we deduce

$$
h\left(u_{1} Ш \cdots ш u_{m}\right) \leq h\left(a_{1}^{\ell_{1}} \amalg \cdots Ш a_{n}^{\ell_{n}}\right)
$$

by Eq. (2)

$$
=\max \left(h\left(a_{1}^{\ell_{1}}\right), \ldots, h\left(a_{n}^{\ell_{n}}\right)\right)
$$

by Prop. 4.2

$$
=\max \left(1+\ell_{1}, \ldots, 1+\ell_{n}\right) .
$$

We may now bound $h\left(u_{1} \amalg u_{2} \amalg \cdots\right)$ as a function of $h\left(u_{1}\right), h\left(u_{2}\right), \ldots$.
Theorem 4.5 (Upper bound for shuffles of words). Assume $|A|=n$.
(1) $h\left(u_{1} \amalg u_{2} \amalg \cdots \amalg u_{m}\right)$ is in $O\left(\left[\sum_{i=1}^{m} h\left(u_{i}\right)\right]^{n}\right)$.
(2) This upper bound is tight: for every $\lambda \in \mathbb{N}$, there exists words $u_{1}, \ldots, u_{m}$ with fixed $m=n$ and such that $h\left(u_{1} \amalg \cdots ш u_{m}\right)=(\lambda+1)^{n}$ and $h\left(u_{1}\right)+\cdots+h\left(u_{m}\right)=$ $n^{2} \lambda+n$.

Proof. (1) By Lemma 4.4,

$$
\begin{aligned}
& h\left(u_{1} \text { Ш } u_{2} \text { Ш } \cdots u_{m}\right)-1 \\
\leq & \max _{a \in A}\left(\left|u_{1}\right|_{a}+\cdots+\left|u_{m}\right|_{a}\right) \leq \sum_{i=1}^{m}\left|u_{i}\right| .
\end{aligned}
$$

On the other hand, [KS16, Prop. 3.8] showed that

$$
|u|<\left(\frac{h(u)}{|A|}+2\right)^{|A|} \text { for any word } u \in A^{*}
$$

Thus, for fixed $A,|u|$ is $O\left(h(u)^{|A|}\right)$ and $\sum_{i}\left|u_{i}\right|$ is $O\left(\left[\sum_{i} h\left(u_{i}\right)\right]^{|A|}\right)$, which establishes the upper bound claim.
(2) We consider $U_{n}$ as defined in the proof of Proposition 4.1 and, for $j=$ $1, \ldots, m$, let $u_{j}$ be $r^{j}\left(U_{n}\right)$ where $r: A^{*} \rightarrow A^{*}$ is the circular renaming that replaces each $a_{i}$ by $a_{i+1}$ (counting modulo $n$ ). Write $\ell$ for $\left|U_{n}\right|$, i.e., $\ell=(\lambda+$ $1)^{n}-1$. We saw that $h\left(u_{j}\right)=h\left(U_{n}\right)=n \lambda+1$ so, fixing $m=n, \sum_{i=1}^{m} h\left(u_{i}\right)=$ $n^{2} \lambda+n$ as claimed. Let $L=u_{1} \amalg u_{2} \amalg \cdots ш u_{n}$. There remains to prove that $h(L)=(\lambda+1)^{n}=\ell+1$.

We first observe that, for any letter $a_{j},\left|u_{1}\right|_{a_{j}}+\cdots+\left|u_{n}\right|_{a_{j}}=\ell$. Indeed, the circular renamings ensure that

$$
\left|r^{1}(u)\right|_{a_{j}}+\cdots+\left|r^{n}(u)\right|_{a_{j}}=|u|_{a_{j-1}}+\cdots+|u|_{a_{j-n}}=|u|
$$

for any word $u \in A^{*}$. We then obtain $h(L) \leq \ell+1$ by Lemma 4.4.
There remains to show $h(L)>\ell$. For this, we observe that, for any $i=$ $1, \ldots, \ell$, the $i$-th letters $u_{1}[i], \ldots, u_{n}[i]$ form a permutation of $\left\{a_{1}, \ldots, a_{n}\right\}$. Thus we can obtain $\left(a_{1} a_{2} \cdots a_{n}\right)^{\ell}$ by shuffling $u_{1}, \ldots, u_{n}$, i.e., $\left(a_{1} a_{2} \cdots a_{n}\right)^{\ell} \in L$. However $\left(a_{1} a_{2} \cdots a_{n}\right)^{\ell} a_{1}$ is not in $L$ (it is too long) and $\left(a_{1} a_{2} \cdots a_{n}\right)^{\ell} a_{1} \sim_{\ell}$ $\left(a_{1} a_{2} \cdots a_{n}\right)^{\ell}$ (both words contain all possible subwords of length $\leq \ell$ ). Thus $L$ is not closed under $\sim_{\ell}$, which concludes the proof.

### 4.3 A general upper bound?

As yet we do not have a good upper bound in the general case.
Recall that the depth of a complete DFA is the maximal length of an acyclic path from some initial to some reachable state. When $L$ is regular, we write $d p(L)$ for the depth of the canonical DFA for $L$. Since $h(L) \leq d p(L)$ holds for all PT languages [KP13], one could try to bound $d p(L Ш w)$ in terms of $d p(L)$ and $w$. This does not seem very promising: First, for $L$ fixed, $d p(L Ш w)$ cannot be bounded in $O(|w|)$. Furthermore, $d p(L)$ can be much larger than $h(L)$ : if $L$ is $k$-PT and $|A|=n$ then the depth of the minimal DFA for $L$ can be as large as $\binom{k+n}{k}-1$ [MT17, Thm. 31]. Finally, this approach would only provide very large upper bounds, far above what we observe in experiments.

## 5 Conclusion

We proved that $L \amalg w$ is piecewise-testable when $L$ is (and when $w$ is a word), relying on a little-used characterization of piecewise-testable languages. This is part of a more general research agenda: identify constructions that produce piecewise-testable languages and compute piecewise complexity modularly. In this direction, an interesting open problem is to identify sufficient conditions that guarantee that a Kleene star $L^{*}$, or a concatenation $L \cdot L^{\prime}$, is piecewisetestable. It is surprising that such questions seem easier for shuffle product than for concatenation.

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[^0]:    ${ }^{1}$ The last four characterizations refer to notions that we do not redefine in this article because we do not use them. See references for details.

[^1]:    ${ }^{2}$ Already given in the long version of [Mas16].

[^2]:    ${ }^{3}$ Indeed nondeterministic and deterministic acyclic automata have different expressive powers, see [STV02].

[^3]:    ${ }^{4}$ The only situation where $A$ is relevant happens for $h_{A}\left(A^{*}\right)=0<h_{A^{\prime}}\left(A^{*}\right)=1$ when $A \subsetneq A^{\prime}$.

