Algorithms for deletion problems on split graphs

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Abstract

In the Split to Block Vertex Deletion and Split to Threshold Vertex Deletion problems the input is a split graph G and an integer k, and the goal is to decide whether there is a set S of at most k vertices such that G - S is a block graph and G - S is a threshold graph, respectively. In this paper we give algorithms for these problems whose running times are $O^*(2.076^k)$ and $O^*(2.733^k)$, respectively.

Keywords graph algorithms, parameterized complexity.

1 Introduction

A graph G is called a *split graph* if its vertex set can be partitioned into two disjoint sets C and I such that C is a clique and I is an independent set. A graph G is a *block graph* if every biconnected component of G is a clique. A graph G is a *threshold graph* if there is a $t \in \mathbb{R}$ and a function $f: V(G) \to \mathbb{R}$ such that for every $u, v \in V(G)$, (u, v) is an edge in G if and only if $f(u) + f(v) \ge t$.

In the Split to Block Vertex Deletion (SBVD) problem the input is a split graph G and an integer k, and the goal is to decide whether there is a set S of at most k vertices such that G-S is a block graph. Similarly, in the Split to Threshold Vertex Deletion (STVD) problem the input is a split graph G and an integer k, and the goal is to decide whether there is a set S of at most k vertices such that G-S is a threshold graph. The SBVD and STVD problems were shown to be NP-hard by Cao et al. [1]. A split graph G is a block graph if and only if G does not contain an induced diamond, where a diamond is a graph with 4 vertices and 5 edges. Additionally, a split graph G is threshold graph if and only if G does not contain an induced path with 4 vertices. Therefore, SBVD and STVD are special cases of the 4-Hitting Set problem. Using the fastest known parameterized algorithm for 4-Hitting Set, due to Fomin et al. [4], the SBVD and STVD problems can be solved in $O^*(3.076^k)$ time. Choudhary et al. [2] gave faster algorithms for SBVD and STVD whose running times are $O^*(2.303^k)$ and $O^*(2.792^k)$, respectively. In this paper we give algorithms for SBVD and STVD whose running times are $O^*(2.076^k)$ and $O^*(2.733^k)$, respectively.

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2 Preliminaries

For a graph G and a vertex $v \in V(G)$, N(v) is the set of vertices that are adjacent to v. For a set S of vertices, G - S is the graph obtained from G by deleting the vertices of S (and incident edges). Let P_4 denote a graph that is a path on 4 vertices.

In the 3-Hitting Set problem the input is a family \mathcal{F} of subsets of size at most 3 of a set U and an integer k, and the goal is to decide whether there is a set $X \subseteq U$ of size at most k such that $X \cap A \neq \emptyset$ for every $A \in \mathcal{F}$.

For two families of sets \mathcal{A} and \mathcal{B} , $\mathcal{A} \circ \mathcal{B} = \{A \cup B : A \in \mathcal{A}, B \in \mathcal{B}\}.$

2.1 Branching algorithm

A branching algorithm (cf. [3]) for a parameterized problem is a recursive algorithm that uses rules. Given an instance (G,k) to the problem, the algorithm applies some rule. In each rule, the algorithm either computes the answer to the instance (G,k), or performs recursive calls on instances $(G_1,k-c_1),\ldots,(G_t,k-c_t)$, where $c_1,\ldots,c_t>0$. The algorithm returns 'yes' if and only if at least one recursive call returned 'yes'. The rule is called a reduction rule if t=1, and a branching rule if $t\geq 2$. To analyze the time complexity of the algorithm, define T(k) to be the maximum number of leaves in the recursion tree of the algorithm when the algorithm is run on an instance with parameter k. Each branching rule corresponds to a recurrence on T(k):

$$T(k) \le T(k - c_1) + \dots + T(k - c_t).$$

The largest real root of $P(x) = 1 - \sum_{i=1}^{t} x^{-c_i}$ is called the *branching number* of the rule. The vector (c_1, \ldots, c_t) is called the *branching vector* of the rule.

Let γ be the maximum branching number over all branching rules. Assuming that the application of a rule takes $O^*(1)$ time, the time complexity of the algorithm is $O^*(\gamma^k)$.

3 Algorithm for SBVD

Lemma 1. Let G be a split graph with a partition C, I of its vertices. G is a block graph if and only if (1) A vertex in I with degree at least 2 is adjacent to all vertices in C, and (2) There is at most one vertex in I with degree at least 2.

Proof. Suppose that G is a block graph. If a vertex $v \in I$ has degree at least 2, let $a_1, a_2 \in C$ be two neighbors of v. For every $b \in C \setminus \{a_1, a_2\}$, we have that v is adjacent o b, otherwise v, a_1, a_2, b induces a diamond, contradicting the assumption that G is a block graph.

Now, suppose conversely that there are $u, v \in I$ with degree at least 2. Let a_1, a_2 be two vertices in C. From the paragraph above, a_1, a_2 are neighbors of u and of v. Therefore, u, v, a_1, a_2 induces a diamond, contradicting the assumption that G is a block graph.

To prove the opposite direction, suppose that G satisfied (1) and (2). Suppose conversely that G is not a block graph. Then there is a set of vertices X that induces

a diamond. Since C is a clique and I is an independent set, $|X \cap I|$ is equal to either 1 or 2. If $|X \cap I| = 1$ then G does not satisfy (1), and if $|X \cap I| = 2$ then G does not satisfy (2), a contradiction. Therefore G is a block graph.

IF (G, k) is a yes instance, let S be a solution for (G, k). By Lemma 1, there is at most one vertex in $I \setminus S$ with degree at least 2 in G - S. Denote this vertex, if it exists, by v^* . For every $v \in I \setminus (\{v^*\} \cup S)$ we have that v has degree at most 1 in G - S.

The algorithm for SBVD goes over all possible choices for the vertex $v^* \in I$. Additionally, the algorithm also inspects the case in which no such vertex exist. For every choice of v^* , the algorithm deletes from G all the vertices of C that are not adjacent to v^* and decreases the value of k by the number of vertices deleted. When the algorithm inspects the case when v^* does not exists, the graph is not modified.

For every choice of v^* , the algorithm generates an instance (\mathcal{F}, k) of 3-Hitting Set as follows: For every vertex $v \in I \setminus \{v^*\}$ that has at least two neighbors, and for every two neighbors $a, b \in C$ of v, the algorithm adds the set $\{v, a, b\}$ to \mathcal{F} . The algorithm then uses the algorithm of Wahlström [5] to solve the instance (\mathcal{F}, k) in $O^*(2.076^k)$ time. If (\mathcal{F}, k) is a yes instance of 3-Hitting Set then the algorithm returns yes. If all the constructed 3-Hitting Set instances, for all choices of v^* , are no instances, the algorithm returns no.

4 Algorithm for STVD

Let I_0 be the set of all vertices in I that have minimum degree (namely, a vertex $u \in I$ is in I_0 if $|N(u)| \leq |N(v)|$ for every $v \in I$). We say that two vertices $u, v \in I$ are twins if N(u) = N(v). Let Twins(v) be a set containing v and all the twins of v. Recall that a split graph G is a threshold graph if and only if G does not contain an induced P_4 . Note that an induced P_4 in G must be of the form u, a, b, v where $u, v \in I$ and $a, b \in C$.

The algorithm for STVD is a branching algorithm. At each step, the algorithm applies the first applicable rule from the rules below. The reduction rules of the algorithm are as follows.

- (R1) If $k \leq 0$ and G is not a threshold graph, return 'no'.
- (R2) If G is an empty graph, return 'yes'.
- (R3) If v is a vertex such that there is no induced P_4 in G that contains v, delete v.

If Rule (R3) cannot be applied we have that for every $a \in C$ there is a vertex $v \in I$ such that $a \notin N(v)$.

We now describe the branching rules of the algorithm. When we say that the algorithm branches on sets S_1, \ldots, S_p , we mean that the algorithm is called recursively on the instances $(G - S_1, k - |S_1|), \ldots, (G - S_p, k - |S_p|)$.

(B1) If there are non-twin vertices $u, v \in I$ such that |N(u)| = |N(v)| = 1, branch on N(u) and N(v).

To show the safeness of Rule (B1), denote $N(u) = \{a\}$ and $N(v) = \{b\}$. If S is a solution for the instance (G, k) then S must contain at least one vertex from

the induced path u, a, b, v. If $u \in S$ then $S' = (S \setminus \{u\}) \cup \{a\}$ is also a solution (since every induced P_4 that contains u also contains a). Additionally, if $v \in S$ then $(S \setminus \{v\}) \cup \{b\}$ is also a solution. Therefore, there is a solution S such that either $a \in S$ or $b \in S$. Thus, Rule (B1) is safe.

The branching vector of Rule (B1) is (1,1).

(B2) If there is a vertex $u \in I$ such that |N(u)| = 1, let $v \in I$ be a vertex such that $N(u) \nsubseteq N(v)$. Branch on $\{v\}$, N(u) and N(v).

Note that the vertex v exists since Rule (R3) cannot be applied. To prove the safeness of Rule (B2), note that if S is a solution for the instance (G, k) then either $u \in S$, $v \in S$, $N(u) \subseteq S$, or $N(v) \subseteq S$. If one of the last three cases occurs we are done. Otherwise (if $u \in S$), $S' = (S \setminus \{u\}) \cup N(u)$ is also a solution. It follows that Rule (B2) is safe.

Since Rule (B1) cannot be applied, $|N(v)| \ge 2$. Therefore, the branching vector of Rule (B2) is at least (1, 1, 2).

Note that if Rule (B2) cannot be applied, every vertex in I has degree at least 2.

(B3) If there are vertices $u, v \in I$ such that $|N(u) \setminus N(v)| \ge 2$ and $|N(v) \setminus N(u)| \ge 2$, branch on $\{u\}, \{v\}, N(u) \setminus N(v), \text{ and } N(v) \setminus N(u).$

If S is a solution for the instance (G, k) then either $u \in S$, $v \in S$, $N(u) \setminus N(v) \subseteq S$, or $N(v) \setminus N(u) \subseteq S$ (If neither of the above cases hold, let $a \in (N(u) \setminus N(v)) \setminus S$ and $b \in (N(v) \setminus N(u)) \setminus S$. Then, u, a, b, v is an induced P_4 in G - S, a contradiction). Therefore, Rule (B3) is safe.

The branching vector of Rule (B3) is at least (1, 1, 2, 2).

Lemma 2. If Rule (B3) cannot be applied and $u, v \in I$ are two vertices such that $|N(u)| \leq |N(v)|$ then $|N(u) \setminus N(v)| \leq 1$.

Proof. Suppose conversely that $|N(u) \setminus N(v)| \ge 2$. Then, $|N(v) \setminus N(u)| \ge |N(u) \setminus N(v)| \ge 2$. Therefore, Rule (B3) can be applied on u, v, a contradiction.

We now consider two cases.

Case 1 In the first case, every two vertices in I_0 are twins. The algorithm picks an arbitrary vertex $u \in I_0$ and vertices $a_1, a_2 \in N(u)$. Since Rule (R3) cannot be applied, there is a vertex $v_1 \in I$ such that $a_1 \notin N(v_1)$ and a vertex $v_2 \in I$ such that $a_2 \notin N(v_2)$. For i = 1, 2 we have that $v_i \notin I_0$ since v_i is not a twin of u. Since $u \in I_0$, it follows that $|N(v_i)| > |N(u)|$. Thus, $|N(v_i) \setminus N(u)| > |N(u) \setminus N(v_i)| \ge 1$. By Lemma 2 and the fact that $a_i \in N(u) \setminus N(v_i)$ we obtain that $N(u) \setminus N(v_i) = \{a_i\}$ and thus $N(u) \setminus \{a_i\} \subseteq N(v_i)$. In particular, $a_2 \in N(v_1)$ and $a_1 \in N(v_2)$. Note that this implies that $v_1 \neq v_2$.

Lemma 3. $|(N(v_1) \cap N(v_2)) \setminus N(u)| \ge 2$.

Proof. Suppose without loss of generality that $|N(v_1)| \leq |N(v_2)|$. By Lemma 2 and the fact that $a_2 \in N(v_1) \setminus N(v_2)$ we have that $N(v_1) \setminus N(u) \subseteq N(v_2)$. Therefore, $(N(v_1) \cap N(v_2)) \setminus N(u) = N(v_2) \setminus N(u)$. We have shown above that $|N(v_2) \setminus N(u)| \geq 2$.

(B4) If Case 1 occurs and $a_1, a_2 \in N(w)$ for every $w \in I \setminus \{v_1, v_2\}$, branch on $\{u\}, (N(v_1) \cap N(v_2)) \setminus N(u), \text{ and } \{v_1, v_2\}.$

We now prove the safeness of Rule (B4). In order to delete the paths of the form u, a_1, b, v_1 or u, a_2, b, v_2 for some $b \in (N(v_1) \cap N(v_2)) \setminus N(u)$, a solution S must satisfy one of the following (1) $u \in S$ (2) $(N(v_1) \cap N(v_2)) \setminus N(u) \subseteq S$, or (3) S contains at least one vertex from $\{a_1, v_1\}$ and at least one vertex from $\{a_2, v_2\}$. Suppose that S is a solution that satisfies (3). Due to the assumption of Rule (B4) and the fact that $a_1 \in N(v_2)$, we have that every vertex in $I \setminus \{v_1\}$ is adjacent to a_1 . Therefore, every induced P_4 that contains a_1 is of the form v_1, x, a_1, y . Thus, if $v_1 \notin S$ then $S' = (S \setminus \{a_1\}) \cup \{v_1\}$ is also a solution. Similarly, if $v_2 \notin S$ then $S' = (S \setminus \{a_2\}) \cup \{v_2\}$ is also a solution. Therefore, if (G, k) is a yes instance, there is a solution S such that either S satisfies (1) or (2) above, or $\{v_1, v_2\} \subseteq S$.

By Lemma 3, the branching vector of Rule (B4) is at least (1, 2, 2).

(B5) If Case 1 occurs, let $w \in I \setminus \{v_1, v_2\}$ be a vertex such that $\{a_1, a_2\} \not\subseteq N(w)$, and without loss of generality assume that $w_1 \notin N(w)$. Branch on $\{u\}$, $(N(v_1) \cap N(v_2)) \setminus N(u)$, and on the sets in $\{\{a_1\}, \{v_1\} \cup (N(w) \setminus N(u)), \{v_1, w\}\} \circ \{\{a_2\}, \{v_2\}\}\}$.

We now show the safeness of Rule (B5). In order to delete the induced paths of the form u, a_1, b, v_1 or u, a_2, b, v_2 for $b \in (N(v_1) \cap N(v_2)) \setminus N(u)$, a solution S must satisfy (1), (2), or (3) above. Suppose that (1) is not satisfied (namely, $u \notin S$) and that (3) is satisfied. Additionally, suppose that $a_1 \notin S$. Therefore, $v_1 \in S$ and S contains at least one vertex from $\{a_2, v_2\}$. In order to delete the induced paths of the form u, a_1, c, w for every $c \in N(w) \setminus N(u)$, either $w \in S$ or $N(w) \setminus N(u) \subseteq S$. We have that $w \notin I_0$ since w is not a twin of u. Since $u \in I_0$, it follows that |N(w)| > |N(u)|. Thus, $|N(w) \setminus N(u)| > |N(u) \setminus N(w)| \ge 1$. From the previous inequality, Lemma 3, and the fact that $v_1, v_2, a_2 \notin N(w) \setminus N(u)$, it follows that the branching vector of Rule (B5) is at least (1, 2, 2, 4, 3, 2, 4, 3).

Case 2 In the second case, there are non-twin vertices in I_0 . Suppose that $u_1, u_2 \in I_0$ are non-twin vertices, where the choice of u_1, u_2 will be given later. By Lemma 2, $|N(u_1) \setminus N(u_2)| = |N(u_2) \setminus N(u_1)| = 1$. Denote $N(u_1) \setminus N(u_2) = \{a_1\}$ and $N(u_2) \setminus N(u_1) = \{a_2\}$. Let $I_1 = I_0 \setminus (\text{Twins}(u_1) \cup \text{Twins}(u_2))$.

Lemma 4. If $I_1 \neq \emptyset$ then either (1) for every $u \in I_1$, N(u) consists of $N(u_1) \cap N(u_2)$ plus an additional vertex that is not in $\{a_1, a_2\}$, or (2) for every $u \in I_1$, N(u) consists of a_1 , a_2 , and all the vertices of $N(u_1) \cap N(u_2)$ except one vertex.

Proof. We first claim that every $u \in I_1$, $|N(u) \cap a_1, a_2|$ is either 0 or 2. Suppose conversely that N(u) contains exactly one vertex from a_1, a_2 and without loss of generality, $a_2 \in N(u)$ and $a_1 \notin N(u)$. By Lemma 2 on u, u_1 we obtain that N(u) contains all the vertices in $N(u_1) \setminus \{a_1\} = N(u_2) \setminus \{a_2\}$. Since we assumed that $a_2 \in N(u)$, we have that $N(u_2) \subseteq N(u)$. From the fact that $|N(u_2)| = |N(u)|$ we obtain that $N(u_2) = N(u)$, contradicting the assumption that u is not a twin of u_2 . Therefore, $|N(u) \cap a_1, a_2|$ is either 0 or 2.

We first assume that there is no vertex $u \in I_1$ such that $a_1, a_2 \in N(u)$. From the claim above we have that $a_1 \notin N(u)$. By Lemma 2 on $u, u_1, N(u)$ contains all the vertices in $N(u_1) \setminus \{a_1\} = N(u_1) \cap N(u_2)$ plus an additional vertex that is not in $\{a_1, a_2\}$.

Now suppose that there is a vertex $u_3 \in I_1$ such that $a_1, a_2 \in N(u_3)$. Consider some $u \in I_1$. We claim that $a_1 \in N(u)$. Suppose conversely that $a_1 \notin N(u)$. From the claim above, $a_2 \notin N(u)$. Therefore, $a_1, a_2 \in N(u_3) \setminus N(u)$, contradicting Lemma 2. Thus, $a_1 \in N(u)$. From the claim above and Lemma 2 we conclude that N(u) contains a_2 and all the vertices in $N(u_1) \cap N(u_2)$ except one vertex.

If $I_1 = \emptyset$ or the first case of Lemma 4 occurs, we say that the vertices of I_0 form a sunflower. Note that if the vertices of I_0 do not form a sunflower, for every vertex $a \in N(u_1) \cup N(u_2)$ there are non-twin vertices $u, u' \in I_0$ that are adjacent to a.

(B6) If there are non-twin vertices $u_1, u_2 \in I_0$ such that $a_1, a_2 \in N(w)$ for every $w \in I \setminus I_0$, then suppose without loss of generality that $|\text{Twins}(u_1)| \leq |\text{Twins}(u_2)|$. Branch on $\text{Twins}(u_1)$ and $\{a_1\}$.

To prove the safeness of Rule (B6), suppose that (G, k) is a yes instance and let S be a solution. If $a_1 \in S$ or $Twins(u_1) \subseteq S$ we are done, so suppose that that $a_1 \notin S$ and Twins $(u_1) \not\subseteq S$. We can assume that $S \cap \text{Twins}(u_1) = \emptyset$ (otherwise, $S' = S \setminus \text{Twins}(u_1)$ is also a solution). Since u'_1, a_1, a_2, u'_2 is an induced path for every $u'_1 \in \text{Twins}(u_1)$ and $u'_2 \in \text{Twins}(u_2)$, either $a_2 \in S$ or $\text{Twins}(u_2) \subseteq S$. Note that we can assume that if S contains at least one vertex from $Twins(u_2)$ then it contains all the vertices of $Twins(u_2)$. Define a set S' by taking the vertices in $S \setminus (\{a_2\} \cup \text{Twins}(u_2))$. Additionally, if $a_2 \in S$, add a_1 to S', and if $\text{Twins}(u_2) \subseteq S$, add Twins (u_1) to S'. We now show that S' is also a solution. Since $|\text{Twins}(u_1)| \leq$ |Twins (u_2) |, we have that $|S'| \leq |S| \leq k$. Suppose conversely that G - S' contains an induced P_4 and denote this path by P'. Create a path P by taking P' and performing the following steps: (1) If a_1 is in P', replace it with a_2 . (2) If a_2 is in P', replace it with a_1 . (3) If P' contains a vertex $u'_1 \in \text{Twins}(u_1)$, replace it with u_2 . (4) If P' contains a vertex $u_2' \in \text{Twins}(u_2)$, replace it with u_1 . Recall that $a_1, a_2 \in N(w)$ for every $w \in I \setminus I_0$. Additionally, for every $u \in I_1$, $a_1, a_2 \notin N(u)$ if the vertices of I_0 form a sunflower, and $a_1, a_2 \in N(u)$ otherwise. Therefore, for every two vertices x', y' in P' and the corresponding vertices x, y in P, we have that (x, y) is an edge if and only if (x', y') is an edge. It follows that P is also an induced path in G. From the assumptions that $a_1 \notin S$ and Twins $(u_1) \cap S = \emptyset$ and from the definition of S' we have that S does not contain a vertex of P. This contradicts the assumption that Sis a solution. Therefore, S' is a solution. The solution S' contains either Twins (u_1) or $\{a_1\}$, and therefore Rule (B6) is safe.

The branching vector of Rule (B6) is at least (1,1).

Now suppose that Rule (B6) cannot be applied. We choose non-twin vertices $u_1, u_2 \in I_0$, a vertex $a \in N(u_1) \cap N(u_2)$, and a vertex $v \in I \setminus I_0$ that is not adjacent to a as follows.

- 1. If the vertices of I_0 form a sunflower, pick arbitrary non-twin vertices $u_1, u_2 \in I_0$. Pick $a \in N(u_1) \cap N(u_2)$. Since Rule (R3) cannot be applied, there is a vertex $v \in I$ such that $a \notin N(v)$. Since $a \in N(u)$ for every $u \in I_0$ (as the vertices of I_0 form a sunflower) it follows that $v \in I \setminus I_0$.
- 2. Otherwise, since Rule (B6) cannot be applied, there is a vertex $v \in I \setminus I_0$ such that $\bigcup_{u \in I_0} N(u) \not\subseteq N(v)$. Pick $a \in (\bigcup_{u \in I_0} N(u)) \setminus N(v)$. Since the vertices of I_0 do not form a sunflower, there are non-twin vertices $u_1, u_2 \in I_0$ such that $a \in N(u_1) \cap N(u_2)$.

Since Rule (B6) cannot be applied, there is a vertex $w \in I \setminus I_0$ such that, without loss of generality, $a_1 \notin N(w)$.

(B7) Branch on $\{u_1\}$, $(N(v) \cap N(w)) \setminus N(u_1)$, and on the sets in $\{\{a\}, \{v\}\}\}$ $\{\{a_1\}, \{w, a_2\}, \{w, u_2\}\}$.

The proof of the safeness of Rule (B7) is similar to the proof for Rule (B5). To bound the branching vector of Rule (B7) we use the following lemma.

Lemma 5. $|(N(v) \cap N(w)) \setminus N(u_1)| \geq 2$.

Proof. Since $|N(v)| > |N(u_1)|$, we have that $|N(v) \setminus N(u_1)| > |N(u_1) \setminus N(v)| \ge 1$. Similarly, $|N(w) \setminus N(u_1)| > |N(u_1) \setminus N(w)| \ge 1$. By Lemma 2 and the fact that $a_1 \in N(u_1) \setminus N(w)$ we have that $a \in N(w)$.

We consider two cases. If |N(v)| > |N(w)| then by Lemma 2 and the fact that $a \in N(w) \setminus N(v)$ we have that $N(w) \setminus N(u_1) \subseteq N(w) \setminus \{a\} \subseteq N(v)$. Therefore, $(N(v) \cap N(w)) \setminus N(u_1) = N(w) \setminus N(u_1)$ and the lemma follows since $|N(w) \setminus N(u_1)| \ge 2$.

If $|N(v)| \leq |N(w)|$ then by Lemma 2 and the fact that $a_1 \in N(v) \setminus N(w)$ we have that $N(v) \setminus N(u_1) \subseteq N(v) \setminus \{a_1\} \subseteq N(w)$. Therefore, $(N(v) \cap N(w)) \setminus N(u_1) = N(v) \setminus N(u_1)$ and the lemma follows since $|N(v) \setminus N(u_1)| \geq 2$.

By Lemma 5, the branching vector of Rule (B7) is at least (1, 2, 2, 2, 3, 3, 3, 3). The rule with largest branching number is Rule (B3) and its branching number is at most 2.733. Therefore, the running time of the algorithm is $O^*(2.733^k)$.

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