# A simple combinatorial algorithm for restricted 2-matchings in subcubic graphs - via half-edges* 

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#### Abstract

We consider three variants of the problem of finding a maximum weight restricted 2 -matching in a subcubic graph $G$. (A 2 -matching is any subset of the edges such that each vertex is incident to at most two of its edges.) Depending on the variant a restricted 2 -matching means a 2 -matching that is either triangle-free or square-free or both triangle- and square-free. While there exist polynomial time algorithms for the first two types of 2-matchings, they are quite complicated or use advanced methodology. For each of the three problems we present a simple reduction to the computation of a maximum weight $b$-matching. The reduction is conducted with the aid of half-edges. A half-edge of edge $e$ is, informally speaking, a half of $e$ containing exactly one of its endpoints. For a subset of triangles of $G$, we replace each edge of such a triangle with two half-edges. Two half-edges of one edge $e$ of weight $w(e)$ may get different weights, not necessarily equal to $\frac{1}{2} w(e)$. In the metric setting when the edge weights satisfy the triangle inequality, this has a geometric interpretation connected to how an incircle partitions the edges of a triangle. Our algorithms are additionally faster than those known before. The running time of each of them is $O\left(n^{2} \log n\right)$, where $n$ denotes the number of vertices in the graph.


## 1 Introduction

A subset $M$ of edges of an undirected simple graph is a 2-matching if every vertex is incident to at most two edges of $M$. 2-matchings belong to a wider class of $b$-matchings, where for every vertex $v$ in the set of vertices $V$ of the graph, we are given a natural number $b(v)$ and a subset of edges is a $b$-matching if every vertex is incident to at most $b(v)$ of its edges. A 2-matching is called $C_{k}$-free if it does not contain any cycle of length at most $k$. Note that every 2 -matching is $C_{2}$-free and the smallest length of a cycle in a 2 -matching is three. A 2 -matching of maximum size can be found in polynomial time by a reduction to a classical matching. The $C_{k}$-free 2 -matching problem consists in finding a $C_{k}$-free 2 -matching of maximum size. Observe that the $C_{k}$-free 2-matching problem for $n / 2 \leq k<n$, where $n$ is the number of vertices in the graph, is equivalent to finding a Hamiltonian cycle, and thus NP-hard. Hartvigsen [9] gave a complicated algorithm for the case of $k=3$. Papadimitriou [5] showed that this problem is NP-hard when $k \geq 5$. The complexity of the $C_{4}$-free 2 -matching problem is unknown.

In the weighted version of the problem, each edge $e$ is associated with a nonnegative weight $w(e)$ and we are interested in finding a $C_{k}$-free 2 -matching of maximum weight, where the weight of a 2 -matching $M$ is defined as the sum of weights of edges belonging to $M$. Vornberger [26] showed that the weighted $C_{4}$-free 2 -matching problem is NP-hard. We refer to cycles of length three and four as triangles and squares, respectively.

[^0]In the paper we consider the following three problems in subcubic graphs: the weighted trianglefree 2 -matching problem (i.e. the weighted $C_{3}$-free 2 -matching problem), the weighted square-free 2 -matching problem, in which we want to find a maximum weight 2 -matching without any squares, but possibly containing triangles and the weighted $C_{4}$-free 2-matching problem. A graph is called cubic if its every vertex has degree 3 and is called subcubic if its every vertex has degree at most 3 .

The weighted triangle-free 2-matching problem in subcubic graphs. The existing two polynomial time algorithms for this problem are the following. Hartvigsen and Li [12] gave a rather complicated primal-dual algorithm with running time $O\left(n^{3}\right)$ and a long analysis. The algorithm uses a type of so-called comb inequality. Kobayashi [15] devised a simpler algorithm using the theory of $M$-concave functions on finite constant-parity jump systems as well as makes $O\left(n^{3}\right)$ computations of a maximum weight $b$-matching for $b \in\{0,1,2\}^{V}$. Its running time is $O\left(n^{5} \log n\right)$.

We present a simple combinatorial algorithm for the problem that uses one computation of a maximum weight $b$-matching for $b \in\{0,1,2\}^{V}$. Given a subcubic graph $G$, we replace some of its triangles with gadgets containing half-edges and define a function $b$ on the set of vertices in such a way that, any $b$-matching in the thus constructed graph $G^{\prime}$ yields a triangle-free 2-matching. A half-edge of edge $e$ is, informally speaking, a half of $e$ containing exactly one of its endpoints. Half-edges have already been introduced in [20] and used in several subsequent papers. Here we use a different weight distribution among half-edges of one edge - two half-edges of one edge $e$ may be assigned different weights and not necessarily equal to $\frac{1}{2} w(e)$. In the metric setting when the edge weights satisfy the triangle inequality, this has a geometric interpretation connected to how an incircle partitions the edges of a triangle. The running time of our algorithm is $O\left(n^{2} \log n\right)$. If the graph is unweighted, then the run time of this algorithm becomes $O\left(n^{3 / 2}\right)$.

Square-free 2-matchings. In bipartite graphs a shortest cycle has length four - a square. Polynomial time algorithms for the $C_{4}$-free 2-matching problem in bipartite graphs were shown by Hartvigsen [10], Pap [21] and analyzed by Király [13]. As for the weighted version of the square-free 2-matching problem in bipartite graphs it was proven to be NP-hard [8, 14] and solved by Makai [18] and Takazawa [23] for the case when the weights of edges are vertex-induced on every square of the graph. When it comes to the square-free 2-matching problem in general graphs, Nam [19] constructed a complex algorithm for it for graphs, in which all squares are vertex-disjoint. Bérczi and Kobayashi [3] showed that the weighted square-free 2 -matching problem is NP-hard for general weights even if the given graph is cubic, bipartite and planar and gave a polynomial algorithm that finds a maximum weight 2-matching that contains no squares (but it can contain triangles). In [3] the square-free 2-matching problem is used for solving the $(n-3)$-connectivity augmentation problem. As regards subcubic graphs, there are two other results besides those mentioned above. Bérczi and Végh [4] considered the problem of finding a maximum $t$-matching (a $b$-matching such that $b(v)=t$ for each vertex $v$ ) which does not contain any subgraph from a given set of forbidden $K_{t, t}$ and $K_{t+1}$ in an undirected graph of degree at most $t+1$. Observe that the square-free 2 -matching problem in subcubic graphs is a special case of this problem for $t=2$.

The $C_{4}$-free 2-matching problem was previously investigated only in the unweighted version by Hartvigsen and Li in [11], who devised an $O\left(n^{3 / 2}\right)$-algorithm. We present combinatorial algorithms for the weighted square-free 2 -matching problem and the weighted $C_{4}$-free 2-matching problem for the case when the weights of edges are vertex-induced on every square of the graph and the graph is subcubic. These algorithms are similar to the one for the weighted triangle-free 2 -matching problem in subcubic graphs and have the same running time.

Related work Some generalizations of the $C_{k}$-free 2-matching problem were investigated. Recently, Kobayashi [16] gave a polynomial algorithm for finding a maximum weight 2-matching that does not contain any triangle from a given set of forbidden edge-disjoint triangles. One can also consider non-simple $b$-matchings, in which every edge $e$ may occur in more than one copy. Problems
connected to non-simple $b$-matchings are usually easier than variants with simple $b$-matchings. Efficient algorithms for triangle-free non-simple 2 -matchings (such 2 -matchings may contain 2 -cycles) were devised by Cornuéjols and Pulleyback [5, 6], Babenko, Gusakov and Razenshteyn [2], and Artamonov and Babenko [1]. Other results for restricted non-simple $b$-matchings appeared in [22, 24, 25].

## 2 Preliminaries

Let $G=(V, E)$ be an undirected graph with vertex set $V$ and edge set $E$. We denote the number of vertices of $G$ by $n$ and the number of edges of $G$ by $m$. We assume that all graphs are simple, i.e., they contain neither loops nor parallel edges. We denote an edge connecting vertices $v$ and $u$ by $(v, u)$. A cycle of graph $G$ is a sequence $c=\left(v_{0}, \ldots, v_{l-1}\right)$ for some $l \geq 3$ of pairwise distinct vertices of $G$ such that $\left(v_{i}, v_{(i+1) \bmod l}\right) \in E$ for every $i \in\{0,1, \ldots, l-1\}$. We refer to $l$ as the length of $c$. For a given cycle $c=\left(v_{0}, \ldots, v_{l-1}\right)$ any edge of $G$, which connects two vertices of $c$ and does not occur in $c$ is called a diagonal (of $c$ ). For a subgraph $H$ of $G$, we denote the edge set of $H$ by $E(H)$. For an edge set $F \subseteq E$ and $v \in V$, we denote by $\operatorname{deg}_{F}(v)$ the number of edges of $F$ incident to $v$.

An instance of each of the three problems that we consider in the paper consists of an undirected subcubic graph $G=(V, E)$ and a weight function $w: E \rightarrow \mathbb{R}_{\geq 0}$. In the weighted triangle-free 2matching problem the goal is to find a maximum weight triangle-free 2-matching of $G$. In the weighted square-free (resp. $C_{4}$-free) 2-matching problem we additionally assume that the weights on the edges are vertex-induced on each square of $G$, i.e. for any square $s=\left(v_{1}, v_{2}, v_{3}, v_{4}\right)$ there exists a function $r:\left\{v_{1}, v_{2}, v_{3}, v_{4}\right\} \rightarrow R$ such that for any edge $e=(u, v)$ connecting two vertices of $s$ it holds that $w(e)=r(u)+r(v)$. The aim in the weighted square-free (resp. $C_{4}$-free) 2-matching problem is to compute a maximum weight square-free (corr. $C_{4}$-free) 2 -matching of $G$.

We will use the classical notion of a $b$-matching, which is a generalization of a matching. For a vector $b \in \mathbb{N}^{V}$, an edge set $M \subseteq E$ is said to be a $b$-matching of $G$ if $\operatorname{deg}_{M}(v) \leq b(v)$ for every $v \in V$. Notice that a $b$-matching with $b(v)=1$ for every $v \in V$ is a classical matching. A $b$-matching of $G$ of maximum weight can be computed in polynomial time. We refer to Lovász and Plummer [17] for further background on $b$-matchings.

We are interested in computing a $b$-matching of a graph $G$ where we are given vectors $l, u \in \mathbb{N}^{V}$ and a weight function $w: E \rightarrow \mathbb{R}$. For a vertex $v \in V,[l(v), u(v)]$ is said to be a capacity interval of $v$. An edge set $M \subseteq E$ is said to be an $(l, u)$-matching if $l(v) \leq \operatorname{deg}_{M}(v) \leq u(v)$ for every $v \in V$. An $(l, u)$-matching $M$ is said to be a maximum weight $(l, u)$-matching if there is no $(l, u)$-matching $M^{\prime}$ of $G$ of weight greater than $w(M)$. A maximum weight $(l, u)$-matching can be computed efficiently.

Theorem 1 ([7]). There is an algorithm that, given a graph $G=(V, E)$, a weight function $w: E \rightarrow \mathbb{R}$ and vectors $l, u \in \mathbb{N}^{V}$, in time $O\left(\left(\sum_{v \in V} u(v)\right) \min \left\{|E(G)| \log |V(G)|,|V(G)|^{2}\right\}\right)$, finds a maximum weight $(l, u)$-matching of $G$.

Given an $(l, u)$-matching $M$ and an edge $e=(u, v) \in M$, we say that $u$ is matched to $v$ in $M$.

## 3 Outline of the Algorithm

The general scheme of the algorithm for each variant of the restricted 2 -matching problem is the same - we give it below.

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\(\overline{\text { Algorithm } 1 \text { Computing a maximum weight restricted 2-matching of a subcubic graph } G \text { given a }}\)
weight function \(w: E \rightarrow \mathbb{R}_{\geq 0}\).
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Step 1. Construct an auxiliary graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ of size $O(n)$ by replacing some triangles and/or squares of $G$ with gadgets containing half-edges. (Both gadgets and half-edges are defined later.)

Step 2. Define a weight function $w^{\prime}: E^{\prime} \rightarrow \mathbb{R}$ and vectors $l, u \in \mathbb{N}^{V^{\prime}}$ such that $u(v) \leq 2$ for every $v \in V^{\prime}$.

Step 3. Compute a maximum weight $(l, u)$-matching $M^{\prime}$ of $G^{\prime}$.
Step 4. Construct a 2 -matching $M$ of $G$ by replacing all half-edges of $M^{\prime}$ with some edges of $G$ in such a way that $w(M) \geq w^{\prime}\left(M^{\prime}\right)$.

Step 5. Remove the remaining triangles and/or squares from $M$ by replacing some of their edges with other ones without decreasing the weight of $M$.

Claim 1. Algorithm $\sqrt{1}$ runs in time $O\left(n^{2} \log n\right)$.
Proof. It will be easy to implement all steps of an Algorithm 1 except Step 3 in linear time. Hence, the running time of our algorithm is equal to the running time of an algorithm for computing a maximum weight $(l, u)$-matching of $G^{\prime}$, i.e., it is equal to $O\left(\left(\sum_{v \in V^{\prime}} u(v)\right) \min \left\{\left|E^{\prime}\right| \log \left|V^{\prime}\right|,\left|V^{\prime}\right|^{2}\right\}\right)$. Recall that $\left|V^{\prime}\right|+\left|E^{\prime}\right|=O(n)$ and $u(v) \leq 2$ for every $v \in V^{\prime}$. Hence, the running time of Step 3 is $O\left(n^{2} \log n\right)$.

Let us also remark that in the unweighted versions of the problem Algorithm 1 runs in $O\left(n^{3 / 2}\right)$.

## 4 Triangle-free 2-matchings in subcubic graphs

In this section we solve a maximum weight triangle-free 2-matching problem in subcubic graphs. We assume that each connected component of $G$ is different from $K_{4}$, i.e., different from a 4-vertex clique.

One can observe that, since $G$ is subcubic, any edge $e$ of $G$ belongs to at most two different triangles. Also, any triangle of $G$ shares an edge with at most one other triangle or, in other words, any triangle of $G$ is not edge-disjoint with at most one other triangle.

Definition 1. A triangle $t$, which has a common edge with some other triangle $t^{\prime}$ such that $w(t) \leq$ $w\left(t^{\prime}\right)$ is said to be unproblematic. Otherwise, $t$ is said to be problematic.

Unproblematic triangles can be easily got rid of from any 2-matching $M$ of $G$ by replacing some of its edges with other ones as explained in more detail in the proof of Theorem 3 ,

Observe that any problematic triangle of $G$ is vertex-disjoint with any other problematic triangle of $G$.

We begin with the following simple fact.
Claim 2. Let $t=(a, b, c)$ be a triangle of $G$, whose edges have weights $w(a, b), w(b, c), w(c, a)$, respectively. Then, there exist real numbers $r_{a}, r_{b}, r_{c}$ such that $w(a, b)=r_{a}+r_{b}, w(b, c)=r_{b}+r_{c}$ and $w(c, a)=r_{c}+r_{a}$.

If the weights of edges of $t$ satisfy the triangle inequality, then Claim 2 has a geometric interpretation connected to how an incircle partitions the edges of a triangle - see Figure 1 .


Figure 1: Partition of the edges of a triangle by its incircle.

If $G$ contains at least one problematic triangle, we build a graph $G^{\prime}=\left(V^{\prime}, E^{\prime}\right)$ together with a weight function $w^{\prime}: E^{\prime} \rightarrow \mathbb{R}$, in which each problematic triangle $t$ is replaced with a subgraph, called a gadget for $t$. The precise construction of $G^{\prime}$ is the following. We start off with $G$.

Let $t=(a, b, c)$ be any problematic triangle of $G$. For each edge $(p, q)$ of $t$ we add two new vertices $v_{q}^{p}$ and $v_{p}^{q}$, called subdivision vertices (of $t$ ), and we replace $(p, q)$ with three new edges: $\left(p, v_{q}^{p}\right),\left(v_{q}^{p}, v_{p}^{q}\right),\left(v_{p}^{q}, q\right)$. Each of the edges $\left(p, v_{q}^{p}\right),\left(v_{p}^{q}, q\right)$ is called a half-edge (of $(p, q)$ and also of $t$ ). The edge $\left(v_{q}^{p}, v_{p}^{q}\right)$ is called an eliminator (of $(p, q)$ ). The half-edges of edges of $t$ get weights equal to values of $r_{a}, r_{b}, r_{c}$ from Claim 2, i.e., $w^{\prime}\left(a, v_{b}^{a}\right)=w^{\prime}\left(a, v_{c}^{a}\right)=r_{a}, w^{\prime}\left(b, v_{a}^{b}\right)=w^{\prime}\left(b, v_{c}^{b}\right)=r_{b}$ and $w^{\prime}\left(c, v_{a}^{c}\right)=w^{\prime}\left(c, v_{b}^{c}\right)=r_{c}$. The weight of each eliminator is 0 . Additionally, we introduce four new vertices $u_{a}, u_{b}, u_{c}, u_{t}$, called global vertices. For every $d \in\{a, b, c\}$ we connect $u_{d}$ with $u_{t}$ and with every subdivision vertex connected to $d$. Every edge incident to a global vertex has weight 0 .

We define vectors $l, u \in \mathbb{N}^{V^{\prime}}$ as follows. We set a capacity interval of every vertex of the original graph $G$ to $[0,2]$ and we set a capacity interval of every other vertex of $G^{\prime}$ to $[1,1]$, i.e., every vertex of $V^{\prime} \backslash V$ is matched to exactly one vertex of $G^{\prime}$ in any $(l, u)$-matching of $G^{\prime}$.

The main ideas behind the gadget for a problematic triangle $t=(a, b, c)$ are the following. An $(l, u)$-matching $M^{\prime}$ of $G^{\prime}$ is to represent roughly a triangle-free 2-matching $M$ of $G$. If $M^{\prime}$ contains both half-edges of some edge $e$, then $e$ is included in $M$. If $M^{\prime}$ contains an eliminator of $e$, then $e$ does not belong to $M$ (is excluded from $M$ ). We want to ensure that at least one edge of $t$ does not belong to $M$. This is done by requiring that two of the global vertices $u_{a}, u_{b}, u_{c}$ are matched to subdivision vertices. In this way two half-edges of $t$ are guaranteed not to belong to $M^{\prime}$ and hence to $M$.


Figure 2: A gadget for a problematic triangle $t=(a, b, c)$.
In the theorem below we show the correspondence between triangle-free 2-matchings of $G$ and
$(l, u)$-matchings of $G^{\prime}$.
Theorem 2. Let $M$ be any triangle-free 2 -matching of $G$. Then we can find an $(l, u)$-matching $M^{\prime}$ of $G^{\prime}$ such that $w^{\prime}\left(M^{\prime}\right)=w(M)$.

Proof. We initialize $M^{\prime}$ as the empty set. We add every edge of $M$ that does not belong to any problematic triangle of $G$ to $M^{\prime}$. Consider any problematic triangle $t=(a, b, c)$ of $G$. Since $M$ is triangle-free, there exists an edge of $t$ that does not belong to $M$. If more than one edge of $t$ does not belong to $M$, we choose one of them. Suppose that we chose $(a, b) \notin M$. Then we add edges ( $v_{b}^{a}, u_{a}$ ), $\left(v_{a}^{b}, u_{b}\right)$ and $\left(u_{t}, u_{c}\right)$ to $M^{\prime}$. For every other edge $e$ of $t$ we proceed as follows. If $e \in M$, we add both half-edges of $e$ to $M^{\prime}$, otherwise we add the eliminator of $e$ to $M^{\prime}$. Since the weight of any edge of $t$ in $G$ is equal to the sum of the weights of its half-edges in $G^{\prime}$, we get that $w^{\prime}\left(M^{\prime}\right)=w(M)$.

Theorem 3. Let $M^{\prime}$ be any ( $l, u$ )-matching of $G^{\prime}$. Then we can find a triangle-free 2-matching $M$ of $G$ such that $w(M) \geq w^{\prime}\left(M^{\prime}\right)$.

Proof. We initialize $M$ as the empty set. We add every edge of $M^{\prime}$ that belongs to $G$ to $M$. For every problematic triangle of $G$ we will add some of its edges to $M$.

Consider any problematic triangle $t=(a, b, c)$ of $G$. Notice that exactly two of the vertices $u_{a}, u_{b}, u_{c}$ are matched to subdivision vertices, because $u_{t}$ is matched to one of $u_{a}, u_{b}, u_{c}$. This corresponds to excluding two half-edges of $t$ from $M$. Since every subdivision vertex is required to be matched to exactly one vertex in $G^{\prime}$, we get that an even number and at most four subdivision vertices of $t$ are matched to the vertices $a, b, c$. This indicates, which half-edges of $t$ are going to be included in $M$. Observe that the two subdivision vertices that are matched in $M^{\prime}$ to vertices $u_{a}, u_{b}, u_{c}$ are adjacent to two different vertices of $t$. Thus, we have:

Claim 3. If $M^{\prime}$ contains exactly four half-edges of $t$, then the two half-edges of that do not belong to $M^{\prime}$ are not adjacent to the same vertex of $t$.

Every other subset of half-edges of $t$ containing an even number of at most four half-edges of $t$ can occur in $M^{\prime}$.

In each of these cases, we proceed as follows (see Figure 3):

1. Exactly zero subdivision vertices of $t$ are matched to $a, b, c$. We do not include any edge of $t$ in M.
2. Exactly two subdivision vertices of $t$ are matched to $a, b, c$.
(a) The two subdivision vertices of $t$ are matched to two different vertices $u, v$ of $t$. Then we include the edge $(u, v)$ in $M$.
(b) The two subdivision vertices of $t$ are matched to the same vertex $u$ of $t$. Then we include in $M$ two edges of $t$ incident to $u$. (This is the only case where $w(M)$ may be greater than $w^{\prime}\left(M^{\prime}\right)$ when it comes to half-edges of $t$. Notice that for any two vertices $u, v$ of $t$ we have that $r_{u}+r_{v} \geq 0$ ).
3. Exactly four subdivision vertices of $t$ are matched to $a, b, c$. Then, by Claim 3, two of these vertices are matched to the same vertex $u$ of $t$ and the other two are matched to the remaining two vertices of $t$. In this case we include in $M$ two edges of $t$ incident to $u$.

Since half-edges incident to the same vertex have the same weight, we get that $w(M) \geq w^{\prime}\left(M^{\prime}\right)$.
The resulting 2 -matching $M$ can contain some unproblematic triangles. We remove them one by one. Let $t=(a, b, c)$ be any such triangle. From Definition 1 there exists another triangle $t^{\prime}=$
$(a, b, d)$, which shares an edge with $t$ and such that $w\left(t^{\prime}\right) \geq w(t)$. Hence, either $w(a, d) \geq w(a, c)$ or $w(b, d) \geq w(b, c)$. Assume that $w(a, d) \geq w(a, c)$. We replace the edge $(a, c)$ with the edge $(a, d)$ without decreasing the weight of $M$.


Figure 3: The construction of a maximum weight triangle-free 2-matching of $G$ from a maximum weight $(l, u)$-matching of $G^{\prime}$.

## 5 Square-free 2-matchings in subcubic graphs

In this section we solve a maximum weight square-free 2-matching problem in subcubic graphs. Recall that this problem is NP-hard for general weights, therefore we assume that weights are vertex-induced on every square, i.e., for any square $s=(a, b, c, d)$ of $G$ there exist real numbers $r_{a}, r_{b}, r_{c}, r_{d}$, called potentials of $s$ such that for any edge $e=(u, v)$ connecting two vertices of $s$ it holds that $w(e)=$ $r_{u}+r_{v}$. (Note that if a given edge $e=(u, v)$ belongs to two different squares $s$ and $s^{\prime}$, then potentials of $s$ and $s^{\prime}$ on $u$ and $v$ may be different.)

We also assume that each connected component of $G$ is different from $K_{4}$.
For a square $s=\left(v_{0}, v_{1}, v_{2}, v_{3}\right)$ of $G$, edges $\left(v_{0}, v_{1}\right),\left(v_{1}, v_{2}\right),\left(v_{2}, v_{3}\right)$ and $\left(v_{3}, v_{0}\right)$ are said to be native edges of $s$.

One can observe that, since $G$ is subcubic, any two different squares of $G$ are vertex-disjoint or have either one or two edges in common.

Definition 2. A square $s$ of $G$ is said to be unproblematic if there exists another square $s^{\prime}$ such that (i) s shares exactly one edge with $s^{\prime}$ or (ii) s shares two edges with $s^{\prime}$ and $w(s) \leq w\left(s^{\prime}\right)$. Otherwise, $s$ is said to be problematic.

Observe that any problematic square of $G$ is vertex-disjoint with any other problematic square of $G$.

The following simple observation shows that squares which have exactly one common edge with another square do not pose any problem for computing a maximum weight square-free 2 -matching of $G$.

Claim 4. Consider any two squares $s=(a, b, c, d)$ and $s^{\prime}=(c, d, e, f)$ of $G$ which share exactly one edge. Let $M_{1}$ be a 2-matching of $G$ that contains $s$. Then there exists a 2 -matching $M_{2}$ of $G$, which does not contain sor any square not already contained in $M_{1}$ and such that $w\left(M_{2}\right) \geq w\left(M_{1}\right)$.

Proof. We set $M_{2}=M_{1} \backslash\{(c, d),(e, f)\} \cup\{(c, f),(d, e)\}$. Note that we can assume that $M_{1}$ contains the edge $(e, f)$, because $G$ is subcubic and $M_{1}$ contains neither $(c, f)$ nor $(d, e)$. It is straightforward to check that $M_{2}$ is a 2-matching of $G$ that does not contain $s$. Furthermore, the given construction
does not introduce any additional squares into $M_{2}$. Observe that $w\left(M_{2}\right) \geq w\left(M_{1}\right)$, since $w$ is vertexinduced on $s^{\prime}$.

We show the construction of a gadget for a problematic square $s=(a, b, c, d)$. We use the notation introduced in Section 4 . For every native edge $(p, q)$ of $s$, we introduce two subdivision vertices $v_{q}^{p}$, $v_{p}^{q}$ and replace $(p, q)$ with two half-edges $\left(p, v_{q}^{p}\right)$ and $\left(v_{p}^{q}, q\right)$ and an eliminator $\left(v_{q}^{p}, v_{p}^{q}\right)$. (We do not replace any diagonal of $s$.) Additionally, we introduce two new global vertices $u_{s}^{1}$ and $u_{s}^{2}$. We connect $u_{s}^{1}$ with all subdivision vertices adjacent to either $a$ or $c$. Symmetrically, we connect $u_{s}^{2}$ with all subdivision vertices adjacent to either $b$ or $d$.

The half-edges incident to $a, b, c$ and $d$ get weight $r_{a}, r_{b}, r_{c}$ and $r_{d}$, respectively, where $r_{a}, \ldots, r_{d}$ are potentials of $s$. All other edges of the gadget get weight 0 . We set a capacity interval of every vertex of $s$ to $[0,2]$ and we set a capacity interval of every other vertex of the gadget to $[1,1]$.


Figure 4: A gadget for a problematic square $s=(a, b, c, d)$.

Theorem 4. Let $M$ be any square-free 2-matching of $G$. Then we can find an $(l, u)$-matching $M^{\prime}$ of $G^{\prime}$ such that $w^{\prime}\left(M^{\prime}\right)=w(M)$.

Proof. We initialize $M^{\prime}$ as the empty set. We add every edge of $M$ that does not belong to any problematic square of $G$ to $M^{\prime}$.

Consider any problematic square $s=(a, b, c, d)$ of $G$. Assume that $(a, b)$ does not belong to $M$. We add edges $\left(v_{b}^{a}, u_{s}^{1}\right)$ and $\left(v_{a}^{b}, u_{s}^{2}\right)$ to $M^{\prime}$. For every other native edge $e$ of $s$ we proceed as follows. If $e \in M$, we add both half-edges of $e$ to $M^{\prime}$, otherwise we add the eliminator of $e$ to $M^{\prime}$.

Theorem 5. Let $M^{\prime}$ be any $(l, u)$-matching of $G^{\prime}$. Then we can find a square-free 2-matching $M$ of $G$ such that $w(M) \geq w^{\prime}\left(M^{\prime}\right)$.

Proof. We initialize $M$ as the empty set. We add every edge of $M^{\prime}$ that belongs to $G$ to $M$. For every problematic square of $G$ we will add some of its edges to $M$. Next we will replace some edges of $M$ with other ones to remove unproblematic squares.

Consider any problematic square $s=(a, b, c, d)$ of $G$. Notice that there exists a native edge $(p, q)$ of $s$ such that $u_{s}^{1}$ and $u_{s}^{2}$ are matched in $M^{\prime}$ to two subdivision vertices, one of which is adjacent to $p$ and the other to $q$. W.l.o.g. assume that $(p, q)=(a, b)$. We consider the following cases:

1. $u_{s}^{1}$ and $u_{s}^{2}$ are matched in $M^{\prime}$ to $v_{b}^{a}$ and $v_{a}^{b}$, respectively. We add every native edge of $s$ whose both half-edges belong to $M^{\prime}$ to $M$. Notice that for every other native edge $e$ of $s$, the eliminator of $e$ belongs to $M^{\prime}$.
2. Either $u_{s}^{1}$ is matched to $v_{b}^{a}$ or $u_{s}^{2}$ is matched to $v_{a}^{b}$ in $M^{\prime}$, but not both of them. Assume that $u_{s}^{1}$ is matched to $v_{b}^{a}$. Therefore, edges $\left(u_{s}^{2}, v_{c}^{b}\right)$ and $\left(b, v_{a}^{b}\right)$ belong to $M^{\prime}$. We replace these two edges with $\left(u_{s}^{2}, v_{a}^{b}\right)$ and $\left(b, v_{c}^{b}\right)$ without changing the weight of $M^{\prime}$. Then we proceed as in case 1 .
3. $u_{s}^{1}$ and $u_{s}^{2}$ are matched to $v_{d}^{a}$ and $v_{c}^{b}$, respectively, in $M^{\prime}$. If $\left(v_{b}^{a}, v_{a}^{b}\right)$ does not belong to $M^{\prime}$, we connect $u_{s}^{1}$ and $u_{s}^{2}$ with $v_{b}^{a}$ and $v_{a}^{b}$, respectively, similarly as in case 2 , and we proceed as in case 1 . Assume now that $\left(v_{b}^{a}, v_{a}^{b}\right)$ belongs to $M^{\prime}$. Notice that $\left(d, v_{a}^{d}\right)$ and $\left(c, v_{b}^{c}\right)$ belong to $M^{\prime}$. We add $(a, d)$ and $(b, c)$ to $M$. Additionally, if both half-edges of $(c, d)$ belong to $M^{\prime}$, we add $(c, d)$ to $M$.

The resulting 2 -matching $M$ can contain some unproblematic squares. We remove squares, which share exactly one edge with another square from $M$ one by one using Claim 4 . We remove the rest of unproblematic squares in a similar way as we got rid of unproblematic triangles in the proof of Theorem 33. Each such removal does not introduce any squares into $M$, therefore $M$ is a square-free 2 -matching in the end.

## $6 C_{4}$-free 2-matchings in subcubic graphs

In this section we solve a maximum weight $C_{4}$-free 2 -matching problem in subcubic graphs. We assume that weights are vertex-induced on every square. We also assume that each connected component of $G$ is different from $K_{4}$.

We say that a cycle $C$ of $G$ is short if it is either a triangle or a square. We say that a short cycle $C$ of $G$ is unproblematic if it shares exactly one edge with some square of $G$ or if it fits Definition 1 or Definition 2. A short cycle, which is not unproblematic is said to be problematic.

We have the analogue of Claim 4, which justifies considering triangles sharing one edge with a square unproblematic:

Claim 5. Consider two short cycles: a triangle $t=(a, c, d)$ and a square $s^{\prime}=(c, d, e, f)$ of $G$ which share exactly one edge. Let $M_{1}$ be a 2-matching of $G$ that contains $t$. Then there exists a 2 -matching $M_{2}$ of $G$, which does not contain $t$ or any short cycle not already contained in $M_{1}$ and such that $w\left(M_{2}\right) \geq w\left(M_{1}\right)$.

Observe that any two different short problematic cycles that are not vertex-disjoint must form a pair consisting of a square $s=(a, c, b, d)$ and a triangle $t_{1}=(a, c, d)$ with exactly two common edges. We call a subgraph induced on vertices of such $s$ and $t_{1}$ a double triangle $T=(a, b, c, d)$. In $G^{\prime}$ we build the following gadget for every double triangle.

Consider any double triangle $T=(a, b, c, d)$. We remove $c$ and $d$ from $G^{\prime}$ and we add a vertex $u_{T}$ to $G^{\prime}$. We connect $v_{T}^{1}$ with $v_{T}^{2}$ and we connect $u_{T}$ with both $a$ and $b$. Let $M_{j}^{i}(T)$ denote the weight of a maximum weight $C_{4}$-free 2 -matching of $T$ in which $a$ has degree $i$ and $b$ has degree $j$. We set the weight of edges $\left(v_{T}^{1}, v_{T}^{2}\right),\left(u_{T}, a\right)$ and $\left(u_{T}, b\right)$ to $M_{1}^{1}(T), M_{1}^{2}(T)-M_{1}^{1}(T)$ and $M_{2}^{1}(T)-M_{1}^{1}(T)$, respectively. We set capacity intervals of $a, b$ and $u_{T}$ to $[0,1]$. We set capacity intervals of $v_{T}^{1}$ and $v_{T}^{2}$ to $[1,1]$.

For every problematic short cycle that is not part of any double triangle we add a corresponding gadget presented in Section 4 or Section 5
Theorem 6. Let $M$ be any $C_{4}$-free 2-matching of $G$. Then we can find an $(l, u)$-matching $M^{\prime}$ of $G^{\prime}$ such that $w^{\prime}\left(M^{\prime}\right) \geq w(M)$.


Figure 5: A gadget for a double triangle $T=(a, b, c, d)$.

Proof. We initialize $M^{\prime}$ as the empty set. We add to $M^{\prime}$ every edge of $M$ that belongs to no problematic short cycle.

Consider any double triangle $T=(a, b, c, d)$ of $G$. We add $\left(v_{T}^{1}, v_{T}^{2}\right)$ to $M^{\prime}$. Let $\hat{M}=M \cap E(T)$. If $\operatorname{deg}_{\hat{M}}(a)=2$, then we add $\left(u_{T}, a\right)$ to $M^{\prime}$. If $\operatorname{deg}_{\hat{M}}(b)=2$, then we add $\left(u_{T}, b\right)$ to $M^{\prime}$. Note that $\operatorname{deg}_{\hat{M}}(a) \leq 1$ or $\operatorname{deg}_{\hat{M}}(b) \leq 1$, therefore $\operatorname{deg}_{M^{\prime}}\left(u_{T}\right) \leq 1$.

For every problematic short cycle that is not part of any double triangle we add edges of a corresponding gadget to $M^{\prime}$ in the same way as we did in the proofs of Theorem 2 and Theorem 4

Theorem 7. Let $M^{\prime}$ be any $(l, u)$-matching of $G^{\prime}$. Then we can find a $C_{4}$-free 2-matching $M$ of $G$ such that $w(M) \geq w^{\prime}\left(M^{\prime}\right)$.

Proof. We initialize $M$ as the empty set. We add to $M$ every edge of $M^{\prime}$ that belongs to $G$.
Consider any double triangle $T=(a, b, c, d)$ of $G$. Let $i$ and $j$ denote the number of edges of the gadget for $T$ incident to $a$ and $b$, respectively. Notice that $i+j \leq 1$. We add to $M$ a maximum weight $C_{4}$-free 2-matching of $T$ in which $a$ has degree $i+1$ and $b$ has degree $j+1$.

For every short cycle that is not part of any double triangle we proceed in the same way as in the proofs of Theorem 3 and Theorem 5

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