INEQUALITIES OF RAFALSON TYPE FOR ALGEBRAIC POLYNOMIALS

K. H. KWON[1] AND D. W. LEE[2]

ABSTRACT. For a positive Borel measure $d\mu$, we prove that the constant

$$\gamma_n(d\nu; d\mu) := \sup_{\pi\in\mathcal{P}_n\setminus\{0\}} \frac{\int_{-\infty}^{\infty}\pi^2(x)d\nu(x)}{\int_{-\infty}^{\infty}\pi^2(x)d\mu(x)},$$

can be represented by the zeros of orthogonal polynomials corresponding to $d\mu$ in case (i) $d\nu(x) = (A + Bx)d\mu(x)$, where $A + Bx$ is nonnegative on the support of $d\mu$ and (ii) $d\nu(x) = (A + Bx^2)d\mu(x)$, where $d\mu$ is symmetric and $A + Bx^2$ is nonnegative on the support of $d\mu$. The extremal polynomials attaining the constant are obtained and some concrete examples are given including Markov type inequality when $d\mu$ is a measure for Jacobi polynomials.

2000 AMS SUBJECT CLASSIFICATION: 41A17, 41A44.
KEY WORDS: INEQUALITIES OF RAFALSON TYPE, ORTHOGONAL POLYNOMIALS.

1. INTRODUCTION

Let $d\mu$ be a positive Borel measure on $\mathbb{R}$ with infinite support whose moments are all finite. Then there exists an orthonormal polynomial system $\{P_n(d\mu; x)\}_{n=0}^{\infty}$ with respect to $d\mu$ such that

$$\int_{-\infty}^{\infty} P_m(d\mu; x)P_n(d\mu; x)\, d\mu(x) = \delta_{mn}, \qquad m, n = 0, 1, 2, \ldots,$$

where δ_{mn} is the Kronecker delta. One of the most important properties for $\{P_n(d\mu; x)\}_{n=0}^{\infty}$ is the three term recurrence relation

$$xP_n(d\mu; x) = a_{n+1}P_{n+1}(d\mu; x) + b_nP_n(d\mu; x) + a_nP_{n-1}(d\mu; x), \quad n = 0, 1, 2, \ldots,$$

where $P_{-1}(x) \equiv 0$, $P_0(d\mu; x) = (\int_{-\infty}^{\infty} d\mu(x))^{\frac{1}{2}}$, and

$$a_n = a_n(d\mu) = \int_{-\infty}^{\infty} xP_n(d\mu; x)P_{n-1}(d\mu; x)d\mu(x), \quad n \geq 1,$$

$$b_n = b_n(d\mu) = \int_{-\infty}^{\infty} xP_n^2(d\mu; x)d\mu(x), \quad n \geq 0.$$

It is interesting to find the best possible constant $\gamma_n = \gamma_n(d\nu; d\mu)$ such that

$$\|\pi\|_{d\nu} \leq \gamma_n\|\pi\|_{d\mu}, \quad \pi \in \mathcal{P}_n, \tag{1.1}$$

where $\mathcal{P}_n$ is the space of all real polynomials of degree at most n , $d\nu$ is another positive Borel measure on $\mathbb{R}$, and

$$\|\pi\|_{d\mu} := \Big\{ \int_{-\infty}^{\infty} \pi^2(x)d\mu(x)\Big\}^{\frac{1}{2}}.$$

The constant γ_n can be redefined by

$$\gamma_n(d\nu; d\mu) = \sup_{\pi \in \mathcal{P}_n} \{\|\pi\|_{d\nu} : \|\pi\|_{d\mu} = 1\}.$$

For $d\mu(x) = (1-x)^\alpha (1+x)^\beta dx$ and $d\nu(x) = (1-x)^\gamma (1+x)^\delta dx$ on $[-1, 1]$, γ_n was estimated in [1, 4] and for $\alpha = \beta = \frac{1}{2}$, $\gamma = \delta = \frac{3}{2}$ or $\alpha = \beta = \frac{3}{2}$, $\gamma = \delta = \frac{1}{2}$, the exact value of$\gamma_n$ was obtained by Refalson [5].

In this paper, we will prove that the constant γ_n can be expressed by the zeros of orthonormal polynomials with respect to $d\mu$ in cases (i) $d\nu(x) = (A + Bx)d\mu(x)$, where $A + Bx$ is nonnegative on the support of $d\mu$ and (ii) $d\nu(x) = (A + Bx^2)d\mu(x)$, where $d\mu$ is symmetric and $A + Bx^2$ is nonnegative on the support of $d\mu$. The extremal polynomial attaining γ_n is obtained and some concrete examples are given including Markov type inequality when $d\mu$ is a measure for Jacobi polynomials.

2. Case $d\nu(x) = (A + Bx)d\mu(x)$

The zeros of orthogonal polynomial $P_n(d\mu; x)$ are denoted by $x_{1n}(d\mu) > x_{2n}(d\mu) > \cdots x_{nn}(d\mu)$. Then by the Gauss quadrature formula, we have

$$x_{1,n+1}(d\mu) = \max_{\pi \in \mathcal{P}_n \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x\pi^2(x) d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x) d\mu(x)} \tag{2.1}$$

and

$$x_{n+1,n+1}(d\mu) = \min_{\pi \in \mathcal{P}_n \setminus \{0\}} \frac{\int_{-\infty}^{\infty} x\pi^2(x) d\mu(x)}{\int_{-\infty}^{\infty} \pi^2(x) d\mu(x)}. \tag{2.2}$$

The maximum and the minimum in (2.1) and (2.2) are attained if and only if $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x - x_{1,n+1}(d\mu)}$ and $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x - x_{n+1,n+1}(d\mu)}$, respectively, where c is a non-zero constant. Using these formula, we can easily prove:

Theorem 2.1. *Let $d\nu(x) = g(x)d\mu(x)$, where $g(x) = A + Bx$ is nonnegative on the support of $d\mu$. Then*

$$\gamma_n(d\nu; d\mu) = \left\{ \max_{k=1,2,\ldots,n+1} g(x_{k,n+1}) \right\}^{\frac{1}{2}} = \begin{cases} \sqrt{g(x_{1,n+1})} & \text{if } B \ge 0 \\ \sqrt{g(x_{n+1,n+1})} & \text{if } B < 0 \end{cases} \tag{2.3}$$

and

$$\gamma_n(d\mu; d\nu) = \left\{ \min_{k=1,2,\ldots,n+1} g(x_{k,n+1}) \right\}^{-\frac{1}{2}} = \begin{cases} g(x_{n+1,n+1})^{-\frac{1}{2}} & \text{if } B \ge 0 \\ g(x_{1,n+1})^{-\frac{1}{2}} & \text{if } B < 0, \end{cases} \tag{2.4}$$

where $x_{k,n+1} = x_{k,n+1}(d\mu)$. The constants $\gamma_n(d\nu; d\mu)$ in (2.3) and $\gamma_n(d\mu; d\nu)$ in (2.4) are attained if and only if $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x - x_{k,n+1}(d\mu)}$, where c is a non-zero constant and

$$k = \begin{cases} 1 & \text{if } B \ge 0 \\ n+1 & \text{if } B < 0 \end{cases} \quad \text{for} \quad \gamma_n(d\nu; d\mu), \qquad k = \begin{cases} n+1 & \text{if } B \ge 0 \\ 1 & \text{if } B < 0 \end{cases} \quad \text{for} \quad \gamma_n(d\mu; d\nu).$$

Proof. By the Gauss quadrature formula, we have for any $\pi \in \mathcal{P}_n$,

$$\begin{aligned}\int_{-\infty}^{\infty} \pi^2(x)d\nu(x) &= \int_{-\infty}^{\infty} (A+Bx)\pi^2(x)d\mu(x)\\ &= \sum_{k=1}^{n+1} \lambda_{k,n+1}(A+Bx_{k,n+1})\pi^2(x_{k,n+1})\\ &\le \max_{k=1,2,\ldots,n+1}(A+Bx_{k,n+1})\sum_{k=1}^{n+1}\lambda_{k,n+1}\ \pi^2(x_{k,n+1})\\ &= \max_{k=1,2,\ldots,n+1} g(x_{k,n+1})\int_{-\infty}^{\infty}\pi^2(x)d\mu(x),\end{aligned} \tag{2.5}$$

where $\lambda_{k,n+1} := \lambda_{k,n+1}(d\mu)$ are the Christoffel numbers for the measure $d\mu$. Now assume $B \ge 0$. Then $\max_{k=1,2,\cdots,n+1} g(x_{k,n+1}) = g(x_{1,n+1})$ and we have the equality in (2.5) for $\pi(x) = \frac{P_{n+1}(x)}{x-x_{1,n+1}}$. Conversely if the equality holds in (1.1) for $\pi(x)$, then the equality holds also in (2.5) so that $\pi(x_{k,n+1}) = 0,\ 2 \le k \le n+1$. Hence $\pi(x) = \frac{cP_{n+1}(x)}{x-x_{1,n+1}},\ c \ne 0$. This proves (2.3) when $B \ge 0$. In case $B < 0$, the proof is similar. Finally the equation (2.4) can be proved by a similar process using (2.2) instead of (2.1) and

$$\gamma_n^2(d\mu;d\nu) = \max_{\pi\in\mathcal{P}_n\backslash\{0\}} \frac{\int_{-\infty}^{\infty}\pi^2(x)d\mu(x)}{\int_{-\infty}^{\infty}\pi^2(x)d\nu(x)} = \left\{\min_{\pi\in\mathcal{P}_n\backslash\{0\}} \frac{\int_{-\infty}^{\infty}\pi^2(x)d\nu(x)}{\int_{-\infty}^{\infty}\pi^2(x)d\mu(x)}\right\}^{-1}. \tag{2.6}$$

□

Corollary 2.2. *Let $d\nu(x) = (1-x)^\alpha(1+x)^\beta dx$, $d\mu(x) = (1-x)^\gamma(1+x)^\delta dx$ on $[-1,1]$, and $\varphi_{\gamma,\delta}^{\alpha,\beta}(n) = \gamma_n(d\nu;d\mu)$, where $\alpha,\beta,\gamma,\delta > -1$. Then*

$$\varphi_{1/2,1/2}^{3/2,1/2}(n) = \varphi_{1/2,1/2}^{1/2,3/2}(n) = \sqrt{2}\cos\frac{\pi}{2(n+2)};$$

$$\varphi_{3/2,1/2}^{1/2,1/2}(n) = \varphi_{1/2,3/2}^{1/2,1/2}(n) = \left(\sqrt{2}\sin\frac{\pi}{2(n+1)}\right)^{-1};$$

$$\varphi_{-1/2,-1/2}^{1/2,-1/2}(n) = \varphi_{-1/2,-1/2}^{-1/2,1/2}(n) = \sqrt{2}\cos\frac{\pi}{4(n+1)};$$

$$\varphi_{1/2,-1/2}^{-1/2,-1/2}(n) = \varphi_{-1/2,1/2}^{-1/2,-1/2}(n) = \left(\sqrt{2}\sin\frac{\pi}{4(n+1)}\right)^{-1}.$$

Proof. Let $g(x) = 1-x$. Since the smallest zero of Chebychev polynomial $U_{n+1}(x)$ of the second kind is $-\cos\frac{\pi}{n+2}$,

$$\varphi_{1/2,1/2}^{3/2,1/2}(n) = \sqrt{1+\cos\frac{\pi}{n+2}} = \sqrt{2}\cos\frac{\pi}{2(n+2)}. \tag{2.7}$$

All others can be proved similarly by Theorem 2.1. □

Example 2.1. Let $d\mu(x) = x^\alpha e^{-x}dx$ and $d\nu(x) = xd\mu(x)$ on $[0, \infty)$, where $\alpha > -1$. Using the asymptotic behavior of the greatest zero $x_{1,n+1}$ of the Laguerre polynomial $L_{n+1}^{(\alpha)}(x)$[6], we can use

$$\lim_{n\to\infty} \frac{\gamma_n(d\nu; d\mu)}{2\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{x_{1,n+1}}}{2\sqrt{n}} = 1.$$

Let $d\nu(x) = g(x)d\mu(x)$, where $g \in \mathcal{P}_\ell$ is nonnegative on $[0, \infty)$. Then by the same process as in the proof of Theorem 2.1, we have for any $\pi \in \mathcal{P}_n$,

$$\int_0^\infty \pi^2(x)d\nu(x) \le \max_{k=1,2,\ldots,n+m} g(x_{k,n+m}) \int_0^\infty \pi^2(x)d\mu(x), \quad m = \left[\frac{\ell+1}{2}\right]$$

and

$$\int_0^\infty \pi^2(x)d\nu(x) \ge \min_{k=1,2,\ldots,n+m} g(x_{k,n+m}) \int_0^\infty \pi^2(x)d\mu(x), \quad m = \left[\frac{\ell+1}{2}\right].$$

Hence, we obtain an estimation for $\gamma_n(d\nu; d\mu)$:

$$\min_{k=1,2,\ldots,n+m} g(x_{k,n+m}) \le \gamma_n^2(d\nu; d\mu) \le \max_{k=1,2,\ldots,n+m} g(x_{k,n+m}). \tag{2.8}$$

But, the estimate (2.8) is not sharp in general if $l \ge 2$.

3. Case $d\nu(x) = (A + Bx^2)d\mu(x)$

In this section $d\mu$ is assumed to be symmetric and so the corresponding orthonormal polynomials satisfy

$$xP_n(d\mu; x) = a_{n+1}P_{n+1}(d\mu; x) + a_nP_{n-1}(d\mu; x), \quad n \ge 0.$$

Lemma 3.1. *Let $d\mu$ be symmetric. Then we have*

$$x_{1,n+2}(d\mu) = \max_{\pi\in\mathcal{P}_n\setminus\{0\}} \frac{\int_{-\infty}^\infty x^2\pi^2(x)d\mu(x)}{\int_{-\infty}^\infty \pi^2(x)d\mu(x)}$$

and equality holds if and only if $\pi(x) = \frac{cP_{n+2}(d\mu;x)}{x^2-x_{1,n+2}^2}$, where c is a nonzero constant.

Proof. See Theorem 2 in [2]. ☐

Lemma 3.2. *For any $(n+1) \times (n+1)$ matrix W $(n \ge 1)$,*

$$W := \begin{pmatrix}
\alpha_0 & 0 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & \alpha_1 & 0 & \beta_2 & 0 & 0 & \cdots & 0 \\
\beta_1 & 0 & \alpha_2 & 0 & \beta_3 & 0 & \cdots & 0 \\
0 & \beta_2 & 0 & \alpha_3 & 0 & \beta_4 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \beta_{n-1} \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \alpha_{n-1} & 0 \\
0 & \cdots & \cdots & \cdots & 0 & \beta_{n-1} & 0 & \alpha_n
\end{pmatrix} \tag{3.1}$$

we have $|W| = |U||V|$, *where* $|W|$ *is the determinant of the matrix* W,

$$U := \begin{pmatrix} \alpha_0 & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 & \beta_3 & 0 & 0 & \cdots & 0 \\ 0 & \beta_3 & \alpha_4 & \beta_5 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2m-2} & \beta_{2m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2m-1} & \alpha_{2m} \end{pmatrix}, \qquad m := \left[\frac{n}{2}\right],$$

and

$$V := \begin{pmatrix} \alpha_1 & \beta_2 & 0 & 0 & 0 & \cdots & 0 \\ \beta_2 & \alpha_3 & \beta_4 & 0 & 0 & \cdots & 0 \\ 0 & \beta_4 & \alpha_5 & \beta_6 & 0 & \cdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2\ell-1} & \beta_{2\ell} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2\ell} & \alpha_{2\ell+1} \end{pmatrix}, \qquad \ell := \left[\frac{n-1}{2}\right].$$

Proof. It can be easily shown by elementary row and column operations. □

Theorem 3.3. *Let* $d\nu(x) = (A + Bx^2)d\mu(x)$, *where* $A + Bx^2$ *is nonnegative on the support of* $d\mu$. *If* $d\mu$ *is symmetric, then*

$$\gamma_n(d\nu; d\mu) = \begin{cases} \sqrt{A + Bx_{1,n+2}^2} & \text{if } B \geq 0 \\ \sqrt{A + Bx_{s+1,n+2}^2} & \text{if } B < 0 \text{ and } n = 2s \\ \sqrt{A + Bx_{s+1,n+1}^2} & \text{if } B < 0 \text{ and } n = 2s + 1 \end{cases} \tag{3.2}$$

and

$$\gamma_n(d\mu; d\nu) = \begin{cases} (A + Bx_{1,n+2}^2)^{-\frac{1}{2}} & \text{if } B \leq 0 \\ (A + Bx_{s+1,n+2}^2)^{-\frac{1}{2}} & \text{if } B > 0 \text{ and } n = 2s \\ (A + Bx_{s+1,n+1}^2)^{-\frac{1}{2}} & \text{if } B > 0 \text{ and } n = 2s + 1. \end{cases} \tag{3.3}$$

Proof. We will prove only (3.2). Then (3.3) can be proved by a similar process with (2.6). When $B = 0$, it is trivial and so we may assume $B \neq 0$. Let $\pi(x) = \sum_{k=0}^{n} c_k P_k(d\mu; x)$. Then by the three term recurrence relation,

$$\begin{aligned} (A + Bx^2)\pi(x) &= \sum_{k=0}^{n} (A + Bx^2) c_k P_k(x) \\ &= \sum_{k=0}^{n} [A + B(a_{k+1}^2 + a_k^2)] c_k P_k(x) \\ &\quad + \sum_{k=2}^{n+2} B a_k a_{k-1} c_{k-2} P_k(x) + \sum_{k=0}^{n-2} B a_{k+2} a_{k+1} c_{k+2} P_k(x), \end{aligned}$$

where $a_k = a_k(d\mu)$ and $P_k(x) = P_k(d\mu; x)$. Hence, by the orthonormality of $\{P_n(x)\}_{n=0}^{\infty}$,

$$\int_{-\infty}^{\infty} \pi^2(x) d\nu(x) = \sum_{k=0}^{n} [A + B(a_{k+1}^2 + a_k^2)] c_k^2 + 2 \sum_{k=0}^{n-2} B a_{k+2} a_{k+1} c_k c_{k+2}.$$

If we assume that $\|\pi\|_{d\mu} = 1$, that is, $\sum_{k=0}^{n} c_k^2 = 1$, then

$$\gamma_n^2(d\nu; d\mu) = \max_{\sum_{k=0}^{n} c_k^2 = 1} \left\{ \sum_{k=0}^{n} [A + B(a_{k+1}^2 + a_k^2)] c_k^2 + 2 \sum_{k=0}^{n-2} B a_{k+2} a_{k+1} c_k c_{k+2} \right\},$$

which is equal to $\max\{|\lambda| : \lambda \text{ is an eigenvalue of } W\}$, where W is the matrix (3.1) with $\alpha_k = A + B(a_k^2 + a_{k+1}^2)$ and $\beta_k = B a_k a_{k+1}$. By Lemma 3.2, $\gamma_n(d\nu : d\mu) = \max\{|\lambda| : U_m(\lambda) = 0$ or $V_\ell(\lambda) = 0\}$, where

$$U_m(\lambda) = \begin{vmatrix} \alpha_0 - \lambda & \beta_1 & 0 & 0 & 0 & \cdots & 0 \\ \beta_1 & \alpha_2 - \lambda & \beta_3 & 0 & 0 & \cdots & 0 \\ 0 & \beta_3 & \alpha_4 - \lambda & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2m-2} - \lambda & \beta_{2m-1} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2m-1} & \alpha_{2m} - \lambda \end{vmatrix} \left(m = \left[\frac{n}{2} \right] \right)$$

and

$$V_\ell(\lambda) = \begin{vmatrix} \alpha_1 - \lambda & \beta_2 & 0 & 0 & 0 & \cdots & 0 \\ \beta_2 & \alpha_3 - \lambda & \beta_4 & 0 & 0 & \cdots & 0 \\ 0 & \beta_4 & \alpha_5 - \lambda & \ddots & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \alpha_{2\ell-1} - \lambda & \beta_{2\ell} \\ 0 & \cdots & \cdots & \cdots & 0 & \beta_{2\ell} & \alpha_{2\ell+1} - \lambda \end{vmatrix} \left(\ell = \left[\frac{n-1}{2} \right] \right).$$

Now zeros of $U_m(\lambda)$ and $V_\ell(\lambda)$ are the zeros of orthonormal polynomials $S_{m+1}(x)$ and $T_{\ell+1}(x)$, respectively, satisfying

$$xS_k = B a_{2k+2} a_{2k+1} S_{k+1} + [A + B(a_{2k+1}^2 + a_{2k}^2)] S_k + B a_{2k} a_{2k-1} S_{k-1}, \tag{3.4}$$

$$xT_k = B a_{2k+3} a_{2k+2} T_{k+1} + [A + B(a_{2k+2}^2 + a_{2k+1}^2)] T_k + B a_{2k+1} a_{2k} T_{k-1}. \tag{3.5}$$

On the other hand, since $d\mu$ is symmetric, if we set

$$Q_k(x^2) = P_{2k}(d\mu; x) \quad \text{and} \quad xR_k(x^2) = P_{2k+1}(d\mu; x), \quad k \geq 0, \tag{3.6}$$

then $\{Q_k(x)\}_{k=0}^{\infty}$ and $\{R_k(x)\}_{k=0}^{\infty}$ are orthonormal polynomials satisfying the three term recurrence relations

$$xQ_k(x) = a_{2k+2} a_{2k+1} Q_{k+1}(x) + (a_{2k+1}^2 + a_{2k}^2) Q_k(x) + a_{2k} a_{2k-1} Q_{k-1}(x), \quad k \geq 0, \tag{3.7}$$

$$xR_k(x) = a_{2k+3}a_{2k+2}R_{k+1}(x) + (a_{2k+2}^2 + a_{2k+1}^2)R_k(x) + a_{2k+1}a_{2k}R_{k-1}(x), \quad k \geq 0. \tag{3.8}$$

Then $\{Q_n(\frac{1}{B}(x-A))\}_{n=0}^{\infty}$ and $\{R_n(\frac{1}{B}(x-A))\}_{n=0}^{\infty}$ satisfy the recurrence relations (3.4) and (3.5), respectively. Hence,

$$S_{m+1}(x) = Q_{m+1}(\frac{1}{B}(x-A)) \text{ and } T_{\ell+1}(x) = R_{\ell+1}(\frac{1}{B}(x-A)).$$

From the relation (3.6), $Q_{m+1}(x_{k,2m+2}^2) = 0$, $k = 1, 2, \ldots, m+1$, and $R_{\ell+1}(x_{k,2\ell+3}^2) = 0$, $k = 1, 2, \ldots, \ell+1$ and so

$$S_{m+1}(A + Bx_{k,2m+2}^2) = 0, \quad k = 1, 2, \ldots, m+1$$

$$T_{\ell+1}(A + Bx_{k,2\ell+3}^2) = 0, \quad k = 1, 2, \ldots, \ell+1.$$

Hence

$$\begin{aligned}\gamma_n^2(d\nu; d\mu) &= \max_{\substack{k=1,2,\ldots,m+1;\\ j=1,2,\ldots,\ell+1}} \{A + Bx_{k,2m+2}^2, A + Bx_{j,2\ell+3}^2\} \\ &= \begin{cases} A + B\max\{x_{1,2m+2}^2, x_{1,2\ell+3}^2\} & \text{if B} > 0 \\ \\ A + B\min\{x_{m+1,2m+2}^2, x_{\ell+1,2\ell+3}^2\} & \text{if B} < 0. \end{cases}\end{aligned}$$

If $B > 0$ and $n = 2s$ is even, then $m = s$ and $\ell = s - 1$ so that

$$\gamma_n^2(d\nu; d\mu) = A + Bx_{1,2s+2}^2 = A + Bx_{1,n+2}^2.$$

If $B > 0$ and $n = 2s + 1$ is odd, then $m = s$ and $\ell = s$ so that

$$\gamma_n^2(d\nu; d\mu) = A + Bx_{1,2s+3}^2 = A + Bx_{1,n+2}^2.$$

If $B < 0$ and $n = 2s$ is even, then $m = s$ and $\ell = s - 1$ so that

$$\gamma_n^2(d\nu; d\mu) = A + Bx_{s+1,2s+2}^2 = A + Bx_{s+1,n+2}^2$$

since $0 < x_{s+1,n+1} < x_{s,n+1}$. If $B < 0$ and $n = 2s + 1$ is odd, then $m = \ell = s$ so that

$$\gamma_n^2(d\nu; d\mu) = A + Bx_{s+1,2s+2}^2 = A + Bx_{s+1,n+1}^2.$$

since $0 < x_{s+1,n+1} < x_{s+1,n+2}$. Hence, the conclusion follows. □

Note that the constant $\gamma_n(d\nu; d\mu)$ in (2.2) is attained if and only if

$$\pi(x) = \begin{cases} \frac{cP_{n+2}(d\mu;x)}{x^2 - x_{1,n+2}^2} & \text{if } B \geq 0 \\ \\ \frac{cQ_{s+1}(x)}{x^2 - x_{s+1,2s+2}^2} & \text{if } B < 0, \end{cases}$$

where c is a non-zero constant.

Corollary 3.4. *Let $d\nu(x) = (1-x)^\alpha(1+x)^\beta dx$, $d\mu(x) = (1-x)^\gamma(1+x)^\delta dx$ on $[-1,1]$, and $\varphi^{\alpha,\beta}_{\gamma,\delta}(n) = \gamma_n(d\nu; d\mu)$, where $\alpha, \beta, \gamma, \delta > -1$. Then*

$$\varphi^{3/2,3/2}_{1/2,1/2}(n) = \begin{cases} \cos\frac{\pi}{2(n+3)} & \text{if } n \text{ is even} \\ \cos\frac{\pi}{2(n+2)} & \text{if } n \text{ is odd} \end{cases};$$

$$\varphi^{1/2,1/2}_{3/2,3/2}(n) = \Big(\sin\frac{\pi}{n+3}\Big)^{-1};$$

$$\varphi^{1/2,1/2}_{-1/2,-1/2}(n) = \begin{cases} \cos\frac{\pi}{2(n+2)} & \text{if } n \text{ is even} \\ \cos\frac{\pi}{2(n+1)} & \text{if } n \text{ is odd} \end{cases};$$

$$\varphi^{-1/2,-1/2}_{1/2,1/2}(n) = \Big(\sin\frac{\pi}{2(n+2)}\Big)^{-1}.$$

Proof. If $\alpha = \beta = \frac{3}{2}$ and $\gamma = \delta = \frac{1}{2}$, then $d\nu(x) = (1-x^2)d\mu(x)$ and the orthonormal polynomials $\{U_n(x)\}_{n=0}^\infty$ with respect to $d\mu$ are the Chebychev polynomials of the second kind, whose zeros are

$$x_{kn}(d\mu) = \cos\frac{k\pi}{n+1}, \quad k = 1, 2, \ldots, n.$$

Hence, by Theorem 3.3, if $n = 2s$, then

$$\varphi^{3/2,3/2}_{1/2,1/2}(n) = \sqrt{1 - \cos^2\frac{(s+1)\pi}{2s+3}} = \cos\frac{\pi}{2(n+3)}$$

and if $n = 2s+1$, then

$$\varphi^{3/2,3/2}_{1/2,1/2}(n) = \sqrt{1 - \cos^2\frac{(s+1)\pi}{2s+3}} = \cos\frac{\pi}{2(n+2)}.$$

All the other cases can be obtained similarly by Theorem 3.3 and the zeros of the Chebychev polynomials of the first and the second kinds. □

Corollary 3.5. *Let $d\mu$ be symmetric. Then we have*

$$\min_{\pi \in \mathcal{P}_n \setminus \{0\}} \frac{\int_{-\infty}^\infty x^2\pi^2(x)d\mu(x)}{\int_{-\infty}^\infty \pi^2(x)d\mu(x)} = \begin{cases} x^2_{s+1,n+2} & \text{if } n = 2s \\ \\ x^2_{s+1,n+1} & \text{if } n = 2s+1. \end{cases} \tag{3.9}$$

The minimum is attained if and only if $\pi(x) = \frac{cP_{n+2}(d\mu;x)}{x^2 - x^2_{s+1,n+2}(d\mu)}$ when $n = 2s$ and $\pi(x) = \frac{cP_{n+1}(d\mu;x)}{x^2 - x^2_{s+1,n+1}(d\mu)}$ when $n = 2s+1$, where c is a nonzero constant.

Proof. Take $A = 0$ and $B = 1$ in Theorem 3.3. Then $\min_{\pi \in \mathcal{P}_n \setminus \{0\}} \frac{\int_{-\infty}^\infty x^2\pi^2(x)d\mu(x)}{\int_{-\infty}^\infty \pi^2(x)d\mu(x)} = \gamma_n^{-2}(d\mu; d\nu)$ and so (3.9) holds by Theorem 3.3. By the Gauss quadrature formula and (2.6), we can show that the minimum is attained only when

$$\pi(x) = \begin{cases} \frac{cP_{n+2}(d\mu;x)}{x^2 - x^2_{s+1,n+2}(d\mu)} & \text{if } n = 2s \\ \\ \frac{cP_{n+1}(d\mu;x)}{x^2 - x^2_{s+1,n+1}(d\mu)} & \text{if } n = 2s+1, \end{cases}$$

where c is a nonzero constant. □

The following sharp inequality was proved in [3](see also [1] for $\alpha = \beta$). If $\pi \in \mathcal{P}_n$ and $\alpha, \beta > -1$, then

$$\|\pi^{(m)}\|_{m+\alpha,m+\beta} \le \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}}\|\pi\|_{\alpha,\beta}, \tag{3.10}$$

where

$$\|\pi\|_{\alpha,\beta} = \Big(\int_{-1}^{1} \pi^2(x)(1-x)^\alpha(1+x)^\beta dx\Big)^{\frac{1}{2}}.$$

Applying Theorem 2.1 iteratively, if $\alpha = \beta + k$, then

$$\begin{aligned}
\|\pi^{(m)}\|_{\alpha+m,\beta+m} &\le \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}}\ \|\pi\|_{\beta+k,\beta} \\
&\le \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}} \prod_{j=0}^{k-1} \varphi_{\beta+j,\beta}^{\beta+j+1,\beta}(n)\ \|\pi\|_{\beta,\beta} \qquad (3.11)\\
&= \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}} \prod_{j=0}^{k-1} \Big(1 - x_{n+1,n+1}^{\beta+j,\beta}\Big)^{\frac{1}{2}}\ \|\pi\|_{\beta,\beta},
\end{aligned}$$

where $\{x_{k,n}^{\alpha,\beta}\}_{k=1}^{n}$ denotes the zeros of Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$. Similarly, if $\alpha = \beta - k$, then

$$\|\pi^{(m)}\|_{\alpha+m,\beta+m} \le \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}} \prod_{j=0}^{k-1} \Big(1 + x_{1,n+1}^{\alpha,\alpha+j}\Big)^{\frac{1}{2}}\ \|\pi\|_{\alpha,\alpha}.$$

Combining Theorem 3.3 and applying Theorem 2.1 again, we obtain a Markov type inequality. More precisely, if $\alpha = \beta + k$, then

$$\begin{aligned}
\|\pi^{(m)}\|_{\alpha+m,\beta+m} &\le D_{n,m}^{\alpha,\beta} \prod_{j=0}^{k-1} \sqrt{1 - x_{n+1,n+1}^{\beta+j,\beta}}\,\|\pi\|_{\beta,\beta} \\
&\le D_{n,m}^{\alpha,\beta} \prod_{j=0}^{k-1} \sqrt{1 - x_{n+1,n+1}^{\beta+j,\beta}} \prod_{j=0}^{m-1} \Big(1 - (x_{1,n+2}^{\beta+j,\beta+j})^2\Big)^{-\frac{1}{2}}\ \|\pi\|_{\beta+k+m,\beta+m} \\
&= D_{n,m}^{\alpha,\beta} \prod_{j=0}^{k-1} \sqrt{\frac{1 - x_{n+1,n+1}^{\beta+j,\beta}}{1 + x_{1,n+1}^{\beta+m+j,\beta+m}}} \prod_{j=0}^{m-1} \Big(1 - (x_{1,n+2}^{\beta+j,\beta+j})^2\Big)^{-\frac{1}{2}}\ \|\pi\|_{\alpha+m,\beta+m}
\end{aligned}$$

and if $\alpha = \beta - k$, then

$$\|\pi^{(m)}\|_{\alpha+m,\beta+m} \le D_{n,m}^{\alpha,\beta} \prod_{j=0}^{k-1} \sqrt{\frac{1 + x_{1,n+1}^{\alpha,\alpha+j}}{1 + x_{n+1,n+1}^{\alpha+m,\alpha+m+j}}} \prod_{j=0}^{m-1} \Big(1-(x_{1,n+2}^{\alpha+j,\alpha+j})^2\Big)^{-\frac{1}{2}}\ \|\pi\|_{\alpha+m,\beta+m}, \tag{3.12}$$

where

$$D_{n,m}^{\alpha,\beta} = \sqrt{\frac{n!\,\Gamma(n+\alpha+\beta+m+1)}{(n-m)!\,\Gamma(n+\alpha+\beta+1)}}.$$

In particular, if $k = 0$, then $\alpha = \beta$ and

$$\|\pi^{(m)}\|_{\alpha+m,\beta+m} \leq \sqrt{\frac{n!\,\Gamma(n+m+2\alpha+1)}{(n-m)!\,\Gamma(n+2\alpha+1)}} \prod_{j=0}^{m-1} \left(1 - (x_{1,n+2}^{\alpha+j,\alpha+j})^2\right)^{-\frac{1}{2}} \|\pi\|_{m+\alpha,m+\alpha},$$

which is a Markov type inequality for ultraspherical polynomials. As a special case, we obtain ($\alpha = \beta = -\frac{1}{2}$ and $m = 1$)

$$\|\pi'\|_{\frac{1}{2},\frac{1}{2}} \leq \frac{n}{\sin\frac{\pi}{2(n+2)}} \|\pi\|_{\frac{1}{2},\frac{1}{2}},$$

which was also found in [5]. In this way, we can obtain various kinds of inequalities using (3.10), (3.11), and (3.12).

Acknowledgment. The first author(KHK) was partially supported by KRF-2002-070-C00004 and the second author(DWL) was supported by the Post-doctoral Fellowship Program of KOSEF.

References

[1] I.K.Daugavet and S.Z.Rafalson, *Some inequalities of Markov-Nikolskii type for algebraic polynomials*, Vestnik Leningrad. Univ. Mekh. Astronom., **1** (1972), 15-25.
[2] G.Freud, *On the greatest zero of orthogonal polynomials I*, Acta Sci. Math.(Szeged), **34** (1973), 91-97.
[3] A.Guessab and G.V.Milovanovic, *Weighted L^2-analogues of Bernstein's inequality and classical orthogonal polynomials*, J. Math. Anal. Appl., **182** (1994), 244-249.
[4] S.V.Konjagin, *On estimates of derivatives of polynomials*, Dokl. Akad. Nauk SSSR, **243** (1978), 1116-1118.
[5] S.Z.Rafalson, *Some sharp inequalities for algebraic polynomials*, J. Approx. Th., **95** (1998), 161-171.
[6] G.Szegö, *Orthogonal Polynomials*, Amer. Math. Soc. Colloquium Publ., N.Y. (1939).

[1]K. H. Kwon
Division of Applied Mathematics
KAIST
Taejon 305-701, Korea
khkwon@jacobi.kaist.ac.kr

[2]D. W. Lee
Department of Mathematics
Teachers College
Kyungpook National University
Taegu 702-701, Korea
dowlee@knu.ac.kr