Approximation by Quasi-projection Operators in Besov Spaces

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† Supported in part by NSERC Canada under Grant OGP 121336

Abstract

In this paper, we investigate approximation of quasi-projection operators in Besov spaces $B^\mu_{p,q}$, $\mu > 0$, $1 \le p, q \le \infty$. Suppose I is a countable index set. Let $(\phi_i)_{i\in I}$ be a family of functions in $L_p(\mathbb{R}^s)$, and let $(\tilde{\phi}_i)_{i\in I}$ be a family of functions in $L_{\tilde{p}}(\mathbb{R}^s)$, where $1/p + 1/\tilde{p} = 1$. Let Q be the quasi-projection operator given by

$$Qf = \sum_{i\in I} \langle f, \tilde{\phi}_i \rangle \phi_i, \quad f \in L_p(\mathbb{R}^s).$$

For $h > 0$, by σ_h we denote the scaling operator given by $\sigma_h f(x) := f(x/h)$, $x \in \mathbb{R}^s$. Let $Q_h := \sigma_h Q \sigma_{1/h}$. Under some mild conditions on the functions ϕ_i and $\tilde{\phi}_i$ $(i \in I)$, we establish the following result: If $0 < \mu < \nu < k$, and if $Qg = g$ for all polynomials of degree at most $k - 1$, then the estimate

$$|f - Q_h f|_{B^\mu_{p,q}} \le C h^{\nu-\mu} |f|_{B^\nu_{p,q}} \quad \forall\, f \in B^\nu_{p,q}(\mathbb{R}^s)$$

is valid for all $h > 0$, where C is a constant independent of h and f. Density of quasi-projection operators in Besov spaces is also discussed.

Key words and phrases. approximation order, moduli of smoothness, quasi-projection, quasi-interpolation, Sobolev spaces, Besov spaces.

Approximation by Quasi-projection Operators in Besov Spaces

§1. Introduction

A quasi-interpolant scheme can be described as follows. Suppose that $(\phi_i)_{i\in I}$ is a family of elements in a Banach space F, where I is a countable index set. Let $(\lambda_i)_{i\in I}$ be a family of continuous functionals on F. The **quasi-interpolant** associated with $(\lambda_i)_{i\in I}$ and $(\phi_i)_{i\in I}$ is the linear operator Q given by

$$Qf := \sum_{i\in I} \lambda_i(f)\phi_i, \quad f \in F.$$

When the ϕ_i's are univariate splines, quasi-interpolants were introduced by de Boor and Fix [3] as efficient schemes of spline approximation. Similar schemes were discussed by Lyche and Schumaker [18]. For L_p approximation, de Boor [1] proposed approximation schemes using linear projectors induced by dual functionals. The idea of de Boor was developed by Jia and Lei [13] and applied to shift-invariant spaces. Furthermore, Lei, Jia, and Cheney [17] investigated approximation with scaled shift-invariant spaces by means of certain integral operators. For L_2 approximation, Jetter and Zhou [8] also employed a projection method to realize the optimal approximation order as given in [2].

In this paper, we are interested in quasi-interpolation schemes in Besov spaces. Before going on, we introduce some notation. Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ denote the set of positive integers, integers, and real numbers, respectively. For a complex-valued (Lebesgue) measurable function f on a measurable subset E of $\mathbb{R}^s$, let

$$\|f\|_p(E) := \left(\int_E |f(x)|^p \, dx\right)^{1/p} \qquad \text{for } \ 1 \le p < \infty,$$

and let $\|f\|_\infty(E)$ denote the essential supremum of $|f|$ on E. When $E = \mathbb{R}^s$, we omit the reference to E. For $1 \le p \le \infty$, by $L_p(\mathbb{R}^s)$ we denote the Banach space of all measurable functions f on $\mathbb{R}^s$ such that $\|f\|_p < \infty$.

Suppose $1 \le p \le \infty$ and $1/p + 1/\tilde{p} = 1$. Let $(\phi_i)_{i\in I}$ be a family of functions in $L_p(\mathbb{R}^s)$, and let $(\tilde{\phi}_i)_{i\in I}$ be a family of functions in $L_{\tilde{p}}(\mathbb{R}^s)$. Each $\tilde{\phi}_i$ induces the continuous functional λ_i as follows:

$$\lambda_i(f) := \langle f, \tilde{\phi}_i\rangle := \int_{\mathbb{R}^s} f(x)\tilde{\phi}_i(x)\, dx, \quad f \in L_p(\mathbb{R}^s).$$

Thus, we have

$$Qf = \sum_{i\in I} \langle f, \tilde{\phi}_i\rangle \phi_i, \quad f \in L_p(\mathbb{R}^s). \tag{1.1}$$

In this paper, Q is called a **quasi-projection operator**. We assume that there exists a constant $M > 0$ such that $\|\phi_i\|_p \le M$ and $\|\tilde{\phi}_i\|_{\tilde{p}} \le M$ for all $i \in I$. Let $(c_i)_{i\in I}$ be a sequence of points in $\mathbb{R}^s$ with the property that there exists a positive integer N such that each cube $\alpha + [0,1]^s$ $(\alpha \in \mathbb{Z}^s)$ contains at most N points c_i. Suppose that there exists a constant $K > 0$ such that, for each $i \in I$, both ϕ_i and $\tilde{\phi}_i$ are supported on the cube $c_i + [-K, K]^s$. Under these assumptions, it can be easily proved that Q is a bounded operator on $L_p(\mathbb{R}^s)$ (see Lemmas 3.1 and 3.2 in [11]). Moreover, for a locally integrable function f on $\mathbb{R}^s$, Qf is well defined.

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. An element of $\mathbb{N}_0^s$ is called a **multi-index**. The length of a mutli-index $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ is given by $|\mu| := \mu_1 + \cdots + \mu_s$. For $\mu = (\mu_1, \ldots, \mu_s) \in \mathbb{N}_0^s$ and $x = (x_1, \ldots, x_s) \in \mathbb{R}^s$, define

$$x^\mu := x_1^{\mu_1} \cdots x_s^{\mu_s}.$$

The function $x \mapsto x^\mu$ ($x \in \mathbb{R}^s$) is called a monomial and its (total) degree is $|\mu|$. A polynomial is a linear combination of monomials. The degree of a polynomial $g = \sum_\mu c_\mu x^\mu$ is defined to be $\deg g := \max\{|\mu| : c_\mu \neq 0\}$. By Π_k we denote the linear space of all polynomials of degree at most k.

Polynomial reproducibility plays a vital role in approximation. Using Fourier analysis, Strang and Fix [21] gave conditions for polynomial reproduction of integer shifts of basis functions. In fact, Schoenberg [20] already obtained the same conditions for the univariate case. See the survey paper [12] for a comprehensive review on the Strang-Fix conditions and related problems of approximation by refinable vectors of functions. In the setting of shift-invariant spaces, the Strang-Fix conditions were used in Lemma 3.2 of [9] to guarantee that

$$Qg = g \quad \forall\, g \in \Pi_{k-1}, \tag{1.2}$$

where k is a positive integer. Our study in this paper is not restricted to shift-invariant spaces. But the polynomial reproducibility in (1.2) will be required throughout the paper.

For a vector $y = (y_1, \ldots, y_s) \in \mathbb{R}^s$, its norm is defined as $|y| := \max_{1 \le j \le s} |y_j|$. We use D_y to denote the differential operator given by

$$D_y f(x) := \lim_{t \to 0} \frac{f(x + ty) - f(x)}{t}, \qquad x \in \mathbb{R}^s.$$

Moreover, we use ∇_y to denote the difference operator given by $\nabla_y f = f - f(\cdot - y)$. Let $e_1, \ldots, e_s$ be the unit coordinate vectors in $\mathbb{R}^s$. For $j = 1, \ldots, s$, we write D_j for D_{e_j}. For a multi-index $\mu = (\mu_1, \ldots, \mu_s)$, D^μ stands for the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. For a measurable subset E in $\mathbb{R}^s$, we define

$$\|f\|_{k,p}(E) := \sum_{j=0}^{k} |f|_{j,p}(E) \quad \text{with} \quad |f|_{j,p}(E) := \sum_{|\mu|=j} \|D^\mu f\|_p(E).$$

When $E = \mathbb{R}^s$, we omit the reference to E. For $1 \le p \le \infty$, the Sobolev space $W_p^k(\mathbb{R}^s)$ consists of all functions $f \in L_p(\mathbb{R}^s)$ such that $\|f\|_{k,p} < \infty$

We are in a position to discuss approximation by quasi-projection operators in Sobolev spaces. For $h > 0$, by σ_h we denote the scaling operator given by $\sigma_h f(x) := f(x/h)$, $x \in \mathbb{R}^s$. Let $Q_h := \sigma_h Q \sigma_{1/h}$. We have

$$Q_h f(x) = \sum_{i \in I} \langle f, h^{-s/\tilde{p}} \tilde{\phi}_i(\cdot/h) \rangle\, h^{-s/p} \phi_i(x/h), \quad x \in \mathbb{R}^s. \tag{1.3}$$

Evidently, $\|Q_h\|_p = \|Q\|_p$. Suppose that $I = \mathbb{Z}^s$ and, for each $i \in \mathbb{Z}^s$, $\phi_i = \phi(\cdot - i)$ and $\tilde{\phi}_i = \tilde{\phi}(\cdot - i)$, where $\phi \in L_p(\mathbb{R}^s)$ and $\tilde{\phi} \in L_{\tilde{p}}(\mathbb{R}^s)$, $1/p + 1/\tilde{p} = 1$. Under some mild decay conditions on ϕ and $\tilde{\phi}$, it was proved in [13] that there exists a positive constant C independent of h and f such that

$$\|f - Q_h f\|_p \le C h^k |f|_{k,p} \quad \forall\, f \in W_p^k(\mathbb{R}^s),$$

provided $Qg = g$ for all $g \in \Pi_{k-1}$. Furthermore, it was shown by Kyriazis [16] that for $0 \le j < k$,

$$|f - Q_h f|_{j,p} \le C h^{k-j} |f|_{k,p} \quad \forall\, f \in W_p^k(\mathbb{R}^s). \tag{1.4}$$

We will see that (1.4) remains true in the general setting.

Let us turn to the study of approximation in Besov spaces. For a positive integer k, the kth **modulus of smoothness** of a function f in $L_p(\mathbb{R}^s)$ is defined by

$$\omega_k(f,h)_p := \sup_{|y|\le h} \left\|\nabla_y^k f\right\|_p, \qquad h \ge 0.$$

In particular, $\omega(f,h)_p := \omega_1(f,h)_p$ is the **modulus of continuity** of f in $L_p(\mathbb{R}^s)$. For $\mu > 0$ and $1 \le p, q \le \infty$, the Besov space $B_{p,q}^\mu = B_{p,q}^\mu(\mathbb{R}^s)$ is the collection of those functions $f \in L_p(\mathbb{R}^s)$ for which the following semi-norm is finite:

$$|f|_{B_{p,q}^\mu} := \begin{cases} \left(\int_0^\infty \left[t^{-\mu}\omega_m(f,t)_p\right]^q \frac{1}{t}\, dt\right)^{1/q}, & \text{for } 1 \le q < \infty, \\ \sup_{t>0}\left\{t^{-\mu}\omega_m(f,t)_p\right\}, & \text{for } q = \infty, \end{cases}$$

where m is an integer greater than μ. It is easily seen that

$$|f|_{B_{p,q}^\mu} \approx \begin{cases} \left(\sum_{j\in\mathbb{Z}} \left[2^{j\mu}\omega_m(f,2^{-j})_p\right]^q\right)^{1/q}, & \text{for } 1 \le q < \infty, \\ \sup_{j\in\mathbb{Z}}\left\{2^{j\mu}\omega_m(f,2^{-j})_p\right\}, & \text{for } q = \infty. \end{cases}$$

In view of this equivalent semi-norm, we have $|f|_{B_{p,\infty}^\mu} \le C|f|_{B_{p,q}^\mu}$ and $B_{p,q}^\mu \subseteq B_{p,\infty}^\mu$. The norm for $B_{p,q}^\mu$ is

$$\|f\|_{B_{p,q}^\mu} := \|f\|_{L_p} + |f|_{B_{p,q}^\mu}.$$

The main result of this paper is the following. Suppose that $0 < \mu < \nu < k$ and $1 \le p, q \le \infty$. Let Q be the quasi-projection operator given in (1.1). If $Qg = g$ for all $g \in \Pi_{k-1}$, then the estimate

$$|f - Q_h f|_{B_{p,q}^\mu} \le C h^{\nu-\mu} |f|_{B_{p,q}^\nu} \quad \forall\, f \in B_{p,q}^\nu(\mathbb{R}^s) \tag{1.5}$$

is valid for all $h > 0$, where C is a constant independent of h and f. This result will be proved in §5. We review local polynomial approximation in §2 and discuss approximation by quasi-projection operators in §3. In order to prove (1.5), in §4 we establish some crucial estimates for moduli of smoothness.

For Triebel-Lizorkin spaces $F_{p,q}^\mu$, results similar to (1.5) were obtained by Kyriazis in [15] and [16]. Recently, DeVore and Ron [5] investigated approximation in Triebel-Lizorkin spaces by using scattered shifts of a multivariate function. For basic properties of Besov spaces and Triebel-Lizorkin spaces the reader is referred to the monograph [6] of Frazier, Jawerth and Weiss.

§2. Preliminaries

In this section we review local polynomial approximation and related properties of moduli of smoothness.

For a measurable function f on $\mathbb{R}^s$, there exists a maximal open set G in $\mathbb{R}^s$ such that f vanishes almost everywhere on G. The complement of G in $\mathbb{R}^s$ is called the **support** of f, and denoted $\operatorname{supp} f$. If $\operatorname{supp} f$ is a compact set in $\mathbb{R}^s$, then we say that f is compactly supported.

By $C(\mathbb{R}^s)$ we denote the space of all continuous functions on $\mathbb{R}^s$. For a nonnegative integer k, we use $C^k(\mathbb{R}^s)$ to denote the linear space of those functions $f \in C(\mathbb{R}^s)$ for which $D^\mu f \in C(\mathbb{R}^s)$ for all $|\mu| \le k$. Moreover, by $C_c^k(\mathbb{R}^s)$ we denote the linear space of all functions in $C^k(\mathbb{R}^s)$ with compact support.

Lemma 2.1. *Let u be a vector in $\mathbb{R}^s$. Then the following inequality is valid for $1 \le p \le \infty$:*

$$\|\nabla_u^k f\|_p \le \|D_u^k f\|_p \quad \forall\, f \in W_p^k(\mathbb{R}^s). \tag{2.1}$$

Proof. It suffices to prove (2.1) for $k = 1$, since the general case can be verified by induction on k. Suppose $f \in W_p^1(\mathbb{R}^s)$, $1 \le p \le \infty$. Choose a function $\rho \in C_c^1(\mathbb{R}^s)$ such that $\rho(x) \ge 0$ for all $x \in \mathbb{R}^s$ and $\int_{\mathbb{R}^s} \rho(x)\,dx = 1$. For $\varepsilon > 0$, let $f_\varepsilon := f * \rho_\varepsilon$, where $\rho_\varepsilon(x) := \rho(x/\varepsilon)/\varepsilon^s$, $x \in \mathbb{R}^s$. Then $f_\varepsilon \in C^1(\mathbb{R}^s)$ and

$$\nabla_u f_\varepsilon(x) = \int_0^1 D_u f_\varepsilon(x - tu)\,dt, \quad x \in \mathbb{R}^s.$$

Applying the Minkowski inequality to the above integral, we obtain

$$\|\nabla_u f_\varepsilon\|_p \le \|D_u f_\varepsilon\|_p, \quad 1 \le p \le \infty.$$

But $\|D_u f_\varepsilon\|_p \le \|D_u f\|_p$. For $1 \le p < \infty$, we have $\lim_{\varepsilon \to 0} \|f_\varepsilon - f\|_p = 0$. Hence,

$$\|\nabla_u f\|_p \le \|D_u f\|_p. \tag{2.2}$$

For $p = \infty$, we have $\lim_{\varepsilon\to 0} f_\varepsilon(x) = f(x)$, whenever x is a Lebesgue point of f (see, *e.g.*, [7, Theorem 10.1]). Therefore, for almost every $x \in \mathbb{R}^s$,

$$|\nabla_u f(x)| = \lim_{\varepsilon \to 0} |\nabla_u f_\varepsilon(x)| \le \|D_u f\|_\infty.$$

This shows that (2.2) is also valid for $p = \infty$. □

For $c \in \mathbb{R}^s$ and $a > 0$, let $E_a(c)$ denote the cube $c + [-a, a]^s$. With a slight modification of the preceding proof we can establish the following inequality for $1 \le p \le \infty$:

$$\|\nabla_u^k f\|_p(E_a(c)) \le \|D_u^k f\|_p(E_{a+kb}(c)) \quad \forall\, f \in W_p^k(\mathbb{R}^s) \quad \text{and} \quad |u| \le b. \tag{2.3}$$

Let ψ be an element of $C_c^k(\mathbb{R}^s)$ such that $\int_{\mathbb{R}^s} \psi(x)\,dx = 1$. For $h > 0$, let $A_{\psi,h}$ be the linear operator on $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$ given by

$$(A_{\psi,h} f)(x) := \int_{\mathbb{R}^s} \big(f - \nabla_u^k f\big)(x) \psi_h(u)\,du, \quad f \in L_p(\mathbb{R}^s),\ x \in \mathbb{R}^s, \tag{2.4}$$

where $\psi_h := \psi(\cdot/h)/h^s$. If there is no ambiguity about ψ, $A_{\psi,h}$ will be abbreviated as A_h.

When the dimension $s = 1$ and ψ is a properly normalized B-spline, these operators were studied in classical approximation theory under the name "generalized Steklov functions" (see [19, p. 50]). These operators were also used to study K-functionals (see [14] and [4, Chap. 6]). In its general form, the following lemma was proved in [10].

Lemma 2.2. *Suppose $f \in L_p(\mathbb{R}^s)$ for $1 \le p < \infty$ or $f \in C(\mathbb{R}^s)$ for $p = \infty$. Then the following two inequalities are valid for $h > 0$:*

$$\|f - A_h f\|_p \le C\omega_k(f,h)_p, \tag{2.5}$$

and

$$|A_h f|_{k,p} \le C\omega_k(f,h)_p / h^k, \tag{2.6}$$

where C is a constant independent of h and f.

We observe that

$$f - \nabla_u^k f = \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} f(\cdot - mu).$$

Hence,

$$A_{\psi,h} f(x) = \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} \int_{\mathbb{R}^s} f(x - mhu)\psi(u)\,du, \quad x \in \mathbb{R}^s.$$

Since $\psi \in C_c^k(\mathbb{R}^s)$, we have $A_{\psi,h} f \in C^k(\mathbb{R}^s)$.

Local polynomial approximation on intervals was studied by Whitney [22]. Whitney's results were extended by Johnen and Scherer [14] to bounded domains in $\mathbb{R}^s$. The following lemma gives an explicit scheme of approximation by polynomials on cubes. In what follows, we use C_j $(j \in \mathbb{N})$ to denote a constant independent of h and f.

Lemma 2.3. *Suppose $f \in W_p^k(\mathbb{R}^s)$ for $k \in \mathbb{N}$ and $1 \le p \le \infty$. For $c \in \mathbb{R}^s$ and $h > 0$, there exists a polynomial $g \in \Pi_{k-1}$ such that*

$$|f - g|_{j,p}(c + [-h,h]^s) \le Ch^{k-j} |f|_{k,p}(c + [-2h,2h]^s), \quad 0 \le j < k, \tag{2.7}$$

where C is a constant independent of h and f.

Proof. Choose $\psi \in C^k(\mathbb{R}^s)$ such that $\int_{\mathbb{R}^s} \psi(x)\,dx = 1$ and $\operatorname{supp}\psi \subseteq \left[-1/k, 1/k\right]^s$. Let $f_h := A_{\psi,h} f$ be as defined in (2.4), and let g be the Taylor polynomial of f_h of degree $k-1$ at c. In order to prove (2.7), it suffices to show that

$$|f - f_h|_{j,p}(c + [-h,h]^s) \le Ch^{k-j} |f|_{k,p}(c + [-2h,2h]^s), \quad 0 \le j < k, \tag{2.8}$$

and

$$|f_h - g|_{j,p}(c + [-h,h]^s) \le Ch^{k-j} |f|_{k,p}(c + [-2h,2h]^s), \quad 0 \le j < k. \tag{2.9}$$

Let us prove (2.8) first. In view of the definition of f_h, we have

$$f(x) - f_h(x) = \int_{\mathbb{R}^s} \nabla_u^k f(x)\psi_h(u)\,du, \quad x \in \mathbb{R}^s.$$

For $\mu \in \mathbb{N}_0^s$ with $|\mu| = j$, we deduce that

$$D^\mu(f - f_h)(x) = \int_{\mathbb{R}^s} \nabla_u^k D^\mu f(x)\psi_h(u)\,du, \quad x \in \mathbb{R}^s.$$

By Minkowski's inequality for integrals, we obtain

$$\|D^\mu(f-f_h)\|_p(E_h(c)) \le \int_{\mathbb{R}^s} \|\nabla_u^k D^\mu f\|_p(E_h(c))|\psi_h(u)|\,du.$$

Note that $\nabla_u^k D^\mu f = \nabla_u^{k-j}\nabla_u^j D^\mu f$. For $u \in \operatorname{supp}\psi_h \subseteq [-h/k, h/k]^s$, (2.3) gives

$$\|\nabla_u^k D^\mu f\|_p(E_h(c)) \le \|D_u^{k-j}\nabla_u^j D^\mu f\|_p(E_{(2-j/k)h}(c)) \le 2^j \|D_u^{k-j} D^\mu f\|_p(E_{2h}(c)).$$

But $D_u = u_1 D_1 + \cdots + u_s D_s$ for $u = (u_1, \ldots, u_s)$. Hence, with $|u| \le h/k$ we have

$$\|D_u^{k-j} D^\mu f\|_p(E_{2h}(c)) \le C_1 h^{k-j} |f|_{k,p}(E_{2h}(c)).$$

Combining the above estimates together, we conclude that

$$\|D^\mu(f-f_h)\|_p(E_h(c)) \le C_2 h^{k-j} |f|_{k,p}(E_{2h}(c)).$$

This is true for every $\mu \in \mathbb{N}_0^s$ with $|\mu| = j$. Therefore, (2.8) is valid.

To prove (2.9) we observe that

$$|f_h - g|_{j,p}(c + [-h,h]^s) \le (2h)^{s/p} |f_h - g|_{j,\infty}(c + [-h,h]^s).$$

The Taylor theorem tells us that

$$|f_h - g|_{j,\infty}(c + [-h,h]^s) \le C_3 h^{k-j} |f_h|_{k,\infty}(c + [-h,h]^s).$$

Recall that $f_h = \sum_{m=1}^k (-1)^{m-1}\binom{k}{m} v_m$, where

$$v_m(x) := \int_{\mathbb{R}^s} f(x - mu)\psi_h(u)\,du, \quad x \in \mathbb{R}^s.$$

By Hölder's inequality, for $|\nu| = k$ we obtain

$$\|D^\nu v_m\|_\infty(c + [-h,h]^s) \le C_4 \|D^\nu f\|_p(c + [-2h, 2h]^s) h^{-s+s/\tilde{p}},$$

where $1/\tilde{p} + 1/p = 1$. Consequently,

$$|f_h|_{k,\infty}(c + [-h,h]^s) \le C_5 |f|_{k,p}(c + [-2h, 2h]^s) h^{-s/p}.$$

Finally, (2.9) follows from the above estimates. $\square$

§3. Quasi-projection Operators

Let Q be the quasi-projection operator defined in (1.1). For $h > 0$, let Q_h be the operator given in (1.3). In this section we study approximation by quasi-projection operators in Sobolev spaces.

Throughout this paper we assume that, for each $i \in I$, both ϕ_i and $\tilde{\phi}_i$ are supported on the cube $c_i + [-K, K]^s$, where $(c_i)_{i\in I}$ is a sequence of points in $\mathbb{R}^s$ with the property that each cube $\alpha + [0,1]^s$ ($\alpha \in \mathbb{Z}^s$) contains at most N points c_i. Consequently, $\sigma_h(\phi_i)$ and $\sigma_h(\tilde{\phi}_i)$ are supported on $E_{Kh}(c_i h)$.

Theorem 3.1. *Let j and k be two integers such that $0 \le j < k$. Suppose that there exists a constant $M > 0$ such that $\|\phi_i\|_{j,p} \le M$ and $\|\tilde{\phi}_i\|_{\tilde{p}} \le M$ for all $i \in I$, where $1 \le p, \tilde{p} \le \infty$ and $1/p + 1/\tilde{p} = 1$. If $Qg = g$ for all $g \in \Pi_{k-1}$, then*

$$|f - Q_h f|_{j,p} \le C h^{k-j} |f|_{k,p} \quad \forall\, f \in W_p^k(\mathbb{R}^s), \tag{3.1}$$

where C is a constant independent of f and h.

Proof. Fix $\alpha \in \mathbb{Z}^s$ for the time being. For $x \in E_h(\alpha h)$ we have

$$Q_h f(x) = \sum_{i \in I_\alpha} \langle f, h^{-s/\tilde{p}} \sigma_h \tilde{\phi}_i \rangle \, h^{-s/p} \sigma_h \phi_i(x),$$

where $I_\alpha := \{i \in I : E_h(\alpha h) \cap E_{Kh}(c_i h) \ne \emptyset\}$. In light of our assumption on the supports of ϕ_i ($i \in I$), there exists an integer $N' > 0$ such that $\# I_\alpha \le N'$ for all $\alpha \in \mathbb{Z}^s$, where $\# I_\alpha$ denotes the cardinality of I_α. Hence, there exists a constant $K'' \ge 1$ such that

$$\bigcup_{i \in I_\alpha} E_{Kh}(c_i h) \subseteq E_{K''h}(\alpha h) \quad \forall\, \alpha \in \mathbb{Z}^s.$$

Let $K' := 2K''$. By Lemma 2.3, there exists some $g_{\alpha,h} \in \Pi_{k-1}$ such that

$$|f - g_{\alpha,h}|_{j,p}(E_{K''h}(\alpha h)) \le C_1 h^{k-j} |f|_{k,p}(E_{K'h}(\alpha h)), \quad 0 \le j < k. \tag{3.2}$$

For $\mu \in \mathbb{N}_0^s$ with $|\mu| = j$, we have

$$D^\mu\big(Q_h(f - g_{\alpha,h})\big)(x) = \sum_{i \in I_\alpha} \langle f - g_{\alpha,h}, h^{-s/\tilde{p}} \sigma_h \tilde{\phi}_i \rangle \, h^{-j} h^{-s/p} D^\mu \phi_i(x/h), \quad x \in E_h(\alpha h).$$

Note that $\operatorname{supp}(\sigma_h \tilde{\phi}_i) \subseteq E_{K''h}(\alpha h)$ for $i \in I_\alpha$. Taking (3.2) into account, we obtain

$$\big|\langle f - g_{\alpha,h}, h^{-s/\tilde{p}} \sigma_h \tilde{\phi}_i \rangle\big| \le M \|f - g_{\alpha,h}\|_p (E_{K''h}(\alpha h)) \le C_1 M h^k |f|_{k,p}(E_{K'h}(\alpha h)).$$

Moreover, $\|h^{-j} h^{-s/p} D^\mu \phi_i(\cdot/h)\|_p \le M h^{-j}$. Therefore,

$$|Q_h(f - g_{\alpha,h})|_{j,p}(E_h(\alpha h)) \le C_2 h^{k-j} |f|_{k,p}(E_{K'h}(\alpha h)). \tag{3.3}$$

Since $Q_h g_{\alpha,h} = g_{\alpha,h}$, we have

$$Q_h f - f = Q_h(f - g_{\alpha,h}) - (f - g_{\alpha,h}).$$

Thus, (3.2) and (3.3) tell us that

$$|Q_h f - f|_{j,p}(E_h(\alpha h)) \le C_3 h^{k-j} |f|_{k,p}(E_{K'h}(\alpha h)). \tag{3.4}$$

This verifies (3.1) for the case $p = \infty$. For $1 \le p < \infty$, we have

$$|Q_h f - f|_{j,p}^p \le \sum_{\alpha \in \mathbb{Z}^s} |Q_h f - f|_{j,p}^p(E_h(\alpha h)) \le \left[C_3 h^{k-j}\right]^p \sum_{\alpha \in \mathbb{Z}^s} |f|_{k,p}^p(E_{K'h}(\alpha h)).$$

Let $\chi_\alpha(x) := 1$ for $x \in E_{K'h}(\alpha h)$ and $\chi_\alpha(x) := 0$ otherwise. For $\nu \in \mathbb{N}_0^s$ with $|\nu| = k$, we have

$$\sum_{\alpha \in \mathbb{Z}^s} \|D^\nu f\|_p^p(E_{K'h}(\alpha h)) = \int_{\mathbb{R}^s} |D^\nu f(x)|^p \sum_{\alpha \in \mathbb{Z}^s} \chi_\alpha(x)\, dx \le (2K'' + 2)^s \int_{\mathbb{R}^s} |D^\nu f(x)|^p\, dx.$$

Consequently, it follows from (3.4) that

$$|Q_h f - f|_{j,p} \le C_3 (2K'' + 2)^{s/p} h^{k-j} |f|_{k,p}.$$

This shows (3.1) for the case $1 \le p < \infty$. $\square$

As a corollary of Theorem 3.1, we have the following result.

Theorem 3.2. *Suppose that $f \in L_p(\mathbb{R}^s)$ for $1 \le p < \infty$ or $f \in C(\mathbb{R}^s)$ for $p = \infty$. If $Qg = g$ for all $g \in \Pi_{k-1}$, then there exists a constant $C > 0$ such that*

$$\|f - Q_h f\|_p \le C \omega_k(f, h)_p \quad \forall\, f \in W_p^k(\mathbb{R}^s). \tag{3.5}$$

Proof. Let $f_h := A_h f$, where $A_h = A_{\psi,h}$ be the operator defined in (2.4). We have

$$f - Q_h f = (f - f_h) + (f_h - Q_h f_h) + (Q_h f_h - Q_h f).$$

By (2.5), $\|f - f_h\|_p \le C_1 \omega_k(f, h)_p$. Moreover,

$$\|Q_h f_h - Q_h f\|_p \le \|Q_h\| \|f_h - f\|_p = \|Q\| \|f_h - f\|_p.$$

Hence, it remains to estimate $\|f_h - Q_h f_h\|_p$. Theorem 3.1 tells us that

$$\|f_h - Q_h f_h\|_p \le C_2 h^k |f_h|_{k,p}.$$

But it follows from (2.6) that $h^k |f_h|_{k,p} \le C_3 \omega_k(f, h)_p$. Combining the above estimates together, we obtain (3.5). $\square$

§4. Estimates of Moduli of Smoothness

In order to study approximation of quasi-projection operators in Besov spaces, we need some estimates of moduli of smoothness.

For $\nu > 0$, the Besov space $B^\nu_{p,\infty}(\mathbb{R}^s)$ is the same as the generalized Lipschitz space $\mathrm{Lip}^*(\nu, L_p(\mathbb{R}^s))$. By the very definition of $B^\nu_{p,\infty}(\mathbb{R}^s)$, we have $\omega_k(\phi, t) \le t^\nu |\phi|_{B^\nu_{p,\infty}}$ for $0 < \nu < k$ and $t > 0$. The following lemma was proved in [11, §3].

Lemma 4.1. *Let $(\phi_i)_{i\in I}$ be a family of functions in $L_p(\mathbb{R}^s)$ such that $\|\phi_i\|_{B^\nu_{p,\infty}} \le M$ for all $i \in I$. Let $E_i := \mathrm{supp}\,\phi_i$, and let χ_i be the function on $\mathbb{R}^s$ given by $\chi_i(x) = 1$ for $x \in E_i$ and $\chi_i(x) = 0$ otherwise. If $\sum_{i\in I} \chi_i(x) \le A$ for all $x \in \mathbb{R}^s$, and if k is an integer greater than ν, then*

$$\omega_k\Big(\sum_{i\in I} b_i\phi_i, t\Big)_p \le \big((k+1)A\big)^{1-1/p}\Big(\sum_{i\in I} |b_i|^p\Big)^{1/p} M t^\nu \quad \forall\, t > 0.$$

Let ϕ be a function defined on $\mathbb{R}^s$. For $k \in \mathbb{N}$, $h > 0$, and $u \in \mathbb{R}^s$, we have

$$\nabla^k_u(\sigma_h\phi)(x) = \sum_{m=0}^k (-1)^m \binom{k}{m} \phi\Big(\frac{x - mu}{h}\Big) = \sigma_h\big(\nabla^k_{u/h}\phi\big)(x), \quad x \in \mathbb{R}^s.$$

It follows that $\|\nabla^k_u(\sigma_h\phi)\|_p = h^{s/p}\|\nabla^k_{u/h}\phi\|_p$. Consequently,

$$\big|h^{-s/p}\sigma_h\phi\big|_{B^\nu_{p,q}} = h^{-\nu}|\phi|_{B^\nu_{p,q}}. \tag{4.1}$$

The following lemma gives an estimate for moduli of smoothness, which will be needed in the next section.

Lemma 4.2. *Suppose $0 < \nu < k$, $1 \le p \le \infty$ and $1/p + 1/\tilde p = 1$. Let Q be the quasi-projection operator given in (1.1), and let $Q_h := \sigma_h Q \sigma_{1/h}$ be as given in (1.3). Suppose that there exists a constant $M > 0$ such that $\|\phi_i\|_{B^\nu_{p,\infty}} \le M$ and $\|\tilde\phi_i\|_{\tilde p} \le M$ for all $i \in I$. Let $f \in L_p(\mathbb{R}^s)$ for $1 \le p < \infty$ or $f \in C(\mathbb{R}^s)$ for $p = \infty$. If $Qg = g$ for all $g \in \Pi_{k-1}$, then*

$$\omega_k(Q_h f, t)_p \le C\omega_k(f, h)_p\Big(\frac{t}{h}\Big)^\nu, \quad 0 < t \le h, \tag{4.2}$$

where C is a constant independent of f, t, and h.

Proof. Let $f_h := A_{\psi,h} f$ be as given in (2.4), where ψ is a function in $C^k(\mathbb{R}^s)$ with $\mathrm{supp}\,\psi \subseteq [-1/k, 1/k]^s$. Write $Q_h f = Q_h(f - f_h) + Q_h f_h$. We have

$$Q_h(f - f_h) = \sum_{i\in I} \langle f - f_h, h^{-s/\tilde p}\sigma_h\tilde\phi_i\rangle\, h^{-s/p}\sigma_h\phi_i.$$

Taking (4.1) into account, we obtain

$$\omega_k\big(h^{-s/p}\sigma_h\phi_i, t\big)_p \le t^\nu\big|h^{-s/p}\sigma_h\phi_i\big|_{B^\nu_{p,\infty}} = t^\nu h^{-\nu}|\phi_i|_{B^\nu_{p,\infty}} \le M\Big(\frac{t}{h}\Big)^\nu. \tag{4.3}$$

Let $b_i := \langle f - f_h, h^{-s/\tilde{p}}\sigma_h\tilde{\phi}_i\rangle$, $i \in I$. An application of Lemma 4.1 gives

$$\omega_k(Q_h(f-f_h),t)_p \le C_1\Big(\sum_{i\in I}|b_i|^p\Big)^{1/p}\Big(\frac{t}{h}\Big)^\nu.$$

By [11, Lemma 3.1] we have

$$\Big(\sum_{i\in I}|b_i|^p\Big)^{1/p} \le C_2\|f-f_h\|_p.$$

But Lemma 2.2 tells us that $\|f-f_h\|_p \le C_3\omega_k(f,h)_p$. Therefore,

$$\omega_k(Q_h(f-f_h),t)_p \le C\omega_k(f,h)_p\Big(\frac{t}{h}\Big)^\nu, \quad 0<t\le h. \tag{4.4}$$

It remains to estimate $\omega_k(Q_hf_h,t)_p$. Fix $\alpha \in \mathbb{Z}^s$ for the time being. For $y \in \mathbb{R}^s$ we have

$$\nabla_y^k(Q_hf_h)(x) = \sum_{i\in I_\alpha}\langle f_h, h^{-s/\tilde{p}}\sigma_h\tilde{\phi}_i\rangle\,\nabla_y^k\big(h^{-s/p}\sigma_h\phi_i\big)(x), \quad x\in E_h(\alpha h),$$

where $I_\alpha := \{i \in I : \operatorname{supp}(\nabla_y^k(\sigma_h\phi_i)) \cap E_h(\alpha h) \ne \emptyset\}$. Suppose $|y| \le t \le h$. Then $\operatorname{supp}(\nabla_y^k(\sigma_h\phi_i)) \subseteq E_{(K+k)h}(c_ih)$. Hence, there exists a positive integer N' such that $\#I_\alpha \le N'$ for all $\alpha \in \mathbb{Z}^s$. Moreover, there exists a constant $K'' \ge 1$ such that

$$\cup_{i\in I_\alpha}E_{(K+k)h}(c_ih) \subseteq E_{K''h}(\alpha h) \quad \forall\,\alpha\in\mathbb{Z}^s.$$

Let $K' := 2K''$. By Lemma 2.3, there exists some $g_{\alpha,h} \in \Pi_{k-1}$ such that

$$\|f_h - g_{\alpha,h}\|_p(E_{K''h}(\alpha h)) \le C_4h^k|f_h|_{k,p}(E_{K'h}(\alpha h)). \tag{4.5}$$

Since $Q_hg_{\alpha,h} = g_{\alpha,h}$ and $\nabla_y^kg_{\alpha,h} = 0$, we have $\nabla_y^k(Q_hf_h) = \nabla_y^k(Q_h(f_h - g_{\alpha,h}))$. Note that

$$\nabla_y^k(Q_h(f_h-g_{\alpha,h}))(x) = \sum_{i\in I_\alpha}\langle f_h-g_{\alpha,h}, h^{-s/\tilde{p}}\sigma_h\tilde{\phi}_i\rangle\,\nabla_y^k\big(h^{-s/p}\sigma_h\phi_i\big)(x), \quad x\in E_h(\alpha h).$$

For $|y| \le t$, it follows from (4.3) that

$$\big\|\nabla_y^k(h^{-s/p}\sigma_h\phi_i)\big\|_p \le \omega_k(h^{-s/p}\sigma_h\phi_i,t)_p \le M\Big(\frac{t}{h}\Big)^\nu.$$

Moreover, for $i \in I_\alpha$ we have $\operatorname{supp}(\sigma_h\tilde{\phi}_i) \subseteq E_{K''h}(\alpha h)$. Hence,

$$\big|\langle f_h-g_{\alpha,h}, h^{-s/\tilde{p}}\sigma_h\tilde{\phi}_i\rangle\big| \le M\|f_h-g_{\alpha,h}\|_p(E_{K''h}(\alpha h)) \le MC_4h^k|f_h|_{k,p}(E_{K'h}(\alpha h)),$$

where (4.5) has been used to derive the last inequality. Consequently,

$$\big\|\nabla_y^k(Q_hf_h)\big\|_p(E_h(\alpha h)) \le C_5h^k|f_h|_{k,p}(E_{K'h}(\alpha h))\Big(\frac{t}{h}\Big)^\nu.$$

Therefore, as was done in the proof of Theorem 3.1, we deduce that

$$\big\|\nabla_y^k(Q_hf_h)\big\|_p \le Ch^k|f_h|_{k,p}\Big(\frac{t}{h}\Big)^\nu, \quad |y|\le t.$$

This in connection with (2.6) yields

$$\omega_k(Q_hf_h,t)_p \le C\omega_k(f,h)_p\Big(\frac{t}{h}\Big)^\nu, \quad 0<t\le h. \tag{4.6}$$

The desired estimate (4.2) follows from (4.4) and (4.6) at once. □

§5. Approximation in Besov Spaces

We are in a position to establish our main results on approximation of quasi-projection operators in Besov spaces.

In this section we assume that $k \in \mathbb{N}$, $0 < \mu < \nu < k$, and $1 \le p, q \le \infty$. Let Q be the quasi-projection operator given in (1.1) and, for $h > 0$, let $Q_h := \sigma_h Q \sigma_{1/h}$ be as given in (1.3). Suppose that there exists a constant $M > 0$ such that $\|\phi_i\|_{B^\nu_{p,\infty}} \le M$ and $\|\tilde{\phi}_i\|_{\tilde{p}} \le M$ for all $i \in I$.

Let f be a function in $B^\nu_{p,q}(\mathbb{R}^s)$. If $Qg = g$ for all $g \in \Pi_{k-1}$, then Theorem 3.2 tells us that $\|f - Q_h f\|_p \le C_1 \omega_k(f, h)_p$. But $\omega_k(f,h)_p \le h^\nu |f|_{B^\nu_{p,\infty}} \le C_2 h^\nu |f|_{B^\nu_{p,q}}$. Hence,

$$\|f - Q_h f\|_p \le C h^\nu |f|_{B^\nu_{p,q}}.$$

In many applications, we need to estimate approximation of $Q_h f$ to f in Besov spaces. The following theorem provides such estimates.

Theorem 5.1. *If $Qg = g$ for all $g \in \Pi_{k-1}$, then*

$$|f - Q_h f|_{B^\mu_{p,q}} \le C h^{\nu-\mu} |f|_{B^\nu_{p,q}} \quad \forall\, f \in B^\nu_{p,q}(\mathbb{R}^s),$$

where C is a constant independent of h and f.

Proof. Suppose $f \in B^\nu_{p,q}(\mathbb{R}^s)$. Let $b_j(f) := 2^{j\mu}\omega_k(f, 2^{-j})_p$, $j \in \mathbb{Z}$. Given $h > 0$, we let j_h be the integer such that $h \le 2^{-j_h} < 2h$. Then $2^{j_h} \le h^{-1}$. By Theorem 3.2 we have

$$\omega_k(f - Q_h f, 2^{-j})_p \le 2^k \|f - Q_h f\|_p \le C_1 \omega_k(f,h)_p.$$

It follows that $\sup_{j \le j_h} b_j(f - Q_h f) \le C_1 h^{-\mu} \omega_k(f,h)_p$. Moreover,

$$\left(\sum_{j=-\infty}^{j_h} \left[b_j(f - Q_h f) \right]^q \right)^{1/q} \le C_1 \omega_k(f,h)_p \left(\sum_{j=-\infty}^{j_h} (2^{j\mu})^q \right)^{1/q}, \quad 1 \le q < \infty.$$

Consequently, for $1 \le q \le \infty$,

$$\left(\sum_{j=-\infty}^{j_h} \left[b_j(f - Q_h f) \right]^q \right)^{1/q} \le C_2 h^{-\mu} \omega_k(f,h)_p \le C_2 h^{\nu-\mu} |f|_{B^\nu_{p,\infty}}. \tag{5.1}$$

Consider the case $j > j_h$. We observe that

$$\omega_k(f - Q_h f, 2^{-j})_p \le \omega_k(f, 2^{-j})_p + \omega_k(Q_h f, 2^{-j})_p.$$

Note that $2^{-j} < h$ for $j > j_h$. By Lemma 4.2 we have

$$\omega_k(Q_h f, 2^{-j})_p \le C_3 \omega_k(f,h)_p \left(\frac{2^{-j}}{h} \right)^\nu.$$

It follows that

$$\Big(\sum_{j=j_h+1}^{\infty}\big[b_j(Q_hf)\big]^q\Big)^{1/q}\le C_3h^{-\nu}\omega_k(f,h)_p\Big(\sum_{j=j_h+1}^{\infty}\big[2^{-j(\nu-\mu)}\big]^q\Big)^{1/q}.$$

Therefore,

$$\Big(\sum_{j=j_h+1}^{\infty}\big[b_j(Q_hf)\big]^q\Big)^{1/q}\le C_4h^{-\mu}\omega_k(f,h)_p\le C_4h^{\nu-\mu}|f|_{B^{\nu}_{p,\infty}}. \tag{5.2}$$

Since $\mu-\nu<0$ and $2^{j_h}\le h^{-1}$, for $j>j_h$ we have $2^{j(\mu-\nu)}\le 2^{j_h(\mu-\nu)}\le h^{\nu-\mu}$. It follows that $2^{j\mu}=2^{j(\mu-\nu)}2^{j\nu}\le h^{\nu-\mu}2^{j\nu}$ for $j>j_h$. Therefore,

$$\Big(\sum_{j=j_h+1}^{\infty}\big[b_j(f)\big]^q\Big)^{1/q}\le h^{\nu-\mu}\Big(\sum_{j=j_h+1}^{\infty}\big[2^{j\nu}\omega_k(f,2^{-j})_p\big]^q\Big)^{1/q}\le C_5h^{\nu-\mu}|f|_{B^{\nu}_{p,q}}. \tag{5.3}$$

Combining the estimates (5.1), (5.2) and (5.3) together, we obtain

$$\Big(\sum_{j=-\infty}^{\infty}\big[2^{j\mu}\omega_k(f-Q_hf,2^{-j})_p\big]^q\Big)^{1/q}\le Ch^{\nu-\mu}|f|_{B^{\nu}_{p,q}}.$$

This completes the proof of the theorem. □

Suppose $\phi\in L_p(\mathbb{R}^s)$ and $\tilde\phi\in L_{\tilde p}(\mathbb{R}^s)$. In the case when $\phi_i=\phi(\cdot-i)$ and $\tilde\phi_i=\tilde\phi(\cdot-i)$, $i\in\mathbb{Z}^s$, Kyriazis in [16, Theorem 4.6] obtained results similar to Theorem 5.1 for the Triebel-Lizorkin spaces $F^{\mu}_{p,q}$. But the case $p=\infty$ was excluded there. Note that for $1\le p<\infty$, $B^{\mu}_{p,p}$ and $F^{\mu}_{p,p}$ are identical and have equivalent norms. In particular, for $p=2$, it is well known that $B^{\mu}_{2,2}(\mathbb{R}^s)$ is the same as the Sobolev space $H^{\mu}(\mathbb{R}^s)$. Here, $H^{\mu}=H^{\mu}(\mathbb{R}^s)$ is defined to be the space of all functions f in $L_2(\mathbb{R}^s)$ such that the semi-norm

$$|f|_{H^{\mu}}:=\Big(\int_{\mathbb{R}^s}|\hat f(\xi)|^2|\xi|^{2\mu}\,d\xi\Big)^{1/2}$$

is finite, where $\hat f$ denotes the Fourier transform of f. The norm in H^{μ} is defined by $\|f\|_{H^{\mu}}:=\|f\|_{L_2}+|f|_{H^{\mu}}$. Moreover, the semi-norms $|\cdot|_{B^{\mu}_{2,2}}$ and $|\cdot|_{H^{\mu}}$ are equivalent. Consequently, the norms $\|\cdot\|_{B^{\mu}_{2,2}}$ and $\|\cdot\|_{H^{\mu}}$ are equivalent.

Approximation by quasi-projection operators in the Lipschitz spaces $\mathrm{Lip}(\mu,L_p(\mathbb{R}^s))$ ($\mu>0$, $1\le p\le\infty$) was discussed in [10]. When $\mu\notin\mathbb{N}$, we have $\mathrm{Lip}(\mu,L_p(\mathbb{R}^s))=\mathrm{Lip}^*(\mu,L_p(\mathbb{R}^s))=B^{\mu}_{p,\infty}(\mathbb{R}^s)$. However, when $\mu\in\mathbb{N}$, the Lipschitz space $\mathrm{Lip}(\mu,L_p(\mathbb{R}^s))$ is a proper subspace of the generalized Lipschitz space $\mathrm{Lip}^*(\mu,L_p(\mathbb{R}^s))$. So the present paper does not completely cover the results in [10].

The following theorem gives a result on density of quasi-projection operators, which will be useful for multiresolution analysis of Besov spaces.

Theorem 5.2. *Suppose $f \in B^{\mu}_{p,q}(\mathbb{R}^s)$, where $0 < \mu < \nu < k$, $1 \le p \le \infty$ and $1 \le q < \infty$. If $Qg = g$ for all $g \in \Pi_{k-1}$, then*

$$\lim_{h\to 0} |Q_h f - f|_{B^{\mu}_{p,q}} = 0.$$

Proof. Let $b_j(f) := 2^{j\mu}\omega_k(f, 2^{-j})_p$, $j \in \mathbb{Z}$. Then

$$|Q_h f - f|_{B^{\mu}_{p,q}} = \left(\sum_{j=-\infty}^{\infty} \big[b_j(Q_h f - f)\big]^q \right)^{1/q}.$$

For $h > 0$, let j_h be the integer such that $h \le 2^{-j_h} < 2h$. We have $\lim_{h\to 0} j_h = \infty$. From the proof of (5.1) we see that

$$\left(\sum_{j=-\infty}^{j_h} \big[b_j(f - Q_h f)\big]^q \right)^{1/q} \le C_1 h^{-\mu}\omega_k(f, h)_p \le C_1 2^{\mu} b_{j_h}(f).$$

For $j > j_h$, we use the inequality $\omega_k(f - Q_h f, 2^{-j})_p \le \omega_k(f, 2^{-j})_p + \omega_k(Q_h f, 2^{-j})_p$. From the proof of (5.2) we deduce that

$$\left(\sum_{j=j_h+1}^{\infty} \big[b_j(Q_h f)\big]^q \right)^{1/q} \le C_2 h^{-\mu}\omega_k(f, h)_p \le C_2 2^{\mu} b_{j_h}(f).$$

Since $f \in B^{\mu}_{p,q}$, we have $\sum_{j=-\infty}^{\infty} \big[b_j(f)\big]^q < \infty$. Consequently,

$$\lim_{j_h\to\infty} b_{j_h}(f) = 0 \quad \text{and} \quad \lim_{j_h\to\infty} \sum_{j=j_h+1}^{\infty} \big[b_j(f)\big]^q = 0.$$

This shows $\lim_{h\to 0} |Q_h f - f|_{B^{\mu}_{p,q}} = 0$, as desired. $\square$

The above theorem is no longer valid when $q = \infty$. This fact will be demonstrated by the following example.

Let $\phi(x) := \max\{1 - |x|, 0\}$ for $x \in \mathbb{R}$, and let $\tilde{\phi}(x) := 4 - 6x$ for $0 < x < 1$ and $\tilde{\phi}(x) := 0$ for $x \in \mathbb{R} \setminus (0,1)$. For $1 \le p < \infty$, we have $\phi \in B^{\nu}_{p,\infty}$ for $\nu := 1 + 1/p$ and $\tilde{\phi} \in B^{\mu}_{p,\infty}$ for $\mu := 1/p$.

For $i \in \mathbb{Z}$, let

$$\phi_i(x) := \phi(x - i) \quad \text{and} \quad \tilde{\phi}_i(x) := \tilde{\phi}(x - i), \quad x \in \mathbb{R}.$$

Then $\langle \phi_i, \tilde{\phi}_j \rangle = \delta_{ij}$, $i, j \in \mathbb{Z}$, where δ_{ij} stands for the Kronecker sign: $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \ne j$.

Consider the quasi-projection operator Q given by

$$Qf := \sum_{i\in\mathbb{Z}} \langle f, \tilde{\phi}_i \rangle \phi_i,$$

where f is a locally integrable function on $\mathbb{R}$. We have $Q\phi_i = \phi_i$ for all $i \in \mathbb{Z}$. In other words, Q is a projection operator. For a polynomial g in Π_1, g can be represented as $g = \sum_{i\in\mathbb{Z}} g(i)\phi_i$. Consequently, $Qg = g$ for all $g \in \Pi_1$.

For $h > 0$, let

$$Q_h f := \sum_{i\in\mathbb{Z}} \langle f, h^{-1}\sigma_h\tilde{\phi}_i\rangle\, \sigma_h\phi_i.$$

Let f be the function given by $f(x) := 1$ for $0 < x < 1$ and $f(x) := 0$ for $x \in \mathbb{R} \setminus (0, 1)$. Then $f \in B^{\mu}_{p,\infty}$ for $\mu = 1/p$. For $i < 0$ and $x > 0$, we have $\sigma_h\tilde{\phi}_i(x) = \tilde{\phi}(x/h - i) = 0$. Since f is supported on $[0, 1]$, we deduce that $\langle f, h^{-1}\sigma_h\tilde{\phi}_i\rangle = 0$ for $i < 0$. Suppose $0 < h < 1/2$. For $i = 0$ or $i = 1$, we have

$$\langle f, h^{-1}\sigma_h\tilde{\phi}_i\rangle = \frac{1}{h}\int_0^1 \tilde{\phi}(x/h - i)\, dx = \int_{-i}^{1/h-i} \tilde{\phi}(y)\, dy = \int_0^1 (4 - 6y)\, dy = 1.$$

For $i \ge 2$ and $x < h$, we have $\sigma_h\phi_i(x) = \phi(x/h - i) = 0$. Consequently,

$$Q_h f(x) = \phi(x/h) + \phi(x/h - 1) \quad \text{for } x < h.$$

It follows that

$$Q_h f(x) = 1 \quad \text{and} \quad Q_h f(x - h) = \phi(x/h - 1) = x/h \quad \text{for } 0 < x < h.$$

Hence,

$$\nabla_h(Q_h f - f)(x) = -x/h \quad \text{for } 0 < x < h.$$

For $1 \le p < \infty$, we thereby obtain

$$\big\|\nabla_h(Q_h f - f)\big\|_p \ge \left[\int_0^h \left(\frac{x}{h}\right)^p dx\right]^{1/p} = (p + 1)^{-1/p} h^{1/p}.$$

Finally, with $\mu = 1/p$, we conclude that

$$|Q_h f - f|_{B^{\mu}_{p,\infty}} \ge \frac{1}{h^{1/p}}\big\|\nabla_h(Q_h f - f)\big\|_p \ge (p + 1)^{-1/p}.$$

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