A product formula for semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type

Product formula for semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type

Toshitaka Matsumoto[a,1,*], Naoki Tanaka[b,2]

[a] *Department of Mathematical and Life Sciences, Graduate School of Science, Hiroshima University, Higashi-Hiroshima 739-8526, Japan*
[b] *Department of Mathematics, Faculty of Science, Shizuoka University, Shizuoka 422-8529, Japan*

Abstract

A product formula for semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type is discussed under a new type of stability condition which admits "error term". The result obtained here is applied to showing the convergence of approximate solutions constructed by a fractional step method to the solution of the complex Ginzburg-Landau equation.

Keywords: Product formula, Semigroup of Lipschitz operators, Semilinear evolution equation of parabolic type, Analytic semigroup, Fractional power, Fractional step method
2000 MSC: 47H14, 47H20, 34G20

1. Introduction

We are concerned with product formulas for semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type. For the linear case Trotter [30] established a formula for products of semigroups and Chernoff [4] extended the formula into more general situation. Product formulas for quasi-contractive nonlinear semigroups were studied by Miyadera-Oharu [25], Brezis-Pazy [2], Miyadera-Kobayashi [24], Kato-Masuda [10], Reich [29] and Kobayashi [11, 12] and applied to the convergence of approximate solutions of a scalar conservation law ([13]). As an extension of quasi-contractive nonlinear semigroups, Kobayashi and Tanaka [14] introduced the notion of semigroups of Lipschitz operators and applied their theory to quasilinear evolution equations. In the case where the infinitesimal generator of such a semigroup is

*Corresponding author
Email addresses: mats@math.sci.hiroshima-u.ac.jp (Toshitaka Matsumoto), sntanak@ipc.shizuoka.ac.jp (Naoki Tanaka)
[1] Partially supported by JSPS Grand in Aid for Scientific Research (C) No. 20540173
[2] Partially supported by JSPS Grand in Aid for Scientific Research (C) No. 22540183

continuous, a generation theorem, a product formula and an application to the convergence of approximate solutions of Kirchhoff equation by Lax-Friedrichs difference scheme were discussed in [14, 15]. Recently, their generation theorem for semigroups of Lipschitz operators has been extended to the case where the infinitesimal generator is not necessarily continuous. For example, we considered in [21] the case where the infinitesimal generator is represented as a relatively continuous perturbation of the infinitesimal generator of an analytic semigroup and gave a characterization for semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type. As an application of the characterization theorem, C^1 well-posedness for the complex Ginzburg-Landau equation was shown there. For extensions to the fully nonlinear case we refer to [16, 17].

In this paper we consider a semilinear evolution equation of the form

$$u'(t) = Au(t) + Bu(t) \quad \text{for } t > 0. \tag{SP}$$

Here A is the infinitesimal generator of an analytic semigroup of class (C_0) on a Banach space $(X, \|\cdot\|)$ and B stands for a continuous operator from a subset C of the domain of a fractional power of $-A$ into X.

Our objective here is to study a product formula for semigroups of Lipschitz operators associated with semilinear evolution equations of parabolic type under a suitable stability condition. We also give an application of the product formula to the convergence of approximate solutions of the complex Ginzburg-Landau equation by using a fractional step method. To establish a product formula, Kobayashi and Tanaka [15] proposed the following stability condition for a family $\{F_h; h \in (0, h_0]\}$ by using a metric-like functional Φ on $X \times X$:

$$\Phi(F_h x, F_h y) \le e^{\omega h}\Phi(x, y) \qquad \text{for } (x, y) \in X \times X \text{ and } h \in (0, h_0]. \tag{1.1}$$

Marsden [20] assumed the similar condition to obtain a product formula on Banach manifolds. We note that if $\Phi(x, y) = \|x - y\|$ then condition (1.1) coincides with the stability condition for quasi-contractive semigroups studied in [2, 10, 11, 12, 25, 29]. In order to construct approximate solutions of (SP) by a fractional step method, we need to apply the product formula with

$$F_h = T_A(h)T_B(h) \qquad \text{for } h \in (0, h_0], \tag{1.2}$$

where $\{T_A(t); t \ge 0\}$ and $\{T_B(t); t \ge 0\}$ stand for operator semigroups generated by A and B, respectively. Since the semigroup $\{T_B(t); t \ge 0\}$ is not quasi-contractive in general, it is difficult to check the stability condition (1.1) for the family $\{F_h; h \in (0, h_0]\}$ defined by (1.2). In this paper we introduce a weaker stability condition which admits "error term"

$$\limsup_{h\downarrow 0}(\sup\{(\Phi(F_h x, F_h y) - \Phi(x, y))/h - \omega\Phi(x, y); x,\ y \in C\}) \le 0, \tag{1.3}$$

and establish a product formula for (SP) under such a stability condition. The use of this stability condition is the feature of our paper.

The paper is organized as follows: Section 2 contains basic assumptions and our main result (Theorem 2.2). The proof of Theorem 2.2 is given in Section 4. An application of the product formula to the complex Ginzburg-Landau equation is discussed in Section 5.

2. Assumptions and main result

Let $(X, \|\cdot\|)$ be a Banach space and D a closed subset of X. We consider a semilinear Cauchy problem in X of the form

$$u'(t) = Au(t) + Bu(t) \quad \text{for } t > 0, \qquad u(0) = u_0 \in D. \qquad \text{(SP; } u_0\text{)}$$

Here A is assumed to be the infinitesimal generator of an analytic semigroup $\{T(t); t \geq 0\}$ of class (C_0) on X such that $\|T(t)\| \leq M_A e^{\omega_A t}$ for all $t \geq 0$, where $M_A \geq 1$ and $\omega_A < 0$ are some constants.

Let $\alpha \in (0, 1)$ and $Y = D((-A)^\alpha)$. Then Y is a Banach space equipped with the norm $\|x\|_Y := \|(-A)^\alpha x\|$ for $x \in Y$. Let $C = D \cap Y$. For the operator B we make the following assumptions:

(B-i) The operator B from C into X is continuous and C is dense in D.

(B-ii) There exists $M_B > 0$ such that $\|Bx\| \leq M_B(1 + \|x\|_Y)$ for $x \in C$.

Let Φ be a nonnegative functional on $X \times X$ satisfying the two conditions below:

(Φ-i) There exists $L \geq 0$ such that

$$|\Phi(x, y) - \Phi(\hat{x}, \hat{y})| \leq L(\|x - \hat{x}\| + \|y - \hat{y}\|) \quad \text{for } (x, y), (\hat{x}, \hat{y}) \in X \times X.$$

(Φ-ii) There exist $M \geq m > 0$ such that

$$m\|x - y\| \leq \Phi(x, y) \leq M\|x - y\| \quad \text{for } (x, y) \in D \times D.$$

Let $\{F_h; h \in (0, h_0]\}$ be a family of nonlinear operators from C into itself which satisfies the following two conditions:

(F-i) There exists $\omega \geq 0$ such that for any null sequence $\{h_n\}$ of positive numbers and any Y-bounded sequences $\{x_n\}$ and $\{y_n\}$ in C,

$$\limsup_{n\to\infty}\{h_n^{-1}(\Phi(F_{h_n}x_n, F_{h_n}y_n) - \Phi(x_n, y_n)) - \omega\Phi(x_n, y_n)\} \leq 0.$$

(F-ii) There exists $\beta \in (0, 1)$ such that for any null sequence $\{h_n\}$ of positive numbers and any convergent sequence $\{x_n\}$ in C with respect to Y norm,

$$\lim_{n\to\infty} h_n^{-1}\|F_{h_n}x_n - J(h_n)x_n\| = 0, \quad \lim_{n\to\infty} h_n^{-\beta}\|F_{h_n}x_n - J(h_n)x_n\|_Y = 0,$$

where

$$J(h)w = T(h)w + \int_0^h T(s)Bw\, ds \qquad \text{for } w \in C \text{ and } h > 0. \qquad (2.1)$$

Definition 2.1. A one-parameter family $\{S(t); t \geq 0\}$ of Lipschitz operators from D into itself is called a *semigroup of Lipschitz operators on D* if the following three conditions are satisfied:

(S1) $S(0)x = x$ for $x \in D$, and $S(t+s)x = S(t)S(s)x$ for $s, t \geq 0$ and $x \in D$.

(S2) For each $x \in D$, $S(\cdot)x : [0, \infty) \to X$ is continuous.

(S3) For each $\tau > 0$ there exists $L_\tau > 0$ such that

$$\|S(t)x - S(t)y\| \leq L_\tau \|x - y\| \quad \text{for } x,\, y \in D \text{ and } t \in [0, \tau].$$

We are now in a position to state our main result.

Theorem 2.2. *Assume that* (B), (Φ) *and* (F) *hold. Then there exists a semigroup* $\{S(t); t \geq 0\}$ *of Lipschitz operators on* D *such that*

$$BS(\cdot)x \in C([0, \infty); X) \qquad \text{for } x \in C,$$
$$BS(\cdot)x \in C((0, \infty); X) \cap L^1_{loc}(0, \infty; X) \qquad \text{for } x \in D,$$

$$S(t)x = T(t)x + \int_0^t T(t-s)BS(s)x\, ds \quad \text{for } x \in D \text{ and } t \geq 0. \tag{2.2}$$

Moreover, the following product formula holds:

$$S(t)x = \lim_{h \downarrow 0} F_h^{[t/h]} x \quad \text{in } X, \text{ for } x \in C \text{ and } t \geq 0, \tag{2.3}$$

where the convergence is uniform on every compact subset of $[0, \infty)$.

The existence of a semigroup $\{S(t); t \geq 0\}$ of Lipschitz operators on D satisfying (2.2) is assured by Remark 2.4 below and [21, Theorem 5.2] with φ defined by $\varphi = 0$ on D and $\varphi = \infty$ on $X \setminus D$. Thus, we have only to prove the product formula (2.3). The proof will be given in the following two sections.

Remark 2.3. It is easily seen that (F-i) and (F-ii) are equivalent to the following conditions, respectively.

(F-i)$'$ There exists $\omega \geq 0$ such that for any Y-bounded set W in C,

$$\limsup_{h \downarrow 0} \left(\sup\{h^{-1}(\Phi(F_h x, F_h y) - \Phi(x, y)) - \omega \Phi(x, y); x,\, y \in W\} \right) \leq 0.$$

(F-ii)$'$ There exists $\beta \in (0, 1)$ such that for any compact set W in C with respect to Y norm,

$$\lim_{h \downarrow 0} h^{-1} \|F_h x - J(h)x\| = 0 \quad \text{uniformly for } x \in W,$$
$$\lim_{h \downarrow 0} h^{-\beta} \|F_h x - J(h)x\|_Y = 0 \quad \text{uniformly for } x \in W.$$

Remark 2.4. Under (Φ-i) and (F), the following condition holds:

There exists $\omega \geq 0$ such that for any null sequence $\{h_n\}$ of positive numbers and x, $y \in C$,

$$\limsup_{n \to \infty} h_n^{-1}(\Phi(J(h_n)x, J(h_n)y) - \Phi(x, y)) \leq \omega \Phi(x, y).$$

Remark 2.5. Without loss of generality, by using the Feller renorming technique [5] if necessary, we may assume that $M_A = 1$ in the proof of Theorem 2.2. We may assume $\beta \in (0, 1 - \alpha]$ in condition (F-ii) as well.

3. Key estimate for product formula

This section is devoted to estimating the difference between the discrete semigroup $\{F_h^k; k \geq 0\}$ and an approximate solution x_j satisfying

$$x_j = T(t_j - t_{j-1})x_{j-1} + \int_{t_{j-1}}^{t_j} T(t_j - s)Bx_{j-1}\, ds + \xi_j$$

for $j = 1, 2, \ldots, N$. We begin by recalling the following result.

Lemma 3.1. ([21, Lemma 3.2]) *There exists $K_0 \geq 1$ such that for any $\tau \in (0, 1]$ and for any finite sequence $\{s_k\}_{k=0}^N$ satisfying $0 \leq s_0 < s_1 < \cdots < s_N \leq \tau$, the following two assertions hold:*

(i) *Let $M_G > 0$ and let G be a measurable function from $[0, \tau)$ into X satisfying $\|G(\xi)\| \leq M_G$ for $\xi \in [0, \tau)$. Then*

$$\int_{s_l}^{s_i} \|T(s_i - \xi)G(\xi)\|_Y\, d\xi \leq K_0 M_G (s_i - s_l)^\beta \quad \text{for } 0 \leq l \leq i \leq N.$$

(ii) *Let $\varepsilon > 0$. Then, for any finite sequence $\{\zeta_i\}_{i=1}^N$ in Y satisfying $\|\zeta_i\| \leq \varepsilon(s_i - s_{i-1})$ and $\|\zeta_i\|_Y \leq \varepsilon(s_i - s_{i-1})^\beta$ for $1 \leq i \leq N$,*

$$\sum_{l=k+1}^{i} \|T(s_i - s_l)\zeta_l\|_Y \leq K_0 \varepsilon (s_i - s_k)^\beta \quad \text{for } 0 \leq k \leq i \leq N.$$

In the rest of this section the symbol K_0 stands for the constant specified in Lemma 3.1.

Lemma 3.2. ([21, Lemma 3.3]) *Let $v_0 \in C$. Assume that $h \in (0, 1]$, $\nu \geq 0$ and positive numbers ρ, M_0 and ε satisfy*

$$\|Bx\| \leq M_0 \quad \text{for } x \in U_Y(v_0, \rho) \cap C,$$
$$K_0(M_0 + \varepsilon + \nu)h^\beta + \sup_{s \in [0,h]} \|T(s)v_0 - v_0\|_Y \leq \rho,$$

where $U_Y(v_0, \rho)$ denotes the closed ball in Y with center v_0 and radius ρ. Let $\delta \in [0, h]$, $w_0 \in C$, $\sigma > 0$ and G be a measurable function from $[0, \delta)$ into X such that

$$\sigma + \delta \leq h, \quad \|w_0 - T(\delta)v_0\| \leq (M_0 + \nu)\delta, \quad \|G(\xi)\| \leq M_0 \text{ for } \xi \in [0, \delta),$$
$$\left\| w_0 - T(\delta)v_0 - \int_0^\delta T(\delta - \xi)G(\xi)\, d\xi \right\|_Y \leq K_0 \nu \delta^\beta.$$

Assume that there exists a sequence $\{(s_i, w_i, \zeta_i)\}_{i=1}^N$ in $[0, \sigma] \times C \times Y$ such that

$$0 = s_0 < s_1 < \cdots < s_N \leq \sigma,$$
$$w_i = T(s_i - s_{i-1})w_{i-1} + \int_{s_{i-1}}^{s_i} T(s_i - \xi)Bw_{i-1}\, d\xi + \zeta_i \quad \text{for } 1 \leq i \leq N,$$
$$\|\zeta_i\| \leq \varepsilon(s_i - s_{i-1}) \quad \text{and} \quad \|\zeta_i\|_Y \leq \varepsilon(s_i - s_{i-1})^\beta \quad \text{for } 1 \leq i \leq N.$$

Then the following assertions hold:

(i-1) $\|T(s_j - s_k)w_k - w_j\| \le (M_0 + \varepsilon)(s_j - s_k)$ *for* $0 \le k \le j \le N$.
(i-2) $\|T(s_j - s_k)w_k - w_j\|_Y \le K_0(M_0 + \varepsilon)(s_j - s_k)^\beta$ *for* $0 \le k \le j \le N$.
(ii-1) $\|w_j - T(s_j + \delta)v_0\| \le (M_0 + \varepsilon + \nu)(s_j + \delta)$ *for* $0 \le j \le N$.
(ii-2) *For each* $j = 0, 1, \ldots, N$, *there exists a measurable function* G_j *from* $[0, s_j + \delta)$ *into* X *with* $\|G_j(\xi)\| \le M_0$ *for* $\xi \in [0, s_j + \delta)$ *such that*

$$\left\| w_j - T(s_j + \delta)v_0 - \int_0^{s_j+\delta} T(s_j + \delta - \xi)G_j(\xi)\, d\xi \right\|_Y \le K_0(\varepsilon + \nu)(s_j + \delta)^\beta.$$

(iii) $w_j \in U_Y(v_0, \rho)$ *and* $\|Bw_j\| \le M_0$ *for* $0 \le j \le N$.

The above lemma is a special version of [21, Lemma 3.3] where φ is a functional on X into $[0, \infty]$ defined by $\varphi = 0$ on D and $\varphi = \infty$ on $X \setminus D$.

For each $h \in (0, h_0]$ we define an operator E_h from C into Y by

$$E_h w = F_h w - J(h)w \quad \text{for } w \in C. \tag{3.1}$$

Lemma 3.3. *Let* $w_0 \in C$. *Assume that* $M_0 > 0$, $\rho > 0$, $\varepsilon > 0$, $\sigma \in (0, 1]$ *and* $\delta \in (0, h_0]$ *satisfy that*

$$\|Bx\| \le M_0 \quad \textit{for } x \in U_Y(w_0, \rho) \cap C,$$

$$K_0(M_0 + \varepsilon)\sigma^\beta + \sup_{0 \le s \le \sigma} \|T(s)w_0 - w_0\|_Y \le \rho\ ,$$

$$\|E_h x\| \le h\varepsilon, \quad \|E_h x\|_Y \le h^\beta \varepsilon \quad \textit{for } h \in (0, \delta] \textit{ and } x \in U_Y(w_0, \rho) \cap C.$$

Then for each $h \in (0, \delta]$ *and nonnegative integer* N *with* $Nh \le \sigma$, *the following are valid:*

(i) $\|T((k - j)h)F_h^j w_0 - F_h^k w_0\| \le (M_0 + \varepsilon)(k - j)h$ *for* $0 \le j \le k \le N$.

(ii) $\|T((k - j)h)F_h^j w_0 - F_h^k w_0\|_Y \le K_0(M_0 + \varepsilon)((k - j)h)^\beta$ *for* $0 \le j \le k \le N$.

(iii) $F_h^k w_0 \in U_Y(w_0, \rho) \cap C$ *for* $0 \le k \le N$.

Proof. Let $h \in (0, \delta]$ and let N be a nonnegative integer with $Nh \le \sigma$. For $k = 0$ conditions (i) through (iii) are obviously valid. Let k_0 be an integer with $1 \le k_0 \le N$ and suppose that for each pair of integers (j, k) with $0 \le j \le k \le k_0 - 1$, conditions (i) through (iii) hold true. Since $F_h^{l-1} w_0 \in C$ for $1 \le l \le k_0$, it follows from (3.1) and (2.1) that

$$F_h^l w_0 = T(h)F_h^{l-1} w_0 + \int_0^h T(s)BF_h^{l-1} w_0\, ds + E_h F_h^{l-1} w_0$$

for $1 \le l \le k_0$. Let $0 \le j \le k_0 - 1$. Applying $T((k_0 - l)h)$ to both sides and summing up the resultant for $l = j + 1, \ldots, k_0$, we have

$$F_h^{k_0} w_0 = T((k_0 - j)h)F_h^j w_0 + \sum_{l=j+1}^{k_0} \int_0^h T((k_0 - l)h + s)BF_h^{l-1} w_0\, ds + \sum_{l=j+1}^{k_0} T((k_0 - l)h)E_h F_h^{l-1} w_0. \tag{3.2}$$

Since $F_h^{l-1}w_0 \in U_Y(w_0,\rho)\cap C$ for $1 \le l \le k_0$ (by the hypothesis of induction), we have $\|BF_h^{l-1}w_0\| \le M_0$, $\|E_hF_h^{l-1}w_0\| \le h\varepsilon$ and $\|E_hF_h^{l-1}w_0\|_Y \le h^\beta\varepsilon$ for $1 \le l \le k_0$. Therefore, since $\{T(t); t \ge 0\}$ may be assumed to be contractive by Remark 2.5, we have $\|F_h^{k_0}w_0 - T((k_0-j)h)F_h^jw_0\| \le (M_0+\varepsilon)(k_0-j)h$ and apply Lemma 3.1 to obtain

$$\begin{aligned}\|F_h^{k_0}w_0 - T((k_0-j)h)F_h^jw_0\|_Y &\le \sum_{l=j+1}^{k_0}\int_{(l-1)h}^{lh}\|T(k_0h-s)BF_h^{l-1}w_0\|_Y\,ds\\ &\quad+\sum_{l=j+1}^{k_0}\|T((k_0-l)h)E_hF_h^{l-1}w_0\|_Y\\ &\le K_0(M_0+\varepsilon)((k_0-j)h)^\beta.\end{aligned}$$

These two inequalities show that assertions (i) and (ii) hold for $0 \le j \le k_0$. Setting $j=0$ in the last inequality, we observe that

$$\begin{aligned}\|F_h^{k_0}w_0 - w_0\|_Y &\le \|F_h^{k_0}w_0 - T(k_0h)w_0\|_Y + \|T(k_0h)w_0 - w_0\|_Y\\ &\le K_0(M_0+\varepsilon)(k_0h)^\beta + \|T(k_0h)w_0 - w_0\|_Y \le \rho.\end{aligned}$$

This means that assertion (iii) is valid for $k = k_0$. The proof is complete. □

Lemma 3.4. *Let $w_0 \in C$. Assume that $M_0 > 0$, $\rho > 0$, $\varepsilon > 0$, $\sigma \in (0,1]$ and $\delta \in (0,h_0]$ satisfy that*

$$\begin{aligned}&\|Bx\| \le M_0 \quad \text{for } x \in U_Y(w_0,\rho)\cap C,\\ &\|Bx - Bw_0\| \le \varepsilon \quad \text{for } x \in U_Y(w_0,\rho)\cap C,\\ &\|E_hx\| \le h\varepsilon, \quad \|E_hx\|_Y \le h^\beta\varepsilon \quad \text{for } h \in (0,\delta] \text{ and } x \in U_Y(w_0,\rho)\cap C,\\ &K_0(M_0+\varepsilon)\sigma^\beta + \sup_{0\le s\le\sigma}\|T(s)w_0 - w_0\|_Y \le \rho.\end{aligned}$$

Then for each $h \in (0,\delta]$ the following holds:

$$\|F_h^{[\sigma/h]}w_0 - J(\sigma)w_0\| \le 2\varepsilon\sigma + M_0h + \sup_{s\in[0,h]}\|T(s)w_0 - w_0\|. \tag{3.3}$$

Proof. Let $h \in (0,\delta]$. By (3.2) we find by a change of variables that

$$\begin{aligned}&F_h^kw_0 - T(kh)w_0 - \int_0^{kh}T(s)Bw_0\,ds\\ &\quad= \sum_{l=1}^{k}T((k-l)h)\left(\int_0^h T(s)(BF_h^{l-1}w_0 - Bw_0)\,ds + E_hF_h^{l-1}w_0\right)\end{aligned} \tag{3.4}$$

for $0 \le k \le [\sigma/h]$. Since $F_h^{l-1}w_0 \in U_Y(w_0,\rho)\cap C$ for $1 \le l \le [\sigma/h]$ (by Lemma 3.3), we have

$$\begin{aligned}\|BF_h^{l-1}w_0 - Bw_0\| &\le \varepsilon,\\ \|E_hF_h^{l-1}w_0\| &\le h\varepsilon,\end{aligned}$$

for $1 \leq l \leq [\sigma/h]$. We use these inequalities to estimate (3.4), so that

$$\left\| F_h^k w_0 - T(kh)w_0 - \int_0^{kh} T(s)Bw_0\, ds \right\| \leq 2\varepsilon(kh)$$

for $0 \leq k \leq [\sigma/h]$. Since

$$\begin{aligned}\|T(\sigma)w_0 - T([\sigma/h]h)w_0\| &= \|T([\sigma/h]h)(T(\sigma - [\sigma/h]h)w_0 - w_0)\| \\ &\leq \|T(\sigma - [\sigma/h]h)w_0 - w_0\|\end{aligned}$$

and

$$\left\| \int_0^{\sigma} T(s)Bw_0\, ds - \int_0^{[\sigma/h]h} T(s)Bw_0\, ds \right\| \leq \int_{[\sigma/h]h}^{\sigma} \|Bw_0\|\, ds \leq M_0 h,$$

the desired inequality (3.3) can be obtained by combining the last three inequalities. □

The next lemma gives the key estimate for the product formula (2.3). We often use the inequality $\|(-A)^\gamma T(t)\| \leq M_\gamma t^{-\gamma}$ for $t > 0$ and $\gamma \in (0,1)$.

Lemma 3.5. *Let $x_0 \in C$ and $\varepsilon \in (0, 1/2]$. Let ω be constant specified in* (F-i)*. Assume that $M_0 > 0$, $\rho_0 > 0$, $\tau_0 \in (0,1]$ and $\delta_0 \in (0, h_0]$ satisfy that*

$$\begin{aligned}&\Phi(F_h x, F_h y) \leq e^{\omega h}(\Phi(x,y) + \varepsilon h) \\ &\qquad \text{for } x,\ y \in U_Y(x_0, 2\rho_0) \cap C \text{ and } h \in (0, \delta_0], &(3.5)\\ &\|Bx\| \leq M_0 \quad \text{for } x \in U_Y(x_0, \rho_0) \cap C, &(3.6)\\ &K_0(M_0+1)\tau_0^\beta + \sup_{0 \leq s \leq \tau_0} \|T(s)x_0 - x_0\|_Y \leq \rho_0, &(3.7)\\ &\|E_h x\| \leq h,\ \|E_h x\|_Y \leq h^\beta \text{ for } h \in (0, \delta_0] \text{ and } x \in U_Y(x_0, \rho_0) \cap C. &(3.8)\end{aligned}$$

Let $\sigma_0 \in (0, \tau_0]$ and $\delta \in (0, \delta_0]$. Let $\{t_j\}_{j=1}^N$, $\{x_j\}_{j=1}^N$ and $\{\xi_j\}_{j=1}^N$ be sequences in $[0, \sigma_0]$, C and Y respectively such that they satisfy the following conditions:

(i) $0 = t_0 < t_1 < \ldots < t_j < \ldots < t_N = \sigma_0$ *and* $t_j - t_{j-1} \leq \varepsilon$ *for* $j = 1, 2, \ldots N$.

(ii) $x_j = T(t_j - t_{j-1})x_{j-1} + \int_{t_{j-1}}^{t_j} T(t_j - s)Bx_{j-1}\, ds + \xi_j$ *for* $j = 1, 2, \ldots, N$.

(iii) $\|\xi_j\| \leq \varepsilon(t_j - t_{j-1})$ *and* $\|\xi_j\|_Y \leq \varepsilon(t_j - t_{j-1})^\beta$ *for* $j = 1, 2, \ldots, N$.

(iv) *If $x \in C$ satisfies*

$$\|x - x_{j-1}\|_Y \leq K_0(M_0+1)(t_j - t_{j-1})^\beta + \sup_{s \in [0, t_j - t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y,$$

then $\|Bx - Bx_{j-1}\| \leq \varepsilon$ for $j = 1, 2, \ldots, N$.

(v) *If* $h \in (0, \delta]$ *and* $x \in C$ *satisfies*

$$\|x - x_{j-1}\|_Y \le K_0(M_0 + 1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y,$$

then $\|E_h x\| \le h\varepsilon$ *and* $\|E_h x\|_Y \le h^\beta \varepsilon$ *for* $j = 1, 2, \ldots, N$.

(vi) $K_0(M_0 + 1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y \le \rho_0$ *for* $j = 1, 2, \ldots, N$.

Define $v(t) = x_{j-1}$ *for* $t \in [t_{j-1}, t_j)$ *and* $j = 1, 2, \ldots, N$, *and* $v(t_N) = x_N$. *Then*

$$\begin{aligned}\Phi(v(t), F_h^n x_0) \le e^{\omega\tau_0}\Big\{&(3L + 1)\tau_0\varepsilon + 4NL(M_0 + 1)h \\ &+ NL \sup_{s\in[0,h]} \max_{1\le j\le N} \|T(s)x_{j-1} - x_{j-1}\| \\ &+ NLM_{1-\alpha}\alpha^{-1}(3h)^\alpha(\|x_0\|_Y + \rho_0)\Big\} \\ &+ L(M_0 + 1)(\varepsilon + 2h) \\ &+ LM_{1-\alpha}\alpha^{-1}(\varepsilon + 2h)^\alpha(\|x_0\|_Y + \rho_0) \qquad (3.9)\end{aligned}$$

for $t \in [0, \sigma_0]$, $n \in \mathbb{N}$ *and* $h \in (0, \delta]$ *with* $nh \le \tau_0$ *and* $|t - nh| \le h$.

PROOF. Let $1 \le j \le N$ and $h \in (0, \delta]$. By Lemma 3.2 we have $\|Bx_{j-1}\| \le M_0$. This and condition (iv) together imply that $\|Bx\| \le M_0 + \varepsilon$ for $x \in U_Y(x_{j-1}, \rho_j) \cap C$, where

$$\rho_j = K_0(M_0 + 1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y.$$

This inequality, the definition of ρ_j and conditions (iv) and (v) assure that all the assumptions in Lemma 3.4 are satisfied with M_0 replaced by $M_0 + \varepsilon$, $w_0 = x_{j-1}$, $\rho = \rho_j$ and $\sigma = t_j - t_{j-1}$; hence

$$\begin{aligned}&\left\| F_h^{[(t_j-t_{j-1})/h]} x_{j-1} - T(t_j - t_{j-1})x_{j-1} - \int_0^{t_j-t_{j-1}} T(s)Bx_{j-1}\, ds \right\| \\ &\qquad \le 2\varepsilon(t_j - t_{j-1}) + (M_0 + \varepsilon)h + \sup_{s\in[0,h]} \|T(s)x_{j-1} - x_{j-1}\|.\end{aligned}$$

Combining this inequality and conditions (ii) and (iii), we obtain

$$\begin{aligned}\|x_j - F_h^{[(t_j-t_{j-1})/h]} x_{j-1}\| \le\ & 3\varepsilon(t_j - t_{j-1}) + (M_0 + 1)h \\ &+ \sup_{s\in[0,h]} \|T(s)x_{j-1} - x_{j-1}\|. \qquad (3.10)\end{aligned}$$

Since all the assumptions in Lemma 3.3 are satisfied with M_0 replaced by $M_0+\varepsilon$, $w_0 = x_{j-1}$, $\rho = \rho_j$ and $\sigma = t_j - t_{j-1}$, we have $F_h^k x_{j-1} \in U_Y(x_{j-1}, \rho_j) \cap C$ for $0 \le k \le [(t_j - t_{j-1})/h]$. Since $x_{j-1} \in U_Y(x_0, \rho_0) \cap C$ by Lemma 3.2 and since $\rho_j \le \rho_0$ by condition (vi), we observe that $F_h^k x_{j-1} \in U_Y(x_0, 2\rho_0) \cap C$ for

$0 \le k \le [(t_j - t_{j-1})/h]$. By (3.6) through (3.8), all the assumptions in Lemma 3.3 are satisfied with $\varepsilon = 1$, $w_0 = x_0$, $\rho = \rho_0$, $\sigma = \tau_0$ and $\delta = \delta_0$; hence $F_h^k x_0 \in U_Y(x_0, \rho_0) \cap C$ for $0 \le k \le [\tau_0/h]$. Therefore, by (3.5) we have

$$\begin{aligned}&\Phi(F_h^{[(t_j-t_{j-1})/h]} x_{j-1}, F_h^{[(t_j-t_{j-1})/h]+[t_{j-1}/h]} x_0)\\ &\quad \le e^{\omega[(t_j-t_{j-1})/h]h}(\Phi(x_{j-1}, F_h^{[t_{j-1}/h]} x_0) + \varepsilon h[(t_j - t_{j-1})/h]). \qquad (3.11)\end{aligned}$$

By (Φ-i), (3.10) and (3.11) we have

$$\begin{aligned}&\Phi(x_j, F_h^{[t_j/h]} x_0)\\ &\quad \le e^{\omega[(t_j-t_{j-1})/h]h}(\Phi(x_{j-1}, F_h^{[t_{j-1}/h]} x_0) + \varepsilon h[(t_j - t_{j-1})/h])\\ &\qquad + L\left(3\varepsilon(t_j - t_{j-1}) + (M_0+1)h + \sup_{s\in[0,h]} \|T(s)x_{j-1} - x_{j-1}\|\right)\\ &\qquad + L\|F_h^{[t_j/h]} x_0 - F_h^{[(t_j-t_{j-1})/h]+[t_{j-1}/h]} x_0\|. \qquad (3.12)\end{aligned}$$

Noting that $[(t_j - t_{j-1})/h] + [t_{j-1}/h] \le [t_j/h]$ and applying Lemma 3.3 with $\varepsilon = 1$, $w_0 = x_0$, $\rho = \rho_0$, $\sigma = \tau_0$ and $\delta = \delta_0$ again, we have

$$\|T((p-q)h)F_h^q x_0 - F_h^p x_0\| \le (M_0+1)(p-q)h,$$

where $p = [t_j/h]$ and $q = [(t_j - t_{j-1})/h] + [t_{j-1}/h]$; hence

$$\begin{aligned}&\|F_h^{[t_j/h]} x_0 - F_h^{[(t_j-t_{j-1})/h]+[t_{j-1}/h]} x_0\|\\ &\quad \le (M_0+1)(p-q)h + \|T((p-q)h)F_h^q x_0 - F_h^q x_0\|\\ &\quad \le 3(M_0+1)h + M_{1-\alpha}\alpha^{-1}(3h)^\alpha(\|x_0\|_Y + \rho_0). \qquad (3.13)\end{aligned}$$

Here we have used the fact that $F_h^q x_0 \in U_Y(x_0, \rho_0) \cap C$ shown above and the inequality that $\|T(t)x - x\| \le M_{1-\alpha}\alpha^{-1}t^\alpha\|x\|_Y$ for $x \in Y$ and $t \ge 0$ to obtain the last inequality. Thus, we find by solving the inequality (3.12) combined with (3.13) that

$$\begin{aligned}\Phi(x_j, F_h^{[t_j/h]} x_0) \le e^{\omega\tau_0}\Big\{&(3L+1)\tau_0\varepsilon\\ &+ 4NL(M_0+1)h + NL \sup_{s\in[0,h]} \max_{1\le l\le N} \|T(s)x_{l-1} - x_{l-1}\|\\ &+ NLM_{1-\alpha}\alpha^{-1}(3h)^\alpha(\|x_0\|_Y + \rho_0)\Big\} \qquad (3.14)\end{aligned}$$

for $0 \le j \le N$.

Now, let $t \in [0, \sigma_0]$ and let $n \in \mathbb{N}$ and $h \in (0, \delta]$ satisfy $nh \le \tau_0$ and $|t - nh| \le h$. Then there exists an integer l with $0 \le l \le N$ such that $|t_l - t| \le \varepsilon$ and $v(t) = x_l$. By a way similar to the deviation of (3.13) we have

$$\|F_h^n x_0 - F_h^{[t_l/h]} x_0\| \le (M_0+1)(\varepsilon + 2h) + M_{1-\alpha}\alpha^{-1}(\varepsilon + 2h)^\alpha(\|x_0\|_Y + \rho_0).$$

Substituting this inequality and (3.14) into the inequality

$$\Phi(v(t), F_h^n x_0) \le \Phi(x_l, F_h^{[t_l/h]} x_0) + L\|F_h^n x_0 - F_h^{[t_l/h]} x_0\|,$$

we obtain the desired inequality (3.9). □

4. Proof of product formula

Let $u_0 \in C$ and $\tau > 0$. Let $\{S(t); t \geq 0\}$ be the semigroup of Lipschitz operators on D obtained by the first part of Theorem 2.2 and put $u(t) = S(t)u_0$ for $t \in [0, \tau]$. By condition (F-ii) one finds $\delta_1 > 0$ and $\rho_1 > 0$ such that

$$\|E_h x\| \leq h \quad \text{and} \quad \|E_h x\|_Y \leq h^\beta \tag{4.1}$$

for $h \in (0, \delta_1]$ and $x \in \bigcup_{t \in [0,\tau]} U_Y(u(t), \rho_1) \cap C$. The continuity of the operator B assures that there exist $M_0 > 0$ and $\rho_2 > 0$ satisfying

$$\|Bx\| \leq M_0 \quad \text{for } x \in \bigcup_{t \in [0,\tau]} U_Y(u(t), \rho_2) \cap C. \tag{4.2}$$

Set $\rho_0 = \min\{1/2,\, \rho_1/2,\, \rho_2/2\}$ and choose $\tau_0 \in (0, 1]$ such that

$$K_0(M_0 + 1)\tau_0^\beta + \sup_{0 \leq s \leq \tau_0} \|T(s)u(t) - u(t)\|_Y \leq \rho_0/3 \quad \text{for } t \in [0, \tau], \tag{4.3}$$

$$K_\gamma(2M_{\gamma,\alpha}(\tau_0))^{1/\gamma} + K_0\tau_0^\beta \leq \rho_0/4, \tag{4.4}$$

where K_0 is the constant specified in Lemma 3.1, $\gamma \in (\alpha, 1)$, K_γ is a positive constant in the moment inequality that

$$\|x\|_Y \leq K_\gamma \|x\|^{(\gamma-\alpha)/\gamma} \|(-A)^\gamma x\|^{\alpha/\gamma} \quad \text{for } x \in D((-A)^\gamma) \tag{4.5}$$

and $M_{\gamma,\alpha}(t)$ is the nondecreasing function on $[0, \infty)$ defined by

$$\begin{aligned} M_{\gamma,\alpha}(t) = {} & M_{\gamma-\alpha}^\alpha t^{(\gamma-\alpha)(1-\alpha)} \Big(\sup\{\|u(s)\|_Y;\, s \in [0, \tau]\} + 1 \Big)^\alpha \\ & + M_\gamma^\alpha M_0^\alpha (1-\gamma)^{-\alpha} t^{\gamma(1-\alpha)} \end{aligned} \tag{4.6}$$

for $t \geq 0$. Since $0 < \alpha < \gamma < 1$, we have $\lim_{t\downarrow 0} M_{\gamma,\alpha}(t) = 0$. This fact guarantees the existence of $\tau_0 \in (0, 1]$ satisfying condition (4.4).

Let $\sigma_0 \in (0, \tau_0)$ and $k_0 \in \mathbb{N}$ satisfy $k_0\sigma_0 = \tau$. Let k be an integer with $0 \leq k \leq k_0 - 1$. Then the proof of the product formula (2.3) is inductively completed once it is shown that if $\lim_{h\downarrow 0} F_h^{[k\sigma_0/h]} u_0 = S(k\sigma_0)u_0$ in X and $\limsup_{h\downarrow 0} \|F_h^{[k\sigma_0/h]} u_0 - S(k\sigma_0)u_0\|_Y \leq \rho_0/4$, then

$$\lim_{h\downarrow 0} (\sup\{\|F_h^{[(t+k\sigma_0)/h]} u_0 - S(t + k\sigma_0)u_0\|; t \in [0, \sigma_0]\}) = 0, \tag{4.7}$$

$$\limsup_{h\downarrow 0} \|F_h^{[(k+1)\sigma_0/h]} u_0 - S((k + 1)\sigma_0)u_0\|_Y \leq \rho_0/4. \tag{4.8}$$

Indeed, assume that the above-mentioned claim is proved for $0 \leq k \leq k_0 - 1$. Since $F_h^{[k\sigma_0/h]} u_0 = u_0 = S(k\sigma_0)u_0$ for $k = 0$, conditions (4.7) and (4.8) are satisfied for $k = 0$. By (4.7) with $k = 0$ we have $\lim_{h\downarrow 0} F_h^{[t/h]} u_0 = S(t)u_0$ in X, uniformly for $t \in [0, \sigma_0]$. In particular, we have $\lim_{h\downarrow 0} F_h^{[\sigma_0/h]} u_0 = S(\sigma_0)u_0$ in

X. This and (4.8) with $k = 0$ together imply that conditions (4.7) and (4.8) are satisfied for $k = 1$. By (4.7) with $k = 1$ we have $\lim_{h\downarrow 0} F_h^{[t/h]}u_0 = S(t)u_0$ in X, uniformly for $t \in [\sigma_0, 2\sigma_0]$. Continuing this procedure up to $k = k_0 - 1$, we have $\lim_{h\downarrow 0} F_h^{[t/h]}u_0 = S(t)u_0$ in X, uniformly for $t \in [0, k_0\sigma_0]$.

Now, let $u_h = F_h^{[k\sigma_0/h]}u_0$ for $h \in (0, h_0]$ and suppose that $\lim_{h\downarrow 0} u_h = S(k\sigma_0)u_0$ in X and $\limsup_{h\downarrow 0} \|u_h - S(k\sigma_0)u_0\|_Y \le \rho_0/4$. Then we want to show (4.7) and (4.8). For this purpose, let $\varepsilon \in (0, 1/2]$. Then we deduce from condition (F-i) that there exists $\delta_2 \in (0, h_0]$ such that

$$\Phi(F_h x, F_h y) \le e^{\omega h}(\Phi(x, y) + \varepsilon h) \tag{4.9}$$

for $h \in (0, \delta_2]$ and $x,\ y \in \bigcup_{t\in[0,\tau]} U_Y(u(t), 1) \cap C$. By the hypothesis that $\limsup_{h\downarrow 0} \|u_h - S(k\sigma_0)u_0\|_Y \le \rho_0/4$, there exists $\delta_3 > 0$ such that

$$\|u_h - S(k\sigma_0)u_0\|_Y \le \rho_0/3 \quad \text{for } h \in (0, \delta_3]. \tag{4.10}$$

Set $\delta_0 = \min\{\delta_1,\ \delta_2,\ \delta_3\}$. Let $\delta \in (0, \delta_0]$. Since $u(k\sigma_0) = S(k\sigma_0)u_0$, we have $U_Y(u_\delta, \rho_0) \subset U_Y(u(k\sigma_0), 2\rho_0)$ by (4.10). It follows from (4.2), (4.3) and (4.1) that

$$\|Bx\| \le M_0 \quad \text{for } x \in U_Y(u_\delta, \rho_0) \cap C, \tag{4.11}$$

$$K_0(M_0 + 1)\tau_0^\beta + \sup_{0\le s\le\tau_0} \|T(s)u_\delta - u_\delta\|_Y \le \rho_0,$$

$$\|E_h x\| \le h, \ \ \|E_h x\|_Y \le h^\beta \quad \text{for } h \in (0, \delta] \text{ and } x \in U_Y(u_\delta, \rho_0) \cap C. \tag{4.12}$$

These three conditions show that all the assumptions in Lemma 3.3 are satisfied with $w_0 = u_\delta$, $\sigma = \tau_0$, $\rho = \rho_0$ and $\varepsilon = 1$; hence $F_h^l u_\delta \in U_Y(u_\delta, \rho_0) \cap C$ for $0 \le l \le [\tau_0/h]$ and $h \in (0, \delta]$. In particular, we have $F_\delta^l u_\delta \in U_Y(u_\delta, \rho_0) \cap C$ for $0 \le l \le [\tau_0/\delta]$. It follows from (4.11), (4.12) and (4.10) that

$$\|BF_h^l u_h\| \le M_0, \tag{4.13}$$

$$\|E_h F_h^l u_h\| \le h,\ \|E_h F_h^l u_h\|_Y \le h^\beta, \tag{4.14}$$

$$F_h^l u_h \in U_Y(u(k\sigma_0), 2\rho_0) \cap C \tag{4.15}$$

for $0 \le l \le [\tau_0/h]$ and $h \in (0, \delta_0]$. By (4.9), (4.2), (4.3) and (4.1) we have

$$\Phi(F_h x, F_h y) \le e^{\omega h}(\Phi(x, y) + \varepsilon h)$$
$$\text{for } x,\ y \in U_Y(u(k\sigma_0), 2\rho_0) \cap C \text{ and } h \in (0, \delta_0], \tag{4.16}$$

$$\|Bx\| \le M_0 \quad \text{for } x \in U_Y(u(k\sigma_0), \rho_0) \cap C, \tag{4.17}$$

$$K_0(M_0 + 1)\tau_0^\beta + \sup_{0\le s\le\tau_0} \|T(s)u(k\sigma_0) - u(k\sigma_0)\|_Y \le \rho_0, \tag{4.18}$$

$$\|E_h x\| \le h,\ \|E_h x\|_Y \le h^\beta \text{ for } h \in (0, \delta_0] \text{ and } x \in U_Y(u(k\sigma_0), \rho_0) \cap C. \tag{4.19}$$

We apply Lemma 3.3 with $\varepsilon = 1$ to obtain the inequality

$$F_h^l u(k\sigma_0) \in U_Y(u(k\sigma_0), \rho_0) \cap C \quad \text{for } 0 \le l \le [\tau_0/h] \text{ and } h \in (0, \delta_0].$$

By this inequality and (4.15) we use the inequality (4.16) to find that

$$\Phi(F_h^l u_h, F_h^l u(k\sigma_0)) \le e^{\omega\tau_0}(\Phi(u_h, u(k\sigma_0)) + \tau_0\varepsilon)$$

for $0 \le l \le [\tau_0/h]$ and $h \in (0, \delta_0]$. Since $([(t + k\sigma_0)/h] - [k\sigma_0/h])h \le t + h \le \sigma_0 + h \le \tau_0$ for $t \in [0, \sigma_0]$ and sufficiently small $h > 0$, we have

$$\lim_{h\downarrow 0}(\sup\{\Phi(F_h^{[(t+k\sigma_0)/h]}u_0, F_h^{[(t+k\sigma_0)/h]-[k\sigma_0/h]}u(k\sigma_0)); t \in [0, \sigma_0]\}) = 0. \quad (4.20)$$

To prove (4.7), it remains to estimate $\|F_h^{[(t+k\sigma_0)/h]-[k\sigma_0/h]}u(k\sigma_0) - S(t)u(k\sigma_0)\|$ for $t \in [0, \sigma_0]$, by applying Lemma 3.5. It should be noticed that assumptions (3.5) through (3.8) with $x_0 = u(k\sigma_0)$ are satisfied by (4.16) through (4.19). By condition (F-ii) one finds $\bar{\delta}_1 > 0$ and $\bar{\rho}_1 > 0$ such that

$$\|E_h x\| \le h\varepsilon \quad \text{and} \quad \|E_h x\|_Y \le h^\beta\varepsilon \quad (4.21)$$

for $h \in (0, \bar{\delta}_1]$ and $x \in \bigcup_{t\in[0,\tau]} U_Y(u(t), \bar{\rho}_1) \cap C$. The continuity of the operator B assures that there exists $\bar{\rho}_2 > 0$ satisfying

$$\|Bx - Bu(t)\| \le \varepsilon \quad \text{for } x \in U_Y(u(t), \bar{\rho}_2) \cap C \text{ and } t \in [0, \tau]. \quad (4.22)$$

Set $\bar{\rho}_0 = \min\{\rho_0, \bar{\rho}_1, \bar{\rho}_2\}$ and choose $\lambda > 0$ so that $\lambda \le \min\{\delta_0, \bar{\delta}_1, \varepsilon\}$ and the following two conditions are satisfied:

If t, $s \in [0, \tau]$ satisfy $|t - s| \le \lambda$, then

$$\|Bu(t) - Bu(s)\| \le (1 + M_\alpha(1 - \alpha)^{-1})^{-1}\varepsilon. \quad (4.23)$$

If $t \in [0, \tau]$, then $K_0(M_0 + 1)\lambda^\beta + \sup_{s\in[0,\lambda]} \|T(s)u(t) - u(t)\|_Y \le \bar{\rho}_0$. (4.24)

Let $\{t_j\}_{j=0}^N$ be a partition of the interval $[0, \sigma_0]$ such that $0 = t_0 < t_1 < \ldots < t_j < \ldots < t_N = \sigma_0$ and $t_j - t_{j-1} \le \lambda$ for $1 \le j \le N$. Put $s_j = k\sigma_0 + t_j$ and $x_j = S(s_j)u_0$ $(= u(s_j))$ for $0 \le j \le N$. In order to apply Lemma 3.5, it suffices to check conditions (ii) through (vi) in Lemma 3.5. Conditions (vi) follows from (4.24), since $\bar{\rho}_0 \le \rho_0$ and $s_{j-1} \le (k + 1)\sigma_0 \le \tau$ for $1 \le j \le N$. Condition (ii) is satisfied by defining

$$\xi_j = x_j - \left(T(t_j - t_{j-1})x_{j-1} + \int_{t_{j-1}}^{t_j} T(t_j - s)Bx_{j-1}\,ds\right)$$

for $1 \le j \le N$. Since we deduce from (2.2) that the right-hand side is written as

$$\int_{s_{j-1}}^{s_j} T(s_j - s)(Bu(s) - Bu(s_{j-1}))\,ds$$

for $1 \le j \le N$, we have by (4.23)

$$\|\xi_j\| \le \int_{s_{j-1}}^{s_j} \|Bu(s) - Bu(s_{j-1})\|\,ds \le (t_j - t_{j-1})\varepsilon,$$

$$\|\xi_j\|_Y \le \int_{s_{j-1}}^{s_j} M_\alpha(s_j - s)^{-\alpha}(1 + M_\alpha(1 - \alpha)^{-1})^{-1}\varepsilon \le (t_j - t_{j-1})^{1-\alpha}\varepsilon$$

for $1 \le j \le N$. Since $t_j - t_{j-1} \le 1$ for $1 \le j \le N$, we observe by these two inequalities and Remark 2.5 that condition (iii) is satisfied. To check the two conditions (iv) and (v), let $1 \le j \le N$ and let $x \in C$ satisfy $\|x - x_{j-1}\|_Y \le K_0(M_0+1)(t_j - t_{j-1})^\beta + \sup_{s\in[0,t_j-t_{j-1}]} \|T(s)x_{j-1} - x_{j-1}\|_Y$. Since $x_{j-1} = u(s_{j-1})$, it follows from (4.24) that $x \in U_Y(u(s_{j-1}), \bar\rho_0) \cap C$. By (4.22) we have $\|Bx - Bu(s_{j-1})\| \le \varepsilon$. This means that condition (iv) is satisfied. In the same way, condition (v) with $\delta = \lambda$ follows from (4.21). Thus, all the conditions in Lemma 3.5 with $x_0 = u(k\sigma_0)$ and $\delta = \lambda$ are proved to be satisfied. Since $nh \le \tau_0$ for sufficiently small $h \in (0, \lambda]$ provided that $t \in [0, \sigma_0]$ and $|t - nh| \le h$ for $h \in (0, h_0]$, we find by Lemma 3.5 that

$$\begin{aligned}
&\limsup_{h\downarrow 0} (\sup\{\Phi(S(t)S(k\sigma_0)u_0, F_h^n S(k\sigma_0)u_0); t \in [0,\sigma_0], |t-nh| \le h\}) \\
&\le L \sup\{\|S(t)S(k\sigma_0)u_0 - S(s)S(k\sigma_0)u_0\|; t,s \in [0,\sigma_0],\ |t-s| \le \lambda\} \\
&\quad + e^{\omega\tau_0}(3L+1)\tau_0\varepsilon + L(M_0+1)\varepsilon + LM_{1-\alpha}\alpha^{-1}\varepsilon^\alpha(\|S(k\sigma_0)u_0\|_Y + \rho_0).
\end{aligned}$$

Letting $\lambda \downarrow 0$ and then letting $\varepsilon \downarrow 0$, we have by condition (Φ-ii)

$$\lim_{h\downarrow 0} (\sup\{\|S(t+k\sigma_0)u_0 - F_h^n S(k\sigma_0)u_0\|; t \in [0,\sigma_0], |t-nh| \le h\}) = 0.$$

This together with (4.20) implies (4.7), since $|([(t+k\sigma_0)/h] - [k\sigma_0/h])h - t| \le h$ for $t \in [0, \sigma_0]$ and $h > 0$.

To prove (4.8), let $l_h = [(k+1)\sigma_0/h] - [k\sigma_0/h]$ for $h \in (0, \delta_0]$ and define

$$v_h = T(l_h h)u_h + \sum_{j=1}^{l_h} \int_0^h T((l_h - j)h + s)BF_h^{j-1}u_h\, ds, \tag{4.25}$$

$$w_h = \sum_{j=1}^{l_h} T((l_h - j)h)E_h F_h^{j-1}u_h \tag{4.26}$$

for $h \in (0, \delta_0]$. Then, by (3.2) we have

$$F^{[(k+1)\sigma_0/h]}u_0 = F_h^{l_h}u_h = v_h + w_h \tag{4.27}$$

for $h \in (0, \delta_0]$. Since $|l_h h - \sigma_0| \le h$ for $h \in (0, \delta_0]$, we have $l_h h \le \tau_0$ for sufficiently small $h \in (0, \delta_0]$. By (4.14) we apply Lemma 3.1 to find that

$$\|w_h\| \le l_h h \quad \text{and} \quad \|w_h\|_Y \le K_0(l_h h)^\beta \tag{4.28}$$

for sufficiently small $h \in (0, \delta_0]$. Since the fact that $\lim_{h\downarrow 0} F_h^{[(k+1)\sigma_0/h]}u_0 = u((k+1)\sigma_0)$ in X is already shown in (4.7), we have by (4.27) and (4.28)

$$\limsup_{h\downarrow 0} \|v_h - u((k+1)\sigma_0)\| \le \sigma_0. \tag{4.29}$$

Let $h \in (0, \delta_0]$ and let $G(s) = BF_h^{j-1}u_h$ for $s \in [(j-1)h, jh)$ and $1 \le j \le l_h$. Then, we observe by (4.13) that $\|G(s)\| \le M_0$ for $s \in [0, l_h h)$. Since the second term on the right-hand side of (4.25) is written as $\int_0^{l_h h} T(l_h h - s)G(s)\,ds$, we find that

$$\|(-A)^\gamma v_h\| \le M_{\gamma-\alpha}(l_h h)^{-(\gamma-\alpha)}\|u_h\|_Y + M_\gamma M_0(1-\gamma)^{-1}(l_h h)^{1-\gamma}. \tag{4.30}$$

It follows from (4.10) and (4.6) that

$$\limsup_{h\downarrow 0} \sigma_0^{\gamma-\alpha}\|(-A)^\gamma v_h\|^\alpha \le M_{\gamma,\alpha}(\sigma_0). \tag{4.31}$$

Here we have used the inequality $(a+b)^\alpha \le a^\alpha + b^\alpha$ for $a, b \ge 0$. By (2.2) and (4.2) we have

$$u((k+1)\sigma_0) = T(\sigma_0)u(k\sigma_0) + \int_0^{\sigma_0} T(\sigma_0 - s)Bu(s + k\sigma_0)\,ds$$

and $\|Bu(s+k\sigma_0)\| \le M_0$ for $s \in [0, \sigma_0]$, respectively. By a way similar to the derivation of (4.30) we observe that $\sigma_0^{\gamma-\alpha}\|(-A)^\gamma u((k+1)\sigma_0)\|^\alpha \le M_{\gamma,\alpha}(\sigma_0)$. Using this inequality, (4.31) and (4.29), we find by the moment inequality (4.5) that $\limsup_{h\downarrow 0}\|v_h - u((k+1)\sigma_0)\|_Y \le K_\gamma(2M_{\gamma,\alpha}(\sigma_0))^{1/\gamma}$. Combining this inequality, (4.27) and (4.28), we have

$$\limsup_{h\downarrow 0}\|F_h^{[(k+1)\sigma_0/h]}u_0 - u((k+1)\sigma_0)\|_Y \le K_\gamma(2M_{\gamma,\alpha}(\sigma_0))^{1/\gamma} + K_0\sigma_0^\beta.$$

By (4.4) this inequality implies the desired inequality (4.8). $\square$

5. Solvability of the complex Ginzburg-Landau equation by a fractional step method

Let $1 < p < \infty$ and let us consider the mixed problem for the complex Ginzburg-Landau equation

$$\text{(CGL)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} - (\lambda + i\mu)\Delta u + (\kappa + i\nu)|u|^{q-2}u - \gamma u = 0 & \text{in } \Omega \times (0,\infty), \\ u = 0 \quad \text{on } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x) \end{cases}$$

in $L^p(\Omega)$ space. Here Ω is a smooth domain in $\mathbb{R}^N$ where $N \ge 1$, and $\lambda > 0$, $\kappa > 0$, $\mu, \nu, \gamma \in \mathbb{R}$. Under the assumption that

$$|\mu|/\lambda < 2\sqrt{p-1}/|p-2| \quad \text{and} \quad 2 \le q \le 2 + 2p/N \tag{5.1}$$

it is shown in [21] that the (CGL) has a unique solution in the class

$$C([0,\infty); L^p(\Omega)) \cap C^1((0,\infty); L^p(\Omega)) \cap C((0,\infty); W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)). \tag{5.2}$$

For further details we refer to [1, 6, 7, 18, 21, 22, 23, 27, 28, 31, 32].

In this section we discuss the solvability of the (CGL) by a fractional step method as an application of Theorem 2.2. For simplicity, we consider the case where $\gamma = 0$. In what follows we assume that $q > 2$.

Following [22, Section 2], we first write (CGL) as the abstract Cauchy problem (SP) in $L^p(\Omega)$ (see [22] for details). Let $X = L^p(\Omega)$ and $\|u\| = \|u\|_{L^p}$ for $u \in X$. Define a linear operator $\mathcal{A}$ in X by

$$\mathcal{A}u = (\lambda + i\mu)\Delta u \qquad \text{for } u \in D(\mathcal{A}) := W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$$

and define $Av = \mathcal{A}v - (\lambda + i\mu)v$ for $v \in D(A) := D(\mathcal{A})$. Then, by (5.1) we deduce from [9, 26] that $\mathcal{A}$ generates an analytic semigroup $\{T_{\mathcal{A}}(z); |\arg z| < \psi_p\}$ of contractions on X and the operator A is the infinitesimal generator of an analytic semigroup $\{T(z)\,(:= e^{-(\lambda+i\mu)z}T_{\mathcal{A}}(z)); |\arg z| < \psi_p\}$ of class (C_0) on X such that $\|T(t)\| \le e^{-\lambda t}$ for $t \ge 0$, where $\psi_p = \tan^{-1}(2\sqrt{p-1}/|p-2|) - \tan^{-1}(|\mu|/\lambda)$. By (5.1) we can choose $\tilde{p}$ such that

$$p < \tilde{p} < p + q - 2, \tag{5.3}$$

$$|\mu|/\lambda < 2\sqrt{\tilde{p}-1}/|\tilde{p}-2|, \tag{5.4}$$

$$\tilde{\theta} := (N/2)(1/p - 1/(\tilde{p}(q-1))) < 1. \tag{5.5}$$

Then, by (5.4) we have

$$\|T_{\mathcal{A}}(t)v\|_{L^{\tilde{p}}} \le \|v\|_{L^{\tilde{p}}} \quad \text{and} \quad \|T(t)v\|_{L^{\tilde{p}}} \le e^{-\lambda t}\|v\|_{L^{\tilde{p}}} \tag{5.6}$$

for $v \in X \cap L^{\tilde{p}}(\Omega)$ and $t \ge 0$. Moreover, we can choose $\alpha \in (0,1)$ such that

$$\tilde{\theta} < \alpha < 1, \tag{5.7}$$

$$D((-A)^\alpha) \subset L^p(\Omega) \cap L^{\tilde{p}}(\Omega) \cap L^{p(q-1)}(\Omega) \cap L^{\tilde{p}(q-1)}(\Omega), \tag{5.8}$$

where the inclusion in (5.8) is continuous (see [22]). Let $Y = D((-A)^\alpha)$. Let $R > 0$ be fixed arbitrarily and let

$$D = \{v \in L^p(\Omega) \cap L^{\tilde{p}}(\Omega); \|v\|_{L^p} + \|v\|_{L^{\tilde{p}}} \le R\}. \tag{5.9}$$

Then, the (CGL) is rewritten as the semilinear Cauchy problem

$$u'(t) = Au(t) + Bu(t) \quad \text{for } t > 0, \qquad u(0) = u_0,$$

by defining a nonlinear operator B from C into X as

$$Bu = -(\kappa + i\nu)|u|^{q-2}u + (\lambda + i\mu)u \qquad \text{for } u \in D(B) = C\ (= D \cap Y).$$

The operator B from C into X is already shown ([22]) to satisfy condition (B) and the locally Lipschitz continuity condition in the following sense: For each $\rho > 0$ there exists $L_B(\rho) > 0$ such that

$$\|Bv - B\hat{v}\| \le L_B(\rho)\|v - \hat{v}\|_Y \quad \text{for } v,\, \hat{v} \in C \text{ with } \|v\|_Y \le \rho,\ \|\hat{v}\|_Y \le \rho.$$

The purpose is to discuss the solvability of the (CGL) through a fractional step method. Namely, we write (CGL) as $u'(t) = \mathcal{A}u(t) + \mathcal{B}u(t)$ for $t > 0$, and $u(0) = u_0$ by using the nonlinear operator $\mathcal{B}$ in X defined by

$$\mathcal{B}u = -(\kappa + i\nu)|u|^{q-2}u \qquad \text{for } u \in D(\mathcal{B}) = L^p(\Omega) \cap L^{p(q-1)}(\Omega).$$

Then we solve the two simpler problems $v'(t) = \mathcal{A}v(t)$ and $w'(t) = \mathcal{B}w(t)$, and obtain the solution u through the formula $u(t) = \lim_{h\downarrow 0}(T_{\mathcal{A}}(h)T_{\mathcal{B}}(h))^{[t/h]}u_0$ for $t \geq 0$, where $\{T_{\mathcal{B}}(t); t \geq 0\}$ is the semigroup generated by $\mathcal{B}$. To do this, we need to investigate some basic properties on the semigroup $\{T_{\mathcal{A}}(t); t \geq 0\}$ and the operator $\mathcal{B}$.

Lemma 5.1. *The following assertions hold.*

(i) *There exists $K > 0$ such that*

$$e^{\lambda t}\|T(t)v\|_{L^{p(q-1)}} = \|T_{\mathcal{A}}(t)v\|_{L^{p(q-1)}} \leq K\|v\|_{L^{p(q-1)}} \tag{5.10}$$

for $v \in X \cap L^{p(q-1)}(\Omega)$ and $t > 0$.

(ii) *There exists $K > 0$ such that*

$$e^{\lambda t}\|T(t)v\|_{L^{p(q-1)}} = \|T_{\mathcal{A}}(t)v\|_{L^{p(q-1)}} \leq Kt^{-(N/p-N/p(q-1))/2}\|v\| \tag{5.11}$$

for $v \in D$ and $t > 0$.

(iii) *There exist $K > 0$ and $\theta_{\mathcal{A}} \in (0,1)$ such that*

$$\|T_{\mathcal{A}}(t)v - v\|_{L^{p(q-1)}} \leq Kt^{\theta_{\mathcal{A}}}\|v\|_Y, \tag{5.12}$$

$$\|\nabla T_{\mathcal{A}}(t)v\|_{L^{p(q-1)}} \leq Kt^{(\theta_{\mathcal{A}}-1)/2}\|v\|_Y \tag{5.13}$$

for $v \in Y$ and $t \in (0,1]$.

(iv) *There exists $K > 0$ such that*

$$\|\mathcal{B}v - \mathcal{B}\hat{v}\| \leq K(\|v\|^{q-2}_{L^{p(q-1)}} + \|\hat{v}\|^{q-2}_{L^{p(q-1)}})\|v - \hat{v}\|_{L^{p(q-1)}} \tag{5.14}$$

for v, $\hat{v} \in D(\mathcal{B})$.

In what follows, the symbol K stands for various constants.

Proof. Assertions (i) and (ii) follow from [19], [26] and L^p-L^q estimates for the heat semigroup. Assertion (iii) will be shown as follows: Since $T_{\mathcal{A}}(t)v - v = \int_0^t (Ae^{(\lambda+i\mu)s}T(s)v + (\lambda + i\mu)e^{(\lambda+i\mu)s}T(s)v)\,ds$ for $v \in Y$ and $t > 0$, we have

$$\|T_{\mathcal{A}}(t)v - v\|_{L^{p(q-1)}} \leq K\int_0^t (\|AT(s)v\|_{L^{p(q-1)}} + \|T(s)v\|_{L^{p(q-1)}})\,ds \tag{5.15}$$

for $v \in Y$ and $t \in (0,1]$. Since $AT(s)v = -T(s/2)(-A)^{1-\alpha}T(s/2)(-A)^{\alpha}v$ for $v \in Y$ and $s > 0$, we find by (5.11) and the inequality $\|(-A)^{\gamma}T(t)\| \leq M_{\gamma}t^{-\gamma}$ for $t > 0$ and $\gamma \in (0,1)$ that

$$\|AT(s)v\|_{L^{p(q-1)}} \leq Ks^{\theta_{\mathcal{A}}-1}\|v\|_Y \tag{5.16}$$

for $v \in Y$ and $s > 0$, where $\theta_{\mathcal{A}} = \alpha - N(q-2)/(2p(q-1))$. By (5.3), (5.5) and (5.7) we have $N(q-2)/(2p(q-1)) < \tilde{\theta} < \alpha < 1$; hence $\theta_{\mathcal{A}} \in (0,1)$. By (5.10) and (5.8) we have

$$\|T(s)v\|_{L^{p(q-1)}} \leq K\|v\|_Y \tag{5.17}$$

for $v \in Y$ and $s > 0$. The inequality (5.12) is obtained by substituting (5.16) and (5.17) into (5.15). By the Gagliardo-Nirenberg inequality

$$\|\nabla w\|_{L^{p(q-1)}} \leq K\|w\|_{L^{p(q-1)}}^{1/2}\|w\|_{W^{2,p(q-1)}}^{1/2} \text{ for } w \in W^{2,p(q-1)}(\Omega),$$

the elliptic estimate $\|w\|_{W^{2,p(q-1)}} \leq K\|Aw\|_{L^{p(q-1)}}$ for $w \in W^{2,p(q-1)}(\Omega)$, and the inequalities (5.16) and (5.17), we have

$$\|\nabla T_{\mathcal{A}}(t)v\|_{L^{p(q-1)}} \leq K\|\nabla T(t)v\|_{L^{p(q-1)}} \leq Kt^{(\theta_{\mathcal{A}}-1)/2}\|v\|_Y$$

for $v \in Y$ and $t \in (0,1]$. Assertion (iv) is shown by using the elementary inequality $||\xi|^{q-2}\xi - |\eta|^{q-2}\eta| \leq K\left(\int_0^1 |\theta\xi + (1-\theta)\eta|^{q-2}\,d\theta\right)|\xi-\eta|$ for $\xi, \eta \in \mathbb{C}$. □

By a direct computation, the Cauchy problem in $\mathbb{C}$

$$\xi'(t) = -(\kappa + i\nu)|\xi(t)|^{q-2}\xi(t) \quad \text{for } t > 0, \quad \xi(0) = \xi_0 \in \mathbb{C} \tag{5.18}$$

has a unique solution ξ given by

$$\begin{aligned}\xi(t) = {} & \left(1 + (q-2)\kappa|\xi_0|^{q-2}t\right)^{-1/(q-2)}\xi_0 \\ & \times \exp\left(-i\frac{\nu}{(q-2)\kappa}\log\left(1 + (q-2)\kappa|\xi_0|^{q-2}t\right)\right)\end{aligned}$$

for $t \geq 0$. By this representation we have

$$|\xi(t)| \leq |\xi_0| \qquad \text{for } t \geq 0. \tag{5.19}$$

By (5.18) and (5.19) we have $|\xi'(t)| = K|\xi(t)|^{q-1} \leq K|\xi_0|^{q-1}$ for $t \geq 0$; hence

$$|\xi(t) - \xi_0| \leq K|\xi_0|^{q-1}t \qquad \text{for } t \geq 0. \tag{5.20}$$

By (5.19) we can define a family $\{T_{\mathcal{B}}(t); t \geq 0\}$ of operators on X by

$$\begin{aligned}(T_{\mathcal{B}}(t)v)(x) = {} & \left(1 + (q-2)\kappa|v(x)|^{q-2}t\right)^{-1/(q-2)}v(x) \\ & \times \exp\left(-i\frac{\nu}{(q-2)\kappa}\log\left(1 + (q-2)\kappa|v(x)|^{q-2}t\right)\right)\end{aligned} \tag{5.21}$$

for $v \in X$.

Lemma 5.2. *The family $\{T_{\mathcal{B}}(t); t \geq 0\}$ has the properties below:*

(i) *For each $v \in X$, $T_{\mathcal{B}}(t)v$ is continuous in $t \geq 0$ and $T_{\mathcal{B}}(t)v \to v$ in X as $t \downarrow 0$. Furthermore, for $s \in [1,\infty)$*

$$\|T_{\mathcal{B}}(t)v\|_{L^s} \leq \|v\|_{L^s} \quad \textit{for } t \geq 0 \textit{ and } v \in X \cap L^s(\Omega). \tag{5.22}$$

(ii) *For each $v \in D(\mathcal{B})$ and $t \geq 0$, $T_{\mathcal{B}}(t)v$ is differentiable with respect to t and $(d/dt)T_{\mathcal{B}}(t)v = \mathcal{B}T_{\mathcal{B}}(t)v$ in X. Moreover,*

$$\|T_{\mathcal{B}}(t)v - v\| \leq Kt\|v\|^{q-1}_{L^{p(q-1)}} \quad \textit{for } t \geq 0 \textit{ and } v \in D(\mathcal{B}). \tag{5.23}$$

(iii) *There exists $\theta_{\mathcal{B}} \in (0,1)$ such that*

$$\|T_{\mathcal{B}}(t)v - v\|_{L^{p(q-1)}} \leq Kt^{1-\theta_{\mathcal{B}}}\|v\|^{\tilde{p}/p}_{L^{\tilde{p}(q-1)}} \tag{5.24}$$

for $t \geq 0$ and $v \in X \cap L^{\tilde{p}(q-1)}(\Omega)$.

Proof. Assertions (i) and (ii) follow from (5.18), (5.19), (5.20) and the dominated convergence theorem. To verify assertion (iii), let $v \in X \cap L^{\tilde{p}(q-1)}(\Omega)$. By (5.21) we find that

$$\begin{aligned}|(T_{\mathcal{B}}(t)v)(x)|^{p(q-1)^2} &\leq \frac{|v(x)|^{(q-1)(p(q-1)-\tilde{p})}|v(x)|^{\tilde{p}(q-1)}}{(1+(q-2)\kappa|v(x)|^{q-2}t)^{(q-1)(p(q-1)-\tilde{p})/(q-2)}} \\ &\leq \frac{|v(x)|^{\tilde{p}(q-1)}}{((q-2)\kappa t)^{(q-1)(p(q-1)-\tilde{p})/(q-2)}}\end{aligned}$$

for almost all $x \in \Omega$ and $t > 0$. Hence $T_{\mathcal{B}}(t)v \in L^{p(q-1)^2}(\Omega)$ for $t > 0$ and $\|T_{\mathcal{B}}(t)v\|_{L^{p(q-1)^2}} \leq Kt^{-(p(q-1)-\tilde{p})/p(q-1)(q-2)}\|v\|^{\tilde{p}/p(q-1)}_{L^{\tilde{p}(q-1)}}$ for $t > 0$. Since $|(\mathcal{B}T_{\mathcal{B}}(t)v)(x)| \leq K|(T_{\mathcal{B}}(t)v)(x)|^{q-1}$ for almost all $x \in \Omega$ and $t > 0$, we have

$$\mathcal{B}T_{\mathcal{B}}(t)v \in L^{p(q-1)}(\Omega) \quad \text{and} \quad \|\mathcal{B}T_{\mathcal{B}}(t)v\|_{L^{p(q-1)}} \leq Kt^{-\theta_{\mathcal{B}}}\|v\|^{\tilde{p}/p}_{L^{\tilde{p}(q-1)}}$$

for $t > 0$, where $\theta_{\mathcal{B}} = (p(q-1)-\tilde{p})/p(q-2)$. By (5.3) and the fact that $p+q-2 < p(q-1)$ we have $\theta_{\mathcal{B}} \in (0,1)$. Thus, the inequality (5.24) holds. □

The following product formula shows the solvability of the (CGL) by a fractional step method.

Theorem 5.3. *Let $u_0 \in C$. Then there exists a unique C^1 solution u to* (CGL) *with the initial value u_0. Moreover, the solution u is obtained through the formula*

$$u(t) = \lim_{h\downarrow 0}(T_{\mathcal{A}}(h)T_{\mathcal{B}}(h))^{[t/h]}u_0 \quad \textit{in } X\textit{, for } t \geq 0, \tag{5.25}$$

where the convergence is uniform on each compact subinterval of $[0,\infty)$.

Proof. The existence and uniqueness of C^1 solutions is known. To prove (5.25) we shall check all the assumptions in Theorem 2.2. Let Φ be the nonnegative functional on $X \times X$ defined by

$$\Phi(u,v) = \exp((b/\kappa p)((\|u\| \wedge R)^p + (\|v\| \wedge R)^p))(\|u-v\| \wedge (2R))$$

for u, $v \in X$, where $a \wedge b = \min\{a, b\}$ for $a, b \in \mathbb{R}$. It is shown ([22, (4.6)]) that assumption (Φ) is satisfied and that there exists $\omega \geq 0$ such that

$$D_+\Phi(u,v)(Au + Bu, Av + Bv) \leq \omega\Phi(u,v) \qquad \text{for } u, v \in D(A) \cap D, \quad (5.26)$$

where

$$D_+\Phi(u,v)(\xi,\eta) = \liminf_{h\downarrow 0}(\Phi(u + h\xi, v + h\eta) - \Phi(u,v))/h$$

for $(u,v), (\xi,\eta) \in X \times X$.

Let $F_h v = T_{\mathcal{A}}(h)T_{\mathcal{B}}(h)v$ for $h > 0$ and $v \in C$. Then we deduce from (5.6) and (5.22) that the operator F_h maps C into itself. By Remark 2.3 we shall check conditions (F-i)$'$ and (F-ii)$'$ in place of conditions (F-i) and (F-ii). To prove that condition (F-ii)$'$ is satisfied, let W be any compact set in C and let ρ be a positive number such that $\|v\|_Y \leq \rho$ for $v \in W$. Put $w(t,v) = F_t v$ for $t > 0$ and $v \in W$. Since

$$\begin{aligned} w'(t,v) &= \mathcal{A}T_{\mathcal{A}}(t)T_{\mathcal{B}}(t)v + T_{\mathcal{A}}(t)\mathcal{B}T_{\mathcal{B}}(t)v \\ &= Aw(t,v) + Bv + f(t,v) \end{aligned}$$

for $t > 0$ and $v \in W$, where

$$f(t,v) = T_{\mathcal{A}}(t)\mathcal{B}T_{\mathcal{B}}(t)v - \mathcal{B}v + (\lambda + i\mu)(w(t,v) - v)$$

for $t > 0$ and $v \in W$, we have

$$F_t v = w(t,v) = J(t)v + \int_0^t T(t-s)f(s,v)\,ds \quad (5.27)$$

for $t > 0$ and $v \in W$. By (5.27) we have

$$\|F_h v - J(h)v\| \leq h \sup_{s\in[0,h]} \|f(s,v)\|, \quad (5.28)$$

$$\|F_h v - J(h)v\|_Y \leq M_\alpha(1-\alpha)^{-1}h^{1-\alpha} \sup_{s\in[0,h]} \|f(s,v)\| \quad (5.29)$$

for $h > 0$ and $v \in W$. To estimate $\|f(s,v)\|$ for $s > 0$ and $v \in W$, we write $f(s,v) = a(s,v) + b(s,v) + c(s,v)$ for $s > 0$ and $v \in W$, where

$$\begin{aligned} a(s,v) &= T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{B}}(s)v - T_{\mathcal{A}}(s)\mathcal{B}v, \\ b(s,v) &= T_{\mathcal{A}}(s)\mathcal{B}v - \mathcal{B}v + (\lambda + i\mu)(T_{\mathcal{A}}(s)v - v), \\ c(s,v) &= (\lambda + i\mu)(T_{\mathcal{A}}(s)T_{\mathcal{B}}(s)v - T_{\mathcal{A}}(s)v) \end{aligned}$$

for $s > 0$ and $v \in W$. Since W is compact in C, the sets $\mathcal{B}(W)$ and W are compact in X. This and the strong continuity of $\{T_{\mathcal{A}}(t); t \geq 0\}$ in $B(X)$ imply that $\{b(s,v)\}$ vanishes in X uniformly for $v \in W$ as $s \downarrow 0$. Since the semigroup $\{T_{\mathcal{A}}(t); t \geq 0\}$ is contractive on X, we find by (5.14), (5.22), (5.24) and (5.8) that

$$\begin{aligned} \|a(s,v)\| &\leq K(\|T_{\mathcal{B}}(s)v\|^{q-2}_{L^{p(q-1)}} + \|v\|^{q-2}_{L^{p(q-1)}})\|T_{\mathcal{B}}(s)v - v\|_{L^{p(q-1)}} \\ &\leq K\rho^{q-2}\rho^{\tilde{p}/p}s^{1-\theta_{\mathcal{B}}} \end{aligned}$$

for $s > 0$ and $v \in W$. By (5.23) we have $\|c(s,v)\| \le K\|T_{\mathcal{B}}(s)v - v\| \le K\rho^{q-1}s$ for $s > 0$ and $v \in W$. Hence $\lim_{h\downarrow 0}\sup_{s\in[0,h]}\|f(s,v)\| = 0$ uniformly for $v \in W$. This together with (5.28) and (5.29) implies that condition (F-ii)$'$ is satisfied.

It remains to show that condition (F-i)$'$ is satisfied. For this purpose, let W be any Y-bounded set in C and let ρ be a positive number such that $\|v\|_Y \le \rho$ for $v \in W$. Put $w(t,v) = T_{\mathcal{A}}(t)T_{\mathcal{B}}(t)v$ for $t > 0$ and $v \in W$. Then we have $w'(t,v) = Aw(t,v) + Bw(t,v) + g(t,v)$ for $t > 0$ and $v \in W$, where $g(t,v) = T_{\mathcal{A}}(t)\mathcal{B}T_{\mathcal{B}}(t)v - \mathcal{B}T_{\mathcal{A}}(t)T_{\mathcal{B}}(t)v$ for $t > 0$ and $v \in W$. By (5.26) we have

$$D_+\Phi(w(t,z),w(t,\hat{z})) \le \omega\Phi(w(t,z),w(t,\hat{z})) + L(\|g(t,z)\| + \|g(t,\hat{z})\|)$$

for $t > 0$ and $z, \hat{z} \in W$, where $D_+\Phi(w(t,z),w(t,\hat{z}))$ is the Dini derivative of the function $t \to \Phi(w(t,z),w(t,\hat{z}))$. This implies that

$$\begin{aligned} &h^{-1}(\Phi(w(h,z),w(h,\hat{z})) - \Phi(z,\hat{z})) \\ &\quad \le h^{-1}(e^{\omega h}-1)\Phi(z,\hat{z}) + h^{-1}L\int_0^h e^{\omega(h-s)}(\|g(s,z)\| + \|g(s,\hat{z})\|)\,ds \end{aligned} \tag{5.30}$$

for $h \in (0,1]$ and $z, \hat{z} \in W$. To verify condition (F-i)$'$ we want to estimate $\|g(s,v)\|$ for $s \in (0,1]$ and $v \in W$. For this purpose, let $s \in (0,1]$ and $v \in W$, and write

$$\begin{aligned} g(s,v) =& (T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{B}}(s)v - T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{A}}(s)v) + (T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{A}}(s)v - \mathcal{B}T_{\mathcal{A}}(s)v) \\ &+ (\mathcal{B}T_{\mathcal{A}}(s)v - \mathcal{B}T_{\mathcal{A}}(s)T_{\mathcal{B}}(s)v). \end{aligned} \tag{5.31}$$

Since $\|v\|_Y \le \rho$ and Y is continuously embedded in the space $L^{p(q-1)}(\Omega) \cap L^{\tilde{p}(q-1)}(\Omega)$ by (5.8), we deduce from Lemmas 5.1 and 5.2 that

$$\begin{aligned} &\|T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{B}}(s)v - T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{A}}(s)v\| \\ &\quad \le K(\|T_{\mathcal{B}}(s)v\|^{q-2}_{L^{p(q-1)}} + \|T_{\mathcal{A}}(s)v\|^{q-2}_{L^{p(q-1)}})\|T_{\mathcal{B}}(s)v - T_{\mathcal{A}}(s)v\|_{L^{p(q-1)}} \\ &\quad \le K\rho^{q-2}(\|T_{\mathcal{B}}(s)v - v\|_{L^{p(q-1)}} + \|T_{\mathcal{A}}(s)v - v\|_{L^{p(q-1)}}) \\ &\quad \le K\rho^{q-2}(\rho^{\tilde{p}/p}s^{1-\theta_{\mathcal{B}}} + \rho s^{\theta_{\mathcal{A}}}). \end{aligned} \tag{5.32}$$

Similarly, we have

$$\begin{aligned} \|\mathcal{B}T_{\mathcal{A}}(s)v - \mathcal{B}T_{\mathcal{A}}(s)T_{\mathcal{B}}(s)v\| &\le K\rho^{q-2}\|v - T_{\mathcal{B}}(s)v\|_{L^{p(q-1)}} \\ &\le K\rho^{q-2+\tilde{p}/p}s^{1-\theta_{\mathcal{B}}}. \end{aligned} \tag{5.33}$$

Since $|(\nabla\mathcal{B}T_{\mathcal{A}}(s)v)(x)| \le K|(T_{\mathcal{A}}(s)v)(x)|^{q-2}|(\nabla T_{\mathcal{A}}(s)v)(x)|$ for almost all $x \in \Omega$, we observe by Lemma 5.1 that $\mathcal{B}T_{\mathcal{A}}(s)v \in W_0^{1,p}(\Omega)$ and

$$\begin{aligned} \|\mathcal{B}T_{\mathcal{A}}(s)v\|_{W^{1,p}} &\le K(\|\mathcal{B}T_{\mathcal{A}}(s)v\| + \|T_{\mathcal{A}}(s)v\|^{q-2}_{L^{p(q-1)}}\|\nabla T_{\mathcal{A}}(s)v\|_{L^{p(q-1)}}) \\ &\le K\rho^{q-1}(1 + s^{(\theta_{\mathcal{A}}-1)/2}). \end{aligned} \tag{5.34}$$

To estimate the second term on the right-hand side of (5.31), let ε be a positive number such that $2\varepsilon < \min\{1-1/p, \theta_{\mathcal{A}}/3\}$. Since $1-2\varepsilon > 1/p$, we notice by [8, Proposition 5.11] that the real interpolation space $(L^p, D(\mathcal{A}))_{1/2-\varepsilon,p}$ between $L^p(\Omega)$ and $D(\mathcal{A})$ is characterized as $\{f \in W^{1-2\varepsilon,p}(\Omega);\ f|_{\partial\Omega} = 0\}$. By this fact, the definition of $(L^p, D(\mathcal{A}))_{1/2-\varepsilon,\infty}$ and the fact that $(L^p, D(\mathcal{A}))_{1/2-\varepsilon,p}$ is continuously embedded in $(L^p, D(\mathcal{A}))_{1/2-\varepsilon,\infty}$ (see [3, Chapter 3]), we find that

$$\begin{aligned}
\|T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{A}}(s)v - \mathcal{B}T_{\mathcal{A}}(s)v\| &\le Ks^{1/2-\varepsilon}\|\mathcal{B}T_{\mathcal{A}}(s)v\|_{(L^p,D(\mathcal{A}))_{1/2-\varepsilon,\infty}} \\
&\le Ks^{1/2-\varepsilon}\|\mathcal{B}T_{\mathcal{A}}(s)v\|_{W^{1-2\varepsilon,p}} \\
&\le Ks^{1/2-\varepsilon}\|\mathcal{B}T_{\mathcal{A}}(s)v\|_{W^{1,p}}.
\end{aligned}$$

This together with (5.34) yields that

$$\|T_{\mathcal{A}}(s)\mathcal{B}T_{\mathcal{A}}(s)v - \mathcal{B}T_{\mathcal{A}}(s)v\| \le K\rho^{q-1}s^{\theta_{\mathcal{A}}/3},$$

since $\theta_{\mathcal{A}}/3 < \theta_{\mathcal{A}}/2 - \varepsilon < 1/2 - \varepsilon$ and $s \in (0,1]$. Combining this inequality, (5.31), (5.32) and (5.33) we find a positive number $K(\rho)$ depending only on ρ such that

$$\|g(s,v)\| \le K(\rho)s^{\theta_0}$$

for $s \in (0,1]$ and $v \in W$, where $\theta_0 = \min\{1-\theta_{\mathcal{B}}, \theta_{\mathcal{A}}/3\}$. By substituting this inequality into (5.30), condition (F-i)$'$ is proved to be satisfied. $\square$

References

[1] I.S. Aranson and L. Kramer, The world of the complex Ginzburg-Landau equation. Reviews of Modern Physics 74 (2002) 99–143.

[2] H. Brezis and A. Pazy, Convergence and approximation of semigroups of nonlinear operators in Banach spaces, J. Funct. Anal. 9 (1972) 63–74.

[3] P. Butzer and H. Berens, Semi-groups of operators and approximation, Die Grundlehren der mathematischen Wissenschaften, Band 145 Springer-Verlag New York Inc., New York 1967.

[4] P. Chernoff, Note on product formulas for operator semigroups, J. Funct. Anal. 2 (1968) 238–242.

[5] W. Feller, On the generation of unbounded semi-groups of bounded linear operators, Ann. of Math. **58** (1953) 166–174.

[6] J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation. I. Compactness methods. Physica D 95 (1996) 191–228.

[7] J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg-Landau equation II. Contraction methods. Commun. Math. Phys. 187 (1997) 45–79.

[8] P. Grisvard, Équations différentielles abstraites, Ann. Scient. Éc. Norm. Sup., 4^e série 2 (1969) 311–395.

[9] D. Henry, Geometric Theory of Semilinear Parabolic Equations, Lecture Notes in Math. 840, Springer-Verlag, Berlin, 1981.

[10] T. Kato and K. Masuda, Trotter's product formula for nonlinear semigroups generated by the subdifferentials of convex functions, J. Math. Soc. Japan 30 (1978) 169–178.

[11] Y. Kobayashi, Product formula for nonlinear semigroups in Hilbert spaces, Proc. Japan Acad. 58 (1982) 425–428.

[12] Y. Kobayashi, Product formula for nonlinear contraction semigroups in Banach spaces, Hiroshima Math. J. 17 (1987) 129–140.

[13] Y. Kobayashi, A product formula approach to first order quasilinear equations, Hiroshima Math. J. 14 (1984) 489–509.

[14] Y. Kobayashi and N. Tanaka, Semigroups of Lipschitz operators, Adv. Differential Equations 6 (2001) 613–640.

[15] Y. Kobayashi and N. Tanaka, Convergence and approximation of semigroups of Lipschitz operators, Nonlinear Anal. T.M.A. 61 (2005) 781–821.

[16] Y. Kobayashi and N. Tanaka, An application of semigroups of locally Lipschitz operators to Carrier equations with acoustic boundary conditions, J. Math. Anal. Appl. 338 (2008) 852–872.

[17] Y. Kobayashi and N. Tanaka, A Lipschitz semigroup approach to two-dimensional Navier-Stokes equations, Nonlinear Anal. TMA 72 (2010) 1820–1828.

[18] C.D. Levermore and M. Oliver, The complex Ginzburg-Landau equation as a model problem. Lect. Appl. Math. 31, Amer. Math. Soc, Providence, RI (1996) 141–190.

[19] A. Lunardi, Analytic Semigroups and Optimal Regularity in Parabolic Problems, Progress in Nonlinear Differential Equations and their Applications 16, Birkhäuser Verlag, Basel, 1995.

[20] J. Marsden, On product formulas for nonlinear semigroups, J. Funct. Anal. 13 (1973) 51–72.

[21] T. Matsumoto and N. Tanaka, Semigroups of locally Lipschitz operators associated with semilinear evolution equations of parabolic type. Nonlinear. Anal. TMA, 69 (2008) 4025–4054.

[22] T. Matsumoto and N. Tanaka, Well-posedness for the complex Ginzburg-Landau equations, *Current Advances in Nonlinear Analysis and Related Topics*, 429–442, GAKUTO Internat. Ser. Math. Sci. Appl. 32, Gakkotosho, Tokyo, 2010.

[23] A. Mielke, The complex Ginzburg-Landau equation on a large and unbounded domains: sharper bounds and attractors. Nonlinearity 10 (1997) 199–222.

[24] I. Miyadera and Y. Kobayashi, Convergence and approximation of nonlinear semigroups, Proceedings of Japan-France Seminar on Functional Analysis and Numerical Analysis, 1978, 277–295.

[25] I. Miyadera and S. Oharu, Approximation of semi-groups of nonlinear operators, Tôhoku Math. J. 22 (1970) 24–47.

[26] N. Okazawa, Sectorialness of second order elliptic operators in divergence form. Proc. Amer. Math. Soc. **113** (1991), 701–706.

[27] N. Okazawa and T. Yokota, Non-contraction semigroups generated by the complex Ginzburg-Landau equation, *Nonlinear Partial Differential Equations and Their Applications* (Shanghai, 2003), 490–504, GAKUTO Internat. Ser. Math. Sci. Appl. **20**, Gakkotosho, Tokyo, 2004.

[28] N. Okazawa and T. Yokota, Monotonicity method applied to the complex Ginzburg-Landau and related equations. J. Math. Anal. Appl. 267 (2002) 247–263.

[29] S. Reich, Product formula, nonlinear semigroups and accretive operators, J. Funct. Anal. 36 (1980) 147–168.

[30] H. Trotter, On the product of semigroups of operators, Proc. Amer. Math. Soc. 10 (1959) 545–551.

[31] T. Yokota and N. Okazawa, Smoothing effect for the complex Ginzburg-Landau equation (general case), Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal. 13B (2006), suppl. 305–316.

[32] T. Yokota and N. Okazawa, The complex Ginzburg-Landau equation (an improvement), In: *Nonlinear Phenomena with Energy Dissipation*, 463–475, GAKUTO Internat. Ser. Math. Sci. Appl., 29, Gakkotosho, Tokyo, 2008.