# Approximation properties of certain operator-induced norms on Hilbert spaces 

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#### Abstract

We consider a class of operator-induced norms, acting as finite-dimensional surrogates to the $L^{2}$ norm, and study their approximation properties over Hilbert subspaces of $L^{2}$. The class includes, as a special case, the usual empirical norm encountered, for example, in the context of nonparametric regression in reproducing kernel Hilbert spaces (RKHS). Our results have implications to the analysis of $M$-estimators in models based on finite-dimensional linear approximation of functions, and also to some related packing problems.


Keywords:
$L^{2}$ approximation, Empirical norm, Quadratic functionals, Hilbert spaces with reproducing kernels, Analysis of $M$-estimators

## 1. Introduction

Given a probability measure $\mathbb{P}$ supported on a compact set $\mathcal{X} \subset \mathbb{R}^{d}$, consider the function class

$$
\begin{equation*}
L^{2}(\mathbb{P}):=\left\{f: \mathcal{X} \rightarrow \mathbb{R} \mid\|f\|_{L^{2}(\mathbb{P})}<\infty\right\} \tag{1}
\end{equation*}
$$

where $\|f\|_{L^{2}(\mathbb{P})}:=\sqrt{\int_{\mathcal{X}} f^{2}(x) d \mathbb{P}(x)}$ is the usual $L^{2}$ norm1 defined with respect to the measure $\mathbb{P}$. It is often of interest to construct approximations

[^0]to this $L^{2}$ norm that are "finite-dimensional" in nature, and to study the quality of approximation over the unit ball of some Hilbert space $\mathcal{H}$ that is continuously embedded within $L^{2}$. For example, in approximation theory and mathematical statistics, a collection of $n$ design points in $\mathcal{X}$ is often used to define a surrogate for the $L^{2}$ norm. In other settings, one is given some orthonormal basis of $L^{2}(\mathbb{P})$, and defines an approximation based on the sum of squares of the first $n$ (generalized) Fourier coefficients. For problems of this type, it is of interest to gain a precise understanding of the approximation accuracy in terms of its dimension $n$ and other problem parameters.

The goal of this paper is to study such questions in reasonable generality for the case of Hilbert spaces $\mathcal{H}$. We let $\Phi_{n}: \mathcal{H} \rightarrow \mathbb{R}^{n}$ denote a continuous linear operator on the Hilbert space, which acts by mapping any $f \in \mathcal{H}$ to the $n$-vector $\left(\left[\Phi_{n} f\right]_{1} \quad\left[\Phi_{n} f\right]_{2} \cdots \quad\left[\Phi_{n} f\right]_{n}\right)$. This operator defines the $\Phi_{n^{-}}$ semi-norm

$$
\begin{equation*}
\|f\|_{\Phi_{n}}:=\sqrt{\sum_{i=1}^{n}\left[\Phi_{n} f\right]_{i}^{2}} \tag{2}
\end{equation*}
$$

In the sequel, with a minor abuse of terminology ${ }^{2}$ we refer to $\|f\|_{\Phi_{n}}$ as the $\Phi_{n}$-norm of $f$. Our goal is to study how well $\|f\|_{\Phi_{n}}$ approximates $\|f\|_{L^{2}}$ over the unit ball of $\mathcal{H}$ as a function of $n$, and other problem parameters. We provide a number of examples of the sampling operator $\Phi_{n}$ in Section 2.2. Since the dependence on the parameter $n$ should be clear, we frequently omit the subscript to simplify notation.

In order to measure the quality of approximation over $\mathcal{H}$, we consider the quantity

$$
\begin{equation*}
R_{\Phi}(\varepsilon):=\sup \left\{\|f\|_{L^{2}}^{2} \mid f \in B_{\mathcal{H}},\|f\|_{\Phi}^{2} \leq \varepsilon^{2}\right\} \tag{3}
\end{equation*}
$$

where $B_{\mathcal{H}}:=\left\{f \in \mathcal{H} \mid\|f\|_{\mathcal{H}} \leq 1\right\}$ is the unit ball of $\mathcal{H}$. The goal of this paper is to obtain sharp upper bounds on $R_{\Phi}$. As discussed in Appendix Appendix C, a relatively straightforward argument can be used to translate such upper bounds into lower bounds on the related quantity

$$
\begin{equation*}
\underline{T}_{\Phi}(\varepsilon):=\inf \left\{\|f\|_{\Phi}^{2} \mid f \in B_{\mathcal{H}},\|f\|_{L^{2}}^{2} \geq \varepsilon^{2}\right\} . \tag{4}
\end{equation*}
$$

[^1]We also note that, for a complete picture of the relationship between the semi-norm $\|\cdot\|_{\Phi}$ and the $L^{2}$ norm, one can also consider the related pair

$$
\begin{align*}
T_{\Phi}(\varepsilon) & :=\sup \left\{\|f\|_{\Phi}^{2} \quad \mid f \in B_{\mathcal{H}},\|f\|_{L^{2}}^{2} \leq \varepsilon^{2}\right\}, \quad \text { and }  \tag{5a}\\
\underline{R}_{\Phi}(\varepsilon) & :=\inf \left\{\|f\|_{L^{2}}^{2} \quad \mid f \in B_{\mathcal{H}},\|f\|_{\Phi}^{2} \geq \varepsilon^{2}\right\} \tag{5b}
\end{align*}
$$

Our methods are also applicable to these quantities, but we limit our treatment to $\left(R_{\Phi}, \underline{T}_{\Phi}\right)$ so as to keep the contribution focused.

Certain special cases of linear operators $\Phi$, and associated functionals have been studied in past work. In the special case $\varepsilon=0$, we have

$$
R_{\Phi}(0)=\sup \left\{\|f\|_{L^{2}}^{2} \mid f \in B_{\mathcal{H}}, \Phi(f)=0\right\}
$$

a quantity that corresponds to the squared diameter of $B_{\mathcal{H}} \cap \operatorname{Ker}(\Phi)$, measured in the $L^{2}$-norm. Quantities of this type are standard in approximation theory (e.g., [1, 2, 3] ), for instance in the context of Kolmogorov and Gelfand widths. Our primary interest in this paper is the more general setting with $\varepsilon>0$, for which additional factors are involved in controlling $R_{\Phi}(\varepsilon)$. In statistics, there is a literature on the case in which $\Phi$ is a sampling operator, which maps each function $f$ to a vector of $n$ samples, and the norm $\|\cdot\|_{\Phi}$ corresponds to the empirical $L^{2}$-norm defined by these samples. When these samples are chosen randomly, then techniques from empirical process theory [4] can be used to relate the two terms. As discussed in the sequel, our results have consequences for this setting of random sampling.

As an example of a problem in which an upper bound on $R_{\Phi}$ is useful, let us consider a general linear inverse problem, in which the goal is to recover an estimate of the function $f^{*}$ based on the noisy observations

$$
y_{i}=\left[\Phi f^{*}\right]_{i}+w_{i}, \quad i=1, \ldots, n
$$

where $\left\{w_{i}\right\}$ are zero-mean noise variables, and $f^{*} \in B_{\mathcal{H}}$ is unknown. An estimate $\widehat{f}$ can be obtained by solving a least-squares problem over the unit ball of the Hilbert space - that is, to solve the convex program

$$
\widehat{f}:=\arg \min _{f \in B_{\mathcal{H}}} \sum_{i=1}^{n}\left(y_{i}-[\Phi f]_{i}\right)^{2} .
$$

For such estimators, there are fairly standard techniques for deriving upper bounds on the $\Phi$-semi-norm of the deviation $\widehat{f}-f^{*}$. Our results in this paper
on $R_{\Phi}$ can then be used to translate this to a corresponding upper bound on the $L^{2}$-norm of the deviation $\widehat{f}-f^{*}$, which is often a more natural measure of performance.

As an example where the dual quantity $\underline{T}_{\Phi}$ might be helpful, consider the packing problem for a subset $\mathcal{D} \subset B_{\mathcal{H}}$ of the Hilbert ball. Let $M\left(\varepsilon ; \mathcal{D},\|\cdot\|_{L^{2}}\right)$ be the $\varepsilon$-packing number of $\mathcal{D}$ in $\|\cdot\|_{L^{2}}$, i.e., the maximal number of function $f_{1}, \ldots, f_{M} \in \mathcal{D}$ such that $\left\|f_{i}-f_{j}\right\|_{L^{2}} \geq \varepsilon$ for all $i, j=1, \ldots, M$. Similarly, let $M\left(\varepsilon ; \mathcal{D},\|\cdot\|_{\Phi}\right)$ be the $\varepsilon$-packing number of $\mathcal{D}$ in $\|\cdot\|_{\Phi}$ norm. Now, suppose that for some fixed $\varepsilon, \underline{T}_{\Phi}(\varepsilon)>0$. Then, if we have a collection of functions $\left\{f_{1}, \ldots, f_{M}\right\}$ which is an $\varepsilon$-packing of $\mathcal{D}$ in $\|\cdot\|_{L^{2}}$ norm, then the same collection will be a $\sqrt{\underline{T_{\Phi}(\varepsilon)}}$-packing of $\mathcal{D}$ in $\|\cdot\|_{\Phi}$. This implies the following useful relationship between packing numbers

$$
M\left(\varepsilon ; \mathcal{D},\|\cdot\|_{L^{2}}\right) \leq M\left(\sqrt{\underline{T_{\Phi}(\varepsilon)}} ; \mathcal{D},\|\cdot\|_{\Phi}\right) .
$$

The remainder of this paper is organized as follows. We begin in Section 2 with background on the Hilbert space set-up, and provide various examples of the linear operators $\Phi$ to which our results apply. Section 3 contains the statement of our main result, and illustration of some its consequences for different Hilbert spaces and linear operators. Finally, Section 4 is devoted to the proofs of our results.

Notation: For any positive integer $p$, we use $\mathbb{S}_{+}^{p}$ to denote the cone of $p \times p$ positive semidefinite matrices. For $A, B \in \mathbb{S}_{+}^{p}$, we write $A \succeq B$ or $B \preceq A$ to mean $A-B \in \mathbb{S}_{+}^{p}$. For any square matrix $A$, let $\lambda_{\text {min }}(A)$ and $\lambda_{\max }(A)$ denote its minimal and maximal eigenvalues, respectively. We will use both $\sqrt{A}$ and $A^{1 / 2}$ to denote the symmetric square root of $A \in \mathbb{S}_{+}^{p}$. We will use $\left\{x_{k}\right\}=$ $\left\{x_{k}\right\}_{k=1}^{\infty}$ to denote a (countable) sequence of objects (e.g. real-numbers and functions). Occasionally we might denote an $n$-vector as $\left\{x_{1}, \ldots, x_{n}\right\}$. The context will determine whether the elements between braces are ordered. The symbols $\ell_{2}=\ell_{2}(\mathbb{N})$ are used to denote the Hilbert sequence space consisting of real-valued sequences equipped with the inner product $\left\langle\left\{x_{k}\right\},\left\{y_{k}\right\}\right\rangle_{\ell_{2}}:=$ $\sum_{k=1}^{\infty} x_{i} y_{i}$. The corresponding norm is denoted as $\|\cdot\|_{\ell_{2}}$.

## 2. Background

We begin with some background on the class of Hilbert spaces of interest in this paper and then proceed to provide some examples of the sampling operators of interest.

### 2.1. Hilbert spaces

We consider a class of Hilbert function spaces contained within $L^{2}(\mathcal{X})$, and defined as follows. Let $\left\{\psi_{k}\right\}_{k=1}^{\infty}$ be an orthonormal sequence (not necessarily a basis) in $L^{2}(\mathcal{X})$ and let $\sigma_{1} \geq \sigma_{2} \geq \sigma_{3} \geq \cdots>0$ be a sequence of positive weights decreasing to zero. Given these two ingredients, we can consider the class of functions

$$
\begin{equation*}
\mathcal{H}:=\left\{f \in L^{2}(\mathbb{P}) \mid f=\sum_{k=1}^{\infty} \sqrt{\sigma_{k}} \alpha_{k} \psi_{k}, \quad \text { for some }\left\{\alpha_{k}\right\}_{k=1}^{\infty} \in \ell_{2}(\mathbb{N})\right\} \tag{6}
\end{equation*}
$$

where the series in (6) is assumed to converge in $L^{2}$. (The series converges since $\sum_{k=1}^{\infty}\left(\sqrt{\sigma_{k}} \alpha_{k}\right)^{2} \leq \sigma_{1}\left\|\left\{\alpha_{k}\right\}\right\|_{\ell_{2}}<\infty$.) We refer to the sequence $\left\{\alpha_{k}\right\}_{k=1}^{\infty} \in \ell_{2}$ as the representative of $f$. Note that this representation is unique due to $\sigma_{k}$ being strictly positive for all $k \in \mathbb{N}$.

If $f$ and $g$ are two members of $\mathcal{H}$, say with associated representatives $\alpha=\left\{\alpha_{k}\right\}_{k=1}^{\infty}$ and $\beta=\left\{\beta_{k}\right\}_{k=1}^{\infty}$, then we can define the inner product

$$
\begin{equation*}
\langle f, g\rangle_{\mathcal{H}}:=\sum_{k=1}^{\infty} \alpha_{k} \beta_{k}=\langle\alpha, \beta\rangle_{\ell_{2}} . \tag{7}
\end{equation*}
$$

With this choice of inner product, it can be verified that the space $\mathcal{H}$ is a Hilbert space. (In fact, $\mathcal{H}$ inherits all the required properties directly from $\ell_{2}$.) For future reference, we note that for two functions $f, g \in \mathcal{H}$ with associated representatives $\alpha, \beta \in \ell_{2}$, their $L^{2}$-based inner product is given by ${ }^{3}\langle f, g\rangle_{L^{2}}=\sum_{k=1}^{\infty} \sigma_{k} \alpha_{k} \beta_{k}$.

We note that each $\psi_{k}$ is in $\mathcal{H}$, as it is represented by a sequence with a single nonzero element, namely, the $k$-th element which is equal to $\sigma_{k}^{-1 / 2}$. It follows from (17) that $\left\langle\sqrt{\sigma_{k}} \psi_{k}, \sqrt{\sigma_{j}} \psi_{j}\right\rangle_{\mathcal{H}}=\delta_{k j}$. That is, $\left\{\sqrt{\sigma_{k}} \psi_{k}\right\}$ is an orthonormal sequence in $\mathcal{H}$. Now, let $f \in \mathcal{H}$ be represented by $\alpha \in \ell_{2}$. We claim that the series in (6) also converges in $\mathcal{H}$ norm. In particular, $\sum_{k=1}^{N} \sqrt{\sigma_{k}} \alpha_{k} \psi_{k}$ is in $\mathcal{H}$, as it is represented by the sequence $\left\{\alpha_{1}, \ldots, \alpha_{N}, 0,0, \ldots\right\} \in \ell_{2}$. It follows from (7) that $\left\|f-\sum_{k=1}^{N} \sqrt{\sigma_{k}} \alpha_{k} \psi_{k}\right\|_{\mathcal{H}}=\sum_{k=N+1}^{\infty} \alpha_{k}^{2}$ which converges to 0 as $N \rightarrow \infty$. Thus, $\left\{\sqrt{\sigma_{k}} \psi_{k}\right\}$ is in fact an orthonormal basis for $\mathcal{H}$.

[^2]We now turn to a special case of particular importance to us, namely the reproducing kernel Hilbert space (RKHS) of a continuous kernel. Consider a symmetric bivariate function $\mathbb{K}: \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R}$, where $\mathcal{X} \subset \mathbb{R}^{d}$ is compact ${ }^{4}$. Furthermore, assume $\mathbb{K}$ to be positive semidefinite and continuous. Consider the integral operator $I_{\mathbb{K}}$ mapping a function $f \in L^{2}$ to the function $I_{\mathbb{K}} f:=$ $\int \mathbb{K}(\cdot, y) f(y) d \mathbb{P}(y)$. As a consequence of Mercer's theorem [5, 6], $I_{\mathbb{K}}$ is a compact operator from $L^{2}$ to $C(\mathcal{X})$, the space of continuous functions on $\mathcal{X}$ equipped with the uniform norm ${ }^{5}$. Let $\left\{\sigma_{k}\right\}$ be the sequence of nonzero eigenvalues of $I_{\mathbb{K}}$, which are positive, can be ordered in nonincreasing order and converge to zero. Let $\left\{\psi_{k}\right\}$ be the corresponding eigenfunctions which are continuous and can be taken to be orthonormal in $L^{2}$. With these ingredients, the space $\mathcal{H}$ defined in equation (6) is the RKHS of the kernel function $\mathbb{K}$. This can be verified as follows.

As another consequence of the Mercer's theorem, $\mathbb{K}$ has the decomposition

$$
\begin{equation*}
\mathbb{K}(x, y):=\sum_{k=1}^{\infty} \sigma_{k} \psi_{k}(x) \psi_{k}(y) \tag{8}
\end{equation*}
$$

where the convergence is absolute and uniform (in $x$ and $y$ ). In particular, for any fixed $y \in \mathcal{X}$, the sequence $\left\{\sqrt{\sigma_{k}} \psi_{k}(y)\right\}$ is in $\ell_{2}$. (In fact, $\sum_{k=1}^{\infty}\left(\sqrt{\sigma_{k}} \psi_{k}(y)\right)^{2}=\mathbb{K}(y, y)<\infty$.) Hence, $\mathbb{K}(\cdot, y)$ is in $\mathcal{H}$, as defined in (6), with representative $\left\{\sqrt{\sigma_{k}} \psi_{k}(y)\right\}$. Furthermore, it can be verified that the convergence in (6) can be taken to be also pointwisf ${ }^{6}$. To be more specific, for any $f \in \mathcal{H}$ with representative $\left\{\alpha_{k}\right\}_{k=1}^{\infty} \in \ell_{2}$, we have $f(y)=\sum_{k=1}^{\infty} \sqrt{\sigma_{k}} \alpha_{k} \psi_{k}(y)$, for all $y \in \mathcal{X}$. Consequently, by definition of the inner product (7), we have

$$
\langle f, \mathbb{K}(\cdot, y)\rangle_{\mathcal{H}}=\sum_{k=1}^{\infty} \alpha_{k} \sqrt{\sigma_{k}} \psi_{k}(y)=f(y)
$$

so that $\mathbb{K}(\cdot, y)$ acts as the representer of evaluation. This argument shows that for any fixed $y \in \mathcal{X}$, the linear functional on $\mathcal{H}$ given by $f \mapsto f(y)$ is

[^3]bounded, since we have
$$
|f(y)|=\left|\langle f, \mathbb{K}(\cdot, y)\rangle_{\mathcal{H}}\right| \leq\|f\|_{\mathcal{H}}\|\mathbb{K}(\cdot, y)\|_{\mathcal{H}},
$$
hence $\mathcal{H}$ is indeed the RKHS of the kernel $\mathbb{K}$. This fact plays an important role in the sequel, since some of the linear operators that we consider involve pointwise evaluation.

A comment regarding the scope: our general results hold for the basic setting introduced in equation (6). For those examples that involve pointwise evaluation, we assume the more refined case of the RKHS described above.

### 2.2. Linear operators, semi-norms and examples

Let $\Phi: \mathcal{H} \rightarrow \mathbb{R}^{n}$ be a continuous linear operator, with co-ordinates $[\Phi f]_{i}$ for $i=1,2, \ldots, n$. It defines the (semi)-inner product

$$
\begin{equation*}
\langle f, g\rangle_{\Phi}:=\langle\Phi f, \Phi g\rangle_{\mathbb{R}^{n}} \tag{9}
\end{equation*}
$$

which induces the semi-norm $\|\cdot\|_{\Phi}$. By the Riesz representation theorem, for each $i=1, \ldots, n$, there is a function $\varphi_{i} \in \mathcal{H}$ such that $[\Phi f]_{i}=\left\langle\varphi_{i}, f\right\rangle_{\mathcal{H}}$ for any $f \in \mathcal{H}$.

Let us illustrate the preceding definitions with some examples.
Example 1 (Generalized Fourier truncation). Recall the orthonormal basis $\left\{\psi_{i}\right\}_{i=1}^{\infty}$ underlying the Hilbert space. Consider the linear operator $\mathbb{T}_{\psi_{1}^{n}}$ : $\mathcal{H} \rightarrow \mathbb{R}^{n}$ with coordinates

$$
\begin{equation*}
\left[\mathbb{T}_{\psi_{1}^{n}} f\right]_{i}:=\left\langle\psi_{i}, f\right\rangle_{L^{2}}, \quad \text { for } i=1,2, \ldots, n \tag{10}
\end{equation*}
$$

We refer to this operator as the (generalized) Fourier truncation operator, since it acts by truncating the (generalized) Fourier representation of $f$ to its first $n$ co-ordinates. More precisely, by construction, if $f=\sum_{k=1}^{\infty} \sqrt{\sigma_{k}} \alpha_{k} \psi_{k}$, then

$$
\begin{equation*}
[\Phi f]_{i}=\sqrt{\sigma_{i}} \alpha_{i}, \quad \text { for } i=1,2, \ldots, n \tag{11}
\end{equation*}
$$

By definition of the Hilbert inner product, we have $\alpha_{i}=\left\langle\psi_{i}, f\right\rangle_{\mathcal{H}}$, so that we can write $[\Phi f]_{i}=\left\langle\varphi_{i}, f\right\rangle_{\mathcal{H}}$, where $\varphi_{i}:=\sqrt{\sigma_{i}} \psi_{i}$.

Example 2 (Domain sampling). A collection $x_{1}^{n}:=\left\{x_{1}, \ldots, x_{n}\right\}$ of points in the domain $\mathcal{X}$ can be used to define the (scaled) sampling operator $\mathbb{S}_{x_{1}^{n}}$ : $\mathcal{H} \rightarrow \mathbb{R}^{n}$ via

$$
\begin{equation*}
\mathbb{S}_{x_{1}^{n}} f:=n^{-1 / 2}\left(f\left(x_{1}\right) \quad \ldots \quad f\left(x_{n}\right)\right), \quad \text { for } f \in \mathcal{H} \tag{12}
\end{equation*}
$$

As previously discussed, when $\mathcal{H}$ is a reproducing kernel Hilbert space (with kernel $\mathbb{K}$ ), the (scaled) evaluation functional $f \mapsto n^{-1 / 2} f\left(x_{i}\right)$ is bounded, and its Riesz representation is given by the function $\varphi_{i}=n^{-1 / 2} \mathbb{K}\left(\cdot, x_{i}\right)$.
Example 3 (Weighted domain sampling). Consider the setting of the previous example. A slight variation on the sampling operator (12) is obtained by adding some weights to the samples

$$
\begin{equation*}
\mathbb{W}_{x_{1}^{n}, w_{1}^{n}} f:=n^{-1 / 2}\left(w_{1} f\left(x_{1}\right) \quad \ldots \quad w_{n} f\left(x_{n}\right)\right), \quad \text { for } f \in \mathcal{H} \tag{13}
\end{equation*}
$$

where $w_{1}^{n}=\left(w_{1}, \ldots, w_{n}\right)$ is chosen such that $\sum_{k=1}^{n} w_{k}^{2}=1$. Clearly, $\varphi_{i}=$ $n^{-1 / 2} w_{i} \mathbb{K}\left(\cdot, x_{i}\right)$.
[As an example of how this might arise, consider approximating $f(t)$ by $\sum_{k=1}^{n} f\left(x_{k}\right) G_{n}\left(t, x_{k}\right)$ where $\left\{G_{n}\left(\cdot, x_{k}\right)\right\}$ is a collection of functions in $L^{2}(\mathcal{X})$ such that $\left\langle G_{n}\left(\cdot, x_{k}\right), G_{n}\left(\cdot, x_{j}\right)\right\rangle_{L^{2}}=n^{-1} w_{k}^{2} \delta_{k j}$. Proper choices of $\left\{G_{n}\left(\cdot, x_{i}\right)\right\}$ might produce better approximations to the $L^{2}$ norm in the cases where one insists on choosing elements of $x_{1}^{n}$ to be uniformly spaced, while $\mathbb{P}$ in (1) is not a uniform distribution. Another slightly different but closely related case is when one approximates $f^{2}(t)$ over $\mathcal{X}=[0,1]$, by say $n^{-1} \sum_{k=1}^{n-1} f^{2}\left(x_{k}\right) W(n(t-$ $\left.x_{k}\right)$ ) for some function $W:[-1,1] \rightarrow \mathbb{R}_{+}$and $x_{k}=k / n$. Again, non-uniform weights are obtained when $\mathbb{P}$ is nonuniform.]

## 3. Main result and some consequences

We now turn to the statement of our main result, and the development of some its consequences for various models.

### 3.1. General upper bounds on $R_{\Phi}(\varepsilon)$

We now turn to upper bounds on $R_{\Phi}(\varepsilon)$ which was defined previously in (3). Our bounds are stated in terms of a real-valued function defined as follows: for matrices $D, M \in \mathbb{S}_{+}^{p}$,

$$
\begin{equation*}
\mathcal{L}(t, M, D):=\max \left\{\lambda_{\max }(D-t \sqrt{D} M \sqrt{D}), 0\right\}, \quad \text { for } t \geq 0 \tag{14}
\end{equation*}
$$

Here $\sqrt{D}$ denotes the matrix square root, valid for positive semidefinite matrices.

The upper bounds on $R_{\Phi}(\varepsilon)$ involve principal submatrices of certain infinite-dimensional matrices - or equivalently linear operators on $\ell_{2}(\mathbb{N})$ that we define here. Let $\Psi$ be the infinite-dimensional matrix with entries

$$
\begin{equation*}
[\Psi]_{j k}:=\left\langle\psi_{j}, \psi_{k}\right\rangle_{\Phi}, \quad \text { for } j, k=1,2, \ldots, \tag{15}
\end{equation*}
$$

and let $\Sigma=\operatorname{diag}\left\{\sigma_{1}, \sigma_{2}, \ldots,\right\}$ be a diagonal operator. For any $p=1,2, \ldots$, we use $\Psi_{p}$ and $\Psi_{\tilde{p}}$ to denote the principal submatrices of $\Psi$ on rows and columns indexed by $\{1,2, \ldots, p\}$ and $\{p+1, p+2, \ldots\}$, respectively. A similar notation will be used to denote submatrices of $\Sigma$.

Theorem 1. For all $\varepsilon \geq 0$, we have:

$$
\begin{equation*}
R_{\Phi}(\varepsilon) \leq \inf _{p \in \mathbb{N}} \inf _{t \geq 0}\left\{\mathcal{L}\left(t, \Psi_{p}, \Sigma_{p}\right)+t\left(\varepsilon+\sqrt{\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right)}\right)^{2}+\sigma_{p+1}\right\} \tag{16}
\end{equation*}
$$

Moreover, for any $p \in \mathbb{N}$ such that $\lambda_{\min }\left(\Psi_{p}\right)>0$, we have

$$
\begin{equation*}
R_{\Phi}(\varepsilon) \leq\left(1-\frac{\sigma_{p+1}}{\sigma_{1}}\right) \frac{1}{\lambda_{\min }\left(\Psi_{p}\right)}\left(\varepsilon+\sqrt{\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right)}\right)^{2}+\sigma_{p+1} \tag{17}
\end{equation*}
$$

Remark (a): These bounds cannot be improved in general. This is most easily seen in the special case $\varepsilon=0$. Setting $p=n$, bound (17) implies that $R_{\Phi}(0) \leq \sigma_{n+1}$ whenever $\Psi_{n}$ is strictly positive definite and $\Psi_{\tilde{n}}=0$. This bound is sharp in a "minimax sense", meaning that equality holds if we take the infimum over all bounded linear operators $\Phi: \mathcal{H} \rightarrow \mathbb{R}^{n}$. In particular, it is straightforward to show that

$$
\begin{equation*}
\inf _{\substack{\Phi: \mathcal{H} \rightarrow \mathbb{R}^{n} \\ \Phi \text { surjective }}} R_{\Phi}(0)=\inf _{\substack{\Phi: \mathcal{H} \rightarrow \mathbb{R}^{n} \\ \Phi \text { surjective }}} \sup _{f \in B_{\mathcal{H}}}\left\{\|f\|_{L^{2}}^{2} \mid \Phi f=0\right\}=\sigma_{n+1}, \tag{18}
\end{equation*}
$$

and moreover, this infimum is in fact achieved by some linear operator. Such results are known from the general theory of $n$-widths for Hilbert spaces (e.g., see Chapter IV in Pinkus [2] and Chapter 3 of [7].)

In the more general setting of $\varepsilon>0$, there are operators for which the bound (17) is met with equality. As a simple illustration, recall the (generalized) Fourier truncation operator $\mathbb{T}_{\psi_{1}^{n}}$ from Example 1. First, it can be


Figure 1: Geometry of Fourier truncation. The plot shows the set $\left\{\left(\|f\|_{L^{2}},\|f\|_{\Phi}\right):\|f\|_{\mathcal{H}} \leq\right.$ $1\} \subset \mathbb{R}^{2}$ for the case of (generalized) Fourier truncation operator $\mathbb{T}_{\psi_{1}^{n}}$.
verified that $\left\langle\psi_{k}, \psi_{j}\right\rangle_{\mathbb{T}_{\psi_{1}^{n}}}=\delta_{j k}$ for $j, k \leq n$ and $\left\langle\psi_{k}, \psi_{j}\right\rangle_{\mathbb{T}_{\psi_{1}^{n}}}=0$ otherwise. Taking $p=n$, we have $\Psi_{n}=I_{n}$, that is, the $n$-by- $n$ identity matrix, and $\Psi_{\widetilde{n}}=0$. Taking $p=n$ in (17), it follows that for $\varepsilon^{2} \leq \sigma_{1}$,

$$
\begin{equation*}
R_{\mathbb{T}_{\psi_{1}^{n}}}(\varepsilon) \leq\left(1-\frac{\sigma_{n+1}}{\sigma_{1}}\right) \varepsilon^{2}+\sigma_{n+1}, \tag{19}
\end{equation*}
$$

As shown in Appendix Appendix E, the bound (19) in fact holds with equality. In other words, the bounds of Theorems 1 are tight in this case. Also, note that (19) implies $R_{\mathbb{T}_{\psi_{1}^{n}}}(0) \leq \sigma_{n+1}$ showing that the (generalized) Fourier truncation operator achieves the minimax bound of (18). Fig 1 provides a geometric interpretation of these results.
Remark (b):. In general, it might be difficult to obtain a bound on $\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \widetilde{\widetilde{p}}_{\widetilde{p}}^{1 / 2}\right)$ as it involves the infinite dimensional matrix $\Psi_{\tilde{p}}$. One may obtain a simple (although not usually sharp) bound on this quantity by noting that for a positive semidefinite matrix, the maximal eigenvalue is bounded by the trace, that is,

$$
\begin{equation*}
\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right) \leq \operatorname{tr}\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right)=\sum_{k>p} \sigma_{k}[\Psi]_{k k} \tag{20}
\end{equation*}
$$

Another relatively easy-to-handle upper bound is

$$
\begin{equation*}
\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right) \leq\left\|\left|\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2} \|_{\infty}=\sup _{k>p} \sum_{r>p} \sqrt{\sigma_{k}} \sqrt{\sigma_{r}}\right|[\Psi]_{k r} \mid\right. \tag{21}
\end{equation*}
$$

These bounds can be used, in combination with appropriate block partitioning of $\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}$, to provide sharp bounds on the maximal eigenvalue. Block partitioning is useful due to the following: for a positive semidefinite matrix $M=\left(\begin{array}{cc}A_{1} & C \\ C^{T} & A_{2}\end{array}\right)$, we have $\lambda_{\max }(M) \leq \lambda_{\max }\left(A_{1}\right)+\lambda_{\max }\left(A_{2}\right)$. We leave the the details on the application of these ideas to examples in Section 3.2.

### 3.2. Some illustrative examples

Theorem has a number of concrete consequences for different Hilbert spaces and linear operators, and we illustrate a few of them in the following subsections.

### 3.2.1. Random domain sampling

We begin by stating a corollary of Theorem 1 in application to random time sampling in a reproducing kernel Hilbert space (RKHS). Recall from equation (12) the time sampling operator $\mathbb{S}_{x_{1}^{n}}$, and assume that the sample points $\left\{x_{1}, \ldots, x_{n}\right\}$ are drawn in an i.i.d. manner according to some distribution $\mathbb{P}$ on $\mathcal{X}$. Let us further assume that the eigenfunctions $\psi_{k}, k \geq 1$ are uniformly bounded ${ }^{7}$ on $\mathcal{X}$, meaning that

$$
\begin{equation*}
\sup _{k \geq 1} \sup _{x \in \mathcal{X}}\left|\psi_{k}(x)\right| \leq C_{\psi} \tag{22}
\end{equation*}
$$

Finally, we assume that $\|\sigma\|_{1}:=\sum_{k=1}^{\infty} \sigma_{k}<\infty$, and that

$$
\begin{gather*}
\sigma_{p k} \leq C_{\sigma} \sigma_{k} \sigma_{p}, \quad \text { for some positive constant } C_{\sigma} \text { and for all large } p,  \tag{23}\\
\sum_{k>p^{m}} \sigma_{k} \leq \sigma_{p}, \quad \text { for some positive integer } m \text { and for all large } p \tag{24}
\end{gather*}
$$

Let $m_{\sigma}$ be the smallest $m$ for which (24) holds. These conditions on $\left\{\sigma_{k}\right\}$ are satisfied, for example, for both a polynomial decay $\sigma_{k}=\mathcal{O}\left(k^{-\alpha}\right)$ with $\alpha>1$ and an exponential decay $\sigma_{k}=\mathcal{O}\left(\rho^{k}\right)$ with $\rho \in(0,1)$. In particular, for the polynomial decay, using the tail bound (B.1) in Appendix Appendix B, we can take $m_{\sigma}=\left\lceil\frac{\alpha}{\alpha-1}\right\rceil$ to satisfy (24). For the exponential decay, we can take $m_{\sigma}=1$ for $\rho \in\left(0, \frac{1}{2}\right)$ and $m_{\sigma}=2$ for $\rho \in\left(\frac{1}{2}, 1\right)$ to satisfy (24).

Define the function

$$
\begin{equation*}
\mathcal{G}_{n}(\varepsilon):=\frac{1}{\sqrt{n}} \sqrt{\sum_{j=1}^{\infty} \min \left\{\sigma_{j}, \varepsilon^{2}\right\}} \tag{25}
\end{equation*}
$$

[^4]as well as the critical radius
\[

$$
\begin{equation*}
r_{n}:=\inf \left\{\varepsilon>0: \mathcal{G}_{n}(\varepsilon) \leq \varepsilon^{2}\right\} \tag{26}
\end{equation*}
$$

\]

Corollary 1. Suppose that $r_{n}>0$ and $64 C_{\psi}^{2} m_{\sigma} r_{n}^{2} \log \left(2 n r_{n}^{2}\right) \leq 1$. Then for any $\varepsilon^{2} \in\left[r_{n}^{2}, \sigma_{1}\right)$, we have

$$
\begin{equation*}
\mathbb{P}\left[R_{\mathbb{S}_{x_{1}^{n}}}(\varepsilon)>\left(\widetilde{C}_{\psi}+\widetilde{C}_{\sigma}\right) \varepsilon^{2}\right] \leq 2 \exp \left(-\frac{1}{64 C_{\psi}^{2} r_{n}^{2}}\right) \tag{27}
\end{equation*}
$$

where $\widetilde{C}_{\psi}:=2\left(1+C_{\psi}\right)^{2}$ and $\widetilde{C}_{\sigma}:=3\left(1+C_{\psi}^{-1}\right) C_{\sigma}\|\sigma\|_{1}+1$.
We provide the proof of this corollary in Appendix Appendix A. As a concrete example consider a polynomial decay $\sigma_{k}=\mathcal{O}\left(k^{-\alpha}\right)$ for $\alpha>1$, which satisfies assumptions on $\left\{\sigma_{k}\right\}$. Using the tail bound (B.1) in Appendix Appendix B, one can verify that $r_{n}^{2}=\mathcal{O}\left(n^{-\alpha /(\alpha+1)}\right)$. Note that, in this case,

$$
r_{n}^{2} \log \left(2 n r_{n}^{2}\right)=\mathcal{O}\left(n^{-\frac{\alpha}{\alpha+1}} \log n^{\frac{1}{\alpha+1}}\right)=\mathcal{O}\left(n^{-\frac{\alpha}{\alpha+1}} \log n\right) \rightarrow 0, \quad n \rightarrow \infty
$$

Hence conditions of Corollary 1 are met for sufficiently large $n$. It follows that for some constants $C_{1}, C_{2}$ and $C_{3}$, we have

$$
R_{\mathbb{S}_{x_{1}^{n}}}\left(C_{1} n^{-\frac{\alpha}{2(\alpha+1)}}\right) \leq C_{2} n^{-\frac{\alpha}{\alpha+1}}
$$

with probability $1-2 \exp \left(-C_{3} n^{\frac{\alpha}{\alpha+1}}\right)$ for sufficiently large $n$.

### 3.2.2. Sobolev kernel

Consider the kernel $\mathbb{K}(x, y)=\min (x, y)$ defined on $\mathcal{X}^{2}$ where $\mathcal{X}=[0,1]$. The corresponding RKHS is of Sobolev type and can be expressed as
$\left\{f \in L^{2}(\mathcal{X}) \mid f\right.$ is absolutely continuous, $f(0)=0$ and $\left.f^{\prime} \in L^{2}(\mathcal{X})\right\}$.
Also consider a uniform domain sampling operator $\mathbb{S}_{x_{1}^{n}}$, that is, that of (12) with $x_{i}=i / n, i \leq n$ and let $\mathbb{P}$ be uniform (i.e., the Lebesgue measure restricted to $[0,1]$ ).

This setting has the benefit that many interesting quantities can be computed explicitly, while also having some practical appeal. The following can
be shown about the eigen-decomposition of the integral operator $I_{\mathbb{K}}$ introduced in Section 2,

$$
\sigma_{k}=\left[\frac{(2 k-1) \pi}{2}\right]^{-2}, \quad \psi_{k}(x)=\sqrt{2} \sin \left(\sigma_{k}^{-1 / 2} x\right), \quad k=1,2, \ldots
$$

In particular, the eigenvalues decay as $\sigma_{k}=\mathcal{O}\left(k^{-2}\right)$.
To compute the $\Psi$, we write

$$
\begin{equation*}
[\Psi]_{k r}=\left\langle\psi_{k}, \psi_{r}\right\rangle_{\Phi}=\frac{1}{n} \sum_{\ell=1}^{n}\left\{\cos \frac{(k-r) \ell \pi}{n}-\cos \frac{(k+r-1) \ell \pi}{n}\right\} . \tag{28}
\end{equation*}
$$

We note that $\Psi$ is periodic in $k$ and $r$ with period $2 n$. It is easily verified that $n^{-1} \sum_{\ell=1}^{n} \cos (q \ell \pi / n)$ is equal to -1 for odd values of $q$ and zero for even values, other than $q=0, \pm 2 n, \pm 4 n, \ldots$. It follows that

$$
[\Psi]_{k r}= \begin{cases}1+\frac{1}{n} & \text { if } k-r=0  \tag{29}\\ -1-\frac{1}{n} & \text { if } k+r=2 n+1 \\ \frac{1}{n}(-1)^{k-r} & \text { otherwise }\end{cases}
$$

for $1 \leq k, r \leq 2 n$. Letting $\mathbb{I}_{s} \in \mathbb{R}^{n}$ be the vector with entries, $\left(\mathbb{I}_{s}\right)_{j}=$ $(-1)^{j+1}, j \leq n$, we observe that $\Psi_{n}=I_{n}+\frac{1}{n} \mathbb{I}_{s} \mathbb{I}_{s}^{T}$. It follows that $\lambda_{\min }\left(\Psi_{n}\right)=$ 1. It remains to bound the terms in (17) involving the infinite sub-block $\Psi_{\tilde{n}}$.

The $\Psi$ matrix of this example, given by (29), shares certain properties with the $\Psi$ obtained in other situations involving periodic eigenfunctions $\left\{\psi_{k}\right\}$. We abstract away these properties by introducing a class of periodic $\Psi$ matrices. We call $\Psi_{\tilde{n}}$ a sparse periodic matrix, if each row (or column) is periodic and in each period only a vanishing fraction of elements are large. More precisely, $\Psi_{\tilde{n}}$ is sparse periodic if there exist positive integers $\gamma$ and $\eta$, and positive constants $c_{1}$ and $c_{2}$, all independent of $n$, such that each row of $\Psi_{\tilde{n}}$ is periodic with period $\gamma n$. and for any row $k$, there exits a subset of elements $S_{k}=\left\{\ell_{1}, \ldots, \ell_{\eta}\right\} \subset\{1, \ldots, \gamma n\}$ such that

$$
\begin{array}{ll}
\left|[\Psi]_{k, n+r}\right| \leq c_{1}, & r \in S_{k} \\
\left|[\Psi]_{k, n+r}\right| \leq c_{2} n^{-1}, & r \in\{1, \ldots, \gamma n\} \backslash S_{k} \tag{30b}
\end{array}
$$

The elements of $S_{k}$ could depend on $k$, but the cardinality of this set should be the constant $\eta$, independent of $k$ and $n$. Also, note that we are indexing rows and columns of $\Psi_{\tilde{n}}$ by $\{n+1, n+2, \ldots\}$; in particular, $k \geq n+1$. For this class, we have the following whose proof can be found in Appendix Appendix B.


Figure 2: Sparse periodic $\Psi$ matrices. Display (a) is a plot of the $N$-by- $N$ leading principal submatrix of $\Psi$ for the Sobolev kernel $(s, t) \mapsto \min \{s, t\}$. Here $n=9$ and $N=6 n$; the period is $2 n=18$. Display (b) is a the same plot for a Fourier-type kernel. The plots exhibit sparse periodic patterns as defined in Section 3.2.2.

Lemma 1. Assume $\Psi_{\overparen{n}}$ to be sparse periodic as defined above and $\sigma_{k}=$ $\mathcal{O}\left(k^{-\alpha}\right), \alpha \geq 2$. Then,
(a) for $\alpha>2, \lambda_{\max }\left(\Sigma_{\widetilde{n}}^{1 / 2} \Psi_{\widetilde{n}} \Sigma_{\widetilde{n}}^{1 / 2}\right)=\mathcal{O}\left(n^{-\alpha}\right), n \rightarrow \infty$,
(b) for $\alpha=2, \lambda_{\max }\left(\Sigma_{\widetilde{n}}^{1 / 2} \Psi_{\widetilde{n}} \Sigma_{\widetilde{n}}^{1 / 2}\right)=\mathcal{O}\left(n^{-2} \log n\right), n \rightarrow \infty$.

In particular (29) implies that $\Psi_{\tilde{n}}$ is sparse periodic with parameters $\gamma=2, \eta=2, c_{1}=2$ and $c_{2}=1$. Hence, part (b) of Lemma 1 applies. Now, we can use (17) with $p=n$ to obtain

$$
\begin{equation*}
R_{\mathbb{S}_{x_{1}^{n}}}(\varepsilon) \leq 2 \varepsilon^{2}+\mathcal{O}\left(n^{-2} \log n\right) \tag{31}
\end{equation*}
$$

where we have also used $(a+b)^{2} \leq 2 a^{2}+2 b^{2}$.

### 3.2.3. Fourier-type kernels

In this example, we consider an RKHS of functions on $\mathcal{X}=[0,1] \subset \mathbb{R}$, generated by a Fourier-type kernel defined as $\mathbb{K}(x, y):=\kappa(x-y), x, y \in[0,1]$, where

$$
\begin{equation*}
\kappa(x)=\zeta_{0}+\sum_{k=1}^{\infty} 2 \zeta_{k} \cos (2 \pi k x), \quad x \in[-1,1] . \tag{32}
\end{equation*}
$$

We assume that $\left(\zeta_{k}\right)$ is a $\mathbb{R}_{+}$-valued nonincreasing sequence in $\ell_{1}$, i.e. $\sum_{k} \zeta_{k}<$ $\infty$. Thus, the trigonometric series in (32) is absolutely (and uniformly) convergent. As for the operator $\Phi$, we consider the uniform time sampling operator $\mathbb{S}_{x_{1}^{n}}$, as in the previous example. That is, the operator defined in (12) with $x_{i}=i / n, i \leq n$. We take $\mathbb{P}$ to be uniform.

This setting again has the benefit of being simple enough to allow for explicit computations while also practically important. One can argue that the eigen-decomposition of the kernel integral operator is given by

$$
\begin{array}{llll}
\psi_{1}=\psi_{0}^{(c)}, & \psi_{2 k}=\psi_{k}^{(c)}, & \psi_{2 k+1}=\psi_{k}^{(s)}, & k \geq 1 \\
\sigma_{1}=\zeta_{0}, & \sigma_{2 k}=\zeta_{k}, & \sigma_{2 k+1}=\zeta_{k}, & k \geq 1 \tag{34}
\end{array}
$$

where $\psi_{0}^{(c)}(x):=1, \psi_{k}^{(c)}(x):=\sqrt{2} \cos (2 \pi k x)$ and $\psi_{k}^{(s)}(t):=\sqrt{2} \sin (2 \pi k x)$ for $k \geq 1$.

For any integer $k$, let $((k))_{n}$ denote $k$ modulo $n$. Also, let $k \mapsto \delta_{k}$ be the function defined over integers which is 1 at $k=0$ and zero elsewhere. Let $\iota:=\sqrt{-1}$. Using the identity $n^{-1} \sum_{\ell=1}^{n} \exp (\iota 2 \pi k \ell / n)=\delta_{((k))_{n}}$, one obtains the following,

$$
\begin{align*}
\left\langle\psi_{k}^{(c)}, \psi_{j}^{(c)}\right\rangle_{\Phi} & =\left[\delta_{((k-j))_{n}}+\delta_{((k+j))_{n}}\right]\left(\frac{1}{\sqrt{2}}\right)^{\delta_{k}+\delta_{j}},  \tag{35a}\\
\left\langle\psi_{k}^{(s)}, \psi_{j}^{(s)}\right\rangle_{\Phi} & =\delta_{((k-j))_{n}}-\delta_{((k+j))_{n}},  \tag{35b}\\
\left\langle\psi_{k}^{(c)}, \psi_{j}^{(s)}\right\rangle_{\Phi} & =0, \quad \text { valid for all } j, k \geq 0 \tag{35c}
\end{align*}
$$

It follows that $\Psi_{n}=I_{n}$ if $n$ is odd and $\Psi_{n}=\operatorname{diag}\{1,1, \ldots, 1,2\}$ if $n$ is even. In particular, $\lambda_{\min }\left(\Psi_{n}\right)=1$ for all $n \geq 1$. It is also clear that the principal submatrix of $\Psi$ on indices $\{2,3, \ldots\}$ has periodic rows and columns with period $2 n$. If follows that $\Psi_{n}$ is sparse periodic as defined in Section 3.2.2 with parameters $\gamma=2, \eta=2, c_{1}=2$ and $c_{2}=0$.

Suppose for example that the eigenvalues decay polynomially, say as $\zeta_{k}=$ $\mathcal{O}\left(k^{-\alpha}\right)$ for $\alpha>2$. Then, applying (17) with $p=n$, in combination with Lemma 1 part (a), we get

$$
\begin{equation*}
R_{\mathbb{S}_{x_{1}^{n}}}(\varepsilon) \leq 2 \varepsilon^{2}+\mathcal{O}\left(n^{-\alpha}\right) \tag{36}
\end{equation*}
$$

As another example, consider the exponential decay $\zeta_{k}=\rho^{k}, k \geq 1$ for some $\rho \in(0,1)$, which corresponds to the Poisson kernel. In this case, the tail sum
of $\left\{\sigma_{k}\right\}$ decays as the sequence itself, namely, $\sum_{k>n} \sigma_{k} \leq 2 \sum_{k>n} \rho^{k}=\frac{2 \rho}{1-\rho} \rho^{k}$. Hence, we can simply use the trace bound (20) together with (17) to obtain

$$
\begin{equation*}
R_{\mathbb{S}_{x_{1}^{n}}}(\varepsilon) \leq 2 \varepsilon^{2}+\mathcal{O}\left(\rho^{n}\right) \tag{37}
\end{equation*}
$$

## 4. Proof of Theorem 1

We now turn to the proof of our main theorem. Recall from Section 2.1 the correspondence between any $f \in \mathcal{H}$ and a sequence $\alpha \in \ell_{2}$; also, recall the diagonal operator $\Sigma: \ell_{2} \rightarrow \ell_{2}$ defined by the matrix $\operatorname{diag}\left\{\sigma_{1}, \sigma_{2}, \ldots\right\}$. Using the definition of (15) of the $\Psi$ matrix, we have

$$
\|f\|_{\Phi}^{2}=\left\langle\alpha, \Sigma^{1 / 2} \Psi \Sigma^{1 / 2} \alpha\right\rangle_{\ell_{2}},
$$

By definition (6) of the Hilbert space $\mathcal{H}$, we have $\|f\|_{\mathcal{H}}^{2}=\sum_{k=1}^{\infty} \alpha_{k}^{2}$ and $\|f\|_{L^{2}}^{2}=\sum_{k} \sigma_{k} \alpha_{k}^{2}$. Letting $B_{\ell_{2}}=\left\{\alpha \in \ell_{2} \mid\|\alpha\|_{\ell_{2}} \leq 1\right\}$ be the unit ball in $\ell_{2}$, we conclude that $R_{\Phi}$ can be written as

$$
\begin{equation*}
R_{\Phi}(\varepsilon)=\sup _{\alpha \in B_{\ell_{2}}}\left\{Q_{2}(\alpha) \mid Q_{\Phi}(\alpha) \leq \varepsilon^{2}\right\} \tag{38}
\end{equation*}
$$

where we have defined the quadratic functionals

$$
\begin{equation*}
Q_{2}(\alpha):=\langle\alpha, \Sigma \alpha\rangle_{\ell_{2}}, \quad \text { and } \quad Q_{\Phi}(\alpha):=\left\langle\alpha, \Sigma^{1 / 2} \Psi \Sigma^{1 / 2} \alpha\right\rangle_{\ell_{2}} \tag{39}
\end{equation*}
$$

Also let us define the symmetric bilinear form

$$
\begin{equation*}
B_{\Phi}(\alpha, \beta):=\left\langle\alpha, \Sigma^{1 / 2} \Psi \Sigma^{1 / 2} \beta\right\rangle_{\ell_{2}}, \quad \alpha, \beta \in \ell^{2} \tag{40}
\end{equation*}
$$

whose diagonal is $B_{\Phi}(\alpha, \alpha)=Q_{\Phi}(\alpha)$.
We now upper bound $R_{\Phi}(\varepsilon)$ using a truncation argument. Define the set

$$
\begin{equation*}
\mathcal{C}:=\left\{\alpha \in B_{\ell_{2}} \mid Q_{\Phi}(\alpha) \leq \varepsilon^{2}\right\} \tag{41}
\end{equation*}
$$

corresponding to the feasible set for the optimization problem (38). For each integer $p=1,2, \ldots$, consider the following truncated sequence spaces

$$
\begin{array}{rlrl}
\mathcal{T}_{p} & :=\left\{\alpha \in \ell_{2} \mid \alpha_{i}=0,\right. & \text { for all } i>p\}, & \text { and } \\
\mathcal{T}_{p}^{\perp} & :=\left\{\alpha \in \ell_{2} \mid \alpha_{i}=0,\right. & \text { for all } i=1,2, \ldots p\} .
\end{array}
$$

Note that $\ell_{2}$ is the direct sum of $\mathcal{T}_{p}$ and $\mathcal{T}_{p}^{\perp}$. Consequently, any fixed $\alpha \in \mathcal{C}$ can be decomposed as $\alpha=\xi+\gamma$ for some (unique) $\xi \in \mathcal{T}_{p}$ and $\gamma \in \mathcal{T}_{p}^{\perp}$. Since $\Sigma$ is a diagonal operator, we have

$$
Q_{2}(\alpha)=Q_{2}(\xi)+Q_{2}(\gamma)
$$

Moreover, since any $\alpha \in \mathcal{C}$ is feasible for the optimization problem (38), we have

$$
\begin{equation*}
Q_{\Phi}(\alpha)=Q_{\Phi}(\xi)+2 B_{\Phi}(\xi, \gamma)+Q_{\Phi}(\gamma) \leq \varepsilon^{2} \tag{42}
\end{equation*}
$$

Note that since $\gamma \in \mathcal{T}_{p}^{\perp}$, it can be written as $\gamma=\left(0_{p}, c\right)$, where $0_{p}$ is a vector of $p$ zeroes, and $c=\left(c_{1}, c_{2}, \ldots\right) \in \ell_{2}$. Similarly, we can write $\xi=(x, 0)$ where $x \in \mathbb{R}^{p}$. Then, each of the terms $Q_{\Phi}(\xi), B_{\Phi}(\xi, \gamma), Q_{\Phi}(\gamma)$ can be expressed in terms of block partitions of $\Sigma^{1 / 2} \Psi \Sigma^{1 / 2}$. For example,

$$
\begin{equation*}
Q_{\Phi}(\xi)=\langle x, A x\rangle_{\mathbb{R}^{p}}, \quad Q_{\Phi}(\gamma)=\langle y, D y\rangle_{\ell_{2}}, \tag{43}
\end{equation*}
$$

where $A:=\Sigma_{p}^{1 / 2} \Psi_{p} \Sigma_{p}^{1 / 2}$ and $D:=\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}$, in correspondence with the block partitioning notation of Appendix Appendix F. We now apply inequality (F.2) derived in Appendix Appendix F. Fix some $\rho^{2} \in(0,1)$ and take

$$
\begin{equation*}
\kappa^{2}:=\rho^{2} \lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right), \tag{44}
\end{equation*}
$$

so that condition (F.5) is satisfied. Then, (F.2) implies

$$
\begin{equation*}
Q_{\Phi}(\xi)+2 B_{\Phi}(\xi, \gamma)+Q_{\Phi}(\gamma) \geq \rho^{2} Q_{\Phi}(\xi)-\frac{\kappa^{2}}{1-\rho^{2}}\|\gamma\|_{2}^{2} \tag{45}
\end{equation*}
$$

Combining (42) and (45), we obtain

$$
\begin{equation*}
Q_{\Phi}(\xi) \leq \frac{\varepsilon^{2}}{\rho^{2}}+\frac{\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right)}{1-\rho^{2}}\|\gamma\|_{2}^{2} \tag{46}
\end{equation*}
$$

We further note that $\|\gamma\|_{2}^{2} \leq\|\gamma\|_{2}^{2}+\|\xi\|_{2}^{2}=\|\alpha\|_{2}^{2} \leq 1$. It follows that

$$
\begin{equation*}
Q_{\Phi}(\xi) \leq \widetilde{\varepsilon}^{2}, \quad \text { where } \quad \widetilde{\varepsilon}^{2}:=\frac{\varepsilon^{2}}{\rho^{2}}+\frac{\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right)}{1-\rho^{2}} \tag{47}
\end{equation*}
$$

Let us define

$$
\begin{equation*}
\widetilde{\mathcal{C}}:=\left\{\xi \in B_{\ell_{2}} \cap \mathcal{T}_{p} \mid Q_{\Phi}(\xi) \leq \widetilde{\varepsilon}^{2}\right\} . \tag{48}
\end{equation*}
$$

Then, our arguments so far show that for $\alpha \in \mathcal{C}$,

$$
\begin{equation*}
Q_{2}(\alpha)=Q_{2}(\xi)+Q_{2}(\gamma) \leq \underbrace{\sup _{\xi \in \widetilde{\mathcal{C}}} Q_{2}(\xi)}_{S_{p}}+\underbrace{\sup _{\gamma \in B_{\ell_{2}} \cap \mathcal{T}_{p}^{\perp}} Q_{2}(\gamma)}_{S_{p}^{\perp}} \tag{49}
\end{equation*}
$$

Taking the supremum over $\alpha \in \mathcal{C}$ yields the upper bound

$$
R_{\Phi}(\varepsilon) \leq S_{p}+S_{p}^{\perp}
$$

It remains to bound each of the two terms on the right-hand side. Beginning with the term $S_{p}^{\perp}$ and recalling the decomposition $\gamma=\left(0_{p}, c\right)$, we have $Q_{2}(\gamma)=\sum_{k=1}^{\infty} \sigma_{k+p} c_{k}^{2}$, from which it follows that

$$
S_{p}^{\perp}=\sup \left\{\sum_{k=1}^{\infty} \sigma_{k+p} c_{k}^{2} \mid \sum_{k=1}^{\infty} c_{k}^{2} \leq 1\right\}=\sigma_{p+1}
$$

since $\left\{\sigma_{k}\right\}_{k=1}^{\infty}$ is a nonincreasing sequence by assumption.
We now control the term $S_{p}$. Recalling the decomposition $\xi=(x, 0)$ where $x \in \mathbb{R}^{p}$, we have

$$
\begin{aligned}
S_{p}=\sup _{\xi \in \widetilde{\mathcal{C}}} Q_{2}(\xi) & =\sup \left\{\left\langle x, \Sigma_{p} x\right\rangle:\langle x, x\rangle \leq 1,\left\langle x, \Sigma_{p}^{1 / 2} \Psi_{p} \Sigma_{p}^{1 / 2} x\right\rangle \leq \widetilde{\varepsilon}^{2}\right\} \\
& =\sup _{\langle x, x\rangle \leq 1} \inf _{t \geq 0}\left\{\left\langle x, \Sigma_{p} x\right\rangle+t\left(\widetilde{\varepsilon}^{2}-\left\langle x, \Sigma_{p}^{1 / 2} \Psi_{p} \Sigma_{p}^{1 / 2} x\right\rangle\right)\right\} \\
& \stackrel{(a)}{\leq} \inf _{t \geq 0}\left\{\sup _{\langle x, x\rangle \leq 1}\left\langle x, \Sigma_{p}^{1 / 2}\left(I_{p}-t \Psi_{p}\right) \Sigma_{p}^{1 / 2} x\right\rangle+t \widetilde{\varepsilon}^{2}\right\}
\end{aligned}
$$

where inequality (a) follows by Lagrange (weak) duality. It is not hard to see that for any symmetric matrix $M$, one has

$$
\sup \{\langle x, M x\rangle:\langle x, x\rangle \leq 1\}=\max \left\{0, \lambda_{\max }(M)\right\}
$$

Putting the pieces together and optimizing over $\rho^{2}$, noting that

$$
\inf _{r \in(0,1)}\left\{\frac{a}{r}+\frac{b}{1-r}\right\}=(\sqrt{a}+\sqrt{b})^{2}
$$

for any $a, b>0$, completes the proof of the bound (16).
We now prove bound (17), using the same decomposition and notation established above, but writing an upper bound on $Q_{2}(\alpha)$ slightly different form (49). In particular, the argument leading to (49), also shows that

$$
\begin{equation*}
R_{\Phi}(\varepsilon) \leq \sup _{\xi \in \mathcal{T}_{p}, \gamma \in \mathcal{T}_{p}^{\perp}}\left\{Q_{2}(\xi)+Q_{2}(\gamma) \mid \xi+\gamma \in B_{\ell_{2}}, Q_{\Phi}(\xi) \leq \widetilde{\varepsilon}^{2}\right\} \tag{50}
\end{equation*}
$$

Recalling the expression (39) for $Q_{\Phi}(\xi)$ and noting that $\Psi_{p} \succeq \lambda_{\min }\left(\Psi_{p}\right) I_{p}$ implies $A=\Sigma_{p}^{1 / 2} \Psi_{p} \Sigma_{p}^{1 / 2} \succeq \lambda_{\text {min }}\left(\Psi_{p}\right) \Sigma_{p}$, we have

$$
\begin{equation*}
Q_{\Phi}(\xi) \geq \lambda_{\min }\left(\Psi_{p}\right) Q_{2}(\xi) \tag{51}
\end{equation*}
$$

Now, since we are assuming $\lambda_{\min }\left(\Psi_{p}\right)>0$, we have

$$
\begin{equation*}
R_{\Phi}(\varepsilon) \leq \sup _{\xi \in \mathcal{T}_{p}, \gamma \in \mathcal{T}_{p}^{\perp}}\left\{Q_{2}(\xi)+Q_{2}(\gamma) \mid \xi+\gamma \in B_{\ell_{2}}, Q_{2}(\xi) \leq \frac{\widetilde{\varepsilon}^{2}}{\lambda_{\min }\left(\Psi_{p}\right)}\right\} \tag{52}
\end{equation*}
$$

The RHS of the above is an instance of the Fourier truncation problem with $\varepsilon^{2}$ replaced with $\widetilde{\varepsilon}^{2} / \lambda_{\text {min }}\left(\Psi_{p}\right)$. That problem is workout in detail in Appendix Appendix E. In particular, applying equation (E.1) in Appendix Appendix E with $\varepsilon^{2}$ changed to $\widetilde{\varepsilon}^{2} / \lambda_{\min }\left(\Psi_{p}\right)$ completes the proof of (17). Figure3 provides a graphical representation of the geometry of the proof.

## 5. Conclusion

We considered the problem of bounding (squared) $L^{2}$ norm of functions in a Hilbert unit ball, based on restrictions on an operator-induced norm acting as a surrogate for the $L^{2}$ norm. In particular, given that $f \in B_{\mathcal{H}}$ and $\|f\|_{\Phi}^{2} \leq \varepsilon^{2}$, our results enable us to obtain, by estimating norms of certain finite and infinite dimensional matrices, inequalities of the form

$$
\|f\|_{L^{2}}^{2} \leq c_{1} \varepsilon^{2}+h_{\Phi, \mathcal{H}}\left(\sigma_{n}\right)
$$

where $\left\{\sigma_{n}\right\}$ are the eigenvalues of the operator embedding $\mathcal{H}$ in $L^{2}, h_{\Phi, \mathcal{H}}(\cdot)$ is an increasing function (depending on $\Phi$ and $\mathcal{H}$ ) and $c_{1} \geq 1$ is some constant. We considered examples of operators $\Phi$ (uniform time sampling and Fourier truncation) and Hilbert spaces $\mathcal{H}$ (Sobolev, Fourier-type RKHSs) and showed


Figure 3: Geometry of the proof of (17). Display (a) is a plot of the set $\mathcal{Q}:=$ $\left\{\left(Q_{2}(\alpha), Q_{\Phi}(\alpha)\right):\|\alpha\|_{\ell_{2}}=1\right\} \subset \mathbb{R}^{2}$. This is a convex set as a consequence of HausdorffToeplitz theorem on convexity of the numerical range and preservation of convexity under projections. Display (b) shows the set $\widetilde{\mathcal{Q}}:=\operatorname{conv}(0, \mathcal{Q})$, i.e., the convex hull of $\{0\} \cup \mathcal{Q}$. Observe that $R_{\Phi}(\varepsilon)=\sup \left\{x:(x, y) \in \widetilde{\mathcal{Q}}, y \leq \varepsilon^{2}\right\}$. For any fixed $r \in(0,1)$, the bound of (17) is a piecewise linear approximation to one side of $\widetilde{\mathcal{Q}}$ as shown in Display (b).
that it is possible to obtain optimal scaling $h_{\Phi, \mathcal{H}}\left(\sigma_{n}\right)=\mathcal{O}\left(\sigma_{n}\right)$ in most of those cases. We also considered random time sampling, under polynomial eigendecay $\sigma_{n}=\mathcal{O}\left(n^{-\alpha}\right)$, and effectively showed that $h_{\Phi, \mathcal{H}}\left(\sigma_{n}\right)=\mathcal{O}\left(n^{-\alpha /(\alpha+1)}\right)$ (for $\varepsilon$ small enough), with high probability as $n \rightarrow \infty$. This last result complements those on related quantities obtained by techniques form empirical process theory, and we conjecture it to be sharp.

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## Appendix A. Analysis of random time sampling

This section is devoted to the proof of Corollary 1 on random time sampling in reproducing kernel Hilbert spaces. The proof is based on an auxiliary result, which we begin by stating. Fix some positive integer $m$ and define

$$
\begin{equation*}
\nu(\varepsilon)=\nu(\varepsilon ; m):=\inf \left\{p: \sum_{k>p^{m}} \sigma_{k} \leq \varepsilon^{2}\right\} . \tag{A.1}
\end{equation*}
$$

With this notation, we have

Lemma 2. Assume $\varepsilon^{2}<\sigma_{1}$ and $32 C_{\psi}^{2} m \nu(\varepsilon) \log \nu(\varepsilon) \leq n$. Then,

$$
\begin{equation*}
\mathbb{P}\left\{R_{\mathbb{S}_{x_{1}^{n}}}(\varepsilon)>\widetilde{C}_{\psi} \varepsilon^{2}+\widetilde{C}_{\sigma} \sigma_{\nu(\varepsilon)}\right\} \leq 2 \exp \left(-\frac{1}{32 C_{\psi}^{2}} \frac{n}{\nu(\varepsilon)}\right) \tag{A.2}
\end{equation*}
$$

We prove this claim in Section Appendix A. 2 below.

## Appendix A.1. Proof of Corollary 1

To apply the lemma, recall that we assume that there exists $m$ such that for all (large) $p$, one has

$$
\begin{equation*}
\sum_{k>p^{m}} \sigma_{k} \leq \sigma_{p} \tag{A.3}
\end{equation*}
$$

and we let $m_{\sigma}$ be the smallest such $m$. We define

$$
\begin{equation*}
\mu(\varepsilon):=\inf \left\{p: \sigma_{p} \leq \varepsilon^{2}\right\} \tag{A.4}
\end{equation*}
$$

and note that by (A.3), we have $\nu\left(\varepsilon ; m_{\sigma}\right) \leq \mu(\varepsilon)$. Then, Lemma 2 states that as long as $\varepsilon^{2}<\sigma_{1}$ and $32 C_{\psi}^{2} m_{\sigma} \mu(\varepsilon) \log \mu(\varepsilon) \leq n$, we have

$$
\begin{equation*}
\mathbb{P}\left\{R_{\mathbb{S}_{x_{1}^{n}}}(\varepsilon)>\left(\widetilde{C}_{\psi}+\widetilde{C}_{\sigma}\right) \varepsilon^{2}\right\} \leq 2 \exp \left(-\frac{1}{32 C_{\psi}^{2}} \frac{n}{\mu(\varepsilon)}\right) . \tag{A.5}
\end{equation*}
$$

Now by the definition of $\mu(\varepsilon)$, we have $\sigma_{j}>\varepsilon^{2}$ for $j<\mu(\varepsilon)$, and hence

$$
\mathcal{G}_{n}^{2}(\varepsilon) \geq \frac{1}{n} \sum_{j<\mu(\varepsilon)} \min \left\{\sigma_{j}, \varepsilon^{2}\right\}=\frac{\mu(\varepsilon)-1}{n} \varepsilon^{2} \geq \frac{\mu(\varepsilon)}{2 n} \varepsilon^{2},
$$

since $\mu(\varepsilon) \geq 2$ when $\varepsilon^{2}<\sigma_{1}$. One can argue that $\varepsilon \mapsto \mathcal{G}_{n}(\varepsilon) / \varepsilon$ is nonincreasing. It follows from definition (26) that for $\varepsilon \geq r_{n}$, we have

$$
\mu(\varepsilon) \leq 2 n\left(\frac{\mathcal{G}(\varepsilon)}{\varepsilon}\right)^{2} \leq 2 n\left(\frac{\mathcal{G}\left(r_{n}\right)}{r_{n}}\right)^{2} \leq 2 n r_{n}^{2}
$$

which completes the proof of Corollary 1 .

## Appendix A.2. Proof of Lemma圆

For $\xi \in \mathbb{R}^{p}$, let $\xi \otimes \xi$ be the rank-one operator on $\mathbb{R}^{p}$ given by $\eta \mapsto$ $\langle\xi,, \eta\rangle_{2} \xi$. For an operator $A$ on $\mathbb{R}^{p}$, let $\|A\|_{2}$ denote its usual operator norm, $\|A\|_{2}:=\sup _{\|x\|_{2} \leq 1}\|A x\|_{2}$. Recall that for a symmetric (i.e., real self-adjoint) operator $A$ on $\mathbb{R}^{p},\|A\|_{2}=\sup \{|\lambda|: \lambda$ an eigenvalue of $A\}$. It follows that $\|A\|_{2} \leq \alpha$ is equivalent to $-\alpha I_{p} \preceq A \preceq \alpha I_{p}$.

Our approach is to first show that $\left\|\Psi_{p}-I_{p}\right\|_{2} \leq \frac{1}{2}$ for some properly chosen $p$ with high probability. It then follows that $\lambda_{\min }\left(\Psi_{p}\right) \geq \frac{1}{2}$ and we can use bound (17)) for that value of $p$. Then, we need to control $\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right)$. To do this, we further partition $\Psi_{\widetilde{p}}$ into blocks. In order to have a consistent notation, we look at the whole matrix $\Psi$ and let $\Psi^{(k)}$ be the principal submatrix indexed by $\{(k-1) p+1, \ldots,(k-1) p+p\}$, for $k=1,2, \ldots, p^{m-1}$. Throughout the proof, $m$ is assumed to be a fixed positive integer. Also, let $\Psi^{(\infty)}$ be the principal submatrix of $\Psi$ indexed by $\left\{p^{m}+1, p^{m}+2, \ldots\right\}$. This provides a full partitioning of $\Psi$ for which $\Psi^{(1)}, \ldots, \Psi^{\left(p^{m-1}\right)}$ and $\Psi^{(\infty)}$ are the diagonal blocks, the first $p^{m-1}$ of which are $p$-by- $p$ matrices and the last an infinite matrix. To connect with our previous notations, we note that $\Psi^{(1)}=\Psi_{p}$ and that $\Psi^{(2)}, \ldots, \Psi^{\left(p^{m-1}\right)}, \Psi^{(\infty)}$ are diagonal blocks of $\Psi_{\tilde{p}}$. Let us also partition the $\Sigma$ matrix and name its diagonal blocks similarly.

We will argue that, in fact, we have $\left\|\Psi^{(k)}-I_{p}\right\|_{2} \leq \frac{1}{2}$ for all $k=$ $1, \ldots, p^{m-1}$, with high probability. Let $\mathcal{A}_{p}$ denote the event on which this claim holds. In particular, on event $\mathcal{A}_{p}$, we have $\Psi^{(k)} \preceq \frac{3}{2} I_{p}$ for $k=$ $2, \ldots, p^{m-1}$; hence, we can write

$$
\begin{align*}
\lambda_{\max }\left(\Sigma_{\widetilde{p}}^{1 / 2} \Psi_{\widetilde{p}} \Sigma_{\widetilde{p}}^{1 / 2}\right) & \leq \sum_{k=2}^{p^{m-1}} \lambda_{\max }\left(\sqrt{\Sigma^{(k)}} \Psi^{(k)} \sqrt{\Sigma^{(k)}}\right)+\lambda_{\max }\left(\sqrt{\Sigma^{(\infty)}} \Psi^{(\infty)} \sqrt{\Sigma^{(\infty)}}\right) \\
& \leq \frac{3}{2} \sum_{k=2}^{p^{m-1}} \lambda_{\max }\left(\Sigma^{(k)}\right)+\operatorname{tr}\left(\sqrt{\Sigma^{(\infty)}} \Psi^{(\infty)} \sqrt{\Sigma^{(\infty)}}\right) \\
& =\frac{3}{2} \sum_{k=2}^{p^{m-1}} \sigma_{(k-1) p+1}+\sum_{k>p^{m}} \sigma_{k}[\Psi]_{k k} . \tag{A.6}
\end{align*}
$$

Using assumptions (23) on the sequence $\left\{\sigma_{k}\right\}$, the first sum can be bounded as

$$
\sum_{k=2}^{p^{m-1}} \sigma_{(k-1) p+1} \leq \sum_{k=2}^{p^{m-1}} \sigma_{(k-1) p} \leq \sum_{k=2}^{p^{m-1}} C_{\sigma} \sigma_{k-1} \sigma_{p} \leq C_{\sigma}\|\sigma\|_{1} \sigma_{p}
$$

Using the uniform boundedness assumption (A.1), we have $[\Psi]_{k k}=n^{-1} \sum_{i=1}^{n} \psi_{k}^{2}\left(x_{i}\right) \leq$ $C_{\psi}^{2}$. Hence the second sum in (A.6) is bounded above by $C_{\psi}^{2} \sum_{k>p^{m}} \sigma_{k}$.

We can now apply Theorem 1. Assume for the moment that $\varepsilon^{2} \geq$ $\sum_{k>p^{m}} \sigma_{k}$ so that the right-hand side of (A.6) is bounded above by $\frac{3}{2} C_{\sigma}\|\sigma\|_{1} \sigma_{p}+$ $C_{\psi}^{2} \varepsilon^{2}$. Applying bound (17), on event $\mathcal{A}_{p}$, with ${ }^{8} r=\left(1+C_{\psi}\right)^{-1}$, we get

$$
\begin{aligned}
R_{\mathbb{S}_{x_{1}^{n}}}\left(\varepsilon^{2}\right) & \leq 2\left\{r^{-1} \varepsilon^{2}+(1-r)^{-1}\left(\frac{3}{2} C_{\sigma}\|\sigma\|_{1} \sigma_{p}+C_{\psi}^{2} \varepsilon^{2}\right)\right\}+\sigma_{p+1} \\
& =2\left(1+C_{\psi}\right)^{2} \varepsilon^{2}+3\left(1+C_{\psi}^{-1}\right) C_{\sigma}\|\sigma\|_{1} \sigma_{p}+\sigma_{p+1} . \\
& \leq \widetilde{C}_{\psi} \varepsilon^{2}+\widetilde{C}_{\sigma} \sigma_{p}
\end{aligned}
$$

where $\widetilde{C}_{\psi}:=2\left(1+C_{\psi}\right)^{2}$ and $\widetilde{C}_{\sigma}:=3\left(1+C_{\psi}^{-1}\right) C_{\sigma}\|\sigma\|_{1}+1$. To summarize, we have shown the following

$$
\begin{equation*}
\text { Event } \mathcal{A}_{p} \text { and } \varepsilon^{2} \geq \sum_{k>p^{m}} \sigma_{k} \Longrightarrow R_{\mathbb{S}_{x_{1}^{n}}}\left(\varepsilon^{2}\right) \leq \widetilde{C}_{\psi} \varepsilon^{2}+\widetilde{C}_{\sigma} \sigma_{p} \tag{A.7}
\end{equation*}
$$

It remains to control the probability of $\mathcal{A}_{p}:=\bigcap_{k=1}^{p^{m-1}}\left\{\left\|\Psi^{(k)}-I_{p}\right\|_{2} \leq \frac{1}{2}\right\}$. We start with the deviation bound on $\Psi^{(1)}-I_{p}$, and then extend by union bound. We will use the following lemma which follows, for example, from the Ahlswede-Winter bound [8], or from [9]. (See also [10, 11, 12].)

Lemma 3. Let $\xi_{1}, \ldots, \xi_{n}$ be i.i.d. random vectors in $\mathbb{R}^{p}$ with $\mathbb{E} \xi_{1} \otimes \xi_{1}=I_{p}$ and $\left\|\xi_{1}\right\|_{2} \leq C_{p}$ almost surely for some constant $C_{p}$. Then, for $\delta \in(0,1)$,

$$
\begin{equation*}
\mathbb{P}\left\{\left\|n^{-1} \sum_{i=1}^{n} \xi_{i} \otimes \xi_{i}-I_{p}\right\|_{2}>\delta\right\} \leq p \exp \left(-\frac{n \delta^{2}}{4 C_{p}^{2}}\right) \tag{A.8}
\end{equation*}
$$

Recall that for the time sampling operator, $\left[\Phi \psi_{k}\right]_{i}=\frac{1}{\sqrt{n}} \psi_{k}\left(x_{i}\right)$ so that from (15),

$$
\Psi_{k \ell}=\frac{1}{n} \sum_{i=1}^{n} \psi_{k}\left(x_{i}\right) \psi_{\ell}\left(x_{i}\right)
$$

[^5]Let $\xi_{i}:=\left(\psi_{k}\left(x_{i}\right), 1 \leq k \leq p\right) \in \mathbb{R}^{p}$ for $i=1, \ldots, n$. Then, $\left\{\xi_{i}\right\}$ satisfy the conditions of Lemma 3. In particular, letting $e_{k}$ denote the $k$-th standard basis vector of $\mathbb{R}^{p}$, we note that

$$
\left\langle e_{k}, \mathbb{E}\left(\xi_{i} \otimes \xi_{i}\right) e_{\ell}\right\rangle_{2}=\mathbb{E}\left\langle e_{k}, \xi_{i}\right\rangle_{2}\left\langle e_{\ell}, \xi_{i}\right\rangle_{2}=\left\langle\psi_{k}, \psi_{\ell}\right\rangle_{L^{2}}=\delta_{k \ell}
$$

and $\left\|\xi_{i}\right\|_{2} \leq \sqrt{p} C_{\psi}$, where we have used uniform boundedness of $\left\{\psi_{k}\right\}$ as in (22). Furthermore, we have $\Psi^{(1)}=n^{-1} \sum_{i=1}^{n} \xi_{i} \otimes \xi_{i}$. Applying Lemma 3 with $C_{p}=\sqrt{p} C_{\psi}$ yields,

$$
\begin{equation*}
\mathbb{P}\left\{\left\|\Psi^{(1)}-I_{p}\right\|_{2}>\delta\right\} \leq p \exp \left(-\frac{\delta^{2}}{4 C_{\psi}^{2}} \frac{n}{p}\right) \tag{A.9}
\end{equation*}
$$

Similar bounds hold for $\Psi^{(k)}, k=2, \ldots, p^{m-1}$. Applying the union bound, we get

$$
\mathbb{P} \bigcup_{k=1}^{p^{m-1}}\left\{\left\|\Psi^{(k)}-I_{p}\right\|_{2}>\delta\right\} \leq \exp \left(m \log p-\frac{\delta^{2}}{4 C_{\psi}^{2}} \frac{n}{p}\right)
$$

For simplicity, let $A=A_{n, p}:=n /\left(4 C_{\psi}^{2} p\right)$. We impose $m \log p \leq \frac{A}{2} \delta^{2}$ so that the exponent in (A.9) is bounded above by $-\frac{A}{2} \delta^{2}$. Furthermore, for our purpose, it is enough to take $\delta=\frac{1}{2}$. It follows that

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{A}_{p}^{c}\right)=\mathbb{P} \bigcup_{k=1}^{p^{m-1}}\left\{\left\|\Psi^{(k)}-I_{p}\right\|_{2}>\frac{1}{2}\right\} \leq \exp \left(-\frac{1}{32 C_{\psi}^{2}} \frac{n}{p}\right) \tag{A.10}
\end{equation*}
$$

if $32 C_{\psi}^{2} m p \log p \leq n$. Now, by (A.7), under $\varepsilon^{2} \geq \sum_{k>p^{m}} \sigma_{k}, R_{\mathbb{S}_{x_{1}^{n}}}\left(\varepsilon^{2}\right)>$ $\widetilde{C}_{\psi} \varepsilon^{2}+\widetilde{C}_{\sigma} \sigma_{p}$ implies $\mathcal{A}_{p}^{c}$. Thus, the exponential bound in (A.10) holds for $\mathbb{P}\left\{R_{\mathbb{S}_{x_{1}^{n}}}\left(\varepsilon^{2}\right)>\widetilde{C}_{\psi} \varepsilon^{2}+\widetilde{C}_{\sigma} \sigma_{p}\right\}$ under the assumptions. We are to choose $p$ and the bound is optimized by making $p$ as small as possible. Hence, we take $p$ to be $\nu(\varepsilon):=\inf \left\{p: \varepsilon^{2} \geq \sum_{k>p^{m}} \sigma_{k}\right\}$ which proves Lemma 2, (Note that, in general, $\nu(\varepsilon)$ takes its values in $\{0,1,2, \ldots\}$. The assumption $\varepsilon^{2}<\sigma_{1}$ guarantees that $\nu(\varepsilon) \neq 0$.)

## Appendix B. Proof of Lemma 1

Assume $\sigma_{k}=C k^{-\alpha}$, for some $\alpha \geq 2$. First, note the following upper bound on the tail sum

$$
\begin{equation*}
\sum_{k>p} \sigma_{k} \leq C \int_{p}^{\infty} x^{-\alpha} d x=C_{1}(\alpha) p^{1-\alpha} \tag{B.1}
\end{equation*}
$$

Furthermore, from the bounds (30a) and (30b), we have, for $k \geq n+1$,

$$
\begin{equation*}
[\Psi]_{k k} \leq \min \left\{c_{1}, c_{2}\right\} \tag{B.2}
\end{equation*}
$$

To simplify notation, let us define $I_{n}:=\{1,2, \ldots, \gamma n\}$.
Consider the case $\alpha>2$. We will use the $\ell_{\infty}-\ell_{\infty}$ upper bound of (21), with $p=n$. Fix some $k \geq n+1$. Note that $\sigma_{k} \leq \sigma_{n+1}$. Then, recalling the assumptions on $\Psi$ and the definition of $S_{k}$, we have

$$
\begin{align*}
\sum_{\ell \geq n+1} \sqrt{\sigma_{k}} \sqrt{\sigma_{\ell}}\left|[\Psi]_{k, \ell}\right| & \leq \sqrt{\sigma_{n+1}} \sum_{q=0}^{\infty} \sum_{r=1}^{\gamma n} \sqrt{\sigma_{n+r+q \gamma n}}\left|[\Psi]_{k, n+r+q \gamma n}\right| \\
& =\sqrt{\sigma_{n+1}} \sum_{q=0}^{\infty} \sum_{r=1}^{\gamma n} \sqrt{\sigma_{n+r+q \gamma n}}\left|[\Psi]_{k, n+r}\right| \\
& \leq \sqrt{\sigma_{n+1}} \sum_{q=0}^{\infty}\left\{c_{1} \sum_{r \in S_{k}} \sqrt{\sigma_{n+r+q \gamma n}}+\frac{c_{2}}{n} \sum_{r \in I_{n} \backslash S_{k}} \sqrt{\sigma_{n+r+q \gamma n}}\right\} \tag{B.3}
\end{align*}
$$

Using (B.1), the second double sum in (B.3) is bounded by

$$
\begin{equation*}
\sum_{q=0}^{\infty} \sum_{r \in I_{n} \backslash S_{k}} \sqrt{\sigma_{n+r+q \gamma n}} \leq \sum_{\ell>n} \sqrt{\sigma_{\ell}} \leq C_{2}(\alpha) n^{1-\alpha / 2} \tag{B.4}
\end{equation*}
$$

Recalling that $S_{k} \subset I_{n}$ and $\left|S_{k}\right|=\eta$, the first double sum in (B.3) can be bounded as follows

$$
\begin{align*}
\sum_{q=0}^{\infty} \sum_{r \in S_{k}} \sqrt{\sigma_{n+r+q \gamma n}} & =\sqrt{C} \sum_{q=0}^{\infty} \sum_{r \in S_{k}}(n+r+q \gamma n)^{-\alpha / 2} \\
& \leq \sqrt{C} \sum_{q=0}^{\infty} \sum_{r \in S_{k}}(n+q \gamma n)^{-\alpha / 2} \\
& \leq \sqrt{C} \eta \sum_{q=0}^{\infty}(1+q \gamma)^{-\alpha / 2} n^{-\alpha / 2} \\
& \leq \sqrt{C} \eta\left(1+\gamma^{-\alpha / 2} \sum_{q=1}^{\infty} q^{-\alpha / 2}\right) n^{-\alpha / 2} \\
& =C_{3}(\alpha, \gamma, \eta) n^{-\alpha / 2} \tag{B.5}
\end{align*}
$$

where in the last line we have used $\sum_{q=1}^{\infty} q^{-\alpha / 2}<\infty$ due to $\alpha / 2>1$. Combining ( (B.3), (B.4) and (B.5) and noting that $\sqrt{\sigma_{n+1}} \leq \sqrt{C} n^{-\alpha / 2}$, we obtain

$$
\begin{equation*}
\sum_{\ell \geq n+1} \sqrt{\sigma_{k}} \sqrt{\sigma_{\ell}}\left|[\Psi]_{k, \ell}\right| \leq \sqrt{C} n^{-\alpha / 2}\left\{c_{1} C_{3}(\alpha, \gamma, \eta) n^{-\alpha / 2}+\frac{c_{2}}{n} C_{2}(\alpha) n^{1-\alpha / 2}\right\}=C_{4}(\alpha, \eta, \gamma) n^{-\alpha} \tag{B.6}
\end{equation*}
$$

Taking supremum over $k \geq 1$ and applying the $\ell_{\infty}-\ell_{\infty}$ bound of (21), with $p=n$, concludes the proof of part (a).

Now, consider the case $\alpha=2$. The above argument breaks down in this case because $\sum_{q=1}^{\infty} q^{-\alpha / 2}$ does not converge for $\alpha=2$. A remedy is to further partition the matrix $\Sigma_{\widetilde{n}}^{1 / 2} \Psi_{\widetilde{n}} \Sigma_{\widetilde{n}}^{1 / 2}$. Recall that the rows and columns of this matrix are indexed by $\{n+1, n+2, \ldots\}$. Let $A$ be the principal submatrix indexed by $\left\{n+1, n+2, \ldots, n^{2}\right\}$ and $D$ be the principal submatrix indexed by $\left\{n^{2}+1, n^{2}+2, \ldots\right\}$. We will use a combination of the bounds (30a) and (30b), and the well-known perturbation bound $\lambda_{\max }\left[\left(\begin{array}{cc}A & C \\ C^{T} & D\end{array}\right)\right] \leq \lambda_{\max }(A)+\lambda_{\max }(D)$, to write

$$
\begin{equation*}
\lambda_{\max }\left(\Sigma_{\overparen{n}}^{1 / 2} \Psi_{\widetilde{n}} \Sigma_{\widetilde{n}}^{1 / 2}\right) \leq \lambda_{\max }(A)+\lambda_{\max }(D) \leq\|A\|_{\infty}+\operatorname{tr}(D) \tag{B.7}
\end{equation*}
$$

The second term is bounded as
$\operatorname{tr}(D)=\sum_{k>n^{2}} \sigma_{k}[\Psi]_{k k} \leq \min \left\{c_{1}, c_{2}\right\} \sum_{k>n^{2}} \sigma_{k}=\min \left\{c_{1}, c_{2}\right\}\left(n^{2}\right)^{1-2}=C_{5}(\gamma) n^{-2}$,
where we have used (B.1) and (B.2). To bound the first term, fix $k \in$ $\left\{n+1, \ldots, n^{2}\right\}$. By an argument similar to that of part (a) and noting that $\gamma \geq 1$, hence $\gamma n^{2} \geq n^{2}$, we have

$$
\begin{align*}
\sum_{\ell=n+1}^{n^{2}} \sqrt{\sigma_{k}} \sqrt{\sigma_{\ell}}\left|[\Psi]_{k, \ell}\right| & \leq \sqrt{\sigma_{n+1}} \sum_{q=0}^{n} \sum_{r=1}^{\gamma n} \sqrt{\sigma_{n+r+q \gamma n}}\left|[\Psi]_{k, n+r}\right| \\
& \leq \sqrt{\sigma_{n+1}} \sum_{q=0}^{n}\left\{c_{1} \sum_{r \in S_{k}} \sqrt{\sigma_{n+r+q \gamma n}}+\frac{c_{2}}{n} \sum_{r \in I_{n} \backslash S_{k}} \sqrt{\sigma_{n+r+q \gamma n}}\right\} \tag{B.9}
\end{align*}
$$

Using $\gamma \geq 1$ again, the second double sum in ( (B.9) is bounded as

$$
\begin{equation*}
\sum_{q=0}^{n} \sum_{r \in I_{n} \backslash S_{k}} \sqrt{\sigma_{n+r+q \gamma n}} \leq \sum_{\ell=n+1}^{3 \gamma n^{2}} \sqrt{\sigma_{\ell}} \leq \sqrt{C} \sum_{\ell=2}^{3 \gamma n^{2}} \frac{1}{\ell} \leq \sqrt{C} \log \left(3 \gamma n^{2}\right) \leq C_{6}(\gamma) \log n \tag{B.10}
\end{equation*}
$$

for sufficiently large $n$. Note that we have used the bound $\sum_{\ell=2}^{p} \ell^{-1} \leq$ $\int_{1}^{p} x^{-1} d x=\log p$. The first double sum in (B.9) is bounded as follows

$$
\begin{align*}
\sum_{q=0}^{\infty} \sum_{r \in S_{k}} \sqrt{\sigma_{n+r+q \gamma n}} & =\sqrt{C} \sum_{q=0}^{n} \sum_{r \in S_{k}}(n+r+q \gamma n)^{-1} \\
& \leq \sqrt{C} \eta \sum_{q=0}^{n}(1+q \gamma)^{-1} n^{-1} \\
& \leq \sqrt{C} \eta\left(1+\gamma^{-1}+\gamma^{-1} \sum_{q=2}^{n} q^{-1}\right) n^{-1} \\
& =C_{7}(\gamma, \eta) n^{-1} \log n, \tag{B.11}
\end{align*}
$$

for $n$ sufficiently large. Combining ( $\overline{\mathrm{B} .9}$ ), ( $(\overline{\mathrm{B} .10})$ and ( $\overline{\mathrm{B} .11})$, taking supremum over $k$ and using the simple bound $\sqrt{\sigma_{n+1}} \leq \sqrt{C} n^{-1}$, we get

$$
\begin{equation*}
\|A\|_{\infty} \leq \sqrt{C} n^{-1}\left\{c_{1} C_{7}(\gamma, \eta) \frac{\log n}{n}+\frac{c_{2}}{n} C_{6}(\gamma) \log n\right\}=C_{8}(\gamma, \eta) \frac{\log n}{n^{2}} \tag{B.12}
\end{equation*}
$$

which in view of (B.8) and (B.7) completes the proof of part (b).

## Appendix C. Relationship between $\boldsymbol{R}_{\Phi}(\varepsilon)$ and $\underline{T}_{\Phi}(\varepsilon)$

In this appendix, we prove the claim made in Section 1 about the relation between the upper quantities $R_{\Phi}$ and $T_{\Phi}$ and the lower quantities $\underline{T}_{\Phi}$ and $\underline{R}_{\Phi}$. We only carry out the proof for $R_{\Phi}$; the dual version holds for $T_{\Phi}$. To simplify the argument, we look at slightly different versions of $R_{\Phi}$ and $\underline{T}_{\Phi}$, defined as

$$
\begin{align*}
& R_{\Phi}^{\circ}(\varepsilon):=\sup \left\{\|f\|_{L^{2}}^{2}: f \in B_{\mathcal{H}},\|f\|_{\Phi}^{2}<\varepsilon^{2}\right\}  \tag{C.1}\\
& \underline{T}_{\Phi}^{\circ}(\delta):=\inf \left\{\|f\|_{\Phi}^{2}: f \in B_{\mathcal{H}},\|f\|_{L^{2}}^{2}>\delta^{2}\right\} \tag{C.2}
\end{align*}
$$

and prove the following

$$
\begin{equation*}
R_{\Phi}^{\circ-1}(\delta)=\underline{T}_{\Phi}^{\circ}(\delta) \tag{C.3}
\end{equation*}
$$

where $R_{\Phi}^{\circ-1}(\delta):=\inf \left\{\varepsilon^{2}: R_{\Phi}^{\circ}(\varepsilon)>\delta^{2}\right\}$ is a generalized inverse of $R_{\Phi}^{\circ}$. To see (C.3), we note that $R_{\Phi}(\varepsilon)>\delta^{2}$ iff there exists $f \in B_{\mathcal{H}}$ such that $\|f\|_{\Phi}^{2}<\varepsilon^{2}$ and $\|f\|_{L^{2}}^{2}>\delta^{2}$. But this last statement is equivalent to $\underline{T}_{\Phi}^{\circ}(\delta)<\varepsilon^{2}$. Hence,

$$
\begin{equation*}
R_{\Phi}^{\circ-1}(\delta)=\inf \left\{\varepsilon^{2}: \underline{T}_{\Phi}^{\circ}(\delta)<\varepsilon^{2}\right\} \tag{C.4}
\end{equation*}
$$

which proves (C.3).
Using the following lemma, we can use relation (C.3) to convert upper bounds on $R_{\Phi}$ to lower bounds on $\underline{T}_{\Phi}$.

Lemma 4. Let $t \mapsto p(t)$ be a nondecreasing function (defined on the real line with values in the extended real line.). Let $q$ be its generalized inverse defined as $q(s):=\inf \{t: p(t)>s\}$. Let $r$ be a properly invertible (i.e., one-to-one) function such that $p(t) \leq r(t)$, for all $t$. Then,
(a) $q(p(t)) \geq t$, for all $t$,
(b) $q(s) \geq r^{-1}(s)$ for all $s$.

Proof. Assume (a) does not hold, that is, $\inf \{\alpha: p(\alpha)>p(t)\}<t$. Then, there exists $\alpha_{0}$ such that $p\left(\alpha_{0}\right)>p(t)$ and $\alpha_{0}<t$. But this contradicts $p(t)$ being nondecreasing. For part (b), note that (a) implies $t \leq q(p(t)) \leq q(r(t))$, since $q$ is nondecreasing by definition. Letting $t:=r^{-1}(s)$ and noting that $r\left(r^{-1}(s)\right)=s$, by assumption, proves (b).

Let $p=R_{\Phi}^{\circ}, q=\underline{T}_{\Phi}^{\circ}$ and $r(t)=A t+B$ for some constant $A>0$. Noting that $R_{\Phi}^{\circ} \leq R_{\Phi}$ and $\underline{T}_{\Phi}(\cdot+\gamma) \geq \underline{T}_{\Phi}^{\circ}$ for any $\gamma>0$, we obtain from Lemma 4 and (C.3) that

$$
\begin{equation*}
R_{\Phi}(\varepsilon) \leq A \varepsilon^{2}+B \quad \Longrightarrow \quad \underline{T}_{\Phi}(\delta+) \geq \frac{\delta^{2}}{A}-B \tag{C.5}
\end{equation*}
$$

where $\underline{T}_{\Phi}(\delta+)$ denotes the right limit of $\underline{T}_{\Phi}$ as $\delta^{2}$. This may be used to translate an upper bound of the form (17) on $R_{\Phi}$ to a corresponding lower bound on $\underline{T}_{\Phi}$.

## Appendix D. The $2 \times 2$ subproblem

The following subproblem arises in the proof of Theorem 1 .

$$
F\left(\varepsilon^{2}\right):=\sup \{\underbrace{\left(\begin{array}{cc}
r & s
\end{array}\right)\left(\begin{array}{cc}
u^{2} & 0  \tag{D.1}\\
0 & v^{2}
\end{array}\right)\binom{r}{s}}_{=: x(r, s)}: r^{2}+s^{2} \leq 1, \underbrace{\left(\begin{array}{cc}
r & s
\end{array}\right)\left(\begin{array}{cc}
a^{2} & 0 \\
0 & d^{2}
\end{array}\right)\binom{r}{s}}_{=: y(r, s)} \leq \varepsilon^{2}\},
$$

where $u^{2}, v^{2}, a^{2}$ and $d^{2}$ are given constants and the optimization is over $(r, s)$. Here, we discuss the solution in some detail; in particular, we provide explicit formulas for $F\left(\varepsilon^{2}\right)$. Without loss of generality assume $u^{2} \geq v^{2}$. Then, it is clear that $F\left(\varepsilon^{2}\right) \leq u^{2}$ and $F\left(\varepsilon^{2}\right)=u^{2}$ for $\varepsilon^{2} \geq u^{2}$. Thus, we are interested in what happens when $\varepsilon^{2}<u^{2}$.

The problem is easily solved by drawing a picture. Let $x(r, s)$ and $y(r, s)$ be as denoted in the last display. Consider the set

$$
\begin{align*}
\mathcal{S} & :=\left\{(x(r, s), y(r, s)): r^{2}+s^{2} \leq 1\right\} \\
& =\left\{r^{2}\left(u^{2}, a^{2}\right)+s^{2}\left(v^{2}, d^{2}\right)+q^{2}(0,0): r^{2}+s^{2}+q^{2}=1\right\} \\
& =\operatorname{conv}\left\{\left(u^{2}, a^{2}\right),\left(v^{2}, d^{2}\right),(0,0)\right\} . \tag{D.2}
\end{align*}
$$

That is, $\mathcal{S}$ is the convex hull of the three points $\left(u^{2}, a^{2}\right),\left(v^{2}, d^{2}\right)$ and the origin $(0,0)$.

Then, two (or maybe three) different pictures arise depending on whether $a^{2}>d^{2}$ (and whether $d^{2} \geq v^{2}$ or $d^{2}<v^{2}$ ) or $a^{2} \leq d^{2}$; see Fig. D.4. It follows that we have two (or three) different pictures for the function $\varepsilon^{2} \mapsto F\left(\varepsilon^{2}\right)$. In particular, for $a^{2}>d^{2}$ and $d^{2}<v^{2}$,

$$
\begin{equation*}
F\left(\varepsilon^{2}\right)=v^{2} \min \left\{\frac{\varepsilon^{2}}{d^{2}}, 1\right\}+\left(u^{2}-v^{2}\right) \max \left\{0, \frac{\varepsilon^{2}-d^{2}}{a^{2}-d^{2}}\right\}, \tag{D.3}
\end{equation*}
$$

for $a^{2}>d^{2}$ and $d^{2} \geq v^{2}, F\left(\varepsilon^{2}\right)=\varepsilon^{2}$, and for $a^{2} \leq d^{2}$,

$$
F\left(\varepsilon^{2}\right)=u^{2} \min \left\{\frac{\varepsilon^{2}}{a^{2}}, 1\right\} .
$$

All the equations above are valid for $\varepsilon^{2} \in\left[0, \sigma_{1}\right]$.


Figure D.4: Top plots illustrate the set $\mathcal{S}$ as defined in (D.2), in various cases. The bottom plots are the corresponding $\varepsilon^{2} \mapsto F\left(\varepsilon^{2}\right)$.

## Appendix E. Details of the Fourier truncation example

Here we establish the claim that the bound (19) holds with equality. Recall that for the (generalized) Fourier truncation operator $\mathbb{T}_{\psi_{1}^{n}}$, we have

$$
R_{\mathbb{T}_{\psi_{1}^{n}}}\left(\varepsilon^{2}\right)=\sup \left\{\sum_{k=1}^{\infty} \sigma_{k} \alpha_{k}^{2}: \sum_{k=1}^{\infty} \alpha_{k}^{2} \leq 1, \sum_{k=1}^{n} \sigma_{k} \alpha_{k}^{2} \leq \varepsilon^{2}\right\}
$$

Let $\alpha=(t \xi, s \gamma)$, where $t, s \in \mathbb{R}, \xi=\left(\xi_{1}, \ldots, \xi_{n}\right) \in \mathbb{R}^{n}, \gamma=\left(\gamma_{1}, \gamma_{2} \ldots\right) \in \ell_{2}$ and $\|\xi\|_{2}=1=\|\gamma\|_{2}$. Let $u^{2}=u^{2}(\xi):=\sum_{k=1}^{n} \sigma_{k} \xi_{k}^{2}$ and $v^{2}=v^{2}(\gamma):=$ $\sum_{k>n} \sigma_{k} \gamma_{k}^{2}$.

Let us fix $\xi$ and $\gamma$ for now and try to optimize over $t$ and $s$. That is, we look at

$$
G\left(\varepsilon^{2} ; \xi, \gamma\right):=\sup \left\{t^{2} u^{2}+s^{2} v^{2}: t^{2}+s^{2} \leq 1, t^{2} u^{2} \leq \varepsilon^{2}\right\}
$$

This is an instance of the 2-by-2 problem (D.1), with $a^{2}=u^{2}$ and $d^{2}=0$. Note that our assumption that $u^{2} \geq v^{2}$ holds in this case, for all $\xi$ and $\gamma$,
because $\left\{\sigma_{k}\right\}$ is a nonincreasing sequence. Hence, we have, for $\varepsilon^{2} \leq \sigma_{1}$,

$$
G\left(\varepsilon^{2} ; \xi, \gamma\right)=v^{2}+\left(u^{2}-v^{2}\right) \frac{\varepsilon^{2}}{u^{2}}=v^{2}(\gamma)+\left(1-\frac{v^{2}(\gamma)}{u^{2}(\xi)}\right) \varepsilon^{2}
$$

Now we can maximize $G\left(\varepsilon^{2} ; \xi, \gamma\right)$ over $\xi$ and then $\gamma$. Note that $G$ is increasing in $u^{2}$. Thus, the maximum is achieved by selecting $u^{2}$ to be $\sup _{\|\xi\|_{2}=1} u^{2}(\xi)=\sigma_{1}$. Thus,

$$
\sup _{\xi} G\left(\varepsilon^{2} ; \xi, \gamma\right)=\left(1-\frac{\varepsilon^{2}}{\sigma_{1}}\right) v^{2}(\gamma)+\varepsilon^{2}
$$

For $\varepsilon^{2}<\sigma_{1}$, the above is increasing in $v^{2}$. Hence the maximum is achieved by setting $v^{2}$ to be $\sup _{\|\gamma\|_{2}=1} v^{2}(\gamma)=\sigma_{n+1}$. Hence, for $\varepsilon^{2} \leq \sigma_{1}$

$$
\begin{equation*}
R_{\mathbb{T}_{\psi_{1}^{n}}}\left(\varepsilon^{2}\right):=\sup _{\xi, \gamma} G\left(\varepsilon^{2} ; \xi, \gamma\right)=\left(1-\frac{\sigma_{n+1}}{\sigma_{1}}\right) \varepsilon^{2}+\sigma_{n+1} \tag{E.1}
\end{equation*}
$$

## Appendix F. An quadratic inequality

In this appendix, we derive an inequality which will be used in the proof of Theorem 1. Consider a positive semidefinite matrix $M$ (possibly infinitedimensional) partitioned as

$$
M=\left(\begin{array}{cc}
A & C \\
C^{T} & D
\end{array}\right) .
$$

Assume that there exists $\rho^{2} \in(0,1)$ and $\kappa^{2}>0$ such that

$$
\left(\begin{array}{cc}
A & C  \tag{F.1}\\
C^{T} & \left(1-\rho^{2}\right) D+\kappa^{2} I
\end{array}\right) \succeq 0 .
$$

Let $(x, y)$ be a vector partitioned to match the block structure of $M$. Then we have the following.

Lemma 5. Under (F.1), for all $x$ and $y$,

$$
\begin{equation*}
x^{T} A x+2 x^{T} C y+y^{T} D y \geq \rho^{2} x^{T} A x-\frac{\kappa^{2}}{1-\rho^{2}}\|y\|_{2}^{2} \tag{F.2}
\end{equation*}
$$

Proof. By assumption (F.1), we have

$$
\left(\begin{array}{ll}
\sqrt{1-\rho^{2}} x^{T} & \frac{1}{\sqrt{1-\rho^{2}}} y^{T}
\end{array}\right)\left(\begin{array}{cc}
A & C  \tag{F.3}\\
C^{T} & \left(1-\rho^{2}\right) D+\kappa^{2} I
\end{array}\right)\binom{\sqrt{1-\rho^{2}} x}{\frac{1}{\sqrt{1-\rho^{2}}} y} \geq 0
$$

Writing (F.1) as a perturbation of the original matrix,

$$
\left(\begin{array}{cc}
A & C  \tag{F.4}\\
C^{T} & D
\end{array}\right)+\left(\begin{array}{cc}
0 & 0 \\
0 & -\rho^{2} D+\kappa^{2} I
\end{array}\right) \succeq 0
$$

we observe that a sufficient condition for (F.1) to hold is $\rho^{2} D \preceq \kappa^{2} I$. That is, it is sufficient to have

$$
\begin{equation*}
\rho^{2} \lambda_{\max }(D) \leq \kappa^{2} . \tag{F.5}
\end{equation*}
$$

Rewriting (F.1) differently, as

$$
\left(\begin{array}{cc}
\left(1-\rho^{2}\right) A & 0  \tag{F.6}\\
0 & \left(1-\rho^{2}\right) D
\end{array}\right)+\left(\begin{array}{cc}
\rho^{2} A & C \\
C^{T} & \kappa^{2} I
\end{array}\right) \succeq 0
$$

we find another sufficient condition for (F.1), namely, $\rho^{2} A-\kappa^{-2} C C^{T} \succeq 0$. In particular, it is also sufficient to have

$$
\begin{equation*}
\kappa^{-2} \lambda_{\max }\left(C C^{T}\right) \leq \rho^{2} \lambda_{\min }(A) \tag{F.7}
\end{equation*}
$$

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    ${ }^{1}$ We also use $L^{2}(\mathcal{X})$ or simply $L^{2}$ to refer to the space (11), with corresponding conventions for its norm. Also, one can take $\mathcal{X}$ to be a compact subset of any separable metric space and $\mathbb{P}$ a (regular) Borel measure.

[^1]:    ${ }^{2}$ This can be justified by identifying $f$ and $g$ if $\Phi f=\Phi g$, i.e. considering the quotient $\mathcal{H} / \operatorname{ker} \Phi$.

[^2]:    ${ }^{3}$ In particular, for $f \in \mathcal{H},\|f\|_{L^{2}} \leq \sqrt{\sigma_{1}}\|f\|_{\mathcal{H}}$ which shows that the inclusion $\mathcal{H} \subset L^{2}$ is continuous.

[^3]:    ${ }^{4}$ Also assume that $\mathbb{P}$ assign positive mass to every open Borel subset of $\mathcal{X}$.
    ${ }^{5}$ In fact, $I_{\mathbb{K}}$ is well defined over $L^{1} \supset L^{2}$ and the conclusions about $I_{\mathbb{K}}$ hold as a operator from $L^{1}$ to $C(\mathcal{X})$.
    ${ }^{6}$ The convergence is actually even stronger, namely it is absolute and uniform, as can be seen by noting that $\sum_{k=n+1}^{m}\left|\alpha_{k} \sqrt{\sigma_{k}} \psi_{k}(y)\right| \leq\left(\sum_{k=n+1}^{m} \alpha_{k}^{2}\right)^{1 / 2}\left(\sum_{k=n+1}^{m} \sigma_{k} \psi_{k}^{2}(y)\right)^{1 / 2} \leq$ $\left(\sum_{k=n+1}^{m} \alpha_{k}^{2}\right)^{1 / 2} \max _{y \in \mathcal{X}} k(y, y)$.

[^4]:    ${ }^{7}$ One can replace $\sup _{x \in \mathcal{X}}$ with essential supremum with respect to $\mathbb{P}$.

[^5]:    ${ }^{8} \mathrm{We}$ are using the alternate form of the bound based on $(\sqrt{A}+\sqrt{B})^{2}=$ $\inf _{r \in(0,1)}\left\{A r^{-1}+B(1-r)^{-1}\right\}$.

