THE BOCHNER-RIESZ MEANS FOR FOURIER-BESSEL EXPANSIONS: NORM INEQUALITIES FOR THE MAXIMAL OPERATOR AND ALMOST EVERYWHERE CONVERGENCE

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Abstract. In this paper, we develop a thorough analysis of the boundedness properties of the maximal operator for the Bochner-Riesz means related to the Fourier-Bessel expansions. For this operator, we study weighted and unweighted inequalities in the spaces $L^p((0,1), x^{2\nu+1}\,dx)$. Moreover, weak and restricted weak type inequalities are obtained for the critical values of p. As a consequence, we deduce the almost everywhere pointwise convergence of these means.

1. Introduction and main results

Let J_ν be the Bessel function of order ν. For $\nu > -1$ we have that

$$\int_0^1 J_\nu(s_j x) J_\nu(s_k x) x\,dx = \frac{1}{2}(J_{\nu+1}(s_j))^2 \delta_{j,k}, \quad j,k = 1,2,\dots$$

where $\{s_j\}_{j\ge 1}$ denotes the sequence of successive positive zeros of J_ν. From the previous identity we can check that the system of functions

$$\psi_j(x) = \frac{\sqrt{2}}{|J_{\nu+1}(s_j)|} x^{-\nu} J_\nu(s_j x), \quad j = 1,2,\dots \tag{1}$$

is orthonormal and complete in $L^2((0,1), d\mu_\nu)$, with $d\mu_\nu(x) = x^{2\nu+1}\,dx$ (for the completeness, see [12]). Given a function f on $(0,1)$, its Fourier series associated with this system, named as Fourier-Bessel series, is defined by

$$f \sim \sum_{j=1}^{\infty} a_j(f)\psi_j, \qquad \text{with} \qquad a_j(f) = \int_0^1 f(y)\psi_j(y)\,d\mu_\nu(y), \tag{2}$$

provided the integral exists. When $\nu = n/2 - 1$, for $n \in \mathbb{N}$ and $n \ge 2$, the functions ψ_j are the eigenfunctions of the radial Laplacian in the multidimensional ball B^n. The eigenvalues are the elements of the sequence $\{s_j^2\}_{j\ge 1}$. The Fourier-Bessel series corresponds with the radial case of the multidimensional Fourier-Bessel expansions analyzed in [1].

For each $\delta > 0$, we define the Bochner-Riesz means for Fourier-Bessel series as

$$\mathcal{B}_R^\delta(f,x) = \sum_{j\ge 1} \left(1 - \frac{s_j^2}{R^2}\right)_+^\delta a_j(f)\psi_j(x),$$

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where $R > 0$ and $(1-s^2)_+ = \max\{1-s^2, 0\}$. Bochner-Riesz means are a regular summation method used oftenly in harmonic analysis. It is very common to analyze regular summation methods for Fourier series when the convergence of the partial sum fails. Cesàro means are other of the most usual summation methods. B. Muckenhoupt and D. W. Webb [14] give inequalities for Cesàro means of Laguerre polynomial series and for the supremum of these means with certain parameters and $1 < p \le \infty$. For $p = 1$, they prove a weak type result. They also obtain similar estimates for Cesàro means of Hermite polynomial series and for the supremum of those means in [15]. An almost everywhere convergence result is obtained as a corollary in [14] and [15]. The result about Laguerre polynomials is an extension of a previous result in [18]. This kind of matters has been also studied by the first author and J. L. Varona in [7] for the Cesàro means of generalized Hermite expansions. The Cesàro means for Jacobi polynomials were analyzed by S. Chanillo and B. Muckenhoupt in [3]. The Bochner-Riesz means themselves have been analyzed for the Fourier transform and their boundedness properties in $L^p(\mathbb{R}^n)$ is an important unsolved problem for $n > 2$ (the case $n = 2$ is well understood, see [2]).

The target of this paper is twofold. First we will analyze the almost everywhere (a. e.) convergence, for functions in $L^p((0,1), d\mu_\nu)$, of the Bochner-Riesz means for Fourier-Bessel expansions. By the general theory [8, Ch. 2], to obtain this result we need to estimate the maximal operator

$$\mathcal{B}^\delta(f,x) = \sup_{R>0} \left|\mathcal{B}_R^\delta(f,x)\right|,$$

in the $L^p((0,1), d\mu_\nu)$ spaces. A deep analysis of the boundedness properties of this operator will be the second goal of our paper. This part of our work is strongly inspired by the results given in [3] for the Fourier-Jacobi expansions.

Before giving our results we introduce some notation. Being $p_0 = \frac{4(\nu+1)}{2\nu+3+2\delta}$ and $p_1 = \frac{4(\nu+1)}{2\nu+1-2\delta}$, we define

$$p_0(\delta) = \begin{cases} 1, & \delta > \nu + 1/2 \text{ or } -1 < \nu \le -1/2, \\ p_0, & \delta \le \nu + 1/2 \text{ and } \nu > -1/2, \end{cases} \tag{3}$$

$$p_1(\delta) = \begin{cases} \infty, & \delta > \nu + 1/2 \text{ or } -1 < \nu \le -1/2, \\ p_1, & \delta \le \nu + 1/2 \text{ and } \nu > -1/2. \end{cases}$$

Concerning to the a. e. convergence of the Bochner-Riesz means, our result reads as follows

Theorem 1. *Let $\nu > -1$, $\delta > 0$, and $1 \le p < \infty$. Then,*

$$\mathcal{B}_R^\delta(f,x) \to f(x) \quad \text{a. e., for } f \in L^p((0,1), d\mu_\nu)$$

if and only if $p_0(\delta) \le p$, where $p_0(\delta)$ is as in (3).

Proof of Theorem 1 is contained in Section 2 and is based on the following arguments. On one hand, to prove the necessity part, we will show the existence of functions in $L^p((0,1), d\mu_\nu)$ for $p < p_0(\delta)$ such that $\mathcal{B}_R^\delta$ diverges for them. In order to do this, we will use a reasoning similar to the one given by C. Meaney in [13] that we describe in Section 2. On the other hand, for the sufficiency, observe that the convergence result follows from the study of the maximal operator $\mathcal{B}^\delta f$. Indeed, it is sufficient to get $(p_0(\delta), p_0(\delta))$-weak type estimates for this operator and this will be the content of Theorem 3.

Regarding the boundedness properties of $\mathcal{B}^\delta f$ we have the following facts. First, a result containing the (p,p)-strong type inequality.

Theorem 2. *Let $\nu > -1$, $\delta > 0$, and $1 < p \le \infty$. Then,*

$$\left\|\mathcal{B}^\delta f\right\|_{L^p((0,1),d\mu_\nu)} \le C\|f\|_{L^p((0,1),d\mu_\nu)}$$

if and only if

$$\begin{cases} 1 < p \le \infty, & \text{for } -1 < \nu \le -1/2 \text{ or } \delta > \nu + 1/2, \\ p_0 < p < p_1, & \text{for } \delta \le \nu + 1/2 \text{ and } \nu > -1/2. \end{cases}$$

In the lower critical value of $p_0(\delta)$ we can prove a $(p_0(\delta), p_0(\delta))$-weak type estimate.

Theorem 3. *Let $\nu > -1$, $\delta > 0$, and $p_0(\delta)$ be the number in* (3)*. Then,*

$$\left\|\mathcal{B}^\delta f\right\|_{L^{p_0(\delta),\infty}((0,1),d\mu_\nu)} \le C\|f\|_{L^{p_0(\delta)}((0,1),d\mu_\nu)},$$

with C independent of f.

Finally, for the upper critical value, when $0 < \delta < \nu + 1/2$ and $\nu > -1/2$, it is possible to obtain a (p_1, p_1)-restricted weak type estimate.

Theorem 4. *Let $\nu > -1/2$ and $0 < \delta < \nu + 1/2$. Then,*

$$\left\|\mathcal{B}^\delta \chi_E\right\|_{L^{p_1,\infty}((0,1),d\mu_\nu)} \le C\|\chi_E\|_{L^{p_1}((0,1),d\mu_\nu)},$$

for all measurable subsets E of $(0,1)$ and C independent of E.

The previous results about norm inequalities are summarized in Figure 1 (case $-1 < \nu \le -1/2$) and Figure 2 (case $\nu > -1/2$).

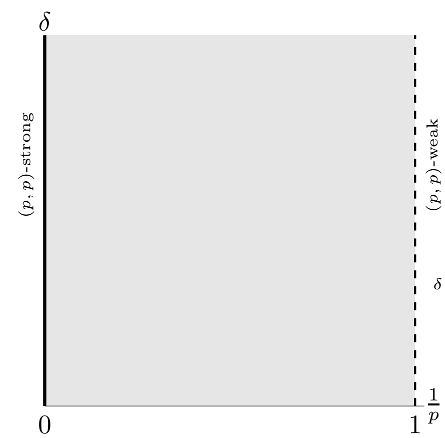

Figure 1: case $-1 < \nu \le -\frac{1}{2}$.

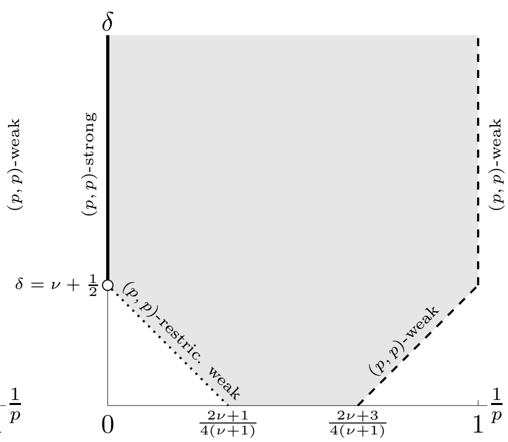

Figure 2: case $\nu > -\frac{1}{2}$.

At this point, a comment is in order. Note that J. E. Gilbert [9] also proves weak type norm inequalities for maximal operators associated with orthogonal expansions. The method used cannot be applied in our case, and the reason is the same as can be read in [3], at the end of Sections 15 and 16 therein. Following the technique in [9] we have to analyze some weak type inequalities for Hardy operator and its adjoint with weights and these inequalities do not hold for $p = p_0$ and $p = p_1$.

The proof of the sufficiency in Theorem 2 will be deduced from a more general result in which we analyze the boundedness of the operator $\mathcal{B}^\delta f$ with potential weights. Before stating it, we need a previous definition. We say that the parameters (b, B, ν, δ) satisfy the C_p conditions if

$$b > \frac{-2(\nu+1)}{p} \quad (\geq \text{ if } p = \infty), \tag{4}$$

$$B < 2(\nu+1)\left(1 - \frac{1}{p}\right) \quad (\leq \text{ if } p = 1), \tag{5}$$

$$b > 2(\nu+1)\left(\frac{1}{2} - \frac{1}{p}\right) - \delta - \frac{1}{2} \quad (\geq \text{ if } p = \infty), \tag{6}$$

$$B \leq 2(\nu+1)\left(\frac{1}{2} - \frac{1}{p}\right) + \delta + \frac{1}{2}, \tag{7}$$

$$B \leq b, \tag{8}$$

and in at least one of each of the following pairs the inequality is strict: (5) and (8), (6) and (8), and (7) and (8) except for $p = \infty$. The result concerning inequalities with potential weights is the following.

Theorem 5. *Let $\nu > -1$, $\delta > 0$, and $1 < p \leq \infty$. If (b, B, ν, δ) satisfy the C_p conditions, then*

$$\left\|x^b \mathcal{B}^\delta f\right\|_{L^p((0,1),d\mu_\nu)} \leq C \|x^B f\|_{L^p((0,1),d\mu_\nu)},$$

with C independent of f.

A result similar to Theorem 5 for the partial sum operator was proved in [10, Theorem 1]. It followed from a weighted version of a general Gilbert's maximal transference theorem, see [9, Theorem 1]. The weighted extension of Gilbert's result given in [10] depended heavily on the A_p theory and it can not be used in our case because it did not capture all the information relative to the weights. On the other hand, it is also remarkable the paper by K. Stempak [19] in which maximal inequalities for the partial sum operator of Fourier-Bessel expansions and divergence and convergence results are discussed.

The necessity in Theorem 2 will follow by showing that the operator $\mathcal{B}^\delta f$ is neither (p_1, p_1)-weak nor (p_0, p_0)-strong for $\nu > -1/2$ and $0 < \delta \leq \nu + 1/2$. This is the content of the next theorems.

Theorem 6. *Let $\nu > -1/2$. Then*

$$\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1} \|\mathcal{B}_R^\delta f\|_{L^{p_1,\infty}((0,1),d\mu_\nu)} \geq C(\log R)^{1/p_0},$$

if $0 < \delta < \nu + 1/2$; and

$$\sup_{\|f\|_{L^\infty((0,1),d\mu_\nu)}=1} \|\mathcal{B}_R^\delta f\|_{L^\infty((0,1),d\mu_\nu)} \geq C \log R,$$

if $\delta = \nu + 1/2$.

Theorem 7. *Let $\nu > -1/2$. Then*

$$\sup_{E \subset (0,1)} \frac{\|\mathcal{B}_R^\delta \chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}{\|\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}} \geq C(\log R)^{1/p_0},$$

if $0 < \delta < \nu + 1/2$*; and*

$$\sup_{\|f\|_{L^1((0,1),d\mu_\nu)}=1} \|\mathcal{B}_R^\delta f\|_{L^1((0,1),d\mu_\nu)} \geq C \log R,$$

if $\delta = \nu + 1/2$.

The paper is organized as follows. In the next section, we give the proof of Theorem 1. In Section 3 we first relate the Bochner-Riesz means $\mathcal{B}_R^\delta$ to the Bochner-Riesz means operator associated with the Fourier-Bessel system in the Lebesgue measure setting. Then, we prove weighted inequalities for the supremum of this new operator. With the connection between these means and the operator $\mathcal{B}_R^\delta$, we obtain Theorem 5 and, as a consequence, the sufficiency of Theorem 2. Sections 4 and 5 will be devoted to the proofs of Theorems 3 and 4, respectively. The proofs of Theorems 6 and 7 are contained in Section 6. One of the main ingredients in the proofs of Theorems 6 and 7 will be Lemma 15, this lemma is rather technical and it will be proved in the Section 7.

Throughout the paper, we will use the following notation: for each $p \in [1, \infty]$, we will denote by p' the conjugate of p, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. We shall write $X \simeq Y$ when simultaneously $X \leq CY$ and $Y \leq CX$.

2. Proof of Theorem 1

The proof of the sufficiency follows from Theorem 3 and standard arguments.

In order to prove the necessity, let us see that, for $0 < \delta < \nu + 1/2$ and $\nu > -1/2$, there exists a function $f \in L^p((0,1), d\mu_\nu)$, $p \in [1, p_0)$, for which $\mathcal{B}_R^\delta(f, x)$ diverges. We follow some ideas contained in [13] and [19].

First, we need a few more ingredients. Recall the well-known asymptotics for the Bessel functions (see [20, Chapter 7])

$$J_\nu(z) = \frac{z^\nu}{2^\nu \Gamma(\nu+1)} + O(z^{\nu+2}), \quad |z| < 1, \quad |\arg(z)| \leq \pi, \tag{9}$$

and

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} \left[\cos\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4} \right) + O(e^{\operatorname{Im}(z)} z^{-1}) \right], \quad |z| \geq 1, \quad |\arg(z)| \leq \pi - \theta, \tag{10}$$

where $D_\nu = -(\nu\pi/2 + \pi/4)$. It will also be useful the fact that (cf. [6, (2.6)])

$$s_j = O(j). \tag{11}$$

For our purposes, we need estimates for the L^p norms of the functions ψ_j. These estimates are contained in the following lemma, whose proof can be read in [5, Lemma 2.1].

Lemma 1. *Let* $1 \leq p \leq \infty$ *and* $\nu > -1$*. Then, for* $\nu > -1/2$,

$$\|\psi_j\|_{L^p((0,1),d\mu_\nu)} \simeq \begin{cases} j^{(\nu+1/2) - \frac{2(\nu+1)}{p}}, & \text{if } p > \frac{2(\nu+1)}{\nu+1/2}, \\ (\log j)^{1/p}, & \text{if } p = \frac{2(\nu+1)}{\nu+1/2}, \\ 1, & \text{if } p < \frac{2(\nu+1)}{\nu+1/2}, \end{cases}$$

and, for $-1 < \nu \leq -1/2$,

$$\|\psi_j\|_{L^p((0,1),d\mu_\nu)} \simeq \begin{cases} 1, & \text{if } p < \infty, \\ j^{\nu+1/2}, & \text{if } p = \infty. \end{cases}$$

We will also use a slight modification of a result by G. H. Hardy and M. Riesz for the Riesz means of order δ, that is contained in [11, Theorem 21]. We present here this result, adapted to the Bochner-Riesz means. We denote by $S_R(f,x)$ the partial sum associated to the Fourier-Bessel expansion, namely

$$S_R(f,x) = \sum_{0<s_j\le R} a_j(f)\psi_j(x).$$

The result reads as follows.

Lemma 2. *Suppose that f can be expressed as a Fourier-Bessel expansion and for some $\delta>0$ and $x\in(0,1)$ its Bochner-Riesz means $\mathcal{B}_R^\delta(f,x)$ converges to c as $R\to\infty$. Then, for $s_n\le R<s_{n+1}$,*

$$|S_R(f,x)-c| \le A_\delta n^\delta \sup_{0<t\le s_{n+1}} |\mathcal{B}_t^\delta(f,x)|.$$

By using this lemma, we can write

$$|a_j(f)\psi_j(x)| = |(S_{s_j}(f,x)-c)-(S_{s_{j-1}}(f,x)-c)| \le A_\delta j^\delta \sup_{0<t\le s_{j+1}} |\mathcal{B}_t^\delta(f,x)|. \tag{12}$$

Let us proceed with the proof of the necessity. Let $1\le p<p_0$. Note that $p_0'=p_1$. Therefore, $p'>p_0'>\frac{2(\nu+1)}{\nu+1/2}$, and $\delta<\nu+1/2-\frac{2(\nu+1)}{p'}:=\lambda$. By Lemma 1, $\|\psi_j\|_{L^{p'}((0,1),d\mu_\nu)}\ge Cj^\lambda$. Then, we have that the mapping $f\mapsto a_j(f)$, where $a_j(f)$ was given in (2), is a bounded linear functional on $L^p((0,1),d\mu_\nu)$ with norm bounded below by a constant multiple of j^λ. By uniform boundedness principle, for p conjugate to p' and each $0\le\varepsilon<\lambda$, there is a function $f_0\in L^p((0,1),d\mu_\nu)$ so that $a_j(f_0)j^{-\varepsilon}\to\infty$ as $j\to\infty$. By taking $\varepsilon=\delta$, we have that

$$a_j(f_0)j^{-\delta}\to\infty \quad \text{as} \quad j\to\infty. \tag{13}$$

Suppose now that $B_R^\delta(f_0,x)$ converges. Then, by Egoroff's theorem, it converges on a subset E of positive measure in $(0,1)$ and, clearly, we can think that $E\subset(\eta,1)$ for some fixed $\eta>0$. For each $x\in E$, we can consider j such that $s_jx\ge1$ and, by (10),

$$\begin{aligned}
|a_j(f_0)\psi_j(x)| &= \Big|a_j(f_0)\Big(\frac{\sqrt{2}}{|J_{\nu+1}(s_j)|}x^{-\nu}J_\nu(s_jx) \\
&\quad - \frac{\sqrt{2}}{|J_{\nu+1}(s_j)|}x^{-\nu}\Big(\frac{2}{\pi s_jx}\Big)^{1/2}\cos(s_jx+D_\nu)\Big) \\
&\quad + a_j(f_0)\frac{\sqrt{2}}{|J_{\nu+1}(s_j)|}x^{-\nu}\Big(\frac{2}{\pi s_jx}\Big)^{1/2}\cos(s_jx+D_\nu)\Big| \\
&= Cs_j^{-1/2}\frac{\sqrt{2}}{|J_{\nu+1}(s_j)|}\big|a_j(f_0)x^{-\nu-1/2}\big(O((s_jx)^{-1})+\cos(s_jx+D_\nu)\big)\big| \\
&\simeq |a_j(f_0)x^{-\nu-1/2}(\cos(s_jx+D_\nu)+O((s_jx)^{-1}))|.
\end{aligned}$$

By (12) on this set E,

$$|a_j(f_0)x^{-\nu-1/2}(\cos(s_jx+D_\nu)+O((j)^{-1}))| \le A_\delta j^\delta \sup_{0<t\le s_{j+1}}|\mathcal{B}_t^\delta(f_0,x)| \le K_Ej^\delta,$$

uniformly on $x\in E$. We also used (11) in the latter. The inequality above is equivalent to

$$|a_j(f_0)(\cos(s_jx+D_\nu)+O(j^{-1}))| \le K_Ex^{\nu+1/2}j^\delta \le K_Ej^\delta.$$

Therefore,

$$|a_j(f_0)j^{-\delta}(\cos(s_j x + D_\nu) + O((j)^{-1}))| \le K_E. \tag{14}$$

Now, taking the functions

$$F_j(x) = a_j(f_0)j^{-\delta}(\cos(s_j x + D_\nu) + O(j^{-1})), \qquad x \in E,$$

and using an argument based on the Cantor-Lebesgue and Riemann-Lebesgue theorems, see [13, Section 1.5] and [21, Section IX.1], we obtain that

$$\int_E |F_j(x)|^2\,dx \ge C|a_j(f_0)j^{-\delta}|^2|E|,$$

where, as usual, $|E|$ denotes the Lebesgue measure of the set E. On the other hand, by (14),

$$\int_E |F_j(x)|^2\,dx \le K_E^2|E|.$$

Then, from the previous estimates, it follows that $|a_j(f_0)j^{-\delta}| \le C$, which contradicts (13).

3. Bochner-Riesz means for Fourier-Bessel expansions in the Lebesgue measure setting. Proof of Theorem 5

For our convenience, we are going to introduce a new orthonormal system. We will take the functions

$$\phi_j(x) = \frac{\sqrt{2x}J_\nu(s_j x)}{|J_{\nu+1}(s_j)|}, \quad j = 1, 2, \dots.$$

These functions are a slight modification of the functions (1); in fact,

$$\phi_j(x) = x^{\nu+1/2}\psi_j(x). \tag{15}$$

The system $\{\phi_j(x)\}_{j\ge1}$ is a complete orthonormal basis of $L^2((0,1), dx)$.

In this case, the corresponding Fourier-Bessel expansion of a function f is

$$f \sim \sum_{j=1}^{\infty} b_j(f)\phi_j(x), \qquad \text{with} \qquad b_j(f) = \left(\int_0^1 f(y)\phi_j(y)\,dy\right)$$

provided the integral exists, and for $\delta > 0$ the Bochner-Riesz means of this expansion are

$$B_R^\delta(f,x) = \sum_{j\ge1}\left(1 - \frac{s_j^2}{R^2}\right)_+^\delta b_j(f)\phi_j(x),$$

where $R > 0$ and $(1-s^2)_+ = \max\{1-s^2, 0\}$. It follows that

$$B_R^\delta(f,x) = \int_0^1 f(y)K_R^\delta(x,y)\,dy$$

where

$$K_R^\delta(x,y) = \sum_{j\ge1}\left(1 - \frac{s_j^2}{R^2}\right)_+^\delta \phi_j(x)\phi_j(y). \tag{16}$$

Our next target is the proof of Theorem 5. Taking into account that

$$\mathcal{B}_R^\delta f(x) = \int_0^1 f(y)\mathcal{K}_R^\delta(x,y)\,d\mu_\nu(y),$$

where

$$\mathcal{K}_R^\delta(x,y)=\sum_{j\geq 1}\left(1-\frac{s_j^2}{R^2}\right)_+^\delta \psi_j(x)\psi_j(y),$$

it is clear, from (15), that $\mathcal{K}_R^\delta(x,y)=(xy)^{-(\nu+1/2)}K_R^\delta(x,y)$. Then, it is verified that the inequality

$$\|x^b\mathcal{B}^\delta(f,x)\|_{L^p((0,1),d\mu_\nu)}\leq C\|x^B f(x)\|_{L^p((0,1),d\mu_\nu)}$$

is equivalent to

$$\|x^{b+(\nu+1/2)(2/p-1)}B^\delta(f,x)\|_{L^p((0,1),dx)}\leq C\|x^{B+(\nu+1/2)(2/p-1)}f(x)\|_{L^p((0,1),dx)},$$

that is, we can focus on the study of a weighted inequality for the operator $B_R^\delta(f,x)$. The first results about convergence of this operator can be found in [4].

We are going to prove an inequality of the form

$$\|x^a B^\delta(f,x)\|_{L^p((0,1),dx)}\leq C\|x^A f(x)\|_{L^p((0,1),dx)}$$

for $\delta>0$, $1<p\leq\infty$, under certain conditions for a,A,ν and δ. Besides, a weighted weak type result for $\sup_{R>0}|B_R^\delta(f,x)|$ will be proved for $p=1$. The abovementioned conditions are the following. Let $\nu>-1$, $\delta>0$ and $1\leq p\leq\infty$; parameters (a,A,ν,δ) will be said to satisfy the c_p conditions provided

$$a>-1/p-(\nu+1/2)\ \ (\geq\ \text{ if } p=\infty), \tag{17}$$

$$A<1-1/p+(\nu+1/2)\ \ (\leq\ \text{ if } p=1), \tag{18}$$

$$a>-\delta-1/p\ \ (\geq\ \text{ if } p=\infty), \tag{19}$$

$$A\leq 1+\delta-1/p, \tag{20}$$

$$A\leq a \tag{21}$$

and in at least one of each of the following pairs the inequality is strict: (18) and (21), (19) and (21), and (20) and (21) except for $p=\infty$.

The main results in this section are the following:

Theorem 8. *Let $\nu>-1$, $\delta>0$ and $1<p\leq\infty$. If (a,A,ν,δ) satisfy the c_p conditions, then*

$$\|x^a B^\delta(f,x)\|_{L^p((0,1),dx)}\leq C\|x^A f(x)\|_{L^p((0,1),dx)},$$

with C independent of f.

Theorem 9. *Let $\nu>-1$ and $\delta>0$. If (a,A,ν,δ) satisfy the c_1 conditions and*

$$E_\lambda=\left\{x\in(0,1)\colon x^a\sup_{R>0}\left(|B_R^\delta(f,x)|\right)>\lambda\right\},$$

then

$$|E_\lambda|\leq C\frac{\|x^A f(x)\|_{L^1((0,1),dx)}}{\lambda},$$

with C independent of f and λ.

Note that, taking $a=b+(\nu+1/2)(2/p-1)$ and $A=B+(\nu+1/2)(2/p-1)$, Theorem 5 follows from Theorem 8.

The proofs of Theorem 8 and Theorem 9 will be achieved by decomposing the square $(0,1)\times(0,1)$ into five regions and obtaining the estimates therein. The regions will be:

$$
\begin{aligned}
A_1 &= \{(x,y) : 0 < x, y \le 4/R\},\\
A_2 &= \{(x,y) : 4/R < \max\{x,y\} < 1,\ |x-y| \le 2/R\},\\
A_3 &= \{(x,y) : 4/R \le x < 1,\ 0 < y \le x/2\},\\
A_4 &= \{(x,y) : 0 < x \le y/2,\ 4/R \le y < 1\},\\
A_5 &= \{(x,y) : 4/R < x < 1,\ x/2 < y < x - 2/R\}\\
&\qquad \cup \{(x,y) : y/2 < x \le y - 2/R,\ 4/R \le y < 1\}.
\end{aligned}
\tag{22}
$$

Theorem 8 and Theorem 9 will follow by showing that, if $1 \le p \le \infty$, then

$$
\left\| \sup_{R>0} \int_0^1 y^{-A} x^a |K_R^\delta(x,y)||f(y)|\chi_{A_j}\, dy \right\|_{L^p((0,1),dx)} \le C\|f(x)\|_{L^p((0,1),dx)}
\tag{23}
$$

holds for $j = 1,3,4$ and that

$$
\int_0^1 y^{-A} x^a |K_R^\delta(x,y)||f(y)|\chi_{A_j}\, dy \le CM(f,x),
\tag{24}
$$

for $j = 2,5$, where M is the Hardy-Littlewood maximal function of f, and C is independent of R, x and f. These results and the fact that M is $(1,1)$-weak and (p,p)-strong if $1 < p \le \infty$ complete the proofs.

To get (23) and (24) we will use a very precise pointwise estimate for the kernel $K_R^\delta(x,y)$, obtained in [4]; there, it was shown that

$$
|K_R^\delta(x,y)| \le C \begin{cases} (xy)^{\nu+1/2} R^{2(\nu+1)}, & (x,y) \in A_1,\\ R, & (x,y) \in A_2\\ \frac{\Phi_\nu(Rx)\Phi_\nu(Ry)}{R^\delta |x-y|^{\delta+1}}, & (x,y) \in A_3 \cup A_4 \cup A_5, \end{cases}
\tag{25}
$$

with

$$
\Phi_\nu(t) = \begin{cases} t^{\nu+1/2}, & \text{if } 0 < t < 2,\\ 1, & \text{if } t \ge 2. \end{cases}
\tag{26}
$$

The proof of (24) follows from the given estimate for the kernel $K_R^\delta(x,y)$ and $y^{-A}x^a \simeq C$ in $A_2 \cup A_5$ because $A \le a$. In the case of A_2, from $|K_R^\delta(x,y)| \le CR$ we deduce easily the required inequality. For A_5 the result is a consequence of $\Phi_\nu(Rx)\Phi_\nu(Ry) \le C$ and of a decomposition of the region in strips such that $R|x-y| \simeq 2^k$, with $k = 0, \dots, [\log_2 R] - 1$; this can be seen in [4, p. 109]

In this manner, to complete the proofs of Theorem 8 and Theorem 9 we only have to show (23) for $j = 1,3,4$ in the conditions c_p for $1 \le p \le \infty$, and this is the content of Corollary 1 in Subsection 3.2. In its turn, Corollary 1 follows from Lemmas 9 and 10 in the same subsection. Previously, Subsection 3.1 contains some technical lemmas that will be used in the proofs of Lemmas 9 and 10.

3.1. **Technical Lemmas.** To prove (23) for $j = 1,3,4$ we will use an interpolation argument based on six lemmas. These are stated below. They are small modifications of the six lemmas contained in Section 3 of [14] where a sketch of their proofs can be found.

Lemma 3. *Let $\xi_0 > 0$, if $r < -1$, $r+t \le -1$ and $r+s+t \le -1$, then for $p = 1$*

$$\left\| x^r \chi_{[1,\infty)}(x) \sup_{\xi_0 \le \xi \le x} \xi^s \int_\xi^x y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dx)} \le C \|f(x)\|_{L^p((0,\infty),dx)}$$

with C independent of f. If $r \le 0$, $r+t \le -1$ and $r+s+t \le -1$ with equality holding in at most one of the first two inequalities, then this holds for $p = \infty$.

Lemma 4. *Let $\xi_0 > 0$, if $t \le 0$, $r+t \le -1$ and $r+s+t \le -1$, with strict inequality in the last two in case of equality in the first, then for $p = 1$*

$$\left\| x^r \chi_{[1,\infty)}(x) \sup_{\xi_0 \le \xi \le x} \xi^s \int_x^\infty y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dx)} \le C \|f(x)\|_{L^p((0,\infty),dx)}$$

with C independent of f. If $t < -1$, $r+t \le -1$ and $r+s+t \le -1$, then this holds for $p = \infty$.

Lemma 5. *If $s < 0$, $s+t \le 0$ and $r+s+t \le -1$,with equality holding in at most one of the last two inequalities, then for $p = 1$*

$$\left\| x^r \chi_{[1,\infty)}(x) \sup_{\xi \ge x} \xi^s \int_x^\xi y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dx)} \le C \|f(x)\|_{L^p((0,\infty),dx)}$$

with C independent of f. If $s < 0$, $s+t \le -1$ and $r+s+t \le -1$ this holds for $p = \infty$.

Lemma 6. *If $t \le 0$, $s+t \le 0$ and $r+s+t \le -1$,with strict inequality holding in the first two in case the third is an equality, then for $p = 1$*

$$\left\| x^r \chi_{[1,\infty)}(x) \sup_{\xi \ge x} \xi^s \int_\xi^\infty y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dx)} \le C \|f(x)\|_{L^p((0,\infty),dx)}$$

with C independent of f. If $t < -1$, $s+t \le -1$ and $r+s+t \le -1$ then this holds for $p = \infty$.

Lemma 7. *If $s < 0$, $r+s < -1$ and $r+s+t \le -1$, then for $p = 1$*

$$\left\| x^r \chi_{[1,\infty)}(x) \sup_{\xi \ge x} \xi^s \int_1^x y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dx)} \le C \|f(x)\|_{L^p((0,\infty),dx)}$$

with C independent of f. If $s < 0$, $r+s \le 0$ and $r+s+t \le -1$, with equality holding in at most one of the last two inequalities, this holds for $p = \infty$.

Lemma 8. *If $r < -1$, $r+s < -1$ and $r+s+t \le -1$, then for $p = 1$*

$$\left\| x^r \chi_{[1,\infty)}(x) \sup_{1 \le \xi \le x} \xi^s \int_1^\xi y^t |f(y)|\, dy \right\|_{L^p((0,\infty),dx)} \le C \|f(x)\|_{L^p((0,\infty),dx)}$$

with C independent of f. If $r \le 0$, $r+s \le 0$ and $r+s+t \le -1$, with equality in at most one of the last two inequalities, this holds for $p = \infty$.

3.2. **Proofs of Theorem 8 and Theorem 9 for regions A_1, A_3 and A_4.** This section contains the proofs of the inequality (23) for regions A_1, A_3 and A_4. The results we will prove are included in the following

Lemma 9. *If $\nu > -1$, $\delta > 0$, $R > 0$, $j = 1, 3, 4$ and (a, A, ν, δ) satisfy the c_1 conditions, then (23) holds for $p = 1$ with C independent of f.*

Lemma 10. *If $\nu > -1$, $\delta > 0$, $R > 0$, $j = 1, 3, 4$ and (a, A, ν, δ) satisfy the c_∞ conditions, then (23) holds for $p = \infty$ with C independent of f.*

Corollary 1. *If $1 \le p \le \infty$, $\nu > -1$, $\delta > 0$, $R > 0$, (a, A, ν, δ) satisfy the c_p conditions and $j = 1, 3, 4$, then (23) holds with C independent of f.*

Proof of Corollary 1. It is enough to observe that if $1 < p < \infty$ and (a, A, ν, δ) satisfy the c_p conditions, then $(a-1+1/p, A-1+1/p, \nu, \delta)$ satisfy the c_1 conditions. So, by Lemma 9

$$\left\| \sup_{R\ge 0} \int_0^1 y^{-A+1-1/p} x^{a-1+1/p} |K_R^\delta(x,y)| \chi_{A_j}(x,y) |f(y)|\, dy \right\|_{L^1((0,1),dx)} \le C \|f(x)\|_{L^1((0,1),dx)},$$

and this is equivalent to

$$\int_0^1 x^{a+1/p} \left(\sup_{R\ge 0} \int_0^1 |K_R^\delta(x,y)| \chi_{A_j}(x,y) |f(y)|\, dy \right) \frac{dx}{x} \le C \int_0^1 x^{A+1/p} |f(x)| \frac{dx}{x},$$

where $j = 1, 3, 4$. Similarly, if (a, A, ν, δ) verify the c_p conditions, then $(a+1/p, A+1/p, \nu, \delta)$ satisfy the c_∞ conditions. Hence, by Lemma 10

$$\left\| x^{a+1/p} \sup_{R\ge 0} \int_0^1 |K_R^\delta(x,y)| \chi_{A_j}(x,y) |f(y)|\, dy \right\|_{L^\infty((0,1),dx)} \le C \|x^{A+1/p} f(x)\|_{L^\infty((0,1),dx)}.$$

Now, we can use the Marcinkiewicz interpolation theorem to obtain the inequality

$$\int_0^1 \left(x^{a+1/p} \left(\sup_{R\ge 0} \int_0^1 |K_R^\delta(x,y)| \chi_{A_j}(x,y) |f(y)|\, dy \right) \right)^p \frac{dx}{x} \le C \int_0^1 \left(x^{A+1/p} |f(x)| \right)^p \frac{dx}{x},$$

for $1 < p < \infty$ and the proof is finished.

Finally, we will prove Lemmas 9 and 10 for A_j, $j = 1, 3$ and 4, separately.

Proof of Lemma 9 and Lemma 10 for A_1. First of all, we have to note that $B_R^\delta(f,x) = 0$ when $0 < R < s_1$, being s_1 the first positive zero of J_ν. Using the estimate (25), the left side of (23) in this case is bounded by

$$C \left\| x^{a+\nu+1/2} \chi_{[0,1]}(x) \sup_{s_1 < R \le 4/x} R^{2(\nu+1)} \int_0^{4/R} y^{-A+\nu+1/2} |f(y)|\, dy \right\|_{L^p((0,1),dx)}.$$

Making the change of variables $x = 4/u$ and $y = 4/v$, we have

$$C \left\| u^{-a-\nu-\frac{1}{2}-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{s_1 \le R \le u} R^{2(\nu+1)} \int_R^\infty v^{A-(\nu+\frac{1}{2})-2+\frac{2}{p}} g(v)\, dv \right\|_{L^p((0,\infty),du)},$$

where $\|\cdot\|_{L^p((0,\infty),du)}$ denotes the L^p norm in the variable u, and

$$g(v) = v^{-2/p}|f(4v^{-1})|.$$

Note that function $g(v)$ is supported in $(1,\infty)$ and $\|g\|_{L^p((0,\infty),du)} = \|f\|_{L^p((0,1),dx)}$. The function g will be used through the subsection, but the value 4 may be changed by another one, at some points, without comment. Now, splitting the inner integral at u, we obtain the sum of

$$C\left\|u^{-a-\nu-\frac{1}{2}-\frac{2}{p}}\chi_{[4,\infty)}(u)\sup_{s_1\le R\le u} R^{2(\nu+1)}\int_R^u v^{A-(\nu+\frac{1}{2})-2+\frac{2}{p}}g(v)\,dv\right\|_{L^p((0,\infty),du)} \tag{27}$$

and

$$C\left\|u^{-a-\nu-\frac{1}{2}-\frac{2}{p}}\chi_{[4,\infty)}(u)\sup_{s_1\le R\le u} R^{2(\nu+1)}\int_u^\infty v^{A-(\nu+\frac{1}{2})-2+\frac{2}{p}}g(v)\,dv\right\|_{L^p((0,\infty),du)}. \tag{28}$$

From Lemma 3 we get the required estimate for (27), using conditions (17) and (21); Lemma 4 is applied to inequality (28), there we need conditions (18) and (21) and the restriction on them. This completes the proof of Lemmas 9 and 10 for $j=1$.

Proof of Lemma 9 and Lemma 10 for A_3. Clearly, the left side of (23) is bounded by

$$C\left\|x^a\chi_{[4/R,1]}(x)\sup_{4/x\le R}\int_0^{x/2} y^{-A}|K_R^\delta(x,y)||f(y)|\,dy\right\|_{L^p((0,1),dx)}.$$

Splitting the inner integral at $2/R$, using the bound for the kernel given in (25) and the definition of Φ_ν, we have this expression majorized by the sum of

$$\left\|x^a\chi_{[0,1]}(x)\sup_{4/x\le R}\int_0^{2/R}|f(y)|\frac{(Ry)^{\nu+1/2}y^{-A}}{R^\delta|x-y|^{\delta+1}}\,dy\right\|_{L^p((0,1),dx)} \tag{29}$$

and

$$\left\|x^a\chi_{[0,1]}(x)\sup_{4/x\le R}\int_{2/R}^{x/2}\frac{|f(y)|y^{-A}}{R^\delta|x-y|^{\delta+1}}\,dy\right\|_{L^p((0,1),dx)}. \tag{30}$$

For (29), taking into account that $|x-y|\simeq x$ in A_3, the changes of variables $x=4/u$, $y=2/v$ give us

$$\left\|u^{-a+(\delta+1)-\frac{2}{p}}\chi_{[4,\infty)}(u)\sup_{u\le R} R^{-\delta+(\nu+1/2)}\int_R^\infty v^{-(\nu+1/2)+A+\frac{2}{p}-2}g(v)\,dv\right\|_{L^p((0,\infty),du)}.$$

Lemma 6 can be used here. The required conditions for $p=1$ are (18), (20) and (21) with the restriction in the pairs therein. For $p=\infty$ the same inequalities are needed.

On the other hand, in (30), using again that $|x-y| \simeq x$, by changing of variables $x = 4/u$ and $y = 2/v$ we have

$$C \left\| u^{-a+(\delta+1)-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{u\le R} R^{-\delta} \int_{2u}^{R} v^{A+\frac{2}{p}-2} g(v)\, dv \right\|_{L^p((0,\infty),du)}$$
$$\le C \left\| u^{-a+(\delta+1)-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{u\le R} R^{-\delta} \int_{u}^{R} v^{A+\frac{2}{p}-2} g(v)\, dv \right\|_{L^p((0,\infty),du)}.$$

Lemma 5 can then be applied. For $p = 1$, we need $\delta > 0$, which is an hypothesis, and (20) and (21) with its corresponding restriction. For $p = \infty$ the inequalities are the same, with the requirement that (20) is strict. This completes the proof of Lemmas 9 and 10 for $j = 3$.

Proof of Lemma 9 and Lemma 10 for A_4. In this case, the left hand side of (23) is estimated by

$$C \left\| x^a \chi_{[0,1/2]}(x) \sup_{R>4} \int_{\max(4/R,2x)}^{1} y^{-A} |K_R^\delta(x,y)| |f(y)|\, dy \right\|_{L^p((0,1),dx)}.$$

To majorize this, we decompose the R-range in two regions: $4 < R \le 2/x$ and $R \ge 2/x$. In this manner, with the bound for the kernel given in (25) and the definition of Φ_ν, the previous norm is controlled by the sum of

$$C \left\| x^a \chi_{[0,1/2]}(x) \sup_{4<R\le 2/x} \int_{4/R}^{1} |f(y)| \frac{(Rx)^{\nu+1/2} y^{-A}}{R^\delta |x-y|^{\delta+1}}\, dy \right\|_{L^p((0,1),dx)}$$

and

$$C \left\| x^a \chi_{[0,1/2]}(x) \sup_{R\ge 2/x} \int_{2x}^{1} \frac{|f(y)| y^{-A}}{R^\delta |x-y|^{\delta+1}}\, dy \right\|_{L^p((0,1),dx)}.$$

Next, using that $|x-y| \simeq y$ in A_4, with the changes of variables $x = 2/u$ and $y = 1/v$ the previous norms are controlled by

$$C \left\| u^{-a-\frac{2}{p}-(\nu+\frac{1}{2})} \chi_{[4,\infty)}(u) \sup_{4<R\le u} R^{-\delta+(\nu+\frac{1}{2})} \int_{1}^{R/4} v^{A+\frac{2}{p}-2+(\delta+1)} g(v)\, dv \right\|_{L^p((0,\infty),du)} \tag{31}$$

and

$$C \left\| u^{-a-\frac{2}{p}} \chi_{[4,\infty)}(u) \sup_{R\ge u} R^{-\delta} \int_{1}^{u/4} v^{A+\frac{2}{p}-2+(\delta+1)} g(v)\, dv \right\|_{L^p((0,\infty),du)}. \tag{32}$$

In (31), we use Lemma 8; for $p = 1$, conditions (17), (19) and (21) are needed; we need the same for $p = \infty$. For (32), Lemma 7 requires the hypothesis $\delta > 0$ and conditions (19) and (21) for $p = 1$ and the same for $p = \infty$ with the restrictions in the pairs therein. This proves Lemmas 9 and 10 for $j = 4$.

4. Proof of Theorem 3

Now we shall prove Theorem 3. First note that, by (15), we can write

$$\mathcal{B}_R^\delta(f,x) = \int_0^1 f(y) \left(\frac{y}{x}\right)^{\nu+1/2} K_R^\delta(x,y)\, dy,$$

where K_R^δ is the kernel in (16). By taking $g(y)=f(y)y^{\nu+1/2}$, to prove the result it is enough to check that

$$\int_E d\mu_\nu(x) \le \frac{C}{\lambda^p}\int_0^1 |g(x)|^p x^{(\nu+1/2)(2-p)}\,dx,$$

where $E=\left\{x\in(0,1):\sup_{R>0}x^{-(\nu+1/2)}\int_0^1|g(y)||K_R^\delta(x,y)|\,dy>\lambda\right\}$ and $p=p_0(\delta)$. We decompose E into four regions, such that $E=\bigcup_{i=1}^4 J_i$, where

$$J_i=\left\{x\in(0,1):\sup_{R>0}x^{-(\nu+1/2)}\int_0^1|g(y)|\chi_{B_i}(x,y)|K_R^\delta(x,y)|\,dy>\lambda\right\}$$

for $i=1,\dots,4$, with $B_1=A_1$, $B_2=A_2\cup A_5$, $B_3=A_3$, and $B_4=A_4$ where the sets A_i were defined in (22). Note also that $\int_E d\mu_\nu(x)\le\sum_{i=1}^4\int_{J_i}d\mu_\nu(x)$, then we need to prove that

$$\int_{J_i} d\mu_\nu(x) \le \frac{C}{\lambda^p}\int_0^1 |g(x)|^p x^{(\nu+1/2)(2-p)}\,dx, \tag{33}$$

for $i=1,\dots,4$ and $p=p_0(\delta)$. At some points along the proof we will use the notation

$$I_p:=\int_0^1|g(y)|^p y^{(\nu+1/2)(2-p)}\,dy. \tag{34}$$

In J_1, by applying (25) and Hölder inequality with $p=p_0$, we have

$$\begin{aligned}
&x^{-(\nu+1/2)}\int_0^1|g(y)|\chi_{B_1}(x,y)|K_R^\delta(x,y)|\,dy\\
&\quad\le Cx^{-(\nu+1/2)}\int_0^{4/R}|g(y)|(xy)^{\nu+1/2}R^{2(\nu+1)}\,dy\\
&\quad\le CR^{2(\nu+1)}\left(\int_0^{4/R}|g(y)|^{p_0}y^{(\nu+1/2)(2-p_0)}\,dy\right)^{1/p_0}\left(\int_0^{4/R}y^{(2\nu+1)}\,dy\right)^{1/p_0'}\\
&\quad= CR^{\frac{2(\nu+1)}{p_0}}\left(\int_0^{4/R}|g(y)|^{p_0}y^{(\nu+1/2)(2-p_0)}\,dy\right)^{1/p_0}\le CR^{\frac{2(\nu+1)}{p_0}}I_{p_0}^{1/p_0}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sup_{R>0}x^{-(\nu+1/2)}\int_0^1|g(y)|\chi_{B_1}(x,y)|K_R^\delta(x,y)|\,dy &\le C\sup_{R>0}\chi_{[0,4/R]}(x)R^{\frac{2(\nu+1)}{p_0}}I_{p_0}^{1/p_0}\\
&\le Cx^{-\frac{2(\nu+1)}{p_0}}I_{p_0}^{1/p_0}.
\end{aligned}$$

In the case $p=1$, it is clear that

$$x^{-(\nu+1/2)}\int_0^1|g(y)|\chi_{B_1}(x,y)|K_R^\delta(x,y)|\,dy\le CR^{2(\nu+1)}I_1$$

and

$$\sup_{R>0}x^{-(\nu+1/2)}\int_0^1|g(y)|\chi_{B_1}(x,y)|K_R^\delta(x,y)|\,dy\le Cx^{-2(\nu+1)}I_1.$$

Hence, for $p=p_0(\delta)$,

$$J_1\subseteq\{x\in(0,1):Cx^{-\frac{2(\nu+1)}{p}}I_p^{1/p}>\lambda\},$$

and this gives (33) for $i = 1$.

In J_3, note first that

$$\begin{aligned}
&\sup_{R>0} x^{-(\nu+1/2)} \int_0^1 |g(y)|\chi_{B_3}(x,y)|K_R^\delta(x,y)|\,dy \\
&= \sup_{R>0} x^{-(\nu+1/2)} \chi_{[4/R,1]}(x) \left(\int_0^{2/R} |g(y)||K_R^\delta(x,y)|\,dy + \int_{2/R}^{x/2} |g(y)||K_R^\delta(x,y)|\,dy \right) \\
&\qquad := R_1 + R_2.
\end{aligned}$$

For R_1, using (25), the inequality $x/2 < x - y$, which holds in B_3, and Hölder inequality with $p = p_0$,

$$\begin{aligned}
R_1 &\le \sup_{R>0} x^{-(\nu+3/2+\delta)} \chi_{[4/R,1]}(x) \int_0^{2/R} R^{\nu+1/2-\delta} y^{\nu+1/2} |g(y)|\,dy \\
&\le \sup_{R>0} x^{-(\nu+3/2+\delta)} \chi_{[4/R,1]}(x) R^{\nu+1/2-\delta} R^{-\frac{2(\nu+1)}{p_0'}} I_{p_0}^{1/p_0} \le C x^{-\frac{2(\nu+1)}{p_0}} I_{p_0}^{1/p_0},
\end{aligned}$$

where I_{p_0} is the same as in (34). In the case $p = 1$, the estimate $R_1 \le C x^{-2(\nu+1)} I_1$ can be obtained easily.

On the other hand, for R_2, by using (25) and Hölder inequality with $p = p_0$ again,

$$\begin{aligned}
R_2 &\le \sup_{R>0} x^{-(\nu+3/2+\delta)} \chi_{[4/R,1]}(x) I_{p_0}^{1/p_0} R^{-\delta} \left(\int_{2/R}^{x/2} y^{-(\nu+1/2)\frac{(2-p_0)p_0'}{p_0}}\,dy \right)^{1/p_0'} \\
&\le \sup_{R>0} x^{-(\nu+3/2+\delta)} \chi_{[4/R,1]}(x) I_{p_0}^{1/p_0} R^{-\delta} \left(\int_{2/R}^{x/2} y^{(\nu+1/2)\frac{2-p_0}{1-p_0}}\,dy \right)^{1/p_0'}.
\end{aligned}$$

Using that $(\nu+1/2)\frac{2-p_0}{1-p_0} < -1$ and $4/R < x < 1$, we have that

$$R^{-\delta} \left(\int_{2/R}^{x/2} y^{(\nu+1/2)\frac{2-p_0}{1-p_0}}\,dy \right)^{1/p_0'} \le C \left(R^{-(\nu+1/2)\frac{2-p_0}{1-p_0}-1} \right)^{1/p_0'} R^{-\delta} = C$$

and the last inequality is true because the exponent of R is zero. Then

$$R_2 \le C x^{\frac{-2(\nu+1)}{p_0}} I_{p_0}^{1/p_0}.$$

In the case $p = 1$ applying Hölder inequality, then

$$R_2 \le \sup_{R>0} x^{-(\nu+3/2+\delta)} \chi_{[4/R,1]}(x) I_1\, R^{-\delta} \sup_{y\in[2/R,x/2]} y^{-(\nu+1/2)}.$$

Now, if $\nu + 1/2 > 0$ and $\nu + 1/2 < \delta$,

$$\begin{aligned}
&\sup_{R>0} \chi_{[4/R,1]}(x) R^{-\delta} \sup_{y\in[2/R,x/2]} y^{-(\nu+1/2)} \\
&\qquad = C \sup_{R>0} \chi_{[4/R,1]}(x) R^{\nu+1/2-\delta} \le C x^{-\nu-1/2+\delta};
\end{aligned}$$

and if $\nu+1/2\le 0$,

$$\sup_{R>0}\chi_{[4/R,1]}(x)R^{-\delta}\sup_{y\in[2/R,x/2]}y^{-(\nu+1/2)} = C\sup_{R>0}\chi_{[4/R,1]}(x)R^{-\delta}x^{-(\nu+1/2)}\le Cx^{-\nu-1/2+\delta}.$$

In this manner

$$R_2\le Cx^{-2(\nu+1)}I_1.$$

Therefore, collecting the estimates for R_1 and R_2 for $p=p_0$ and $p=1$, we have shown that

$$J_3\subseteq\{x\in(0,1):Cx^{\frac{-2(\nu+1)}{p}}(x)I^{1/p}>\lambda\},$$

hence we can deduce (33) for $i=3$.

For the region J_4, we proceed as follows

$$\begin{aligned}
&\sup_{R>0}x^{-(\nu+1/2)}\int_0^1|g(y)|\chi_{B_4}(x,y)|K_R^\delta(x,y)|\,dy\\
&\qquad\le\sup_{R>0}x^{-(\nu+1/2)}\chi_{[0,2/R]}(x)\int_{4/R}^1|g(y)||K_R^\delta(x,y)|\,dy\\
&\qquad\quad+\sup_{R>0}x^{-(\nu+1/2)}\chi_{[2/R,1]}(x)\int_{2x}^1|g(y)||K_R^\delta(x,y)|\,dy\\
&\qquad\le C\sup_{R>0}x^{-(\nu+1/2)}\chi_{[0,2/R]}(x)(Rx)^{\nu+1/2}\int_{4/R}^1\frac{|g(y)|}{R^\delta|x-y|^{\delta+1}}\,dy\\
&\qquad\quad+C\sup_{R>0}x^{-(\nu+1/2)}\chi_{[2/R,1]}(x)\int_{2x}^1\frac{|g(y)|}{R^\delta|x-y|^{\delta+1}}\,dy:=S_1+S_2.
\end{aligned}$$

We first deal with S_1, we use that $y-x>y/2$, then

$$\begin{aligned}
S_1\le&C\sup_{R>0}\chi_{[0,2/R]}(x)R^{\nu+1/2-\delta}\int_{4/R}^1\frac{|g(y)|}{y^{\delta+1}}\,dy\\
&\le C\sup_{R>0}\chi_{[0,2/R]}(x)R^{\nu+1}\int_{4/R}^1\frac{|g(y)|}{\sqrt{y}}\,dy\le Cx^{-(\nu+1)}\int_x^1\frac{|g(y)|}{\sqrt{y}}\,dy.
\end{aligned}$$

Now for $p=p_0$ or $p=1$, we have that $2\nu+1-p(\nu+1)>-1$ and Hardy's inequality [17, Lemma 3.14, p. 196] is applied in the following estimate

$$\begin{aligned}
\int_0^1|S_1(x)|^px^{2\nu+1}\,dx&\le C\int_0^1\left(\int_x^1\frac{|g(y)|}{\sqrt{y}}\,dy\right)^px^{2\nu+1-p(\nu+1)}\,dx\\
&\le C\int_0^1\left|\frac{g(y)}{\sqrt{y}}\right|^py^{2\nu+1-p\nu}\,dy=C\int_0^1|g(y)|^py^{(\nu+1/2)(2-p)}\,dy.
\end{aligned}$$

Concerning S_2, observe that $\sup_{R>0}\chi_{[2/R,1]}(x)R^{-\delta}\le Cx^\delta$, thus

$$S_2\le Cx^{-\nu-1/2+\delta}\int_x^1\frac{|g(y)|}{y^{\delta+1}}\,dy.$$

Since for $p = p_0$ or $p = 1$ we have that $2\nu + 1 - p(\nu + 1/2 - \delta) > -1$, we can use again Hardy's inequality to complete the required estimate. Indeed,

$$\begin{aligned}\int_0^1 |S_2(x)|^p x^{2\nu+1}\,dx &\le C\int_0^1 \left(\int_x^1 \frac{|g(y)|}{y^{\delta+1}}\,dy\right)^p x^{2\nu+1-p(\nu+1/2-\delta)}\,dx\\ &\le C\int_0^1 \left|\frac{g(y)}{y^{\delta+1}}\right|^p y^{2\nu+1-p(\nu+1/2-\delta)+p}\,dy\\ &= C\int_0^1 |g(y)|^p y^{(\nu+1/2)(2-p)}\,dy.\end{aligned}$$

With the inequalities for S_1 and S_2, we can conclude (33) for $i = 4$.

To prove (33) for $i = 2$ we define, for k a nonnegative integer, the intervals

$$I_k = [2^{-k-1}, 2^{-k}], \qquad N_k = [2^{-k-3}, 2^{-k+2}]$$

and the function $g_k(y) = |g(y)|\chi_{I_k}(y)$. By using (25) for $x/2 < y < 2x$, with $x \in (0,1)$, we have the bound

$$|K_R^\delta(x,y)| \le \frac{C}{R^\delta(|x-y| + 2/R)^{\delta+1}}.$$

Then

$$J_2 \subset \left\{x \in (0,1) : \sup_{R>0} \sum_{k=0}^\infty \int_{x/2}^{\min\{2x,1\}} \frac{g_k(t)}{R^\delta(|x-y|+2/R)^{\delta+1}}\,dy > C\lambda x^{\nu+1/2}\right\}.$$

Since at most three of these integrals are not zero for each $x \in (0,1)$

$$\begin{aligned}J_2 &\subset \bigcup_{k=0}^\infty \left\{x \in (0,1) : 3\sup_{R>0} \int_{x/2}^{\min\{2x,1\}} \frac{g_k(t)}{R^\delta(|x-y|+2/R)^{\delta+1}}\,dy > C\lambda x^{\nu+1/2}\right\}\\ &\subset \bigcup_{k=0}^\infty \left\{x \in N_k : M(g_k, x) > C\lambda x^{\nu+1/2}\right\}\end{aligned}$$

where in the las step we have used that

$$\sup_{R>0} \int_{x/2}^{\min\{2x,1\}} \frac{g_k(t)}{R^\delta(|x-y|+2/R)^{\delta+1}}\,dy \le CM(g_k, x).$$

By using the estimate $x \simeq 2^{-k}$ for $x \in N_k$, we can check easily that

$$J_2 \subset \bigcup_{k=1}^\infty \left\{x \in N_k : M(g_k, x) > C\lambda 2^{-k(\nu+1/2)}\right\}.$$

Finally by using again that $x \simeq 2^{-k}$ for $x \in I_k, N_k$ and the weak type norm inequality for the Hardy-Littlewood maximal function we have

$$\begin{aligned}\int_{J_2} x^{2\nu+1}\,dx &\le C\sum_{k=0}^\infty 2^{-k(2\nu+1)} \int_{\left\{x\in N_k : M(g_k,x) > C\lambda 2^{-k(\nu+1/2)}\right\}} dx\\ &\le C\sum_{k=0}^\infty \frac{2^{pk(\nu+1/2)-k(2\nu+1)}}{\lambda^p} \int_{I_k} |g(y)|^p\,dy\\ &\le \frac{C}{\lambda^p}\int_0^1 |g(y)|^p y^{(\nu+1/2)(2-p)}\,dy\end{aligned}$$

and the proof is complete.

5. Proof of Theorem 4

To conclude the result we have to prove (33) with $g(x) = \chi_E(x)$ and $p = p_1$. For J_1 and J_2 the result follows by using the steps given in the proof of Theorem 3 for the same intervals. To analyze J_3 we proceed as we did for J_4 in the proof of Theorem 3. In this case we obtain that

$$\sup_{R>0} x^{-(\nu+1/2)} \int_0^1 |g(y)|\chi_{B_3}(x,y)|K_r^\delta(x,y)| \\ \le C\left(x^{-(\nu+1)}\int_0^x \frac{|g(y)|}{\sqrt{y}}\,dy + x^{-(\nu+3/2+\delta)}\int_0^x |g(y)|y^\delta\,dy\right).$$

Now taking into account that for $p = p_1$ we have $2\nu+1-p(\nu+1) < -1$ and $2\nu+1-p(\nu+3/2+\delta) < -1$ we can apply Hardy's inequalities to obtain that

$$\int_0^1 \left(x^{-(\nu+1)}\int_0^x \frac{|g(y)|}{\sqrt{y}}\,dy\right)^p x^{2\nu+1}\,dx \le C\int_0^1 |g(y)|^p y^{(\nu+1/2)(2-p)}\,dy$$

and

$$\int_0^1 \left(x^{-(\nu+3/2+\delta)}\int_0^x |g(y)|y^\delta\,dy\right)^p x^{2\nu+1}\,dx \le C\int_0^1 |g(y)|^p y^{(\nu+1/2)(2-p)}\,dy,$$

with these two inequalities we can deduce that (33) holds for J_3 with $p = p_1$ in this case.

The main difference with the previous proof appears in the analysis of J_4. To deal with this case, we have to use the following lemma [3, Lemma 16.5]

Lemma 11. *If $1 < p < \infty$, $a > -1$, and $E \subset [0,\infty)$, then*

$$\left(\int_E x^a\,dx\right)^p \le 2^p(a+1)^{1-p}\int_E x^{(a+1)p-1}\,dx.$$

In this case, it is enough to prove that

$$\int_{\mathcal{J}} d\mu_\nu(x) \le \frac{C}{\lambda^p}\int_0^1 \chi_E(y)\,d\mu_\nu(y),$$

where

$$\mathcal{J} = \left\{x \in (0,1) : \sup_{R>0} x^{-(\nu+1/2)}\int_0^1 \chi_E(y)\chi_{B_4}(x,y)y^{\nu+1/2}|K_R^\delta(x,y)|\,dy > \lambda\right\},$$

and this can be deduced immediately by using the inclusion

$$\mathcal{J} \subseteq [0, \min\{1, H\}] \tag{35}$$

with

$$H^{2(\nu+1)} = \frac{C}{\lambda^p}\int_0^1 \chi_E(y)\,d\mu_\nu(y).$$

Let's prove (35). By using (16) and the estimate $y-x>y/2$, we have

$$\begin{aligned}\sup_{R>0} x^{-(\nu+1/2)} \int_0^1 \chi_E(y)\chi_{B_4}(x,y)y^{\nu+1/2}|K_R^\delta(x,y)|\,dy\\ \le C\sup_{R>0} R^{-\delta+\nu+1/2}\chi_{[0,2/R]}(x)\int_{4/R}^1 \chi_E(y)y^{-\delta+\nu-1/2}\,dy\\ + C\sup_{R>0} R^{-\delta}x^{-(\nu+1/2)}\chi_{[2/R,1]}(x)\int_{2x}^1 \chi_E(y)y^{-\delta+\nu-1/2}\,dy.\end{aligned}$$

In the first summand we can use that $R^{-\delta+\nu+1/2}\le Cx^{\delta-\nu-1/2}$ and in the second one that $R^{-\delta}\le x^\delta$. Moreover observing that with $p=p_1$ it holds $-\delta+\nu+1/2=2(\nu+1)/p$ we obtain that

$$\begin{aligned}\sup_{R>0} x^{-(\nu+1/2)} \int_0^1 \chi_E(y)\chi_{B_4}y^{\nu+1/2}|K_R^\delta(x,y)|\,dy \le Cx^{-2(\nu+1)/p}\int_E y^{-1+2(\nu+1)/p}\,dy\\ \le Cx^{-2(\nu+1)/p}\int_E d\mu_\nu(y),\end{aligned}$$

where in the last step we have used Lemma 11, and this is enough to deduce the inclusion in (35).

6. Proofs of Theorem 6 and Theorem 7

This section will be devoted to the proofs of Theorem 6 and Theorem 7. To this end we need a suitable identity for the kernel and in order to do that we have to introduce some notation. $H_\nu^{(1)}$ will denote the Hankel function of the first kind, and it is defined as follows

$$H_\nu^{(1)}(z)=J_\nu(z)+iY_\nu(z),$$

where Y_ν denotes the Weber's function, given by

$$Y_\nu(z)=\frac{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)}{\sin\nu\pi},\ \nu\notin\mathbb{Z},\ \text{and}\ Y_n(z)=\lim_{\nu\to n}\frac{J_\nu(z)\cos\nu\pi-J_{-\nu}(z)}{\sin\nu\pi}.$$

From these definitions, we have

$$H_\nu^{(1)}(z)=\frac{J_{-\nu}(z)-e^{-\nu\pi i}J_\nu(z)}{i\sin\nu\pi},\ \nu\notin\mathbb{Z},\ \text{and}\ H_n^{(1)}(z)=\lim_{\nu\to n}\frac{J_{-\nu}(z)-e^{-\nu\pi i}J_\nu(z)}{i\sin\nu\pi}.$$

For the function $H_\nu^{(1)}$, the asymptotic

$$H_\nu^{(1)}(z)=\sqrt{\frac{2}{\pi z}}e^{i(z-\nu\pi/2-\pi/4)}[A+O(z^{-1})],\quad |z|>1,\quad -\pi<\arg(z)<2\pi, \tag{36}$$

holds for some constant A.

In [4, Lemma 1] the following lemma was proved

Lemma 12. *For $R>0$ the following holds:*

$$K_R^\delta(x,y)=I_{R,1}^\delta(x,y)+I_{R,2}^\delta(x,y)$$

with

$$I_{R,1}^\delta(x,y)=(xy)^{1/2}\int_0^R z\left(1-\frac{z^2}{R^2}\right)^\delta J_\nu(zx)J_\nu(zy)\,dz$$

and

$$I_{R,2}^{\delta}(x,y) = \lim_{\varepsilon\to 0} \frac{(xy)^{1/2}}{2} \int_{\mathbf{S}_\varepsilon} \left(1-\frac{z^2}{R^2}\right)^{\delta} \frac{zH_\nu^{(1)}(z)J_\nu(zx)J_\nu(zy)}{J_\nu(z)}\,dz,$$

where, for each $\varepsilon>0$, $\mathbf{S}_\varepsilon$ is the path of integration given by the interval $R+i[\varepsilon,\infty)$ in the direction of increasing imaginary part and the interval $-R+i[\varepsilon,\infty)$ in the opposite direction.

Then, by Lemma 12 we have

$$\mathcal{K}_R^{\delta}(x,y) = \mathcal{I}_{R,1}^{\delta}(x,y) + \mathcal{I}_{R,2}^{\delta}(x,y)$$

where $\mathcal{I}_{R,j}^{\delta}(x,y) = (xy)^{-(\nu+1/2)} I_{R,j}^{\delta}(x,y)$ for $j=1,2$. The main tool to deduce our negative results will be the following lemma

Lemma 13. *For $\nu>-1/2$, $\delta>0$, and $R>0$ it is verified that*

$$\mathcal{K}_R^{\delta}(0,y) = \frac{2^{\delta-\nu}\Gamma(\delta+1)}{\Gamma(\nu+1)} R^{2(\nu+1)} \frac{J_{\nu+\delta+1}(yR)}{(yR)^{\nu+\delta+1}} + \mathcal{I}_{R,2}^{\delta}(0,y),$$

where

$$\left|\mathcal{I}_{R,2}^{\delta}(0,y)\right| \le C \begin{cases} R^{2\nu-\delta+1}, & yR\le 1,\\ R^{\nu-\delta+1/2}y^{-(\nu+1/2)}, & yR>1. \end{cases} \tag{37}$$

Proof. From (9), it is clear that

$$\mathcal{I}_{R,1}^{\delta}(0,y) = \frac{y^{-\nu}}{2^\nu\Gamma(\nu+1)} \int_0^R z^{\nu+1}\left(1-\frac{z^2}{R^2}\right)^{\delta} J_\nu(zy)\,dz.$$

Now, by using Sonine's identity [20, Ch. 12, 12.11, p. 373]

$$\int_0^1 s^{\nu+1}\left(1-s^2\right)^{\delta} J_\nu(sy)\,ds = 2^{\delta}\Gamma(\delta+1)\frac{J_{\nu+\delta+1}(y)}{y^{\delta+1}}, \qquad \nu,\delta>-1,$$

we deduce the leading term of the expression for $\mathcal{K}_R^{\delta}(0,y)$.

To control the term

$$\mathcal{I}_{R,2}^{\delta}(0,y) = \lim_{\varepsilon\to 0}\frac{y^{-(\nu+1/2)}}{2}\int_{\mathbf{S}_\varepsilon}\left(1-\frac{z^2}{R^2}\right)^{\delta}\frac{z^{\nu+1/2}H_\nu^{(1)}(z)(zy)^{1/2}J_\nu(zy)}{J_\nu(z)}\,dz,$$

we start by using the asymptotic expansions given in (36) and (10) for $H_\nu^{(1)}(z)$ and $J_\nu(z)$. We see that on $\mathbf{S}_\varepsilon$, the path of integration described in Lemma 12, for $t=\operatorname{Im}(z)$ the estimate

$$\left|\frac{H_\nu(z)}{J_\nu(z)}\right| \le Ce^{-2t},$$

holds for $t>0$. Now, from (9) and (10), it is clear that for $z=\pm R+it$

$$|\sqrt{zy}J_\nu(zy)| \le Ce^{yt}\Phi_\nu((R+t)y)$$

where Φ_ν is the function in (26). Then

$$|\mathcal{I}_{R,2}^{\delta}(0,y)| \le Cy^{-(\nu+1/2)}R^{-2\delta}\int_0^\infty t^{\delta}(R+t)^{\nu+\delta+1/2}\Phi_\nu((R+t)y)e^{-(2-y)t}\,dt.$$

If $y > 1/R$ we have the inequality $\Phi_\nu((R+t)y) \le C$, then

$$\begin{aligned}|\mathcal{I}_{R,2}^{\delta}(0,y)| &\le Cy^{-(\nu+1/2)}R^{-2\delta}\int_0^\infty t^\delta (R+t)^{\nu+\delta+1/2}e^{-(2-y)t}\,dt\\ &\le Cy^{-(\nu+1/2)}R^{-\delta}(R^{\nu+1/2}+R^{-\delta}) \le CR^{\nu-\delta+1/2}y^{-(\nu+1/2)}\end{aligned}$$

and (37) follows in this case. If $y \le 1/R$ we obtain the bound in (37) with the estimate $\Phi_\nu((R+t)y) \le C(\Phi_\nu(yR) + (yt)^{\nu+1/2})$. Indeed,

$$\begin{aligned}|\mathcal{I}_{R,2}^{\delta}(0,y)| \le{}& Cy^{-(\nu+1/2)}R^{-2\delta}\Phi_\nu(yR)\int_0^\infty t^\delta (R+t)^{\nu+\delta+1/2}e^{-(2-y)t}\,dt\\ &+CR^{-2\delta}\int_0^\infty t^{\nu+\delta+1/2}(R+t)^{\nu+\delta+1/2}e^{-(2-y)t}\,dt\\ &\le C(R^{2\nu-\delta+1}+R^{\nu-2\delta+1/2}+R^{\nu-\delta+1/2}+R^{-2\delta}) \le R^{2\nu-\delta+1}.\end{aligned}$$

□

Lemma 14. *For $\nu > -1/2$ and $0 < \delta \le \nu + 1/2$, the estimate*

$$\|\mathcal{K}_R^\delta(0,y)\|_{L^{p_0}((0,1),d\mu_\nu)} \ge CR^{\nu-\delta+1/2}(\log R)^{1/p_0}$$

holds.

Proof. We will use the decomposition in Lemma 13. By using (9) and (10) as was done in [5, Lemma 2.1] we obtain that

$$\left\|R^{2(\nu+1)}\frac{J_{\nu+\delta+1}(yR)}{(yR)^{\nu+\delta+1}}\right\|_{L^{p_0}((0,1),d\mu_\nu)} \ge CR^{\nu-\delta+1/2}(\log R)^{1/p_0}.$$

With the bound (37) it can be deduced that

$$\left\|\mathcal{I}_{R,2}^{\delta}(0,y)\right\|_{L^{p_0}((0,1),d\mu_\nu)} \le CR^{\nu-\delta+1/2}.$$

With the previous estimates the proof is completed. □

Finally, the last element that we need to prove Theorems 6 and 7 is the norm inequality for finite linear combinations of the functions $\{\psi_j\}_{j\ge1}$ contained in the next lemma. Its proof is long and technical and it will be done in the last section.

Lemma 15. *For $\nu > -1/2$, $R > 0$, $1 < p < \infty$ and f a linear combination of the functions $\{\psi_j\}_{1\le j\le N(R)}$ with $N(R)$ a positive integer such that $N(R) \simeq R$, the inequality*

$$\|f\|_{L^\infty((0,1),d\mu_\nu)} \le CR^{2(\nu+1)/p}\|f\|_{L^{p,\infty}((0,1),d\mu_\nu)}$$

holds.

Proof of Theorem 6. With the bound in Lemma 14 we have

$$\begin{aligned}(\log R)^{1/p_0} &\le CR^{-2(\nu+1)/p_1}\left\|\mathcal{K}_R^\delta(0,y)\right\|_{L^{p_0}((0,1),d\mu_\nu)}\\ &= CR^{-2(\nu+1)/p_1}\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1}\left|\int_0^1 \mathcal{K}_R^\delta(0,y)f(y)\,d\mu_\nu\right|\\ &= CR^{-2(\nu+1)/p_1}\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1}\left|\mathcal{B}_R^\delta f(0)\right|.\end{aligned}$$

From the previous estimate the result for $\delta=\nu+1/2$ follows. In the case $\delta<\nu+1/2$ it is obtained by using Lemma 15 because

$$R^{-2(\nu+1)/p_1}\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1}\left|\mathcal{B}_R^\delta f(0)\right| \\ \leq C\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1}\left\|\mathcal{B}_R^\delta f(x)\right\|_{L^{p_1,\infty}((0,1),d\mu_\nu)}$$

since $\mathcal{B}_R^\delta f(x)$ is a linear combination of the functions $\{\psi_j\}_{1\leq j\leq N(R)}$ with $N(R)\simeq R$. $\square$

Proof of Theorem 7. In the case $\delta<\nu+1/2$, the result follows from Theorem 6 by using a duality argument. Indeed, it is clear that

$$\begin{aligned}\sup_{E\subset(0,1)}\frac{\|\mathcal{B}_R^\delta\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}{\|\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}&=\sup_{E\subset(0,1)}\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1}\frac{\left|\int_0^1 f(y)\mathcal{B}_R^\delta\chi_E(y)\,d\mu_\nu\right|}{\|\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}\\ &=\sup_{\|f\|_{L^{p_1}((0,1),d\mu_\nu)}=1}\sup_{E\subset(0,1)}\frac{\left|\int_0^1 \chi_E(y)\mathcal{B}_R^\delta f(y)\,d\mu_\nu\right|}{\|\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}.\end{aligned}\tag{38}$$

By Theorem 6 it is possible to choose a function g such that $\|g\|_{L^{p_1}((0,1),d\mu_\nu)}=1$ and

$$\|\mathcal{B}_R^\delta g(x)\|_{L^{p_1,\infty}((0,1),d\mu_\nu)}\geq C(\log R)^{1/p_0}.$$

Then, with the notation

$$\mu_\nu(E)=\int_E d\mu_\nu,$$

we have

$$\lambda^{p_1}\mu_\nu(A)\geq C(\log R)^{p_1/p_0},\tag{39}$$

for some positive λ and $A=\{x\in(0,1):|B_R^\delta g(x)|>\lambda\}$. Now, we consider the subsets of A

$$A_1=\{x\in(0,1):B_R^\delta g(x)>\lambda\}\qquad\text{and}\qquad A_2=\{x\in(0,1):B_R^\delta g(x)<-\lambda\}$$

and we define $D=A_1$ if $\mu_\nu(A_1)\geq\mu_\nu(A)/2$ and $D=A_2$ otherwise. Then, by (39), we deduce that

$$\lambda\geq C\frac{(\log R)^{1/p_0}}{\mu_\nu(D)^{1/p_1}}.\tag{40}$$

Taking $f=g$ and $E=D$ in (38) and using (40), we see that

$$\sup_{E\subset(0,1)}\frac{\|\mathcal{B}_R^\delta\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}{\|\chi_E\|_{L^{p_0}((0,1),d\mu_\nu)}}\geq C\lambda\frac{\mu_\nu(D)}{\|\chi_D\|_{L^{p_0}((0,1),d\mu_\nu)}}\geq C(\log R)^{1/p_0}$$

and the proof is complete in this case. For $\delta=\nu+1/2$ the result follows from Theorem 6 with a standard duality argument. $\square$

7. Proof of Lemma 15

To proceed with the proof of Lemma 15 we need some auxiliary results that are included in this section.

We start by defining a new operator. For each non-negative integer r, we consider the vector of coefficients $\alpha = (\alpha_1, \dots, \alpha_{r+1})$ and we define

$$T_{r,R,\alpha} f(x) = \sum_{\ell=1}^{r+1} \alpha_\ell \mathcal{B}^r_{\ell R} f(x).$$

This new operator is an analogous of the *generalized delayed means* considered in [16]. In [16] the operator is defined in terms of the Cesàro means instead of the Bochner-Riesz means. The properties of $T_{r,R,\alpha}$ that we need are summarized in the next lemma

Lemma 16. *For each non-negative integer r and $\nu \geq -1/2$, the following statements hold*

a) *$T_{r,R,\alpha} f$ is a linear combination of the functions $\{\psi_j\}_{1\leq j\leq N((r+1)R)}$, where $N((r+1)R)$ is a non-negative integer such that $N((r+1)R) \simeq (r+1)R$;*

b) *there exists a vector of coefficients α, verifying that $|\alpha_\ell| \leq A$, for $\ell = 1, \dots, r+1$, with A independent of R and such that $T_{r,R,\alpha} f(x) = f(x)$ for each linear combination of the functions $\{\psi_j\}_{1\leq j\leq N(R)}$ where $N(R)$ is a positive integer. Moreover, in this case, for $r > \nu + 1/2$,*

$$\|Tf_{r,R,\alpha}\|_{L^1((0,1),d\mu_\nu)} \leq C\|f\|_{L^1((0,1),d\mu_\nu)}$$

and

$$\|T_{r,R,\alpha} f\|_{L^\infty((0,1),d\mu_\nu)} \leq C\|f\|_{L^\infty((0,1),d\mu_\nu)},$$

with C independent of R and f.

Proof. Part a) is a consequence of the definition of $T_{r,R,\alpha}$ and the fact that the m-th zero of the Bessel function J_ν, with $\nu \geq -1/2$, is contained in the interval $(m\pi + \nu\pi/2 + \pi/2, m\pi + \nu\pi/2 + 3\pi/4)$.

To prove b) we consider $f(x) = \sum_{j=1}^{N(R)} a_j \psi_j(x)$. In order to obtain the vector of coefficients such that $T_{r,R,\alpha} f(x) = f(x)$ the equations

$$\sum_{\ell=1}^{r+1} \alpha_\ell \left(1 - \frac{s_k^2}{(\ell R)^2}\right)^r = 1,$$

for all $k = 1, \dots, N(R)$, should be verified. After some elementary manipulations each one of the previous equations can be written as

$$\sum_{j=0}^{r} s_k^{2j} \binom{r}{j} \frac{(-1)^j}{R^{2j}} \sum_{\ell=1}^{r+1} \frac{\alpha_\ell}{\ell^{2j}} = 1$$

and this can be considered as a polynomial in s_k^2 which must be equal 1, therefore we have the system of equations

$$\sum_{\ell=1}^{r+1} \frac{\alpha_\ell}{\ell^{2j}} = \delta_{j,0}, \qquad j = 0, \dots, r.$$

This system has an unique solution because the determinant of the matrix of coefficients is a Vandermonde's one. Of course for each $\ell = 1, \dots, r+1$, it is verified that $|\alpha_\ell| \leq A$, with A a constant depending on r but not on $N(R)$.

The norm estimates are consequence of the uniform boundedness

$$\|\mathcal{B}_R^{\delta} f\|_{L^p((0,1),d\mu_\nu)} \le C\|f\|_{L^p((0,1),d\mu_\nu)},$$

for $p = 1$ and $p = \infty$ when $\delta > \nu + 1/2$ (see [4]). □

In the next lemma we will control the L^∞-norm of a finite linear combination of the functions $\{\psi_j\}_{j\ge 1}$ by its L^1-norm.

Lemma 17. *If $\nu > -1/2$ and $f(x)$ is a linear combination of the functions $\{\psi_j\}_{1\le j\le N(R)}$ with $N(R)$ a positive integer such that $N(R) \simeq R$, the inequality*

$$\|f\|_{L^\infty((0,1),d\mu_\nu)} \le CR^{2(\nu+1)}\|f\|_{L^1((0,1),d\mu_\nu)}$$

holds.

Proof. It is clear that

$$f(x) = \sum_{j=1}^{N(R)} \psi_j(x) \int_0^1 f(y)\psi_j(y)\,d\mu_\nu(y).$$

Now, using Hölder inequality and Lemma 1 we have

$$\begin{aligned}\|f\|_{L^\infty((0,1),d\mu_\nu)} &\le C \sum_{j=1}^{N(R)} \|\psi_j\|^2_{L^\infty((0,1),d\mu_\nu)}\|f\|_{L^1((0,1),d\mu_\nu)}\\ &\le C\|f\|_{L^1((0,1),d\mu_\nu)} \sum_{j=1}^{N(R)} j^{2\nu+1} \le CR^{2(\nu+1)}\|f\|_{L^1((0,1),d\mu_\nu)}.\end{aligned}$$

□

The following lemma is a version in the space $((0,1), d\mu_\nu)$ of Lemma 19.1 in [3]. The proof can be done in the same way, with the appropriate changes, so we omit it.

Lemma 18. *Let $\nu > -1$, $1 < p < \infty$ and T be a linear operator defined for functions in $L^1((0,1), d\mu_\nu)$ and such that*

$$\|Tf\|_{L^\infty((0,1),d\mu_\nu)} \le A\|f\|_{L^1((0,1),d\mu_\nu)} \text{ and } \|Tf\|_{L^\infty((0,1),d\mu_\nu)} \le B\|f\|_{L^\infty((0,1),d\mu_\nu)},$$

then

$$\|Tf\|_{L^\infty((0,1),d\mu_\nu)} \le CA^{1/p}B^{1/p'}\|f\|_{L^{p,\infty}((0,1),d\mu_\nu)}.$$

Now, we are prepared to conclude the proof of Lemma 15.

Proof of Lemma 15. We consider the operator $T_{r,R,\alpha}f$ given in Lemma 16 b) with $r > \nu + 1/2$. By Lemma 16 and Lemma 17 we have

$$\begin{aligned}\|T_{r,R,\alpha}f\|_{L^\infty((0,1),d\mu_\nu)} &\le C((r+1)R)^{2(\nu+1)}\|T_{r,R,\alpha}f\|_{L^1((0,1),d\mu_\nu)}\\ &\le CR^{2(\nu+1)}\|f\|_{L^1((0,1),d\mu_\nu)}.\end{aligned}$$

From b) in Lemma 16 we obtain the estimate

$$\|T_{r,R,\alpha}f\|_{L^\infty((0,1),d\mu_\nu)} \le C\|f\|_{L^\infty((0,1),d\mu_\nu)}.$$

So, by using Lemma 18, we obtain the inequality

$$\|T_{r,R,\alpha}f\|_{L^\infty((0,1),d\mu_\nu)} \le CR^{2(\nu+1)/p}\|f\|_{L^{p,\infty}((0,1),d\mu_\nu)}$$

for any $f \in L^1((0,1), d\mu_\nu)$. Now, since $T_{r,R,\alpha}f(x) = f(x)$ for a linear combination of the functions $\{\psi_j\}_{1\le j\le N(R)}$, the proof is complete. $\square$

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