# Approximation in Hermite spaces of smooth functions

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#### Abstract

We consider  $\mathbb{L}_2$ -approximation of elements of a Hermite space of analytic functions over  $\mathbb{R}^s$ . The Hermite space is a weighted reproducing kernel Hilbert space of real valued functions for which the Hermite coefficients decay exponentially fast. The weights are defined in terms of two sequences  $\boldsymbol{a} = \{a_j\}$  and  $\boldsymbol{b} = \{b_j\}$  of positive real numbers. We study the *n*th minimal worst-case error  $e(n, \text{APP}_s; \Lambda^{\text{std}})$  of all algorithms that use *n* information evaluations from the class  $\Lambda^{\text{std}}$  which only allows function evaluations to be used.

We study (uniform) exponential convergence of the *n*th minimal worst-case error, which means that  $e(n, \text{APP}_s; \Lambda^{\text{std}})$  converges to zero exponentially fast with increasing *n*. Furthermore, we consider how the error depends on the dimension *s*. To this end, we study the minimal number of information evaluations needed to compute an  $\varepsilon$ -approximation by considering several notions of tractability which are defined with respect to *s* and  $\log \varepsilon^{-1}$ . We derive necessary and sufficient conditions on the sequences *a* and *b* for obtaining exponential error convergence, and also for obtaining the various notions of tractability. It turns out that the conditions on the weight sequences are almost the same as for the information class  $\Lambda^{\text{all}}$  which uses all linear functionals. The results are also constructive as the considered algorithms are based on tensor products of Gauss-Hermite rules for multivariate integration. The obtained results are compared with the analogous results for integration in the same Hermite space. This allows us to give a new sufficient condition for EC-weak tractability for integration.

Keywords: Multivariate Approximation, Exponential Convergence, Tractability, Hermite spaces

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# 1 Introduction

In this paper we study  $\mathbb{L}_2$ -approximation of functions belonging to a certain reproducing kernel Hilbert space  $\mathcal{H}(K_s)$  of s-variate functions defined on  $\mathbb{R}^s$  with reproducing kernel

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 $K_s : \mathbb{R}^s \times \mathbb{R}^s \to \mathbb{R}$ . We are interested in approximating the embedding operators  $APP_s : \mathcal{H}(K_s) \to \mathbb{L}_2(\mathbb{R}^s, \varphi_s)$  with

$$\operatorname{APP}_{s}(f) = f,$$

where  $\varphi_s$  denotes the density of the s-dimensional standard Gaussian measure.

We consider the worst-case setting. In this case it follows from general results on information-based complexity, see, e.g., [21] or [15, Section 4], that linear algorithms are optimal. So we approximate APP<sub>s</sub> by a linear algorithm  $A_{n,s}$  using *n* information evaluations either from the class  $\Lambda^{\text{std}}$  of standard information which consists of only function evaluations or from the class  $\Lambda^{\text{all}}$  of all continuous linear functionals. That is,

$$A_{n,s}(f) = \sum_{j=1}^{n} \alpha_j L_j(f)$$
 for all  $f \in \mathcal{H}(K_s)$ ,

where  $L_j$  belongs to the dual space of  $\mathcal{H}(K_s)$ , i.e.,  $L_j \in \mathcal{H}(K_s)^*$ , for the class  $\Lambda^{\text{all}}$ , whereas  $L_j(f) = f(\boldsymbol{x}_j)$  for all  $f \in \mathcal{H}_s$ , with  $\boldsymbol{x}_j \in \mathbb{R}^s$  for the class  $\Lambda^{\text{std}}$ , and  $\alpha_j \in \mathbb{L}_2(\mathbb{R}^s, \varphi_s)$  for all  $j = 1, 2, \ldots, n$ . Since  $\mathcal{H}(K_s)$  is a reproducing kernel Hilbert space we obviously have  $\Lambda^{\text{std}} \subseteq \Lambda^{\text{all}}$ . In this paper we will mainly concentrate on the approximation problem with respect to the class  $\Lambda^{\text{std}}$ , because the problem for the class  $\Lambda^{\text{all}}$  is covered by [8].

We measure the error of an algorithm  $A_{n,s}$  in terms of the *worst-case error*, which is defined as

$$e^{\operatorname{app}}(\mathcal{H}(K_s), A_{n,s}) := \sup_{\substack{f \in \mathcal{H}(K_s) \\ \|f\|_{K_s} \le 1}} \left\|\operatorname{APP}_s(f) - A_{n,s}(f)\right\|_{\mathbb{L}_2}, \tag{1}$$

where  $\|\cdot\|_{K_s}$  denotes the norm in  $\mathcal{H}(K_s)$ , and  $\|\cdot\|_{\mathbb{L}_2}$  denotes the norm in  $\mathbb{L}_2(\mathbb{R}, \varphi_s)$  which is given by

$$\|g\|_{\mathbb{L}_2} = \left(\int_{\mathbb{R}^s} |g(\boldsymbol{x})|^2 \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}\right)^{1/2} \quad \text{for } g \in \mathbb{L}_2(\mathbb{R}, \varphi_s).$$

The *nth minimal (worst-case) error* is given by

$$e(n, APP_s; \Lambda) := \inf_{A_{n,s}} e^{app}(\mathcal{H}(K_s), A_{n,s}),$$
(2)

where the infimum is taken over all admissible algorithms  $A_{n,s}$  using information from the class  $\Lambda \in {\Lambda^{\text{all}}, \Lambda^{\text{std}}}$ .

For n = 0, we consider algorithms that do not use any information evaluation, and therefore we use  $A_{0,s} \equiv 0$ . The error of  $A_{0,s}$  is called the *initial (worst-case) error* and is given by

$$e(0, \operatorname{APP}_{s}) := \sup_{\substack{f \in \mathcal{H}(K_{s}) \\ \|f\|_{K_{s}} \leq 1}} \left\|\operatorname{APP}_{s}(f)\right\|_{\mathbb{L}_{2}} = \left\|\operatorname{APP}_{s}\right\|.$$
(3)

When studying algorithms  $A_{n,s}$ , we do not only want to control how their errors depend on n, but also how they depend on the dimension s. This is of particular importance for high-dimensional problems. To this end, we define, for  $\varepsilon \in (0,1)$  and  $s \in \mathbb{N}$ , the *information complexity* by

$$n(\varepsilon, \operatorname{APP}_s; \Lambda) := \min \left\{ n : e(n, \operatorname{APP}_s; \Lambda) \le \varepsilon \right\}$$

as the minimal number of information evaluations needed to obtain an  $\varepsilon$ -approximation to APP<sub>s</sub>. In this case, we speak of the *absolute error criterion*. Alternatively, we can also define the information complexity as

$$n(\varepsilon, APP_s; \Lambda) := \min \{ n : e(n, APP_s; \Lambda) \le \varepsilon e(0, APP_s) \},\$$

i.e., as the minimal number of information evaluations needed to reduce the initial error by a factor of  $\varepsilon$ . In this case we speak of the *normalized error criterion*.

The specific problem considered in this paper has the convenient property that the initial error is one, and the absolute and normalized error criteria coincide.

### **1.1** Exponential convergence and tractability

Since the particular weighted function space we are going to define in Section 1.2 is such that its elements are infinitely many times differentiable and even analytic, it is natural to expect that the *n*th minimal error converges to zero very quickly as *n* increases. Indeed, we would like to achieve exponential convergence of the *n*th minimal errors, and we first define this type of convergence in detail.

By exponential convergence we mean that there exist functions  $q : \mathbb{N} = \{1, 2, \ldots\} \rightarrow (0, 1)$  and  $p, C : \mathbb{N} \rightarrow (0, \infty)$  such that

$$e(n, APP_s; \Lambda) \le C(s) q(s)^{n^{p(s)}}$$
 for all  $s, n \in \mathbb{N}$ .

Obviously, the functions  $q(\cdot)$  and  $p(\cdot)$  are not uniquely defined. For instance, we can take an arbitrary number  $q \in (0, 1)$ , define the function  $C_1$  as

$$C_1(s) = \left(\frac{\log q}{\log q(s)}\right)^{1/p(s)}$$

and then

$$C(s) q(s)^{n^{p(s)}} = C(s) q^{(n/C_1(s))^{p(s)}}.$$

We prefer to work with the latter bound which was also considered in [4, 7, 10].

**Definition 1.** We say that we achieve exponential convergence (EXP) if there exist a number  $q \in (0, 1)$  and functions  $p, C, C_1 : \mathbb{N} \to (0, \infty)$  such that

$$e(n, APP_s; \Lambda) \le C(s) q^{(n/C_1(s))^{p(s)}} \quad \text{for all} \quad s, n \in \mathbb{N}.$$
(4)

If (4) holds, then the largest possible rate of exponential convergence is defined as

$$p^*(s) = \sup\{ p \in (0,\infty) : \exists C, C_1 : \mathbb{N} \to (0,\infty) \text{ such that} \\ \forall n \in \mathbb{N} : e(n, \operatorname{APP}_s; \Lambda) \le C(s)q^{(n/C_1(s))^p} \}.$$

**Definition 2.** We say that we achieve *uniform exponential convergence* (UEXP) if the function p in (4) can be taken as a constant function, i.e., p(s) = p > 0 for all  $s \in \mathbb{N}$ . Furthermore, let

$$p^* = \sup\{p \in (0,\infty) : \exists C, C_1 : \mathbb{N} \to (0,\infty) \text{ such that} \\ \forall n, s \in \mathbb{N} : e(n, \operatorname{APP}_s; \Lambda) \le C(s)q^{(n/C_1(s))^p}\}$$

denote the largest rate of uniform exponential convergence.

We note, see [4, 5], that if (4) holds and  $e(0, APP_s) = 1$  then

$$n(\varepsilon, \operatorname{APP}_{s}; \Lambda) \leq \left\lceil C_{1}(s) \left( \frac{\log C(s) + \log \varepsilon^{-1}}{\log q^{-1}} \right)^{1/p(s)} \right\rceil \quad \text{for all } s \in \mathbb{N} \text{ and } \varepsilon \in (0, 1).$$
(5)

Conversely, if (5) holds then

 $e(n+1, \operatorname{APP}_s; \Lambda) \le C(s) q^{(n/C_1(s))^{p(s)}}$  for all  $s, n \in \mathbb{N}$ .

This means that (4) and (5) are practically equivalent. Note that 1/p(s) determines the power of log  $\varepsilon^{-1}$  in the information complexity, whereas log  $q^{-1}$  only affects the multiplier of  $\log^{1/p(s)} \varepsilon^{-1}$ . From this point of view, p(s) is more important than q.

From (5) we learn that exponential convergence implies that asymptotically, with respect to  $\varepsilon$  tending to zero, we need  $\mathcal{O}(\log^{1/p(s)} \varepsilon^{-1})$  information evaluations to obtain an  $\varepsilon$  approximation. However, it is not clear how long we have to wait to see this nice asymptotic behavior especially for large s. This, of course, depends on how  $C(s), C_1(s)$ and p(s) depend on s, and this is the subject of tractability. The following tractability notions were already considered in [4, 5, 7, 8, 10, 11]. The nomenclature was introduced in [11]. In this paper we define  $\log 0 = 0$  for convention.

**Definition 3.** We say that we have:

(a) Exponential Convergence-Weak Tractability (EC-WT) if

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \operatorname{APP}_s; \Lambda)}{s+\log \varepsilon^{-1}} = 0.$$

(b) Exponential Convergence-Polynomial Tractability (EC-PT) if there exist non-negative numbers  $c, \tau_1, \tau_2$  such that

$$n(\varepsilon, \operatorname{APP}_s; \Lambda) \le c \, s^{\,\tau_1} \, (1 + \log \, \varepsilon^{-1})^{\,\tau_2}$$
 for all  $s \in \mathbb{N}, \, \varepsilon \in (0, 1).$ 

(c) Exponential Convergence-Strong Polynomial Tractability (EC-SPT) if there exist nonnegative numbers c and  $\tau$  such that

$$n(\varepsilon, \operatorname{APP}_s; \Lambda) \leq c (1 + \log \varepsilon^{-1})^{\tau}$$
 for all  $s \in \mathbb{N}, \varepsilon \in (0, 1)$ .

The exponent  $\tau^*$  of EC-SPT is defined as the infimum of  $\tau$  for which the above relation holds.

EC-WT means that we rule out the cases for which  $n(\varepsilon, \text{APP}_s; \Lambda)$  depends exponentially on s and log  $\varepsilon^{-1}$ . EC-PT means that the information complexity depends at most polynomially on s and log  $\varepsilon^{-1}$  whereas EC-SPT means that  $n(\varepsilon, \text{APP}_s; \Lambda)$  is bounded at most polynomially in log  $\varepsilon^{-1}$ , independently of s.

We remark that in many papers tractability has been studied for problems where we do not have exponential but usually polynomial error convergence. For this kind of problems, tractability has been defined by studying how the information complexity depends on sand  $\varepsilon^{-1}$ , for a detailed survey of such results we refer to [15, 16, 17]. With the notions of EC-tractability considered in [4, 5, 7, 10, 11] and in the present paper, however, we study how the information complexity depends on s and log  $\varepsilon^{-1}$ . We remark that log  $\varepsilon^{-1}$  also corresponds to the number of bits of desired accuracy, cf. [18].

We collect some well-known relations:

**Proposition 1.** We have:

- (i) EC-SPT  $\Rightarrow$  EC-PT  $\Rightarrow$  EC-WT.
- (ii) EC-PT (and therefore also EC-SPT) implies UEXP.
- (iii) EC-WT implies that  $e(n, APP_s; \Lambda)$  converges to zero faster than any power of  $n^{-1}$ as n goes to infinity, i.e.,

$$\lim_{n \to \infty} n^{\alpha} e(n, \operatorname{APP}_s; \Lambda) = 0 \quad \text{for all } \alpha \in \mathbb{R}^+ \text{ and all } s \in \mathbb{N}.$$

(iv) If we have UEXP,  $e(n, APP_s; \Lambda) \leq C(s) q^{(n/C_1(s))^p}$ , then:

- $C(s) = \exp(\exp(o(s)))$  and  $C_1(s) = \exp(o(s)) \Rightarrow \text{EC-WT};$
- $C(s) = \exp(\mathcal{O}(s^{\vartheta}))$  and  $C_1(s) = \mathcal{O}(s^{\eta})$  for some  $\vartheta, \eta > 0 \Rightarrow \text{EC-PT};$
- $C(s) = \mathcal{O}(1)$  and  $C_1(s) = \mathcal{O}(1) \Rightarrow \text{EC-SPT}.$

*Proof.* (i) is clear. A proof of (ii) can be found in [4, 11] and (iii) and (iv) are shown in [11].  $\Box$ 

Of course Point (ii) of Proposition 1 is the motivation for the use of the prefix EC (exponential convergence) in our notation.

The goal of this paper is to find relations between the concepts EXP, UEXP, and the various tractability notions, as well as necessary and sufficient conditions on the weights of the considered function space for which these concepts hold, mostly for the class  $\Lambda^{\text{std}}$ .

## **1.2** Hermite spaces with infinite smoothness

We briefly summarize some facts on *Hermite polynomials*; for further details, we refer to [9] and the references therein. For  $k \in \mathbb{N}_0 = \{0, 1, 2, ...\}$  the kth Hermite polynomial is given by

$$H_k(x) = \frac{(-1)^k}{\sqrt{k!}} \exp(x^2/2) \frac{\mathrm{d}^k}{\mathrm{d}x^k} \exp(-x^2/2),$$

which is sometimes also called normalized probabilistic Hermite polynomial. Here we follow the definition given in [2], but we remark that there are slightly different ways to introduce Hermite polynomials, see, e.g., [20]. For  $s \ge 2$ ,  $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$  and  $\mathbf{x} = (x_1, \ldots, x_s) \in \mathbb{R}^s$ , we define s-dimensional Hermite polynomials by

$$H_{\boldsymbol{k}}(\boldsymbol{x}) = \prod_{j=1}^{s} H_{k_j}(x_j).$$

It is well known, see again [2], that the sequence of Hermite polynomials  $\{H_k\}_{k \in \mathbb{N}_0^s}$  forms an orthonormal basis of the function space  $\mathbb{L}_2(\mathbb{R}^s, \varphi_s)$ , where  $\varphi_s$  denotes the density of the *s*-dimensional standard Gaussian measure,

$$arphi_s(oldsymbol{x}) = rac{1}{(2\pi)^{s/2}} \exp\left(-rac{1}{2}\,oldsymbol{x}\cdotoldsymbol{x}
ight) \quad ext{for all }oldsymbol{x}\in\mathbb{R}^s,$$

where "." is the standard Euclidean inner product in  $\mathbb{R}^s$ . We write  $\varphi := \varphi_1$ .

Similarly to what has been done in [9], we are now going to define function spaces based on Hermite polynomials. These spaces are Hilbert spaces with a *reproducing kernel*. For details on reproducing kernel Hilbert spaces, we refer to [1].

Let  $r: \mathbb{N}_0^s \to \mathbb{R}^+$  be a summable function, i.e.,  $\sum_{k \in \mathbb{N}_0^s} r(k) < \infty$ . Define a kernel function

$$K_r(oldsymbol{x},oldsymbol{y}) = \sum_{oldsymbol{k} \in \mathbb{N}_0^s} r(oldsymbol{k}) H_{oldsymbol{k}}(oldsymbol{x}) H_{oldsymbol{k}}(oldsymbol{y}) \quad ext{ for } \quad oldsymbol{x},oldsymbol{y} \in \mathbb{R}^s$$

and an inner product

$$\langle f,g 
angle_{K_r} = \sum_{oldsymbol{k} \in \mathbb{N}_0^s} \frac{1}{r(oldsymbol{k})} \widehat{f}(oldsymbol{k}) \widehat{g}(oldsymbol{k})$$

where

$$\widehat{f}(\boldsymbol{k}) = \int_{\mathbb{R}^s} f(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

is the kth Hermite coefficient of f. Note that  $K_r(x, y)$  is well defined for all  $x, y \in \mathbb{R}^s$ , since

$$|K_r(\boldsymbol{x}, \boldsymbol{y})| \leq \sum_{\boldsymbol{k}} r(\boldsymbol{k}) |H_{\boldsymbol{k}}(\boldsymbol{x})| |H_{\boldsymbol{k}}(\boldsymbol{y})| \leq \frac{1}{\sqrt{\varphi_s(\boldsymbol{x})\varphi_s(\boldsymbol{y})}} \sum_{\boldsymbol{k}} r(\boldsymbol{k}) < \infty,$$

since Cramer's bound for Hermite polynomials, see, e.g., [19, p. 324], states that

$$|H_k(x)| \le \frac{1}{\sqrt{\varphi(x)}}$$
 for all  $k \in \mathbb{N}_0$ .

Let  $\mathcal{H}(K_r)$  be the reproducing kernel Hilbert space corresponding to  $K_r$ , which we will call a Hermite space. The norm in  $\mathcal{H}(K_r)$  is given by  $||f||_{K_r}^2 = \langle f, f \rangle_{K_r}$ . From this we see that the functions in  $\mathcal{H}(K_r)$  are characterized by the decay rate of their Hermite coefficients, which is regulated by the function r. Roughly speaking, the faster r decreases as  $\mathbf{k}$  grows, the faster the Hermite coefficients of the elements of  $\mathcal{H}(K_r)$  decrease. In [9], the case of polynomially decreasing r as well as exponentially decreasing r was considered. In [7] further results were obtained for numerical integration for exponentially decreasing r, and in [8] for approximation using information from  $\Lambda^{\text{all}}$ . In this paper, we continue the work on exponentially decreasing r for approximation using information from  $\Lambda^{\text{std}}$ , thereby extending the results of [7, 8, 9].

To define our function r, we first introduce two weight sequences of positive real numbers,  $\boldsymbol{a} = \{a_i\}$  and  $\boldsymbol{b} = \{b_i\}$  such that

$$0 < a_1 \le a_2 \le \cdots$$
 and  $b_* := \inf_j b_j \ge 1$ .

Furthermore, we fix a parameter  $\omega \in (0, 1)$ . For a vector  $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ , we consider

$$r(\boldsymbol{k}) = \omega_{\boldsymbol{k}} := \omega^{\sum_{j=1}^{s} a_j k_j^{b_j}}.$$

For simplicity we assume without loss of generality that  $a_1 \ge 1$ , because we can always modify  $\omega$  in such a way that  $a_1$  is greater than or equal to 1.

We modify the notation for the kernel function to

$$K_r(\boldsymbol{x}, \boldsymbol{y}) = K_{s, \boldsymbol{a}, \boldsymbol{b}, \omega}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \omega_{\boldsymbol{k}} H_{\boldsymbol{k}}(\boldsymbol{x}) H_{\boldsymbol{k}}(\boldsymbol{y}).$$

From now on, we deal with the corresponding reproducing kernel Hilbert space  $\mathcal{H}(K_{s,a,b,\omega})$ . Our concrete choice of r now decreases exponentially fast as  $\mathbf{k}$  grows, which influences the smoothness of the elements in  $\mathcal{H}(K_{s,a,b,\omega})$ . Indeed, if  $b_* \geq 1$  it can be shown that functions  $f \in \mathcal{H}(K_{s,a,b,\omega})$  are analytic, see [7]. More precisely, we have that for all  $\mathbf{x} \in \mathbb{R}^s$ the Taylor expansion of f centered at  $\mathbf{x}$  converges in a ball with radius  $\rho(\omega) > 0$  around  $\mathbf{x}$ . It can also be shown that this radius  $\rho(\omega)$  is independent of  $\mathbf{x}$  and  $\lim_{\omega \to 0} \rho(\omega) = \infty$ and  $\lim_{\omega \to 1} \rho(\omega) = 0$ .

**Remark 1.** Apparently the assumption  $b_* \geq 1$  has technical reasons, see, e.g., the footnote on page 15, and pages 17 and 25. However, the assumption  $b_* \geq 1$  is also essential in showing that functions  $f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  are analytic, see [7]. For the moment it must remain an open question whether our results are also correct if  $b_* \in (0,1)$ . However, in the case  $b_* \in (0,1)$ , we can show that functions  $f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  belong to the Gevrey class of index  $1/b_*$ , which is work in progress.

We remark that reproducing kernel Hilbert spaces of a similar flavor were previously studied in [4, 5, 10, 11], but the functions considered there were one-periodic functions defined on the unit cube  $[0,1]^s$ . Here, we study functions which are defined on the  $\mathbb{R}^s$ , which is a major difference. Obviously,  $\mathcal{H}(K_{s,a,b,\omega})$  contains all polynomials on the  $\mathbb{R}^s$ , but there are further functions of practical interest which belong to such spaces. For example, it is easy to verify, see again [7], that  $f(\boldsymbol{x}) = \exp(\boldsymbol{\lambda} \cdot \boldsymbol{x})$  is an element of the Hilbert space  $\mathcal{H}(K_{s,a,1,\omega})$  for any weight sequence  $\boldsymbol{a}$  and any  $\boldsymbol{\lambda} \in \mathbb{R}^s$ . Functions of a similar form occur in problems of financial derivative pricing, see, e.g., [13].

Multivariate integration in  $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  has been studied in [7] and will be discussed further in Section 3 of this paper.

# 2 $\mathbb{L}_2$ -approximation in $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$

Let  $APP_s : \mathcal{H}(K_{s,a,b,\omega}) \to \mathbb{L}_2(\mathbb{R}^s, \varphi_s)$  with  $APP_s(f) = f$ . In order to approximate  $APP_s$  in the norm  $\|\cdot\|_{\mathbb{L}_2}$  we use linear algorithms  $A_{n,s}$ , which use *n* information evaluations and which are of the form

$$A_{n,s}(f) = \sum_{k=1}^{n} \alpha_k L_k(f) \quad \text{for } f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}),$$

where each  $\alpha_k$  is a function from  $\mathbb{L}_2(\mathbb{R}^s, \varphi_s)$  and each  $L_k$  is a continuous linear functional defined on  $\mathcal{H}(K_{s,a,b,\omega})$  from a permissible class  $\Lambda \in \{\Lambda^{\text{all}}, \Lambda^{\text{std}}\}$  of information.

The worst-case error  $e^{\text{app}}$  of an algorithm  $A_{n,s}$  is defined as in (1) and the *n*th minimal worst-case error for the information class  $\Lambda$  is given by (2). The initial error, defined by (3), is

$$e(0, \operatorname{APP}_{s}) = \|\operatorname{APP}_{s}\| = \sup_{\substack{f \in \mathcal{H}(K_{s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}}) \\ \|f\|_{K_{s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}} \leq 1}}} \|f\|_{\mathbb{L}_{2}} = \sup_{\substack{f \in \mathcal{H}(K_{s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}}) \\ \|f\|_{K_{s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}} \leq 1}}} \|f\|_{K_{s, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\omega}} \leq 1} = 1,$$

since we always have  $||f||_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}} \geq ||f||_{\mathbb{L}_2}$  and equality is obtained for the constant function 1 which certainly belongs to  $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$ . This means that the approximation problem is well normalized and that the absolute and the normalized error criteria coincide, i.e., the *information complexity* is

$$n(\varepsilon, \operatorname{APP}_s; \Lambda) := \min \{n : e(n, \operatorname{APP}_s; \Lambda) \le \varepsilon\}.$$

## 2.1 Results for $\mathbb{L}_2$ -approximation for the class $\Lambda^{\text{all}}$

 $\mathbb{L}_2$ -approximation for the class  $\Lambda^{\text{all}}$  defined over very general Hilbert spaces with exponential weights is discussed in [8]. Since the Hermite space  $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  with weight sequences  $\boldsymbol{a}$  and  $\boldsymbol{b}$  fits into the setting of [8], we know that the following results hold for the class  $\Lambda^{\text{all}}$ :

- 1. EXP holds for arbitrary **a** and **b** and  $p^*(s) = 1/B(s)$  with  $B(s) := \sum_{j=1}^{s} b_j^{-1}$ .
- 2. UEXP holds iff **a** is an arbitrary sequence and **b** such that  $B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty$ . If so then  $p^* = 1/B$ .
- 3. We have

EC-WT 
$$\Leftrightarrow \lim_{j \to \infty} a_j = \infty,$$
  
EC-PT  $\Leftrightarrow B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \text{ and } \alpha^* := \liminf_{j \to \infty} \frac{\log a_j}{j} > 0,$   
EC-SPT  $\Leftrightarrow B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty \text{ and } \alpha^* := \liminf_{j \to \infty} \frac{\log a_j}{j} > 0.$ 

Then the exponent  $\tau^*$  of EC-SPT satisfies  $\max\left(B, \frac{\log 2}{\alpha^*}\right) \leq \tau^* \leq B + \frac{\log 2}{\alpha^*}$ . In particular, we have EC-PT  $\Leftrightarrow$  EC-SPT.

4. The following notions are equivalent:

## **2.2** Results for $\mathbb{L}_2$ -approximation for the class $\Lambda^{\text{std}}$

We present the main results of this paper in the following theorem:

**Theorem 1.** Consider  $\mathbb{L}_2$ -approximation defined over the Hermite space  $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  with weight sequences  $\boldsymbol{a}$  and  $\boldsymbol{b}$  satisfying  $0 < a_1 \leq a_2 \leq a_3 \leq \ldots$  and  $\inf_j b_j \geq 1$ . The following results hold for the class  $\Lambda^{\text{std}}$ .

1. EXP holds for arbitrary  $\boldsymbol{a}$  and  $\boldsymbol{b}$  and

$$p^*(s) = \frac{1}{B(s)}$$
 with  $B(s) := \sum_{j=1}^s \frac{1}{b_j}$ .

2. UEXP holds iff a is an arbitrary sequence and b is such that

$$B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty.$$

If this is the case then  $p^* = 1/B$ .

- 3. We have
  - a. EC-WT iff  $\lim_{j\to\infty} a_j = \infty$ ,
  - b. EC-PT iff EC-SPT,
  - c. EC-SPT iff  $B := \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty$  and  $\alpha^* := \liminf_{j \to \infty} \frac{\log a_j}{j} > 0$ . Then the exponent  $\tau^*$  of EC-SPT satisfies

$$\max\left(B, \frac{\log 2}{\alpha^*}\right) \le \tau^* \le B + \frac{\log 3}{\alpha^*}.$$

In particular,  $\alpha^* = \infty$  implies  $\tau^* = B$ .

The results we achieve for the information class  $\Lambda^{\text{std}}$  match those for the class  $\Lambda^{\text{all}}$ , although the upper bound on the exponent of EC-SPT is slightly different. From Theorem 1 we see once more that EC-PT implies UEXP, cf. Proposition 1.

We cannot determine the exponent of EC-SPT exactly but we get an upper and a lower bound such that we know  $\tau^* \in [\max(B, (\log 2)/\alpha^*), B + (\log 3)/\alpha^*].$ 

The proof of Theorem 1 will be given in Section 2.4. First we collect some auxiliary results in the following section.

## 2.3 Auxiliary results for the proof of Theorem 1

### 2.3.1 Gauss-Hermite rules

A one-dimensional Gauss-Hermite rule of order n is a linear integration rule  $Q_n$  of the form

$$Q_n(f) = \sum_{i=1}^n \alpha_i f(x_i)$$

that is exact for all polynomials p of degree less than 2n,

$$\int_{\mathbb{R}} p(x)\varphi(x) \, \mathrm{d}x = \sum_{i=1}^{n} \alpha_i p(x_i)$$

The nodes  $x_1, \ldots, x_n \in \mathbb{R}$  are the zeros of the *n*th Hermite polynomial  $H_n$  and the integration weights  $\alpha_i$  are given by

$$\alpha_i = \frac{1}{nH_{n-1}^2(x_i)}$$
 for  $i = 1, 2, \dots, n$ ,

see [6]. We stress that the weights  $\alpha_i$  are all positive. The following lemma summarizes a few basic facts on Gauss-Hermite rules.

**Lemma 1.** Let  $n \in \mathbb{N}$ . Then we have:

- 1.  $Q_n(H_0) = \sum_{i=1}^n \alpha_i = 1;$ 2. for  $k \in \{1, 2, ..., 2n - 1\}$  we have  $Q_n(H_k) = 0;$
- 3. for  $k \in \{2n, 2n + 1, \ldots\}$  we have

$$|Q_n(H_k)| \le \begin{cases} \sqrt[4]{8\pi} & \text{if } k \text{ is even,} \\ 0 & \text{if } k \text{ is odd.} \end{cases}$$

*Proof.* See [7, Proof of Proposition 1].

For integration in the multivariate case, we use the tensor product of one-dimensional Gauss-Hermite rules. Let  $m_1, \ldots, m_s \in \mathbb{N}$  and let  $n = m_1 m_2 \cdots m_s$ . For  $j = 1, 2, \ldots, s$  let

$$Q_{m_j}^{(j)}(f) = \sum_{i=1}^{m_j} \alpha_i^{(j)} f(x_i^{(j)})$$

be one-dimensional Gauss-Hermite rules of order  $m_j$  with nodes  $x_1^{(j)}, \ldots, x_{m_j}^{(j)}$  and with weights  $\alpha_1^{(j)}, \ldots, \alpha_{m_j}^{(j)}$ , respectively. Then we apply the *s*-dimensional tensor product rule

$$Q_{n,s} = Q_{m_1}^{(1)} \otimes \cdots \otimes Q_{m_s}^{(s)},$$

i.e.,

$$Q_{n,s}(f) = \sum_{i_1=1}^{m_1} \dots \sum_{i_s=1}^{m_s} \alpha_{i_1}^{(1)} \cdots \alpha_{i_s}^{(s)} f(x_{i_1}^{(1)}, \dots, x_{i_s}^{(s)}).$$
(6)

By  $\mathcal{G}_{n,s}^{\perp}$  we denote the set

 $\mathcal{G}_{n,s}^{\perp} = \{ \boldsymbol{v} \in \mathbb{N}_0^s : \text{ for all } j = 1, \dots, s \text{ either } v_j = 0, \text{ or } v_j \ge 2m_j \text{ and } v_j \text{ even} \}.$ 

We will make use of the following result.

**Lemma 2.** Let  $Q_{n,s}$  be as in (6). For any g of the form

$$g(\boldsymbol{x}) = \sum_{\boldsymbol{v} \in \mathbb{N}_0^s} \widehat{g}(\boldsymbol{v}) H_{\boldsymbol{v}}(\boldsymbol{x}) \quad for \ all \ \ \boldsymbol{x} \in \mathbb{R}^s,$$

we have

$$\left|\int_{\mathbb{R}^s} g(\boldsymbol{x})\varphi_s(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} - Q_{n,s}(g)\right| \leq \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} |\widehat{g}(\boldsymbol{v})| (\sqrt[4]{8\pi})^{|\boldsymbol{v}|_*},$$

where we put  $|v|_* := |\{j : v_j \neq 0\}|$  for  $v = (v_1, ..., v_s) \in \mathbb{N}_0^s$ .

*Proof.* Using the results from Lemma 1 as well as the orthonormality of the Hermite polynomials we have

$$\begin{split} \left| \int_{\mathbb{R}^s} g(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - Q_{n,s}(g) \right| &= \left| \sum_{\boldsymbol{v} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} \widehat{g}(\boldsymbol{v}) Q_{n,s}(H_{\boldsymbol{v}}) \right| \\ &\leq \sum_{\boldsymbol{v} \in \mathbb{N}_0^s \setminus \{\boldsymbol{0}\}} |\widehat{g}(\boldsymbol{v})| \prod_{j=1}^s |Q_{m_j}^{(j)}(H_{v_j})| \\ &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^\perp \setminus \{\boldsymbol{0}\}} |\widehat{g}(\boldsymbol{v})| \, (\sqrt[4]{8\pi})^{|\boldsymbol{v}|_*}, \end{split}$$

as desired.

## **2.3.2** Error analysis in $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$

We proceed in a similar way as in [4, 12]. Let M > 1, and define

$$\mathcal{A}(s,M) := \left\{ \boldsymbol{h} \in \mathbb{N}_0^s : \omega_{\boldsymbol{h}}^{-1} < M \right\}.$$
(7)

For  $f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  and  $\boldsymbol{h} \in \mathbb{N}_0^s$  define

$$f_{\boldsymbol{h}}(\boldsymbol{x}) := f(\boldsymbol{x})H_{\boldsymbol{h}}(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^{s}.$$

We approximate  $f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  by algorithms of the form

$$A_{n,s,M}(f)(\boldsymbol{x}) = \sum_{\boldsymbol{h} \in \mathcal{A}(s,M)} Q_{n,s}(f_{\boldsymbol{h}}) H_{\boldsymbol{h}}(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in \mathbb{R}^{s},$$
(8)

where  $Q_{n,s}$  is a Gauss-Hermite rule of the form (6). The choice of M will be given below. Then we have

$$(f - A_{n,s,M}(f))(\boldsymbol{x}) = \sum_{\boldsymbol{h} \notin \mathcal{A}(s,M)} \widehat{f}(\boldsymbol{h}) H_{\boldsymbol{h}}(\boldsymbol{x}) + \sum_{\boldsymbol{h} \in \mathcal{A}(s,M)} (\widehat{f}(\boldsymbol{h}) - Q_{n,s}(f_{\boldsymbol{h}})) H_{\boldsymbol{h}}(\boldsymbol{x}).$$

Using Parseval's identity we obtain

$$\|f - A_{n,s,M}(f)\|_{\mathbb{L}_{2}}^{2} = \sum_{\boldsymbol{h} \notin \mathcal{A}(s,M)} |\widehat{f}(\boldsymbol{h})|^{2} + \sum_{\boldsymbol{h} \in \mathcal{A}(s,M)} |\widehat{f}(\boldsymbol{h}) - Q_{n,s}(f_{\boldsymbol{h}})|^{2}$$
$$= \sum_{\boldsymbol{h} \notin \mathcal{A}(s,M)} |\widehat{f}(\boldsymbol{h})|^{2} + \sum_{\boldsymbol{h} \in \mathcal{A}(s,M)} \left| \int_{\mathbb{R}^{s}} f_{\boldsymbol{h}}(\boldsymbol{x}) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - Q_{n,s}(f_{\boldsymbol{h}}) \right|^{2}.$$
(9)

We have

$$\sum_{\boldsymbol{h}\notin\mathcal{A}(s,M)} |\widehat{f}(\boldsymbol{h})|^2 = \sum_{\boldsymbol{h}\notin\mathcal{A}(s,M)} |\widehat{f}(\boldsymbol{h})|^2 \omega_{\boldsymbol{h}} \, \omega_{\boldsymbol{h}}^{-1} \le \frac{1}{M} \|f\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}}^2.$$
(10)

Now we estimate the second term in (9). Unfortunately, in general,  $f \in \mathcal{H}(K_{s,a,b,\omega})$ does not imply  $f_h \in \mathcal{H}(K_{s,a,b,\omega})$ . However, we can show the following result.

**Lemma 3.** The function  $f_h$  can be pointwise represented as a Hermite series

$$f_{\boldsymbol{h}}(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \widehat{f}_{\boldsymbol{h}}(\boldsymbol{k}) H_{\boldsymbol{k}}(\boldsymbol{x}) \qquad ext{for all } \boldsymbol{x} \in \mathbb{R}^s.$$

For technical reasons, we defer the proof of this lemma to the end of this subsection. With the help of Lemma 3 we can estimate the integration error of  $Q_{n,s}$  for functions of the form  $f_h$ .

**Lemma 4.** For f in the unit ball of  $\mathcal{H}(K_{s,a,b,\omega})$  and  $h \in \mathcal{A}(s, M)$  we have

$$\left| \int_{\mathbb{R}^s} f_{\boldsymbol{h}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - Q_{n,s}(f_{\boldsymbol{h}}) \right|^2$$
  
$$\leq 2^s \left( \prod_{j=1}^{\min(s,j(x_M))} \left\lceil \left( \frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right\rceil \right) M^K \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_*} \omega^{\frac{1}{2} \sum_{j=1}^s a_j (v_j/2)^{b_j}},$$

where

$$x_M := \frac{\log M}{\log \omega^{-1}},$$
  

$$j(x) := \sup\{j \in \mathbb{N} : x > a_j\},$$
  

$$K = K(\omega) := 3k - 1 + \frac{2\log(1 + \omega^k)}{\log \omega^{-1}}, \quad with \ k := \max\left(1, \left\lceil \frac{\log(\omega^{-1/8} - 1)}{\log \omega} \right\rceil\right).$$
(11)

Proof. According to Lemma 3 we can apply Lemma 2 to the second term in (9) and obtain

$$\left| \int_{\mathbb{R}^s} f_{\boldsymbol{h}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - Q_{n,s}(f_{\boldsymbol{h}}) \right|^2 \leq \left( \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} |\widehat{f}_{\boldsymbol{h}}(\boldsymbol{v})| (\sqrt[4]{8\pi})^{|\boldsymbol{v}|_*} \right)^2.$$
(12)

For fixed  $\boldsymbol{h} \in \mathcal{A}(s, M)$  and  $\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}$ , we have

$$\widehat{f}_{h}(\boldsymbol{v}) = \int_{\mathbb{R}^{s}} f(\boldsymbol{x}) H_{h}(\boldsymbol{x}) H_{\boldsymbol{v}}(\boldsymbol{x}) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$
$$= \int_{\mathbb{R}^{s}} f(\boldsymbol{x}) \left( \prod_{j=1}^{s} H_{h_{j}}(x_{j}) H_{v_{j}}(x_{j}) \right) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}.$$

Now we write the product of two Hermite polynomials as a linear combination of Hermite polynomials. To this end we write  $t_j = \min(v_j, h_j)$  and  $T_j = \max(v_j, h_j)$  for  $j \in \{1, \ldots, s\}$ . With this notation we have, using a result from [3, p. 1],

$$H_{h_j}(x_j)H_{v_j}(x_j) = \sum_{r_j=0}^{t_j} \left(\frac{t_j!}{T_j!}\right)^{1/2} \binom{T_j}{t_j - r_j} \frac{\left(\left(|h_j - v_j| + 2r_j\right)!\right)^{1/2}}{r_j!} H_{|h_j - v_j| + 2r_j}(x_j).$$

Hence

$$\widehat{f}_{h}(v) =$$

For  $j \in \{1, \ldots, s\}$ , and given  $v_j, h_j$ , and  $r_j$ , we now write

$$h_j \oplus_{r_j} v_j := |h_j - v_j| + 2r_j,$$

and by  $h \oplus_r v$  we denote the same operation applied component-wise to vectors. With this notation,

$$\widehat{f}_{\boldsymbol{h}}(\boldsymbol{v}) = \sum_{r_1=0}^{t_1} \cdots \sum_{r_s=0}^{t_s} \left( \prod_{j=1}^s \left( \frac{t_j!}{T_j!} \right)^{1/2} \binom{T_j}{t_j - r_j} \frac{((h_j \oplus_{r_j} v_j)!)^{1/2}}{r_j!} \right) \\ \times \int_{\mathbb{R}^s} f(\boldsymbol{x}) H_{\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}}(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\ = \sum_{r_1=0}^{t_1} \cdots \sum_{r_s=0}^{t_s} \left( \prod_{j=1}^s \left( \frac{t_j!}{T_j!} \right)^{1/2} \binom{T_j}{t_j - r_j} \frac{((h_j \oplus_{r_j} v_j)!)^{1/2}}{r_j!} \right) \widehat{f}(\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}).$$

Therefore, from (12),

$$\begin{split} \int_{\mathbb{R}^{s}} f_{h}(\boldsymbol{x})\varphi_{s}(\boldsymbol{x}) \,\mathrm{d}\boldsymbol{x} - Q_{n,s}(f_{h}) \Big|^{2} \\ &\leq \left( \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\mathbf{0}\}} (\sqrt[4]{8\pi})^{|\boldsymbol{v}|_{*}} \Big| \sum_{r_{1}=0}^{t_{1}} \cdots \sum_{r_{s}=0}^{t_{s}} \left( \prod_{j=1}^{s} \left( \frac{t_{j}!}{T_{j}!} \right)^{1/2} \binom{T_{j}}{t_{j} - r_{j}} \frac{((h_{j} \oplus_{r_{j}} v_{j})!)^{1/2}}{r_{j}!} \right) \\ &\qquad \times \widehat{f}(\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}) \Big| \right)^{2} \\ &\leq \left( \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\mathbf{0}\}} (\sqrt[4]{8\pi})^{|\boldsymbol{v}|_{*}} \sum_{r_{1}=0}^{t_{1}} \cdots \sum_{r_{s}=0}^{t_{s}} \left( \prod_{j=1}^{s} \left( \frac{t_{j}!}{T_{j}!} \right)^{1/2} \binom{T_{j}}{t_{j} - r_{j}} \frac{((h_{j} \oplus_{r_{j}} v_{j})!)^{1/2}}{r_{j}!} \right) \\ &\qquad \times |\widehat{f}(\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v})| \omega_{\boldsymbol{h}\oplus_{\boldsymbol{r}}\boldsymbol{v}}^{-1/2} \omega_{\boldsymbol{h}\oplus_{\boldsymbol{r}}\boldsymbol{v}}^{1/2} \right)^{2}. \end{split}$$

Using the Cauchy-Schwarz inequality we obtain

$$\left| \int_{\mathbb{R}^{s}} f_{\boldsymbol{h}}(\boldsymbol{x}) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - Q_{n,s}(f_{\boldsymbol{h}}) \right|^{2} \\ \leq \left( \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} \sum_{r_{1}=0}^{t_{1}} \cdots \sum_{r_{s}=0}^{t_{s}} |\widehat{f}(\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v})|^{2} \omega_{\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}}^{-1} \right) \\ \times \left( \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \sum_{r_{1}=0}^{t_{1}} \cdots \sum_{r_{s}=0}^{t_{s}} \left( \prod_{j=1}^{s} \frac{t_{j}!}{T_{j}!} {T_{j}!} {T_{j}!} {T_{j}!} {T_{j}!} {T_{j}!} {Y_{j}!} \right)^{2} \frac{(h_{j} \oplus_{r_{j}} v_{j})!}{(r_{j}!)^{2}} \omega_{\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}} \right) \\ = \Theta_{1} \times \Theta_{2}, \tag{13}$$

where

$$\Theta_1 := \sum_{r_1=0}^{t_1} \cdots \sum_{r_s=0}^{t_s} \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} |\widehat{f}(\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v})|^2 \omega_{\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}}^{-1},$$
(14)

and

$$\Theta_2 := \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_*} \sum_{r_1=0}^{t_1} \cdots \sum_{r_s=0}^{t_s} \left( \prod_{j=1}^s \frac{t_j!}{T_j!} \binom{T_j}{t_j - r_j}^2 \frac{(h_j \oplus_{r_j} v_j)!}{(r_j!)^2} \right) \omega_{\boldsymbol{h} \oplus_{\boldsymbol{r}} \boldsymbol{v}}.$$

Now we estimate  $\Theta_1$  and  $\Theta_2$  from above.

Upper bound on  $\Theta_1$ : For given  $\boldsymbol{h} = (h_1, \ldots, h_s) \in \mathcal{A}(s, M), \, \boldsymbol{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ , and

$$\boldsymbol{r} = (r_1, \ldots, r_s) \in \bigotimes_{j=1}^s \{0, \ldots, t_j\},$$

the system of equations

$$h_1 \oplus_{r_1} v_1 = k_1,$$
  

$$h_2 \oplus_{r_2} v_2 = k_2,$$
  

$$\vdots$$
  

$$h_s \oplus_{r_s} v_s = k_s$$

has at most  $2^s$  solutions  $(v_1, \ldots, v_s) \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}$ . Hence,

$$\Theta_1 \le \sum_{r_1=0}^{t_1} \cdots \sum_{r_s=0}^{t_s} 2^s \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} |\widehat{f}(\boldsymbol{k})|^2 \omega_{\boldsymbol{k}}^{-1} \le 2^s \|f\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}}^2 \prod_{j=1}^s (h_j+1),$$

where we used that  $t_j \leq h_j$ .

Note that  $\boldsymbol{h} \in \mathcal{A}(s, M)$  means by definition that  $\omega_{\boldsymbol{h}}^{-1} < M$ , and this implies  $\omega^{-a_j h_j^{b_j}} < M$  for each  $j \in \{1, \ldots, s\}$ . Hence we obtain, for  $j \in \{1, \ldots, s\}$ ,

$$h_j \le \left\lceil \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j} \right\rceil - 1,$$

and so

$$\prod_{j=1}^{s} (h_j+1) \le \prod_{j=1}^{s} \left\lceil \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j} \right\rceil = \prod_{j=1}^{\min(s,j(x_M))} \left\lceil \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j} \right\rceil,$$

where  $x_M = \log M / (\log \omega^{-1})$ , and  $j(x_M) = \sup\{j \in \mathbb{N} : x_M > a_j\}$ . Overall we have

$$\Theta_1 \le 2^s \left\| f \right\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}}^2 \prod_{j=1}^{\min(s,j(x_M))} \left\lceil \left( \frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right\rceil.$$
(15)

**Upper bound on**  $\Theta_2$ : Note that  $h_j \oplus_{r_j} v_j = T_j - t_j + 2r_j$  and therefore

$$\frac{t_j!}{T_j!} \binom{T_j}{(t_j - r_j)^2} \frac{(h_j \oplus_{r_j} v_j)!}{(r_j!)^2} = \binom{t_j}{r_j} \binom{T_j - t_j + 2r_j}{r_j} \binom{T_j}{(t_j - r_j)^2}.$$

Hence,

$$\Theta_2 = \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_*} \prod_{j=1}^s \sum_{r_j=0}^{t_j} \binom{t_j}{r_j} \binom{T_j - t_j + 2r_j}{r_j} \binom{T_j}{t_j - r_j} \omega_{h_j \oplus_{r_j} v_j}.$$

Since  $a_j, b_j \ge 1$ , we have<sup>1</sup>

$$\omega_{h_j \oplus_{r_j} v_j} = \omega^{a_j (|h_j - v_j| + 2r_j)^{b_j}} \le \omega^{a_j |h_j - v_j|^{b_j} + a_j (2r_j)^{b_j}} = \omega_{|h_j - v_j|} \omega_{2r_j} \le \omega_{|h_j - v_j|} \omega^{2r_j}$$

Thus,

$$\begin{aligned} \Theta_{2} &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \sum_{r_{j}=0}^{t_{j}} \omega^{2r_{j}} \binom{t_{j}}{r_{j}} \binom{T_{j}-t_{j}+2r_{j}}{r_{j}} \binom{T_{j}}{t_{j}-r_{j}} \\ &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \left(\sum_{r_{j}=0}^{t_{j}} \binom{t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}+t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}}{t_{j}-r_{j}}\right) \\ &= \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \left(\sum_{r_{j}=0}^{t_{j}} \binom{t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}+t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}}{r_{j}}\right) \\ &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \omega^{-t_{j}} \left(\sum_{r_{j}=0}^{t_{j}} \binom{t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}+t_{j}}{r_{j}}\right) \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}}{r_{j}} \omega^{r_{j}}\right) \\ &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \omega^{-t_{j}} \left(\sum_{r_{j}=0}^{t_{j}} \binom{t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}+t_{j}}{r_{j}} \omega^{r_{j}}\right) \left(\sum_{r_{j}=0}^{t_{j}} \binom{T_{j}}{r_{j}} \omega^{r_{j}}\right) \left$$

<sup>1</sup>Here we require  $b_j \ge 1$  since for  $b_j \in (0,1)$  we would have, according to Jensen's inequality, that  $(|h_j - v_j| + 2r_j)^{b_j} \le |h_j - v_j|^{b_j} + (2r_j)^{b_j}$ .

Now, let  $k = k(\omega)$  be the smallest positive integer such that

$$k \ge \frac{\log(\omega^{-1/8} - 1)}{\log \omega}.$$

We then get

$$\begin{split} \Theta_{2} &\leq \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\backslash\{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \left[ \omega_{|h_{j}-v_{j}|} \omega^{-t_{j}} \left( \sum_{r_{j}=0}^{t_{j}} \binom{t_{j}}{r_{j}} \omega^{kr_{j}} \omega^{-(k-1)r_{j}} \right) \\ &\times \left( \sum_{r_{j}=0}^{t} \binom{T_{j}+t_{j}}{r_{j}} \omega^{kr_{j}} \omega^{-(k-1)r_{j}} \right) \left( \sum_{r_{j}=0}^{t_{j}} \binom{T_{j}}{r_{j}} \omega^{kr_{j}} \omega^{-(k-1)r_{j}} \right) \right] \\ &\leq \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\backslash\{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \left[ \omega_{|h_{j}-v_{j}|} \omega^{-(3k-2)t_{j}} \\ &\times \left( \sum_{r_{j}=0}^{t} \binom{t_{j}}{r_{j}} \omega^{kr_{j}} \right) \left( \sum_{r_{j}=0}^{t_{j}} \binom{T_{j}+t_{j}}{r_{j}} \omega^{kr_{j}} \right) \left( \sum_{r_{j}=0}^{t_{j}} \binom{T_{j}}{r_{j}} \omega^{kr_{j}} \right) \right] \\ &\leq \sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\backslash\{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \left[ \omega_{|h_{j}-v_{j}|} \omega^{-(3k-2)t_{j}} \\ &\times \left( \sum_{r_{j}=0}^{t} \binom{t_{j}}{r_{j}} \omega^{kr_{j}} \right) \left( \sum_{r_{j}=0}^{T_{j}+t_{j}} \binom{T_{j}+t_{j}}{r_{j}} \omega^{kr_{j}} \right) \left( \sum_{r_{j}=0}^{T_{j}} \binom{T_{j}}{r_{j}} \omega^{kr_{j}} \right) \right] \end{split}$$

Using the binomial theorem we obtain

$$\Theta_{2} \leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \omega^{-(3k-2)t_{j}} (1+\omega^{k})^{2T_{j}+2t_{j}}$$
$$= \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \omega^{-(3k-2)t_{j}} (1+\omega^{k})^{2h_{j}+2v_{j}}.$$

.

Using again  $t_j = \min(v_j, h_j) \le h_j$ , we conclude

$$\Theta_{2} \leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \omega^{-(3k-2)h_{j}} (1+\omega^{k})^{2h_{j}+2v_{j}}$$
$$= \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega_{|h_{j}-v_{j}|} \omega^{-(3k-2)h_{j}} \omega^{-(2h_{j}+2v_{j})\frac{\log(1+\omega^{k})}{\log\omega^{-1}}}.$$

We now use

$$|v_j|^{b_j} \le 2^{b_j} (|v_j \pm h_j|^{b_j} + |h_j|^{b_j}),$$

i.e.,

$$\frac{|v_j|^{b_j}}{2^{b_j}} - |h_j|^{b_j} \le |v_j \pm h_j|^{b_j}$$

for any  $b_j \ge 1$  and any  $v_j, h_j \in \mathbb{Z}$ . Consequently,

$$\begin{split} \Theta_{2} &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega^{2^{-b_{j}} a_{j} v_{j}^{b_{j}}} \omega^{-a_{j} h_{j}^{b_{j}}} \omega^{-h_{j} \left((3k-2)+\frac{2\log(1+\omega^{k})}{\log\omega^{-1}}\right)} \omega^{-v_{j} \frac{2\log(1+\omega^{k})}{\log\omega^{-1}}} \\ &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \prod_{j=1}^{s} \omega^{2^{-b_{j}} a_{j} v_{j}^{b_{j}}} \omega^{-a_{j} h_{j}^{b_{j}} \left(3k-1+\frac{2\log(1+\omega^{k})}{\log\omega^{-1}}\right)} \omega^{-v_{j} \frac{2\log(1+\omega^{k})}{\log\omega^{-1}}} \\ &= \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} (\omega_{h}^{-1})^{3k-1+\frac{2\log(1+\omega^{k})}{\log\omega^{-1}}} \prod_{j=1}^{s} \omega^{2^{-b_{j}} a_{j} v_{j}^{b_{j}} - v_{j} \frac{2\log(1+\omega^{k})}{\log\omega^{-1}}} \\ &\leq \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} M^{K} \prod_{j=1}^{s} \omega^{2^{-b_{j}} a_{j} v_{j}^{b_{j}} - v_{j} \frac{2\log(1+\omega^{k})}{\log\omega^{-1}}}, \end{split}$$

where we used  $a_j, b_j \ge 1$  for the second inequality, and  $\mathbf{h} \in \mathcal{A}(s, M)$  for the last inequality, and where  $K = K(\omega) := 3k - 1 + \frac{2\log(1+\omega^k)}{\log \omega^{-1}}$ . For  $j \in \{1, \ldots, s\}$  we now study the term

$$\omega^{2^{-b_j}a_jv_j^{b_j}-v_j\frac{2\log(1+\omega^k)}{\log\omega^{-1}}} = \omega^{2^{-b_j-1}a_jv_j^{b_j}}\omega^{2^{-b_j-1}a_jv_j^{b_j}-v_j\frac{2\log(1+\omega^k)}{\log\omega^{-1}}}.$$

We show that

$$2^{-b_j - 1} a_j v_j^{b_j} - v_j \frac{2\log(1 + \omega^k)}{\log \omega^{-1}} \ge 0.$$
(16)

Indeed, (16) is trivially fulfilled if  $v_j = 0$ . If  $v_j > 0$ , this implies  $v_j \ge 2$ , since  $\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp}$ . In this case, (16) is fulfilled if and only if

$$v_j^{b_j-1} \ge 8 \frac{1}{a_j} \frac{\log(1+\omega^k)}{\log \omega^{-1}} 2^{b_j-1}.$$
(17)

Since  $v_j \ge 2$ , and since  $a_j \ge 1$ , (17) is certainly fulfilled if

$$2^{b_j - 1} \ge 8 \frac{\log(1 + \omega^k)}{\log \omega^{-1}} 2^{b_j - 1}.$$

However, k was chosen exactly such that the latter condition holds true. Hence, (16) is satisfied, and we have

$$\omega^{2^{-b_j}a_jv_j^{b_j}-v_j\frac{2\log(1+\omega^k)}{\log\omega^{-1}}} \le \omega^{2^{-b_j-1}a_jv_j^{b_j}}.$$

Consequently,

$$\Theta_2 \leq M^K \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_*} \omega^{\frac{1}{2}\sum_{j=1}^s a_j (v_j/2)^{b_j}}.$$

Now we insert our upper bounds for  $\Theta_1$  and  $\Theta_2$  into (13). For f in the unit ball of  $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  we obtain

$$\left| \int_{\mathbb{R}^{s}} f_{h}(\boldsymbol{x}) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} - Q_{n,s}(f_{h}) \right|^{2} \\ \leq 2^{s} \left( \prod_{j=1}^{\min(s,j(x_{M}))} \left\lceil \left( \frac{\log M}{a_{j} \log \omega^{-1}} \right)^{1/b_{j}} \right\rceil \right) M^{K} \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \omega^{\frac{1}{2} \sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}},$$
aimed.

as claimed.

Next we show the following proposition.

**Proposition 2.** We have

$$[e^{\operatorname{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}), A_{n,s,M})]^2 \le \frac{1}{M} + M^{2B(s)+K} D(s,\omega,\boldsymbol{b}) F_n,$$
(18)

where  $B(s) := \sum_{j=1}^{s} b_{j}^{-1}$ ,  $K = K(\omega)$  as in (11),

$$F_n := \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_*} \omega^{\frac{1}{2} \sum_{j=1}^s a_j (v_j/2)^{b_j}},$$
(19)

and where

$$D(s,\omega,\boldsymbol{b}) := 8^{s} \prod_{j=1}^{s} \left( 1 + \log^{-1/b_{j}} \omega^{-1} \right)^{2}.$$
 (20)

*Proof.* Let  $f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  with  $\|f\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}} \leq 1$ . Using (9), (10), and Lemma 4, we have

$$\begin{split} \|f - A_{n,s,M}(f)\|_{\mathbb{L}_{2}}^{2} &\leq \frac{1}{M} + \sum_{\boldsymbol{h} \in \mathcal{A}(s,M)} 2^{s} \left( \prod_{j=1}^{\min(s,j(x_{M}))} \left\lceil \left(\frac{\log M}{a_{j}\log\omega^{-1}}\right)^{1/b_{j}} \right\rceil \right) \right. \\ &\times M^{K} \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \omega^{\frac{1}{2}\sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}} \\ &= \frac{1}{M} + |\mathcal{A}(s,M)| 2^{s} \left( \prod_{j=1}^{\min(s,j(x_{M}))} \left\lceil \left(\frac{\log M}{a_{j}\log\omega^{-1}}\right)^{1/b_{j}} \right\rceil \right) \right. \\ &\times M^{K} \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \omega^{\frac{1}{2}\sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}}. \end{split}$$

Since  $|\mathcal{A}(s, M)| \leq \prod_{j=1}^{s} (1 + (\log M/(a_j \log \omega^{-1}))^{1/b_j})$  due to [8, Lemma 1] we have

$$\|f - A_{n,s,M}(f)\|_{\mathbb{L}_{2}}^{2} \leq \frac{1}{M} + 2^{s} M^{K} \left( \prod_{j=1}^{\min(s,j(x_{M}))} \left\lceil \left( \frac{\log M}{a_{j} \log \omega^{-1}} \right)^{1/b_{j}} \right\rceil \right) \times \left( \prod_{j=1}^{s} \left( 1 + \left( \frac{\log M}{a_{j} \log \omega^{-1}} \right)^{1/b_{j}} \right) \right) \sum_{\mathbf{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\mathbf{v}|_{*}} \omega^{\frac{1}{2} \sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}} ds^{j}$$

This means that

$$[e^{\text{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}), A_{n,s,M})]^2 \le \frac{1}{M} + 2^s M^K \left(\prod_{j=1}^s \left(1 + \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j}\right)\right)^2 F_n, \quad (21)$$

where  $F_n$  is as in (19).

Furthermore, we estimate

$$\begin{split} \prod_{j=1}^{s} \left( 1 + \left( \frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right) &\leq \prod_{j=1}^{s} \left( 1 + \left( \frac{\log M}{\log \omega^{-1}} \right)^{1/b_j} \right) \\ &\leq \prod_{j=1}^{s} \left( 1 + \log^{-1/b_j} \omega^{-1} \right) \prod_{j=1}^{s} \left( 1 + \log^{1/b_j} M \right). \end{split}$$

Since M is assumed to be at least 1, we can bound  $1 + \log^{1/b_j} M \le 2M^{1/b_j}$ , and obtain

$$\prod_{j=1}^{s} \left( 1 + \left( \frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right) \le 2^s M^{B(s)} \prod_{j=1}^{s} \left( 1 + \log^{-1/b_j} \omega^{-1} \right).$$

Plugging this into (21), we obtain

$$[e^{\operatorname{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}),A_{n,s,M})]^2 \leq \frac{1}{M} + M^{2B(s)+K}D(s,\omega,\boldsymbol{b})F_n,$$

where  $D(s, \omega, \boldsymbol{b})$  is as in (20).

We now give the proof of Lemma 3:

*Proof.* To show that  $f_h$  can be pointwise represented by its Hermite series, due to [9, Proposition 2.6] it is sufficient to verify that

$$\sum_{oldsymbol{v}\in\mathbb{N}_0^s} |\widehat{f}_{oldsymbol{h}}(oldsymbol{v})| <\infty.$$

To this end we proceed quite similarly to what we did when we estimated

$$\left(\sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}}|\widehat{f}_{\boldsymbol{h}}(\boldsymbol{v})|(\sqrt[4]{8\pi})^{|\boldsymbol{v}|_{*}}\right)^{2},$$

see (12), in the proof of Lemma 4. By going through analogous steps, we see that

$$\left(\sum_{\boldsymbol{v}\in\mathbb{N}_0^s} |\widehat{f}_{\boldsymbol{h}}(\boldsymbol{v})|\right)^2 \leq \|f\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}}^2 2^s \left(\prod_{j=1}^{\min(s,j(x_M))} \left\lceil \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j} \right\rceil\right) M^K$$
$$\times \sum_{\boldsymbol{v}\in\mathbb{N}_0^s} \prod_{j=1}^s \omega^{2^{-b_j} a_j v_j^{b_j} - v_j \frac{2\log(1+\omega^k)}{\log \omega^{-1}}}.$$

However,

$$\begin{split} \sum_{v \in \mathbb{N}_0^s} \prod_{j=1}^s \omega^{2^{-b_j} a_j v_j^{b_j} - v_j \frac{2\log(1+\omega^k)}{\log \omega^{-1}}} &= \prod_{j=1}^s \sum_{v_j=0}^\infty \omega^{2^{-b_j} a_j v_j^{b_j} - v_j \frac{2\log(1+\omega^k)}{\log \omega^{-1}}} \\ &= \prod_{j=1}^s \left( 1 + \omega^{2^{-b_j} a_j - \frac{2\log(1+\omega^k)}{\log \omega^{-1}}} + \sum_{v_j=2}^\infty \omega^{2^{-b_j} a_j v_j^{b_j} - v_j \frac{2\log(1+\omega^k)}{\log \omega^{-1}}} \right). \end{split}$$

In the derivation of (16), it was sufficient that  $v_j \ge 2$ . Hence we can proceed analogously for the sum in the latter expression to see that this sum is finite. Hence we derive that

$$\left(\sum_{oldsymbol{v}\in\mathbb{N}_0^s} |\widehat{f_{oldsymbol{h}}}(oldsymbol{v})|
ight)^2 <\infty.$$

## 2.4 The proof of Theorem 1

We now prove Theorem 1. To this end, we need the following proposition.

**Proposition 3.** For  $s \in \mathbb{N}$  and  $\varepsilon \in (0, 1)$  define

$$m = \max_{j=1,2,\dots,s} \left[ \left( \frac{2^{b_j+1}}{a_j} \frac{\log\left(1 + \frac{s\sqrt{8\pi}}{(1-\omega^{1/2})\log(1+\eta^2)}\right)}{\log\omega^{-1}} \right)^{B(s)} \right],$$

where

$$\eta = \left(\frac{\varepsilon^2}{2D(s,\omega,\boldsymbol{b})^{\frac{1}{2B(s)+K+1}}}\right)^{\frac{2B(s)+K+1}{2}}$$

and  $K = K(\omega)$  as in (11). Let  $m_1, m_2, \ldots, m_s$  be given by

$$m_j := \lfloor m^{1/(B(s)b_j)} \rfloor$$
 for  $j = 1, 2, \dots, s$  and  $n = \prod_{j=1}^s m_j$ .

Then for  $M = 2/\varepsilon^2$  we have

 $e^{\operatorname{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}), A_{n,s,M}) \leq \varepsilon \quad and \quad n = \mathcal{O}(\log^{B(s)}(1 + \varepsilon^{-1}))$ 

with the factor in the  $\mathcal{O}$  notation independent of  $\varepsilon^{-1}$  but dependent on s. Proof. From (19) we have

$$F_n = \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_*} \omega^{\frac{1}{2}\sum_{j=1}^s a_j (v_j/2)^{b_j}}$$

$$= -1 + \prod_{j=1}^{s} \left( 1 + \sqrt{8\pi} \sum_{h=m_{j}}^{\infty} \omega^{\frac{1}{2}a_{j}h^{b_{j}}} \right)$$
  
$$= -1 + \prod_{j=1}^{s} \left( 1 + \omega^{\frac{1}{2}a_{j}m_{j}^{b_{j}}} \sqrt{8\pi} \sum_{h=m_{j}}^{\infty} \omega^{\frac{1}{2}a_{j}(h^{b_{j}} - m_{j}^{b_{j}})} \right)$$
  
$$\leq -1 + \prod_{j=1}^{s} \left( 1 + \omega^{\frac{1}{2}a_{j}m_{j}^{b_{j}}} \sqrt{8\pi} \sum_{h=0}^{\infty} \omega^{\frac{1}{2}h} \right)$$
  
$$= -1 + \prod_{j=1}^{s} \left( 1 + \omega^{\frac{1}{2}a_{j}m_{j}^{b_{j}}} \frac{\sqrt{8\pi}}{1 - \omega^{1/2}} \right),$$

where we used that  $a_j(h^{b_j} - m_j^{b_j}) \ge h - m_j$ , since  $a_j, b_j \ge 1$ . Since  $\lfloor x \rfloor \ge x/2$  for all  $x \ge 1$ , we have

$$(2m_j)^{b_j} \ge m^{1/B(s)}$$
 for all  $j = 1, 2, \dots, s$ .

Hence,

$$F_n \le -1 + \prod_{j=1}^s \left( 1 + \omega^{m^{1/B(s)} a_j 2^{-b_j - 1}} \frac{\sqrt{8\pi}}{1 - \omega^{1/2}} \right).$$

From the definition of m we have

$$\omega^{m^{1/B(s)}a_j 2^{-b_j - 1}} \frac{\sqrt{8\pi}}{1 - \omega^{1/2}} \le \frac{\log(1 + \eta^2)}{s} \quad \text{for all} \quad j = 1, 2, \dots, s.$$

This proves

$$F_n \le -1 + \left(1 + \frac{\log(1+\eta^2)}{s}\right)^s \le -1 + \exp(\log(1+\eta^2)) = \eta^2.$$
(22)

Now, plugging this into (18), we obtain

$$[e^{\operatorname{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}), A_{n,s,M})]^2 \le \frac{1}{M} + M^{2B(s)+K} D(s,\omega,\boldsymbol{b})\eta^2.$$
(23)

Note that

$$(D(s,\omega,\boldsymbol{b})\eta^2)^{-\frac{1}{2B(s)+K+1}} = \frac{2}{\varepsilon^2} \ge 1.$$

Hence we are allowed to choose

$$M = (D(s, \omega, \boldsymbol{b})\eta^2)^{-\frac{1}{2B(s)+K+1}},$$

which yields, inserting into (23),

$$[e^{\operatorname{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}),A_{n,s,M})]^2 \le 2(D(s,\omega,\boldsymbol{b})\eta^2)^{\frac{1}{2B(s)+K+1}} = \varepsilon^2,$$

as claimed.

It remains to verify that n is of the order stated in the proposition. Note that

$$n = \prod_{j=1}^{s} m_j = \prod_{j=1}^{s} \left\lfloor m^{1/(B(s)b_j)} \right\rfloor \le m^{\frac{1}{B(s)}\sum_{j=1}^{s} 1/b_j} = m.$$

However,

$$m = \mathcal{O}(\log^{B(s)}(1+\eta^{-1})),$$

as  $\eta$  tends to zero. From this, it is easy to see that we indeed have

$$n = \mathcal{O}(\log^{B(s)}(1 + \varepsilon^{-1})),$$

which concludes the proof of Proposition 3.

We now prove the successive points of Theorem 1.

#### Proof of Point 1 (Exponential Convergence)

We conclude from Proposition 3 that

$$n(\varepsilon, \operatorname{APP}_s, \Lambda^{\operatorname{std}}) = \mathcal{O}(\log^{B(s)}(1 + \varepsilon^{-1})).$$

This implies that we indeed have EXP for all  $\boldsymbol{a}$  and  $\boldsymbol{b}$ , with p(s) = 1/B(s), and thus  $p^*(s) \ge 1/B(s)$ . On the other hand, note that obviously  $e(n, \text{APP}_s, \Lambda^{\text{std}}) \ge e(n, \text{APP}_s, \Lambda^{\text{all}})$ , hence the rate of EXP for  $\Lambda^{\text{std}}$  cannot be larger than for  $\Lambda^{\text{all}}$  which is 1/B(s). Thus, we have  $p^*(s) = 1/B(s)$ .

#### Proof of Point 2 (Uniform Exponential Convergence)

Suppose first that a is an arbitrary sequence and that b is such that

$$B = \sum_{j=1}^{\infty} \frac{1}{b_j} < \infty$$

Then we can replace B(s) by B in Proposition 3, and we obtain

$$n(\varepsilon, \operatorname{APP}_{s}, \Lambda^{\operatorname{std}}) = \mathcal{O}\left(\log^{B}\left(1 + \varepsilon^{-1}\right)\right),$$

hence UEXP with  $p^* \geq 1/B$  holds. On the other hand, if we have UEXP for  $\Lambda^{\text{std}}$ , this implies UEXP for  $\Lambda^{\text{all}}$ , which in turn implies that  $B < \infty$  and that  $p^* \leq 1/B$ .

### Proof of Point 3 (EC-Weak Tractability)

Assume that EC-WT holds for the class  $\Lambda^{\text{std}}$ . Then EC-WT also holds for the class  $\Lambda^{\text{all}}$  and this implies that  $\lim_{i} a_{i} = \infty$ , as claimed.

Assume now that  $\lim_{j} a_{j} = \infty$ . We consider the operator

$$W_s := \operatorname{APP}^*_s \operatorname{APP}_s : \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}) \to \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}),$$

which is given by

$$W_s f = \sum_{\boldsymbol{k} \in \mathbb{N}_0^s} \omega_{\boldsymbol{k}} \langle f, e_{\boldsymbol{k}} \rangle_{K_{s, \boldsymbol{a}, \boldsymbol{b}, \omega}} e_{\boldsymbol{k}} \quad \text{for } f \in \mathcal{H}(K_{s, \boldsymbol{a}, \boldsymbol{b}, \omega}),$$

where  $e_{k} = \sqrt{\omega_{k}} H_{k}$  and  $\langle e_{k}, e_{l} \rangle = \delta_{k,l}$ . We then have

$$W_s e_{\boldsymbol{k}} = \omega_{\boldsymbol{k}} e_{\boldsymbol{k}}$$
 for all  $\boldsymbol{k} \in \mathbb{N}_0^s$ 

so the eigenpairs of  $W_s$  are  $(\omega_k, e_k)$  for  $k \in \mathbb{N}_0^s$ ; see [8, Section 3] for more details.

We use [17, Theorem 26.18] which states that if the ordered eigenvalues  $\lambda_{s,n}$  of  $W_s$  satisfy

$$\lambda_{s,n} \le \frac{M_{s,\tau}^2}{n^{2\tau}} \qquad \text{for all} \qquad n \in \mathbb{N},\tag{24}$$

for some positive  $M_{s,\tau}$  and  $\tau > \frac{1}{2}$  then there is a semi-constructive algorithm<sup>2</sup> such that

$$e(n+2, \operatorname{APP}_s; \Lambda^{\operatorname{std}}) \le \frac{M_{s,\tau} C(\tau)}{n^{\tau(2\tau/(2\tau+1))}} \quad \text{for all} \quad n \in \mathbb{N}$$
 (25)

where  $C(\tau)$  is given explicitly in [17, Theorem 26.18]. However, the form of  $C(\tau)$  is not important for our consideration.

For  $\eta \in (0, 1)$ , let  $\tau = 1/(2\eta) > \frac{1}{2}$ . We stress that  $\tau$  can be arbitrarily large if we take sufficiently small  $\eta$ . Now we have

$$n\lambda_{s,n}^{\eta} \leq \sum_{j=1}^{\infty} \lambda_{s,j}^{\eta} = \sum_{\boldsymbol{h} \in \mathbb{N}_0^s} \omega_{\boldsymbol{h}}^{\eta} = \prod_{j=1}^s \left( 1 + \sum_{h=1}^{\infty} \omega^{\eta \, a_j \, h^{b_j}} \right).$$

Note that

$$\sum_{h=1}^{\infty} \omega^{\eta \, a_j \, h^{b_j}} \le \sum_{h=1}^{\infty} \omega^{\eta \, a_j \, h} = \omega^{\eta \, a_j} \sum_{h=1}^{\infty} \omega^{\eta \, a_j \, (h-1)} \le \omega^{\eta \, a_j} A_{\eta},$$

where

$$A_{\eta} := \sum_{h=0}^{\infty} \omega^{\eta h} = \frac{1}{1 - \omega^{\eta}} < \infty.$$
 (26)

This proves that

$$\lambda_{s,n} \le \frac{\prod_{j=1}^{s} \left(1 + \omega^{\eta a_j} A_{\eta}\right)^{1/\eta}}{n^{1/\eta}}.$$

Hence, we can take

$$M_{s,\tau} = \prod_{j=1}^{s} (1+c_j)^{\tau} < \infty \quad \text{with} \quad c_j = \omega^{a_j/(2\tau)} A_{\frac{1}{2\tau}},$$

 $<sup>^{2}</sup>$ By semi-constructive we mean that this algorithm can be constructed after a few random selections of sample points, more can be found in [17, Section 24.3].

where  $A_{\frac{1}{2\tau}}$  is defined as in (26). Furthermore, we know that  $\lim_{j \to \infty} a_j = \infty$  implies that  $\lim_{s \to j=1}^{s} c_j/s = 0$ .

From (25) we obtain

$$n(\varepsilon, \text{APP}_s; \Lambda^{\text{std}}) \le 3 + (M_{s,\tau} C(\tau))^{(1+1/(2\tau))/\tau} \varepsilon^{-(1+1/(2\tau))/\tau}.$$

This yields that

$$\limsup_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \operatorname{APP}_s; \Lambda^{\operatorname{std}})}{s+\log \varepsilon^{-1}} \le \left(1+\frac{1}{2\tau}\right) \frac{1}{\tau} \left(1+\limsup_{s\to\infty} \frac{\log M_{s,\tau}}{s}\right)$$

Since  $(\log M_{s,\tau})/s \leq \tau \sum_{j=1}^{s} c_j/s$  tends to zero as  $s \to \infty$ , we have

$$\limsup_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \operatorname{APP}_s; \Lambda^{\operatorname{std}})}{s+\log \varepsilon^{-1}} \le \left(1+\frac{1}{2\tau}\right) \frac{1}{\tau}$$

Since  $\tau$  can be arbitrarily large this proves that

$$\lim_{s+\varepsilon^{-1}\to\infty} \frac{\log n(\varepsilon, \operatorname{APP}_s; \Lambda^{\operatorname{std}})}{s+\log \varepsilon^{-1}} = 0.$$

This means that EC-WT holds for the class  $\Lambda^{\text{std}}$ , as claimed.

#### Proof of Point 3 (EC-Polynomial Tractability)

Suppose that EC-PT holds for the class  $\Lambda^{\text{std}}$ . Then EC-PT holds for the class  $\Lambda^{\text{all}}$ . From [8] we know that this implies EC-SPT for the class  $\Lambda^{\text{all}}$  which is equivalent to  $B < \infty$  and  $\alpha^* > 0$ . If the conditions  $B < \infty$  and  $\alpha^* > 0$  hold, we will show in the following that this implies EC-SPT and therefore we also have EC-PT.

### Proof of Point 3 (EC-Strong Polynomial Tractability)

The necessity of the conditions for EC-SPT on **b** and **a** follows from the same conditions for the class  $\Lambda^{\text{all}}$  and the fact that the information complexity for  $\Lambda^{\text{std}}$  cannot be smaller than for  $\Lambda^{\text{all}}$ .

To prove the sufficiency of the conditions for EC-SPT on  $\boldsymbol{a}$  and  $\boldsymbol{b}$  stated in Point 3 we analyze the algorithm  $A_{n,s,M}$  given by (8), where the sample points  $\boldsymbol{x}_k$  come from a Gauss-Hermite rule with

$$m_j = 2 \left[ \left( \frac{\log M}{a_j^\beta \log \widetilde{\omega}^{-1}} \right)^{1/b_j} \right] - 1 \quad \text{for all} \quad j = 1, 2, \dots, s,$$

where M > 1,  $\beta \in (0, 1)$ , and  $\tilde{\omega} := \omega^{\frac{1}{2K+2}}$  with  $K = K(\omega)$ , defined in (11). Note that  $m_j \ge 1$  and is always an odd number. Furthermore  $m_j = 1$  if  $a_j \ge ((\log M)/(\log \tilde{\omega}^{-1}))^{1/\beta}$ . We know that  $\alpha^* \in (0, \infty]$ . Since for all  $\delta \in (0, \alpha^*)$  we have

$$a_j \ge \exp(\delta j)$$
 for all  $j \ge j_{\delta}^*$ ,

we conclude that

$$j \ge j^*_{\beta,\delta,M} := \max\left(j^*_{\delta}, \frac{\log(((\log M)/(\log \widetilde{\omega}^{-1}))^{1/\beta})}{\delta}\right) \text{ implies } m_j = 1.$$

For given  $\boldsymbol{h} \in \mathcal{A}(s, M)$  and  $\boldsymbol{r} \in \mathbb{N}_0^s$  suppose that

$$oldsymbol{h} \oplus_{oldsymbol{r}} oldsymbol{v}^{(1)} = oldsymbol{h} \oplus_{oldsymbol{r}} oldsymbol{v}^{(2)}$$

for some  $\boldsymbol{v}^{(1)}, \boldsymbol{v}^{(2)} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}, \boldsymbol{v}^{(1)} \neq \boldsymbol{v}^{(2)}$ . This means that for all  $j \in \{1, \ldots, s\}$  we must have  $|h_j - v_j^{(1)}| + 2r_j = |h_j - v_j^{(2)}| + 2r_j$ , which is equivalent to  $|h_j - v_j^{(1)}| = |h_j - v_j^{(2)}|$ . As  $\boldsymbol{v}^{(1)} \neq \boldsymbol{v}^{(2)}$ , there must be at least one  $j \in \{1, \ldots, s\}$  such that  $v_j^{(1)} \neq v_j^{(2)}$ . For this j, the condition  $|h_j - v_j^{(1)}| = |h_j - v_j^{(2)}|$  is then equivalent to  $2h_j = v_j^{(1)} + v_j^{(2)}$ . From the choice of  $\boldsymbol{h}, \boldsymbol{v}^{(1)}$  and  $\boldsymbol{v}^{(2)}$  it follows that for this j we must have

$$2h_j = v_j^{(1)} + v_j^{(2)} \ge \max(v_j^{(1)}, v_j^{(2)}) \ge 2m_j$$

and hence for this j we have  $h_j \ge m_j$ . This leads to a contradiction, because if  $h_j$  is the jth component of  $\mathbf{h} \in \mathcal{A}(s, M)$ , we must have

$$m_j \le h_j < \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j} \le \left(\frac{\log M}{a_j^\beta \log \widetilde{\omega}^{-1}}\right)^{1/b_j} \le \frac{m_j + 1}{2} \le m_j$$

for each  $j \in \{1, \ldots, s\}$ .

Consequently, each coefficient  $\widehat{f}(\mathbf{h} \oplus_{\mathbf{r}} \mathbf{v})$  occurs at most once in (14), and so we get rid of the factor  $2^s$  in the upper bound (15) of  $\Theta_1$ . This way we obtain the improved bound

$$\Theta_1 \le \|f\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega}}^2 \prod_{j=1}^{\min(s,j(x_M))} \left\lceil \left(\frac{\log M}{a_j \log \omega^{-1}}\right)^{1/b_j} \right\rceil.$$

Together with our previous upper bounds on  $\Theta_2$  we obtain

$$e_{n,s}^{2} := [e(A_{n,s,M}, \operatorname{APP}_{s}; \Lambda^{\operatorname{std}})]^{2}$$

$$\leq \frac{1}{M} + \sum_{\boldsymbol{h} \in \mathcal{A}(s,M)} \left( \prod_{j=1}^{\min(s,j(x_{M}))} \left\lceil \left( \frac{\log M}{a_{j} \log \omega^{-1}} \right)^{1/b_{j}} \right\rceil \right)$$

$$\times M^{K} \sum_{\boldsymbol{v} \in \mathcal{G}_{n,s}^{\perp} \setminus \{\mathbf{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \omega^{\frac{1}{2} \sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}}.$$

For  $s \in \mathbb{N}$  we use the notation  $[s] = \{1, \ldots, s\}$ . We now estimate

$$\sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \omega^{\frac{1}{2}\sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}} = \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \prod_{j\in\mathfrak{u}} \sum_{\ell=0}^{\infty} \sqrt{8\pi} \omega^{\frac{1}{2}a_{j}((2m_{j}+2\ell)/2)^{b_{j}}},$$

where we separated the cases for  $v_j \neq 0$  and  $v_j = 0$ .

Note that, as  $m_j, b_j \ge 1$ ,

$$((2m_j + 2\ell)/2)^{b_j} \ge \left(\frac{m_j + 1}{2} + \ell\right)^{b_j} \ge \left(\frac{m_j + 1}{2}\right)^{b_j} + \ell^{b_j}.$$

Hence,

$$\sum_{\ell=0}^{\infty} \sqrt{8\pi} \omega^{\frac{1}{2}a_j((2m_j+2\ell)/2)^{b_j}} \leq \sum_{\ell=0}^{\infty} \sqrt{8\pi} \omega^{\frac{1}{2}a_j\left(\left(\frac{m_j+1}{2}\right)^{b_j}+\ell^{b_j}\right)} \\ = \omega^{\frac{1}{2}a_j\left(\frac{m_j+1}{2}\right)^{b_j}} \sum_{\ell=0}^{\infty} \sqrt{8\pi} \omega^{\frac{1}{2}a_j\ell^{b_j}} \\ \leq \omega^{\frac{1}{2}a_j\left(\frac{m_j+1}{2}\right)^{b_j}} A,$$

where  $A = \frac{\sqrt{8\pi}}{1 - \sqrt{\omega}}$ . Consequently,

$$\sum_{\boldsymbol{v}\in\mathcal{G}_{n,s}^{\perp}\setminus\{\boldsymbol{0}\}} (\sqrt{8\pi})^{|\boldsymbol{v}|_{*}} \omega^{\frac{1}{2}\sum_{j=1}^{s} a_{j}(v_{j}/2)^{b_{j}}} \leq \sum_{\emptyset\neq\mathfrak{u}\subseteq[s]} \prod_{j\in\mathfrak{u}} \omega^{\frac{1}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}} A$$
$$= -1 + \prod_{j=1}^{s} \left(1 + \omega^{\frac{1}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}} A\right).$$

Furthermore,

$$e_{n,s}^{2} \leq \frac{1}{M} + M^{K} |\mathcal{A}(s,M)| \left( \prod_{j=1}^{\min(s,j(x_{M}))} \left\lceil \left( \frac{\log M}{a_{j}\log \omega^{-1}} \right)^{1/b_{j}} \right\rceil \right) \times \left( -1 + \prod_{j=1}^{s} \left( 1 + \omega^{\frac{1}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}} A \right) \right).$$

Using  $\log(1+x) \le x$  we obtain

$$\log\left[\prod_{j=1}^{s} \left(1 + \omega^{\frac{1}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}}A\right)\right] \le A\sum_{j=1}^{\infty} \omega^{\frac{1}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}} =: \gamma.$$

From the definition of  $m_j$  we have  $a_j[(m_j+1)/2]^{b_j} \ge a_j^{1-\beta} (\log M)/\log \tilde{\omega}^{-1}$ . Therefore

$$\omega^{\frac{1}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}} = \omega^{\frac{1}{2K+2}\frac{2K+2}{2}a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}} = \widetilde{\omega}^{a_{j}\left(\frac{m_{j}+1}{2}\right)^{b_{j}}(K+1)}$$
$$\leq \widetilde{\omega}^{a_{j}^{1-\beta}(K+1)(\log M)/\log\widetilde{\omega}^{-1}} = \left(\frac{1}{M^{K+1}}\right)^{a_{j}^{1-\beta}}$$

Without loss of generality, we assume  $M \ge e$ . Since  $a_j \ge 1$  for  $j \le j^*_{\beta,\delta,M} - 1$  and  $a_j \ge \exp(\delta j)$  for  $j \ge j^*_{\beta,\delta,M}$  we obtain

$$\gamma \le A \left( \frac{j_{\beta,\delta,M}^* - 1}{M^{K+1}} + \sum_{j=j_{\beta,\delta,M}^*}^{\infty} \left( \frac{1}{M^{K+1}} \right)^{\exp((1-\beta)\delta j)} \right)$$

Note that there exists a constant  ${\cal C}>0$  such that

$$j^*_{\beta,\delta,M} \le (\log \log M) j^*_{\beta,\delta}$$

with

$$j_{\beta,\delta}^* := C \max\left(j_{\delta}^*, \frac{1 - \log \log \tilde{\omega}^{-1}}{\delta \beta}\right).$$

Thus

$$\gamma \le A \left( \frac{(\log \log M) j_{\beta,\delta}^* - 1}{M^{K+1}} + \sum_{j=0}^{\infty} \left( \frac{1}{M^{K+1}} \right)^{\exp((1-\beta)\delta j)} \right) \le \frac{C_{\beta,\delta}}{M^K},$$

with

$$C_{\beta,\delta} := A\left(j_{\beta,\delta}^* - 1 + \sum_{j=0}^{\infty} \left(\frac{1}{e}\right)^{\exp((1-\beta)\delta j) - 1}\right) < \infty,$$

where we made use of  $M \ge e$ . Note that for  $M \ge C_{\beta,\delta}^{1/K}$  we have  $\gamma \le 1$ . Using convexity we easily check that  $-1 + \exp(\gamma) \le (e - 1)\gamma$  for all  $\gamma \in [0, 1]$ . Thus for  $M \ge C_{\beta,\delta}^{1/K}$  we obtain

$$-1 + \prod_{j=1}^{s} \left( 1 + \omega^{\frac{1}{2}a_j \left(\frac{m_j+1}{2}\right)^{b_j}} A \right) \le -1 + \exp(\gamma) \le (e-1)\gamma \le \frac{C_{\beta,\delta}(e-1)}{M^K}$$

We now turn to  $|\mathcal{A}(s, M)|$ . From the proof of [8, Theorem 9] we get that

$$|\mathcal{A}(s,M)| \le 2^{j^*_{\beta,\delta,M}} \left(1 + \frac{\log M}{\log \omega^{-1}}\right)^{B + (\log 2)/\delta}$$

as well as

$$\prod_{j=1}^{\min(s,j(x_M))} \left\lceil \left( \frac{\log M}{a_j \log \omega^{-1}} \right)^{1/b_j} \right\rceil \le 2^{j^*_{\beta,\delta,M}} \left( 1 + \frac{\log M}{\log \omega^{-1}} \right)^{B + (\log 2)/\delta}$$

Therefore

$$e_{n,s}^2 \le \frac{1}{M} \left[ 1 + C_{\beta,\delta}(\mathbf{e} - 1) 4^{j_{\beta,\delta,M}^*} \left( 1 + \frac{\log M}{\log \omega^{-1}} \right)^{2B + (2\log 2)/\delta} \right] \le \frac{D_{\beta,\delta}}{M^{1/2}},$$

where

$$D_{\beta,\delta} := \sup_{x \ge C_{\beta,\delta}} \left( \frac{1}{x^{1/2}} + \frac{C_{\beta,\delta}(e-1)(\log x)^{j^*_{\beta,\delta}\log 4}}{x^{1/2}} \left( 1 + \frac{\log x}{\log \omega^{-1}} \right)^{B + (\log 2)/\delta} \right) < \infty.$$

Hence for  $M = \max(C_{\beta,\delta}^{1/K}, D_{\beta,\delta}^2 \varepsilon^{-4}, \mathbf{e})$  we have

$$e_{n,s} \leq \varepsilon.$$

We estimate the number n of function values used by the algorithm  $A_{n,s,M}$ . We have

$$n = \prod_{j=1}^{s} m_j = \prod_{j=1}^{\min(s, j^*_{\beta, \delta, M})} m_j \le \prod_{j=1}^{\min(s, j^*_{\beta, \delta, M})} \left( 1 + 2\left(\frac{\log M}{a_j^\beta \log \widetilde{\omega}^{-1}}\right)^{1/b_j} \right)$$

$$\leq 3^{j^*_{\beta,\delta,M}} \left( \frac{\log M}{\log \widetilde{\omega}^{-1}} \right)^B = \mathcal{O}((1 + \log \varepsilon^{-1})^{B + (\log 3)/(\beta \delta)}),$$

where the factor in the big  $\mathcal{O}$  notation depends only on  $\beta$  and  $\delta$ . This proves EC-SPT with

$$\tau = B + \frac{\log 3}{\beta \,\delta}.$$

Since  $\beta$  can be arbitrarily close to one, and  $\delta$  can be arbitrarily close to  $\alpha^*$ , the exponent  $\tau^*$  of EC-SPT is at most

$$B + \frac{\log 3}{\alpha^*},$$

where for  $\alpha^* = \infty$  we have  $\frac{\log 3}{\alpha^*} = 0$ . This completes the proof of Theorem 1.

# 3 Relations to multivariate integration

Multivariate integration

$$\operatorname{INT}_{s}(f) = \int_{\mathbb{R}^{s}} f(\boldsymbol{x}) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$$

for f from the Hermite space  $\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})$  was studied in [7]. It is easy to see that multivariate approximation using information from  $\Lambda^{\text{std}}$  is not easier than multivariate integration, see e.g., [14]. More precisely, for any algorithm  $A_{n,s}(f) = \sum_{k=1}^{n} \alpha_k f(\boldsymbol{x}_k)$  for multivariate approximation using the nodes  $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n \in [0, 1)^s$  and  $\alpha_k \in \mathbb{L}_2(\mathbb{R}^s, \varphi_s)$ , define  $\beta_k := \int_{\mathbb{R}^s} \alpha_k(\boldsymbol{x}) \varphi_s(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x}$  and the algorithm

$$Q_{n,s}(f) = \sum_{k=1}^{n} \beta_k f(\boldsymbol{x}_k)$$

for multivariate integration. Then

$$|\operatorname{INT}_{s}(f) - Q_{n,s}(f)| = \left| \int_{\mathbb{R}^{s}} \left( f(\boldsymbol{x}) - \sum_{k=1}^{n} \alpha_{k}(\boldsymbol{x}) f(\boldsymbol{x}_{k}) \right) \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right|$$
$$\leq \left( \int_{\mathbb{R}^{s}} \left( f(\boldsymbol{x}) - \sum_{k=1}^{n} \alpha_{k}(\boldsymbol{x}) f(\boldsymbol{x}_{k}) \right)^{2} \varphi_{s}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \right)^{1/2}$$
$$= \|f - A_{n,s}(f)\|_{\mathbb{L}_{2}}.$$

This proves that for the worst-case error of integration we have

$$e^{\operatorname{int}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}),Q_{n,s}) := \sup_{\substack{f \in \mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega})\\ \|f\|_{K_{s,\boldsymbol{a},\boldsymbol{b},\omega} \leq 1}}} |\operatorname{INT}_{s}(f) - Q_{n,s}(f)| \le e^{\operatorname{app}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}),A_{n,s}).$$

Since this holds for all linear approximation algorithms  $A_{n,s}$  we conclude that

$$e(n, \text{INT}_s) := \inf_{Q_{n,s}} e^{\text{int}}(\mathcal{H}(K_{s,\boldsymbol{a},\boldsymbol{b},\omega}), Q_{n,s}) \le e(n, \text{APP}_s; \Lambda^{\text{std}}),$$
(27)

where  $e(n, \text{INT}_s)$  denotes the *n*th minimal (worst-case) error of integration.

Furthermore for n = 0 we have equality,

$$e(0, \mathrm{INT}_s) = e(0, \mathrm{APP}_s) = 1.$$

From these observations it follows that for  $\varepsilon \in (0, 1)$  and  $s \in \mathbb{N}$  we have

$$n(\varepsilon, \text{INT}_s) \le n(\varepsilon, \text{APP}_s; \Lambda^{\text{std}}),$$
(28)

where  $n(\varepsilon, \text{INT}_s)$  is the information complexity for the integration problem.

The inequalities (27) and (28) mean that all positive results for multivariate approximation also hold for multivariate integration.

In [7] the following results were proved:

**Theorem 2** ([7, Theorem 1]). For the integration problem over the Hermite space  $\mathcal{H}(K_{s,a,b,\omega})$  we have:

1. EXP holds for all **a** and **b** considered, and

$$p^*(s) = \frac{1}{B(s)}$$
 with  $B(s) := \sum_{j=1}^s \frac{1}{b_j}$ .

- 2. The following assertions are equivalent:
  - (a) The  $b_j^{-1}$ 's are summable, i.e.,  $B := \sum_j b_j^{-1} < \infty$ ;
  - (b) we have UEXP;
  - (c) we have EC-PT;
  - (d) we have EC-SPT.

If one of the assertions holds then  $p^* = 1/B$  and the exponent  $\tau^*$  of EC-SPT is B.

- 3. EC-WT implies that  $\lim_{j\to\infty} a_j 2^{b_j} = \infty$ .
- 4. A sufficient condition for EC-WT is that there exist  $\eta > 0$  and  $\beta > 0$  such that

$$a_j 2^{b_j} \ge \beta j^{1+\eta} \quad for all \quad j \in \mathbb{N}.$$

Compared with our results for approximation from Theorem 1 we have:

- The conditions for EXP and for UEXP are the same for both problems.
- For the integration problem UEXP and EC-SPT are equivalent and these properties only depend on b but not on a. This makes a difference to the approximation problem where we have the same condition on b as for the integration problem in order to achieve UEXP. However, to obtain also EC-SPT for approximation we must require that the sequence a grows at an exponential rate.
- For the integration problem there is a gap between the necessary and sufficient condition for EC-WT whereas for the approximation problem we have an *if and only if* condition. With the help of our result for EC-WT for approximation and with our previous considerations we can present a different sufficient condition for EC-WT for integration as compared to [7, Theorem 1] (see Point 4 of Theorem 2).

**Theorem 3.** A sufficient condition for EC-WT for integration in  $\mathcal{H}(K_{s,a,b,\omega})$  is that  $\lim_{j\to\infty} a_j = \infty$ .

*Proof.* Assume that we have  $\lim_{j\to\infty} a_j = \infty$ . Then Theorem 1 implies that we have EC-WT for the approximation problem. But now it follows easily from (28) that we also have EC-WT for the integration problem.

Although Theorem 3 is in some cases an improvement of the sufficient condition for EC-WT for integration from [7, Theorem 1] there still remains a small gap to the necessary condition.

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