On alternative quantization for doubly weighted approximation and integration over unbounded domains

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Abstract

It is known that for a ϱ-weighted L_q-approximation of single variable functions f with the rth derivatives in a ψ-weighted L_p space, the minimal error of approximations that use n samples of f is proportional to $\|\omega^{1/\alpha}\|_{L_1}^{\alpha}\|f^{(r)}\psi\|_{L_p}n^{-r+(1/p-1/q)_+}$, where $\omega=\varrho/\psi$ and $\alpha=r-1/p+1/q$. Moreover, the optimal sample points are determined by quantiles of $\omega^{1/\alpha}$. In this paper, we show how the error of best approximations changes when the sample points are determined by a quantizer κ other than ω. Our results can be applied in situations when an alternative quantizer has to be used because ω is not known exactly or is too complicated to handle computationally. The results for $q=1$ are also applicable to ϱ-weighted integration over unbounded domains.

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1 Introduction

In various applications, continuous objects (signals, images, etc.) are represented (or approximated) by their discrete counterparts. That is, we deal with *quantization*. From a pure mathematics point of view, quantization often leads to approximating functions from a given space by step functions or, more generally, by (quasi-)interpolating piecewise polynomials of certain degree. Then it is important to know which quantizer should be used, or how to select n break points (knots) to make the error of approximation as small as possible.

It is well known that for L_q approximation on a compact interval $D=[a,b]$ in the space $F_p^r(D)$ of real-valued functions f such that $f^{(r)}\in L_p(D)$, the choice of an optimal quantizer is not a big issue, since equidistant knots lead to approximations with optimal L_q error

$$c\,(b-a)^{\alpha}\|f^{(r)}\|_{L_q}n^{-r+(1/p-1/q)_+} \qquad \text{with} \quad \alpha:=r-\frac{1}{p}+\frac{1}{q}, \tag{1}$$

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where c depends only on r, p, and q, and where $x_+ := \max(x, 0)$. The problem becomes more complicated if we switch to weighted approximation on unbounded domains. A generalization of (1) to this case was given in [5], and it reads as follows. Assume for simplicity that the domain $D = \mathbb{R}_+ := [0, +\infty)$. Let $\psi, \varrho : D \to (0, +\infty)$ be two positive and integrable *weight* functions. For a positive integer r and $1 \le p, q \le +\infty$, consider the ϱ-weighted L_q approximation in the linear space $F^r_{p,\psi}(D)$ of functions $f : D \to \mathbb{R}$ with absolutely (locally) continuous $(r-1)$st derivative and such that the ψ-weighted L_p norm of $f^{(r)}$ is finite, i.e., $\|f^{(r)}\psi\|_{L_p} < +\infty$. Note that the spaces $F^r_{p,\psi}(D)$ have been introduced in [7], and the role of ψ is to moderate their size.

Denote

$$\omega := \frac{\varrho}{\psi}, \tag{2}$$

and suppose that ω and ψ are nonincreasing on D, and that

$$\|\omega^{1/\alpha}\|_{L_1} := \int_D \omega^{1/\alpha}(x)\,\mathrm{d}x < +\infty. \tag{3}$$

It was shown in [5, Theorem 1] that then one can construct approximations using n knots with ϱ-weighted L_q error at most

$$c_1\, \|\omega^{1/\alpha}\|^{\alpha}_{L_1}\, \|f^{(r)}\psi\|_{L_p} n^{-r+(1/p-1/q)_+}.$$

This means that if (3) holds true, then the upper bound on the worst-case error is proportional to $\|\omega^{1/\alpha}\|^{\alpha}_{L_1}\, n^{-r+(1/p-1/q)_+}$. The convergence rate $n^{-r+(1/p-1/q)_+}$ is optimal and a corresponding lower bound implies that if (3) is not satisfied then the rate $n^{-r+(1/p-1/q)_+}$ cannot be reached (see [5, Theorem 3]).

The optimal knots

$$0 = x_0^* < x_1^* < \ldots < x_{n-1}^* < x_n^* = +\infty$$

are determined by quantiles of $\omega^{1/\alpha}$, to be more precise,

$$\int_0^{x_i^*} \omega^{1/\alpha}(t)\,\mathrm{d}t = \frac{i}{n}\, \|\omega^{1/\alpha}\|_{L_1}. \tag{4}$$

In order to use the optimal quantizer (4) one has to know ω; otherwise he has to rely on some approximations of ω. Moreover, even if ω is known, it may be a complicated and/or non-monotonic function and therefore difficult to handle computationally. Driven by this motivation, the purpose of the present paper is to generalize the results of [5] even further to see how the quality of best approximations will change if the optimal quantizer ω is replaced in (4) by another quantizer κ.

A general answer to the aforementioned question is given in Theorems 1 and 3 of Section 2. They show, respectively, tight (up to a constant) upper and lower bounds for the error when a quantizer κ with $\|\kappa^{1/\alpha}\|_{L_1} < +\infty$ instead of ω is used to determine the knots. To be more specific, define

$$\mathcal{E}^q_p(\omega, \kappa) \;=\; \left\|\frac{\omega}{\kappa}\right\|_{L_\infty} \qquad \text{for } p \le q, \tag{5}$$

and

$$\mathcal{E}^q_p(\omega, \kappa) \;=\; \left(\int_D \frac{\kappa^{1/\alpha}(x)}{\|\kappa^{1/\alpha}\|_{L_1}} \left(\frac{\omega(x)}{\kappa(x)}\right)^{\frac{1}{1/q-1/p}} \mathrm{d}x\right)^{1/q-1/p} \qquad \text{for } p \ge q. \tag{6}$$

(Note that (5) and (6) are consistent for $p = q$.) If $\mathcal{E}_p^q(\omega,\kappa) < +\infty$ then the best achievable error is proportional to

$$\|\kappa^{1/\alpha}\|_{L_1}^{\alpha}\, \mathcal{E}_p^q(\omega,\kappa)\, \|f^{(r)}\psi\|_{L_p} n^{-r+(1/p-1/q)_+}.$$

This means, in particular, that for the error to behave as $n^{-r+(1/p-1/q)_+}$ it is sufficient (but not necessary) that $\kappa(x)$ decreases no faster than $\omega(x)$ as $|x| \to +\infty$. For instance, if the optimal quantizer is Gaussian, $\omega(x) = \exp(-x^2/2)$, then the optimal rate is still preserved if its exponential substitute $\kappa(x) = \exp(-a|x|)$ with arbitrary $a > 0$ is used. It also shows that, in case ω is not exactly known, it is much safer to overestimate than underestimate it, see also Example 5.

The use of a quantizer κ as above results in approximations that are worse than the optimal approximations by the factor of

$$\mathrm{FCTR}(p,q,\omega,\kappa) \;=\; \frac{\|\kappa^{1/\alpha}\|_{L_1}^{\alpha}}{\|\omega^{1/\alpha}\|_{L_1}^{\alpha}}\, \mathcal{E}_p^q(\omega,\kappa) \;\ge\; 1.$$

In Section 3, we calculate the exact values of this factor for various combinations of weights ϱ, ψ, and κ, including: Gaussian, exponential, log-normal, logistic, and t-Student. It turns out that in many cases $\mathrm{FCTR}(p,q,\omega,\kappa)$ is quite small, so that the loss in accuracy of approximation is well compensated by simplification of the weights.

The results for $q = 1$ are also applicable for problems of approximating ϱ-weighted integrals

$$\int_D f(x)\,\varrho(x)\,\mathrm{d}x \quad \text{for} \quad f \,\in\, F_{p,\psi}^r(D).$$

More precisely, the worst case errors of quadratures that are integrals of the corresponding piecewise interpolation polynomials approximating functions $f \in F_{p,\psi}^r(D)$ are the same as the errors for the ϱ-weighted $L_1(D)$ approximations. Hence their errors, proportional to n^{-r}, are (modulo a constant) the best possible among all quadratures. These results are especially important for unbounded domains, e.g., $D = \mathbb{R}_+$ or $D = \mathbb{R}$. For such domains, the integrals are often approximated by Gauss-Laguerre rules and Gauss-Hermite rules, respectively, see, e.g., [1, 3, 6]; however, their efficiency requires smooth integrands and the results are asymptotic. Moreover, it is not clear which Gaussian rules should be used when ψ is not a constant function. But, even for $\psi \equiv 1$, it is likely that the worst case errors (with respect to $F_{p,\psi}^r$) of Gaussian rules are much larger than $O(n^{-r})$, since the Weierstrass theorem holds only for compact D. A very interesting extension of Gaussian rules to functions with singularities has been proposed in [2]. However, the results of [2] are also asymptotic and it is not clear how the proposed rules behave for functions from spaces $F_{p,\psi}^r$. In the present paper, we deal with functions of bounded smoothness ($r < +\infty$) and provide worst-case error bounds that are minimal. We stress here that the regularity degree r is a fixed but arbitrary positive integer. The paper [4] proposes a different approach to the weighted integration over unbounded domains; however, it is restricted to regularity $r = 1$ only.

The paper is organized as follows. In the following section, we present ideas and results about alternative quantizers. The main results are Theorems 1 and 3. In Section 3, we apply our results to some specific cases for which numerical values of $\mathrm{FCTR}(p,q,\omega,\kappa)$ are calculated.

2 Optimal versus alternative quantizers

We consider ϱ-weighted L_q approximation in the space $F^r_{p,\psi}(D)$ as defined in the introduction; however, in contrast to [5], we do not assume that the weights ψ and ω are nonincreasing. Although the results of this paper pertain to domains D being an arbitrary interval, to begin with we assume that

$$D = \mathbb{R}_+.$$

We will explain later what happens in the general case including $D = \mathbb{R}$.

Let the knots $0 = x_0 < \ldots < x_n = +\infty$ be determined by a nonincreasing function (quantizer) $\kappa : D \to (0, +\infty)$ satisfying $\|\kappa^{1/\alpha}\|_{L_1} < +\infty$, i.e.,

$$\int_0^{x_i} \kappa^{1/\alpha}(t)\,\mathrm{d}t = \frac{i}{n}\,\|\kappa^{1/\alpha}\|_{L_1} \quad \text{with} \quad \alpha \,=\, r - \frac{1}{p} + \frac{1}{q}. \tag{7}$$

Let $\mathcal{T}_n f$ be a piecewise Taylor approximation of $f \in F^r_{p,\psi}(D)$ with break-points (7),

$$\mathcal{T}_n f(x) = \sum_{i=1}^{n} \mathbf{1}_{[x_{i-1},x_i)}(x) \sum_{k=0}^{r-1} \frac{f^{(k)}(x_{i-1})}{k!}(x - x_{i-1})^k.$$

We remind the reader of the definition of the quantity $\mathcal{E}^q_p(\omega, \kappa)$ in (5) and (6), which will be of importance in the following theorem.

Theorem 1 *Suppose that*

$$\mathcal{E}^q_p(\omega, \kappa) < +\infty.$$

Then for every $f \in F^q_{p,\psi}(D)$ we have

$$\|(f - \mathcal{T}_n f)\varrho\|_{L_q} \;\le\; c_1\, \|\kappa^{1/\alpha}\|^{\alpha}_{L_1}\, \mathcal{E}^q_p(\omega, \kappa)\, \|f^{(r)}\psi\|_{L_p}\, n^{-r+(1/p-1/q)_+}, \tag{8}$$

where

$$c_1 \;=\; \frac{1}{(r-1)!\,((r-1)p^* + 1)^{1/p^*}}.$$

Proof. We proceed as in the proof of [5, Theorem 1] to get that for $x \in [x_{i-1}, x_i)$

$$\begin{aligned}
\varrho(x)|f(x) - \mathcal{T}_n f(x)| \;&=\; \varrho(x)\left| \int_{x_{i-1}}^{x_i} f^{(r)}(t)\frac{(x-t)^{r-1}_+}{(r-1)!}\mathrm{d}t \right| \\
&\le\; c_1\,\frac{\omega(x)}{\kappa(x)} \left(\int_{x_{i-1}}^{x_i} |f^{(r)}(t)\psi(t)|^p \mathrm{d}t \right)^{1/p} \kappa(x)(x - x_{i-1})^{r-1/p}.
\end{aligned}$$

Since (cf. [5, p.36])

$$\kappa(x)(x - x_i)^{r-1/p} \le (\kappa^{1/\alpha}(x))^{1/q} \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n} \right)^{r-1/p},$$

the error is upper bounded as follows:

$$\begin{aligned}
\|(f - \mathcal{T}_n f)\varrho\|_{L_q} &= \left(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \varrho^q(x)|f(x) - \mathcal{T}_n f(x)|^q \mathrm{d}x \right)^{1/q} \\
&\le c_1 \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n} \right)^{r-1/p} \left(\sum_{i=1}^{n} \left(\int_{x_{i-1}}^{x_i} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)} \right)^q \mathrm{d}x \right) \left(\int_{x_{i-1}}^{x_i} |f^{(r)}(t)\psi(t)|^p \mathrm{d}t \right)^{q/p} \right)^{1/q}.
\end{aligned} \tag{9}$$

Now we maximize the right hand side of (9) subject to

$$\|f^{(r)}\psi\|_{L_p}^p = \sum_{i=1}^n \int_{x_{i-1}}^{x_i} |f^{(r)}(t)\psi(t)|^p \mathrm{d}t = 1.$$

After the substitution

$$A_i := \int_{x_{i-1}}^{x_i} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)}\right)^q \mathrm{d}x, \qquad B_i := \left(\int_{x_{i-1}}^{x_i} |f^{(r)}(t)\psi(t)|^p \mathrm{d}t\right)^{q/p},$$

this is equivalent to

$$\text{maximizing} \quad \textstyle\sum_{i=1}^n A_i B_i \quad \text{subject to} \quad \sum_{i=1}^n B_i^{p/q} = 1.$$

We have two cases:

For $p \le q$, we set $i^* = \arg\max_{1\le i\le n} A_i$, and use Jensen's inequality to obtain

$$\sum_{i=1}^n A_i B_i \le A_{i^*} \sum_{i=1}^n B_i \le A_{i^*} \left(\sum_{i=1}^n B_i^{p/q}\right)^{q/p} = A_{i^*}.$$

Hence the maximum equals A_{i^*} and it is attained at $B_i^* = 1$ for $i = i^*$, and $B_i^* = 0$ otherwise. In this case, the maximum is upper bounded by $\|\omega/\kappa\|_{L_\infty}^q \|\kappa^{1/\alpha}\|_{L_1}/n$, which means that

$$\|(f - \mathcal{T}_n f)\varrho\|_{L_q} \le c_1 \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n}\right)^\alpha \left\|\frac{\omega}{\kappa}\right\|_{L_\infty} \|f^{(r)}\psi\|_{L_p}.$$

For $p > q$ we use the method of Lagrange multipliers and find this way that the maximum equals

$$\left(\sum_{i=1}^n A_i^{\frac{1}{1-q/p}}\right)^{1-q/p} = \left(\sum_{i=1}^n \left(\int_{x_{i-1}}^{x_i} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)}\right)^q \mathrm{d}x\right)^{\frac{1}{1-q/p}}\right)^{1-q/p},$$

and is attained at

$$B_i^* = \left(\frac{A_i^{\frac{1}{1-q/p}}}{\sum_{j=1}^n A_j^{\frac{1}{1-q/p}}}\right)^{q/p}, \qquad 1 \le i \le n.$$

Since $1/(1-q/p) > 1$, by the probabilistic version of Jensen's inequality with density $n\,\kappa^{1/\alpha}/\|\kappa^{1/\alpha}\|_{L_1}$, we have

$$\left(\int_{x_{i-1}}^{x_i} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)}\right)^q \mathrm{d}x\right)^{\frac{1}{1-q/p}} \le \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n}\right)^{\frac{1}{p/q-1}} \int_{x_{i-1}}^{x_i} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)}\right)^{\frac{1}{1/q-1/p}} \mathrm{d}x.$$

This implies that

$$\left(\sum_{i=1}^n A_i^{\frac{1}{1-q/p}}\right)^{1-q/p} \le \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n}\right)^{q/p} \left(\int_0^{+\infty} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)}\right)^{\frac{1}{1/q-1/p}} \mathrm{d}x\right)^{1-q/p},$$

and finally

$$\|(f - \mathcal{T}_n f)\varrho\|_{L_q} \le c_1 \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n}\right)^r \left(\int_0^{+\infty} \kappa^{1/\alpha}(x) \left(\frac{\omega(x)}{\kappa(x)}\right)^{\frac{1}{1/q-1/p}} \mathrm{d}x\right)^{1/q-1/p} \|f^{(r)}\psi\|_{L_p},$$

as claimed since $1/q - 1/p = \alpha - r$. $\square$

Remark 2 If derivatives of f are difficult to compute or to sample, a piecewise Lagrange interpolation $\mathcal{L}_n$ can be used, as in [5]. Then the result is slightly weaker than that of the present Theorem 1; namely (cf. [5, Theorem 2]), there exists $c_1' > 0$ depending only on p, q, and r, such that

$$\limsup_{n\to\infty} \sup_{f\in F^r_{p,\psi}(D)} \frac{\|(f-\mathcal{L}_n f)\varrho\|_{L_q}}{\|f^{(r)}\psi\|_{L_p}}\, n^{r+(1/p-1/q)_+} \;\le\; c_1'\, \|\kappa^{1/\alpha}\|^{\alpha}_{L_1}\, \mathcal{E}^q_p(\omega,\kappa).$$

We now show that the error estimate of Theorem 1 cannot be improved.

Theorem 3 *There exists $c_2 > 0$ depending only on p, q, and r with the following property. For any approximation $\mathcal{A}_n$ that uses only information about function values and/or its derivatives (up to order $r-1$) at the knots $x_0,\ldots,x_n$ given by (7), we have*

$$\liminf_{n\to\infty} \sup_{f\in F^r_{p,\psi}(D)} \frac{\|(f-\mathcal{A}_n f)\varrho\|_{L_q}}{\|f^{(r)}\psi\|_{L_p}}\, n^{r-(1/p-1/q)_+} \;\ge\; c_2\, \|\kappa^{1/\alpha}\|^{\alpha}_{L_1}\, \mathcal{E}^q_p(\omega,\kappa). \tag{10}$$

Proof. We fix n and consider first the weighted L_q approximation on $[0,x_{n-1})$ assuming that in this interval the weights are step functions with break points x_i given by (7). Let ψ_i, ϱ_i, $\omega_i = \varrho_i/\psi_i$, and κ_i be correspondingly the values of ψ, ϱ, ω, and κ on successive intervals $[x_{i-1},x_i)$. Then we clearly have that $(x_i - x_{i-1})\kappa_i^{1/\alpha} = \|\kappa^{1/\alpha}\|_{L_1(0,x_{n-1})}/(n-1)$.

For simplicity, we write $I_i := (x_{i-1},x_i)$. Let f_i, $1\le i\le n-1$, be functions supported on I_i, such that $f_i^{(j)}(x_{i-1}) = 0 = f_i^{(j)}(x_i)$ for $0\le j\le r-1$, and

$$\|f_i\|_{L_q(I_i)} \ge c_2 (x_i - x_{i-1})^{\alpha} \|f_i^{(r)}\|_{L_p(I_i)}. \tag{11}$$

We also normalize f_i so that $\|f_i^{(r)}\|_{L_p(I_i)} = 1/\psi_i$. We stress that a positive c_2 in (11) exists and depends only on r, p, and q.

Since all $f_i^{(j)}$ nullify at the knots x_k, the 'sup' (worst case error) in (10) is bounded from below by

$$\mathrm{Sup}(n) \;:=\; \sup\left\{ \|f\varrho\|_{L_q} :\; f = \sum_{i=1}^{n-1} \beta_i f_i,\; \sum_{i=1}^{n-1} |\beta_i|^p = 1 \right\},$$

where we used the fact that $\|f^{(r)}\psi\|_{L_p} = \left(\sum_{i=1}^{n-1} |\beta_i|^p\right)^{1/p}$. For such f we have

$$\begin{aligned}
\|f\varrho\|_{L_q} &= \left(\sum_{i=1}^{n-1} \beta_i^q \|f_i\varrho\|^q_{L_q(I_i)}\right)^{1/q} = \left(\sum_{i=1}^{n-1} \left(|\beta_i|\varrho_i \|f_i\|_{L_q(I_i)}\right)^q\right)^{1/q} \\
&\ge c_2 \left(\sum_{i=1}^{n-1} \left(|\beta_i|\varrho_i (x_i - x_{i-1})^{\alpha} \|f_i^{(r)}\|_{L_p(I_i)}\right)^q\right)^{1/q} \\
&= c_2 \left(\sum_{i=1}^{n-1} \left(|\beta_i|\, \frac{\omega_i}{\kappa_i}\, \kappa_i (x_i - x_{i-1})^{\alpha}\right)^q\right)^{1/q} \\
&= c_2 \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)^{\alpha} \left(\sum_{i=1}^{n-1} |\beta_i|^q \left(\frac{\omega_i}{\kappa_i}\right)^q\right)^{1/q}.
\end{aligned}$$

Thus we arrive at a maximization problem that we already had in the proof of Theorem 1.

For $p \le q$ we have

$$\mathrm{Sup}(n) \;=\; c_2\left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)^{\alpha}\max_{1\le i\le n-1}\frac{\omega_i}{\kappa_i} \;=\; c_2\left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)^{\alpha}\operatorname*{ess\,sup}_{0\le x<x_{n-1}}\frac{\omega(x)}{\kappa(x)},$$

while for $p > q$ we have

$$\begin{aligned}
\mathrm{Sup}(n) &= c_2\left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)^{\alpha}\left(\sum_{i=1}^{n-1}\left(\frac{\omega_i}{\kappa_i}\right)^{\frac{1}{\alpha-r}}\right)^{\alpha-r}\\
&= c_2\left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)^{r}\left(\sum_{i=1}^{n-1}\left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)\left(\frac{\omega_i}{\kappa_i}\right)^{\frac{1}{\alpha-r}}\right)^{\alpha-r}\\
&= c_2\left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n-1}\right)^{r}\left(\int_0^{x_{n-1}}\kappa^{1/\alpha}(x)\left(\frac{\omega(x)}{\kappa(x)}\right)^{\frac{1}{\alpha-r}}\mathrm{d}x\right)^{\alpha-r},
\end{aligned}$$

as claimed.

For arbitrary weights, we replace ψ, ϱ, and κ with the corresponding step functions with

$$\psi_i = \operatorname*{ess\,sup}_{x\in(x_{i-1},x_i)}\psi(x),\quad \varrho_i = \operatorname*{ess\,inf}_{x\in(x_{i-1},x_i)}\varrho(x),\quad \kappa_i = \left(\frac{\|\kappa^{1/\alpha}\|_{L_1}}{n(x_i-x_{i-1})}\right)^{\alpha},\qquad 1\le i\le n-1,$$

and go with n to $+\infty$. □

We now comment on what happens when the domain is different from $\mathbb{R}_+$. It is clear that Theorems 1 and 3 remain valid for D being a compact interval, say $D=[0,c]$ with $c<+\infty$. Consider

$$D=\mathbb{R}.$$

In this case, we assume that κ is nonincreasing on $[0,+\infty)$ and nondecreasing on $(-\infty,0]$. We have $2n+1$ knots x_i, which are determined by the condition

$$\int_0^{x_i}\kappa^{1/\alpha}(t)\,\mathrm{d}t \;=\; \frac{i}{2n}\,\|\kappa^{1/\alpha}\|_{L_1(\mathbb{R})},\qquad |i|\le n \tag{12}$$

(where $\int_0^{-a} = -\int_a^0$). Note that (12) automatically implies $x_0=0$. The piecewise Taylor approximation is also correspondingly defined for negative arguments. With these modifications, the corresponding Theorems 1 and 3 have literally the same formulation for $D=\mathbb{R}$ and for $D=\mathbb{R}_+$.

Observe that the error estimates of Theorems 1 and 3 for arbitrary κ differ from the error for optimal $\kappa=\omega$ by the factor

$$\mathrm{FCTR}(p,q,\omega,\kappa) \;:=\; \frac{\|\kappa^{1/\alpha}\|_{L_1}^{\alpha}}{\|\omega^{1/\alpha}\|_{L_1}^{\alpha}}\,\mathcal{E}_p^q(\omega,\kappa).$$

From this definition it is clear that for any $s,t>0$ we have

$$\mathrm{FCTR}(p,q,s\,\omega,t\,\kappa) = \mathrm{FCTR}(p,q,\omega,\kappa).$$

This quantity satisfies the following estimates.

Proposition 4 *We have*

$$1 \;=\; \mathrm{FCTR}(p,q,\omega,\omega) \;\leq\; \mathrm{FCTR}(p,q,\omega,\kappa) \;\leq\; \frac{\|\kappa^{1/\alpha}\|_{L_1}^{\alpha}}{\|\omega^{1/\alpha}\|_{L_1}^{\alpha}} \left\|\frac{\omega}{\kappa}\right\|_{L_\infty}. \tag{13}$$

The rightmost inequality is actually an equality whenever $p \leq q$.

Proof. Assume without loss of generality that $\|\kappa^{1/\alpha}\|_{L_1} = \|\omega^{1/\alpha}\|_{L_1} = 1$, so that $\mathrm{FCTR}(p,q,\omega,\kappa) = \mathcal{E}_p^q(\omega,\kappa)$. Then for any p and q

$$1 = \|\omega^{1/\alpha}\|_{L_1}^{\alpha} \leq \|\kappa^{1/\alpha}\|_{L_1}^{\alpha} \left\|\frac{\omega^{1/\alpha}}{\kappa^{1/\alpha}}\right\|_{L_\infty}^{\alpha} = \left\|\frac{\omega}{\kappa}\right\|_{L_\infty},$$

which equals $\mathcal{E}_p^q(\omega,\kappa)$ for $p \leq q$. For $p > q$ we have $(1/q - 1/p)/\alpha = 1 - r/\alpha < 1$, so that we can use Jensen's inequality to get

$$\mathcal{E}_p^q(\omega,\kappa) = \left(\int_D \kappa^{1/\alpha}(x)\left(\frac{\omega^{1/\alpha}(x)}{\kappa^{1/\alpha}(x)}\right)^{\frac{\alpha}{\alpha-r}} \mathrm{d}x\right)^{\left(\frac{\alpha-r}{\alpha}\right)\alpha} \geq \left(\int_D \kappa^{1/\alpha}(x)\left(\frac{\omega^{1/\alpha}(x)}{\kappa^{1/\alpha}(x)}\right)\mathrm{d}x\right)^{\alpha} = 1.$$

The remaining inequality $\mathcal{E}_p^q(\omega,\kappa) \leq \left\|\frac{\omega}{\kappa}\right\|_{L_\infty}$ is obvious. □

Although the main idea of this paper is to replace ω by another function κ that is easier to handle, our results allow a further interesting observation that is illustrated in the following example.

Example 5 Let $D = \mathbb{R}$,

$$r = 1, \qquad p = +\infty, \qquad q = 1,$$

and the weights

$$\varrho(x) = \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right), \qquad \psi(x) = 1.$$

Then $\alpha = 2$ and $1/q - 1/p = 1$, and $\omega(x) = \varrho(x)$. Suppose that instead of ω we use

$$\kappa_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \qquad \text{with} \qquad \sigma^2 > 0.$$

Since $p > q$, we have

$$\mathrm{FCTR}(p,q,\omega,\kappa_\sigma) \;=\; \frac{\|\kappa_\sigma^{1/2}\|_{L_1}^2}{\|\omega^{1/2}\|_{L_1}^2} \int_{\mathbb{R}} \frac{\kappa_\sigma^{1/2}(x)}{\|\kappa_\sigma^{1/2}\|_{L_1}} \frac{\omega(x)}{\kappa_\sigma(x)}\,\mathrm{d}x \;=\; \begin{cases} +\infty & \text{if} \quad \sigma^2 \leq 1/2, \\ \frac{\sigma^2}{\sqrt{2\sigma^2-1}} & \text{if} \quad \sigma^2 > 1/2. \end{cases}$$

The graph of $\mathrm{FCTR}(p,q,\omega,\kappa_\sigma)$ is drawn in Fig. 1. It follows that it is safer to overestimate the actual variance $\sigma^2 = 1$ than to underestimate it.

3 Special cases

Below we apply our results to specific weights ϱ, ψ, and specific values of p and q.

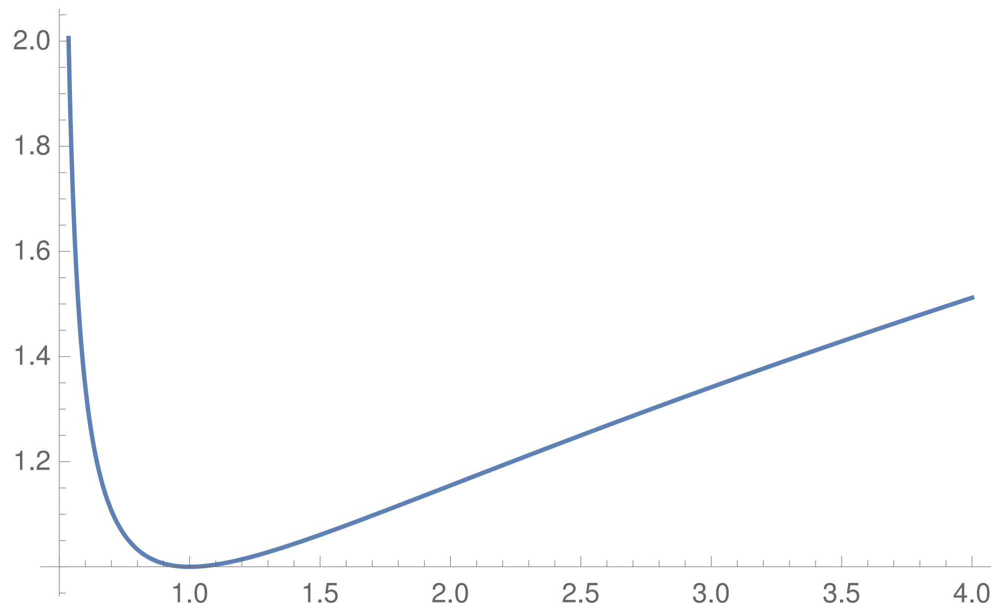

Figure 1: Plot of $\mathrm{FCTR}(p, q, \omega, \kappa_\sigma)$ versus σ^2 from Example 5

3.1 Gaussian ϱ and ψ

Consider $D = \mathbb{R}$,

$$\varrho(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right) \quad \text{and} \quad \psi(x) = \exp\left(\frac{-x^2}{2\lambda^2}\right)$$

for positive σ and λ. Since

$$\omega(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}(\sigma^{-2} - \lambda^{-2})\right),$$

for $\|\omega^{1/\alpha}\|_{L_1} < \infty$ we have to have $\lambda > \sigma$, and then

$$\|\omega^{1/\alpha}\|_{L_1}^{\alpha} = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{\alpha\, 2\pi}{\sigma^{-2} - \lambda^{-2}}\right)^{\alpha/2}.$$

We propose using

$$\kappa(x) = \kappa_a(x) = \exp(-|x|\, a) \quad \text{for} \quad a > 0.$$

Then $\|\kappa_a^{1/\alpha}\|_{L_1(D)} = 2\alpha/a$ and the points $x_{-n}, \ldots, x_n$ satisfying (12),

$$\int_0^{x_i} \kappa_a^{1/\alpha}(t)\, \mathrm{d}t = \frac{i}{2n} \int_{-\infty}^{\infty} \kappa_a^{1/\alpha}(t)\, \mathrm{d}t \quad \text{for} \quad |i| \le n,$$

are given by

$$x_i = -x_{-i} = -\frac{\alpha}{a} \ln\left(1 - \frac{i}{n}\right) \quad \text{for } 0 \le i \le n. \tag{14}$$

In particular, we have

$$x_{-n} = -\infty, \quad x_0 = 0, \quad \text{and} \quad x_n = \infty.$$

We now consider the two cases $p \le q$ and $p > q$ separately:

3.1.1 Case of $p \le q$

Clearly

$$\mathcal{E}_p^q(\omega,\kappa_a) \;=\; \left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} \;=\; \frac{1}{\sigma\sqrt{2\,\pi}}\,\exp\left(\frac{a^2}{2\,(\sigma^{-2}-\lambda^{-2})}\right)$$

and

$$\min_{a>0}\|\kappa_a^{1/\alpha}\|_{L_1(D)}^\alpha\left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty}\quad\text{is attained at}\quad a_* \;=\; \sqrt{\alpha\,\left(\frac{1}{\sigma^2}-\frac{1}{\lambda^2}\right)}.$$

Hence, for $p \le q$ we have that

$$\mathrm{FCTR}(p,q,\omega,\kappa_{a_*}) \;=\; \left(\frac{2\,\mathrm{e}}{\pi}\right)^{\alpha/2}.$$

Note that $\mathrm{FCTR}(p,q,\omega,\kappa_{a_*})$ does not depend on σ and λ (as long as $\lambda>\sigma$). For instance, we have the following rounded values:

α	1	2	3	4
$\mathrm{FCTR}(p,q,\omega,\kappa_{a_*})$	1.315	1.731	2.276	2.995

3.1.2 Case of $p > q$

We have now

$$\mathcal{E}_p^q(\omega,\kappa_a) \;=\; \left(\frac{a}{\alpha}\right)^{\alpha-r}\,\frac{1}{\sigma\sqrt{2\,\pi}}\,A^{\alpha-r},$$

where

$$\begin{aligned}
A \;&=\; \int_0^\infty \exp\left(-\frac{x^2\,(\sigma^{-2}-\lambda^{-2})}{2\,(\alpha-r)}+\frac{a\,x\,r}{\alpha\,(\alpha-r)}\right)\,\mathrm{d}x\\
&=\; \int_0^\infty \exp\left(-\frac{\sigma^{-2}-\lambda^{-2}}{2\,(\alpha-r)}\,\left(x-\frac{a\,r}{\alpha\,(\sigma^{-2}-\lambda^{-2})}\right)^2+\frac{a^2\,r^2}{2\,\alpha^2\,(\alpha-r)\,(\sigma^{-2}-\lambda^{-2})}\right)\,\mathrm{d}x\\
&=\; \exp\left(\frac{a^2\,r^2}{2\,\alpha^2\,(\alpha-r)\,(\sigma^{-2}-\lambda^{-2})}\right)\,\int_{-\frac{a\,r}{\alpha\,(\sigma^{-2}-\lambda^{-2})}}^\infty \exp\left(-\frac{(\sigma^{-2}-\lambda^{-2})\,t^2}{2\,(\alpha-r)}\right)\,\mathrm{d}t\\
&=\; \exp\left(\frac{a^2\,r^2}{2\,\alpha^2\,(\alpha-r)\;(\sigma^{-2}-\lambda^{-2})}\right)\,\sqrt{\frac{\pi\,(\alpha-r)}{2\,(\sigma^{-2}-\lambda^{-2})}}\,\left[1+\mathrm{erf}\left(\frac{a\,r}{\alpha\,\sqrt{2\,(\alpha-r)\,(\sigma^{-2}-\lambda^{-2})}}\right)\right],
\end{aligned}$$

where $\mathrm{erf}(z) := \frac{2}{\sqrt{\pi}}\int_0^z \mathrm{e}^{-t^2}\,\mathrm{d}t$. This gives

$$\begin{aligned}
\mathcal{E}_p^q(\omega,\kappa_a) \;&=\; \left(\frac{a^2\,\pi\,(\alpha-r)}{\alpha^2\,2\,(\sigma^{-2}-\lambda^{-2})}\right)^{(\alpha-r)/2}\,\frac{1}{\sigma\sqrt{2\,\pi}}\,\exp\left(\frac{a^2\,r^2}{\alpha^2\,2\,(\sigma^{-2}-\lambda^{-2})}\right)\\
&\qquad\times\left[1+\mathrm{erf}\left(\frac{a\,r}{\alpha\,\sqrt{2\,(\alpha-r)\,(\sigma^{-2}-\lambda^{-2})}}\right)\right]^{\alpha-r}.
\end{aligned}$$

Since

$$\frac{\|\kappa_a^{1/\alpha}\|_{L_1(D)}^\alpha}{\|\omega^{1/\alpha}\|_{L_1(D)}^\alpha} = \sigma\sqrt{2\pi}\left(\frac{2\alpha(\sigma^{-2}-\lambda^{-2})}{\pi a^2}\right)^{\alpha/2}$$

we obtain

$$\mathrm{FCTR}(p,q,\omega,\kappa_a) \;=\; \left(\frac{2\alpha(\sigma^{-2}-\lambda^{-2})}{\pi a^2}\right)^{r/2} \left(\frac{\alpha-r}{\alpha}\right)^{(\alpha-r)/2} \exp\left(\frac{a^2\,r^2}{2\,\alpha^2(\sigma^{-2}-\lambda^{-2})}\right)$$
$$\times \left[1+\mathrm{erf}\left(\frac{a\,r}{\alpha\,\sqrt{2\,(\alpha-r)\,(\sigma^{-2}-\lambda^{-2})}}\right)\right]^{\alpha-r}.$$

We provide some numerical tests for $q=1$ and $p=2$ or $p=\infty$. Then $\alpha=r+1/2$ or $\alpha=r+1$, respectively. Recall that results for $q=1$ are also applicable to the ϱ-integration problem.

For $r\in\{1,2\}$, $p\in\{2,\infty\}$, $\lambda=2$ and $\sigma=1$, we vary a and obtain the following rounded values:

a	1	2	3	4		
$\mathrm{FCTR}(2,1,\omega,\kappa_a)$	1.135	1.476	4.361	26.036	$r=1$	$p=2$
$\mathrm{FCTR}(2,1,\omega,\kappa_a)$	1.645	1.552	5.836	65.061	$r=2$	
$\mathrm{FCTR}(\infty,1,\omega,\kappa_a)$	1.172	1.179	1.979	4.920	$r=1$	$p=\infty$
$\mathrm{FCTR}(\infty,1,\omega,\kappa_a)$	1.733	1.269	2.617	11.826	$r=2$	

3.2 Gaussian ϱ and Exponential ψ

Consider $D=\mathbb{R}$,

$$\varrho(x) \;=\; \frac{1}{\sigma\,\sqrt{2\,\pi}}\exp\left(\frac{-x^2}{2\,\sigma^2}\right) \quad \text{and} \quad \psi(x) \;=\; \exp\left(\frac{-|x|}{\lambda}\right)$$

for positive λ and σ. Now

$$\omega(x)=\frac{\varrho(x)}{\psi(x)}=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{x^2}{2\sigma^2}+\frac{|x|}{\lambda}\right), \tag{15}$$

and

$$\begin{aligned}
\|\omega^{1/\alpha}\|_{L_1(D)}^{\alpha} &= \frac{1}{\sigma\,\sqrt{2\,\pi}}\left(2\int_0^\infty \exp\left(\frac{-x^2}{2\,\sigma^2\,\alpha}+\frac{x}{\lambda\,\alpha}\right)\mathrm{d}x\right)^{\alpha}\\
&= \frac{1}{\sigma\,\sqrt{2\,\pi}}\left(2\int_0^\infty \exp\left(\frac{-(x/\sigma-\sigma/\lambda)^2}{2\,\alpha}+\frac{\sigma^2}{2\,\lambda^2\,\alpha}\right)\mathrm{d}x\right)^{\alpha}\\
&= \frac{1}{\sigma\,\sqrt{2\,\pi}}\exp\left(\frac{\sigma^2}{2\,\lambda^2}\right)\left(\sigma\,\sqrt{2\,\pi\,\alpha}\,\frac{2}{\sqrt{\pi}}\int_{-\sigma/(\lambda\,\sqrt{2\,\alpha})}^{\infty}\exp(-y^2)\,\mathrm{d}y\right)^{\alpha}\\
&= \frac{1}{\sigma\,\sqrt{2\,\pi}}\exp\left(\frac{\sigma^2}{2\,\lambda^2}\right)\left(\sigma\,\sqrt{2\,\pi\,\alpha}\left(1+\mathrm{erf}\left(\frac{\sigma}{\lambda\,\sqrt{2\,\alpha}}\right)\right)\right)^{\alpha}.
\end{aligned}$$

As before, we propose using $\kappa_a(x)=\exp(-|x|\,a)$. Hence $\|\kappa_a^{1/\alpha}\|_{L_1}=2\,\alpha/a$ and the points x_i are given by (14).

3.2.1 Case of $p\le q$

We have

$$\mathcal{E}_p^q(\omega,\kappa_a) \;=\; \left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} \;=\; \frac{1}{\sigma\,\sqrt{2\,\pi}}\exp\left(\frac{\sigma^2\,(a+\lambda^{-1})^2}{2}\right).$$

It is easy to verify that the minimum over $a > 0$ satisfies

$$\min_{a>0} \|\kappa_a^{1/\alpha}\|_{L_1(D)}^{\alpha} \left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} = \frac{1}{\sigma\sqrt{2\pi}} \left(\frac{2\alpha}{a_*}\right)^{\alpha} \exp\left(\frac{\sigma^2 (a_* + \lambda^{-1})^2}{2}\right)$$

for

$$a_* = \frac{\sqrt{1 + 4\alpha\lambda^2/\sigma^2} - 1}{2\lambda}.$$

Therefore

$$\mathrm{FCTR}(p, q, \omega, \kappa_{a_*}) = \left(\sqrt{\frac{2\alpha}{\pi}} \frac{1}{a_*\sigma\left(1 + \mathrm{erf}(\sigma/\sqrt{2\alpha\lambda})\right)}\right)^{\alpha} \exp\left(\frac{\sigma^2 a_*(a_* + 2/\lambda)}{2}\right).$$

Note that the value of FCTR depends on p and q only via α. Rounded values of FCTR for $\alpha \in \{1, 2\}$ and $\sigma = 1$ and various λ's are[1]:

λ	1	5	10	20	30	100	
FCTR	1.723	1.183	1.162	1.174	1.188	1.231	$\alpha = 1$
FCTR	2.468	1.460	1.436	1.465	1.491	1.573	$\alpha = 2$

3.2.2 Case of $p > q$

We have

$$\mathcal{E}_p^q(\omega, \kappa_a) = \left(\frac{a}{\alpha}\right)^{\alpha - r} \frac{1}{\sigma\sqrt{2\pi}} A^{\alpha - r},$$

where now

$$\begin{aligned}
A &= \int_0^\infty \exp\left(-\frac{x^2}{2\sigma^2(\alpha - r)} + x\left(\frac{a}{\alpha - r} + \frac{1}{\lambda(\alpha - r)} - \frac{a}{\alpha}\right)\right) \mathrm{d}x \\
&= \int_0^\infty \exp\left(-\frac{1}{2\sigma^2(\alpha - r)}\left(x^2 - 2x\sigma^2\left(\frac{ar}{\alpha} + \frac{1}{\lambda}\right)\right)\right) \mathrm{d}x \\
&= \exp\left(\frac{\sigma^2 (\frac{ar}{\alpha} + \frac{1}{\lambda})^2}{2(\alpha - r)}\right) \int_0^\infty \exp\left(-\frac{(x - \sigma^2(\frac{ar}{\alpha} + \frac{1}{\lambda}))^2}{2\sigma^2(\alpha - r)}\right) \mathrm{d}x \\
&= \exp\left(\frac{\sigma^2 (\frac{ar}{\alpha} + \frac{1}{\lambda})^2}{2(\alpha - r)}\right) \sqrt{\frac{\sigma^2\pi(\alpha - r)}{2}} \left[1 + \mathrm{erf}\left(\frac{\sigma(\frac{ar}{\alpha} + \frac{1}{\lambda})}{\sqrt{2(\alpha - r)}}\right)\right].
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{E}_p^q(\omega, \kappa) = \frac{1}{\sigma\sqrt{2\pi}} &\left(\frac{a^2\pi\sigma^2(\alpha - r)}{2\alpha^2}\right)^{(\alpha - r)/2} \exp\left(\frac{\sigma^2}{2}\left(\frac{ar}{\alpha} + \frac{1}{\lambda}\right)^2\right) \\
&\times \left[1 + \mathrm{erf}\left(\frac{\sigma(\frac{ar}{\alpha} + \frac{1}{\lambda})}{\sqrt{2(\alpha - r)}}\right)\right]^{\alpha - r}.
\end{aligned}$$

Since

$$\frac{\|\kappa_a^{1/\alpha}\|_{L_1(D)}^{\alpha}}{\|\omega^{1/\alpha}\|_{L_1(D)}^{\alpha}} = \left(\frac{2\alpha}{a}\right)^{\alpha} \sigma\sqrt{2\pi} \exp\left(-\frac{\sigma^2}{2\lambda^2}\right) \left(\sigma\sqrt{2\pi\alpha}\left(1 + \mathrm{erf}\left(\frac{\sigma}{\lambda\sqrt{2\alpha}}\right)\right)\right)^{-\alpha}$$

[1]Computed with MATHEMATICA

we obtain

$$\begin{aligned}\mathrm{FCTR}(p,q,\omega,\kappa_a) &= \left(\frac{1}{a\sigma}\sqrt{\frac{2\alpha}{\pi}}\right)^{\alpha}\left(\frac{a^2\,\pi\,\sigma^2\,(\alpha-r)}{2\,\alpha^2}\right)^{(\alpha-r)/2}\exp\left(\frac{\sigma^2}{2}\left(\left(\frac{a\,r}{\alpha}+\frac{1}{\lambda}\right)^2-\frac{1}{\lambda^2}\right)\right)\\ &\quad\times\frac{\left[1+\mathrm{erf}\left(\frac{\sigma\,(\frac{a\,r}{\alpha}+\frac{1}{\lambda})}{\sqrt{2\,(\alpha-r)}}\right)\right]^{\alpha-r}}{\left[1+\mathrm{erf}\left(\frac{\sigma}{\lambda\sqrt{2\alpha}}\right)\right]^{\alpha}}.\end{aligned}$$

We again provide numerical results, first for the case $p=2$ and $q=1$, i.e., $\alpha=r+1/2$. For $r\in\{1,2\}$ and varying a, we obtain the following rounded values:

a	1	2	3	4		
$\mathrm{FCTR}(2,1,\omega,\kappa_a)$	1.273	2.426	9.570	66.233	$\lambda=1,\ \sigma=1$	$r=1$
$\mathrm{FCTR}(2,1,\omega,\kappa_a)$	1.181	1.642	4.652	23.070	$\lambda=2,\ \sigma=1$	
$\mathrm{FCTR}(2,1,\omega,\kappa_a)$	1.747	2.546	12.473	146.677	$\lambda=1,\ \sigma=1$	$r=2$
$\mathrm{FCTR}(2,1,\omega,\kappa_a)$	1.747	1.729	5.683	44.797	$\lambda=2,\ \sigma=1$	

We now change p to $p=\infty$, and choose again $q=1$, which implies $\alpha=r+1$. For $r\in\{1,2\}$ and varying a we obtain the following rounded values:

a	1	2	3	4		
$\mathrm{FCTR}(\infty,1,\omega,\kappa_a)$	1.203	1.512	3.156	9.409	$\lambda=1,\ \sigma=1$	$r=1$
$\mathrm{FCTR}(\infty,1,\omega,\kappa_a)$	1.199	1.242	2.081	4.888	$\lambda=2,\ \sigma=1$	
$\mathrm{FCTR}(\infty,1,\omega,\kappa_a)$	1.724	1.700	4.509	23.434	$\lambda=1,\ \sigma=1$	$r=2$
$\mathrm{FCTR}(\infty,1,\omega,\kappa_a)$	1.827	1.366	2.647	9.897	$\lambda=2,\ \sigma=1$	

3.3 Log-Normal ϱ and constant ψ

Consider $D=\mathbb{R}_+$, $\psi(x)=1$ and

$$\varrho(x) \;=\; \omega(x) \;=\; \frac{1}{x\,\sigma\,\sqrt{2\,\pi}}\,\exp\left(-\frac{(\ln x-\mu)^2}{2\,\sigma^2}\right) \tag{16}$$

for given $\mu\in\mathbb{R}$ and $\sigma>0$.

For κ we take

$$\kappa_c(x) \;=\; \begin{cases} 1 & \text{if } x\in[0,\mathrm{e}^{\mu}],\\ \exp(c\,(\mu-\ln x)) & \text{if } x>\mathrm{e}^{\mu},\end{cases}$$

for positive c. For $\kappa_c^{1/\alpha}$ to be integrable we have to restrict c so that

$$c>\alpha.$$

It can be checked that

$$\|\kappa_c^{1/\alpha}\|_{L_1(D)}^{\alpha} \;=\; \left(\frac{c}{c-\alpha}\right)^{\alpha}\,\mathrm{e}^{\alpha\,\mu}.$$

Then the points x_i for $i=0,1,\ldots,n$ that satisfy (7) are given by

$$x_i=\begin{cases} \frac{c}{c-\alpha}\,\mathrm{e}^{\mu}\,\frac{i}{n} & \text{for } i\le n\,\frac{c-\alpha}{c},\\ \mathrm{e}^{\mu}\,\left(\frac{\alpha}{c}\,\frac{n}{n-i}\right)^{\alpha/(c-\alpha)} & \text{otherwise.}\end{cases}$$

3.3.1 Case of $p \le q$

We determine $\|\omega/\kappa_c\|_{L_\infty(D)}$. For $x \le \mathrm{e}^\mu$ we have

$$\frac{\omega(x)}{\kappa_c(x)} = \omega(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2} - t\right) \quad \text{with} \quad t = \ln x \le \mu.$$

Its maximum is attained at $t = \mu - \sigma^2$ and

$$\max_{x \le \mathrm{e}^\mu} \frac{\omega(x)}{\kappa_c(x)} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{\sigma^2}{2} - \mu\right).$$

For $x > \mathrm{e}^\mu$,

$$\frac{\omega(x)}{\kappa_c(x)} = \frac{1}{\exp(c\,\mu)\,\sigma\sqrt{2\pi}} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2} + t\,(c-1)\right) \quad \text{with } t = \ln x > \mu.$$

The maximum of the expression above is attained at $t = \mu + \sigma^2\,(c-1)$ and

$$\begin{aligned}
\sup_{x > \mathrm{e}^\mu} \frac{\omega(x)}{\kappa_c(x)} &= \frac{1}{\exp(c\,\mu)\,\sigma\sqrt{2\pi}} \exp\left((c-1)\,\mu + \frac{(c-1)^2\,\sigma^2}{2}\right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\mu + \frac{(c-1)^2\,\sigma^2}{2}\right).
\end{aligned}$$

This yields that

$$\left\|\frac{\omega}{\kappa_c}\right\|_{L_\infty(D)} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\mu + \frac{\sigma^2}{2} \max(1, (c-1)^2)\right).$$

To find the optimal value of c, note that

$$\left\|\frac{\omega}{\kappa_c}\right\|_{L_\infty(D)} \|\kappa_c^{1/\alpha}\|_{L_1(D)}^\alpha = \frac{\mathrm{e}^{(\alpha-1)\,\mu}}{\sigma\sqrt{2\pi}} \,(f(c))^\alpha,$$

where $f(c)$ is given by

$$f(c) = \exp\left(\frac{\sigma^2 \max(1, (c-1)^2)}{2\,\alpha}\right) \left(1 + \frac{\alpha}{c-\alpha}\right).$$

Consider first $\alpha \ge 2$ and recall the restriction $c > \alpha$. For such values of c we have

$$f(c) = \exp\left(\frac{\sigma^2\,(c-1)^2}{2\,\alpha}\right) \left(1 + \frac{\alpha}{c-\alpha}\right)$$

and hence

$$f'(c) = \frac{\sigma^2}{\alpha\,(c-\alpha)^2} \exp\left(\frac{\sigma^2}{2\alpha}\,(c-1)^2\right) \left(c\,(c-1)\,(c-\alpha) - \frac{\alpha^2}{\sigma^2}\right).$$

Therefore,

$$\min_{c > \alpha} f(c) = f(c_*) = \exp\left(\frac{\sigma^2\,(c_*-1)^2}{2\,\alpha}\right) \frac{c_*}{c_*-\alpha}$$

for c_* such that

$$c_* > \alpha \quad \text{and} \quad c_* \, (c_* - 1) \, (c_* - \alpha) \, = \, \frac{\alpha^2}{\sigma^2}. \tag{17}$$

Consider next $\alpha \in (0, 2)$. Then for $c \le 2$, the minimum of $f(c)$ is attained in $c = 2$, and it is a global minimum if $2(2-\alpha) \ge \alpha^2/\sigma^2$. Otherwise, the minimum is at c_* given by (17).

In summary, for $\alpha > 0$, we have

$$\min_{c>\alpha} \left\| \frac{\omega}{\kappa_c} \right\|_{L_\infty(D)} \|\kappa_c^{1/\alpha}\|_{L_1(D)}^\alpha \, = \, \frac{\mathrm{e}^{(\alpha-1)\,\mu}}{\sigma\,\sqrt{2\,\pi}} \times \begin{cases} \exp\left(\frac{\sigma^2\,(c_*-1)^2}{2}\right) \left(\frac{c_*}{c_*-\alpha}\right)^\alpha & \text{if } \alpha \, \ge \, 2 \\ & \text{or } 2\,(2-\alpha) \, \le \, \frac{\alpha^2}{\sigma^2}, \\ \exp\left(\frac{\sigma^2}{2}\right) \left(\frac{2}{2-\alpha}\right)^\alpha & \text{otherwise.} \end{cases}$$

To derive the value of the L_1 norm of $\omega^{1/\alpha}$, we will use the following well-known facts: If $\mathbf{X}_{\sigma,\mu}$ is a log-normally distributed random variable with parameters σ and μ, then the mean value and the variance of $\mathbf{X}_{\sigma,\mu}$ are, respectively, equal to

$$\mathbb{E}(\mathbf{X}_{\sigma,\mu}) \, = \, \exp\left(\sigma^2/2 + \mu\right) \quad \text{and} \quad \mathbb{E}\left(\mathbf{X}_{\sigma,\mu} - \mathbb{E}(\mathbf{X}_{\sigma,\mu})\right)^2 \, = \, \left(\exp\left(\sigma^2\right) - 1\right) \, \exp\left(\sigma^2 + 2\,\mu\right).$$

Hence

$$\mathbb{E}\left(\mathbf{X}_{\sigma,\mu}^2\right) \, = \, \exp\left(2\sigma^2 + 2\,\mu\right). \tag{18}$$

If $\alpha = 1$, then $\|\omega^{1/\alpha}\|_{L_1(D)}^\alpha = 1$, and then

$$\mathrm{FCTR}(p, q, \omega, \kappa_{c_*}) \, = \, \frac{1}{\sigma\,\sqrt{2\,\pi}} \begin{cases} \frac{c_*}{c_*-1} \, \exp\left(\frac{\sigma^2\,(c_*-1)^2}{2}\right) & \text{if } 2 \, \le \, \frac{1}{\sigma^2}, \\ 2 \, \exp\left(\frac{\sigma^2}{2}\right) & \text{otherwise.} \end{cases}$$

For $\alpha \in (1, 2)$, to simplify the notation, we will use, in the following, parameters s and γ given by

$$s \, = \, \frac{2\,\alpha}{\alpha - 1} \quad \text{and} \quad \gamma \, = \, \frac{\sigma\,\sqrt{\alpha}}{s}.$$

The change of the variable $x = t^s$ gives

$$\begin{aligned} (\sigma\,\sqrt{2\,\pi})^{1/\alpha} \, \|\omega^{1/\alpha}\|_{L_1(D)} \, &= \, \int_0^\infty \frac{1}{x^{1/\alpha}} \, \exp\left(\frac{-(\ln x - \mu)^2}{2\,\alpha\,\sigma^2}\right) \mathrm{d}x \\ &= \, s \int_0^\infty t^{s - s/\alpha - 1} \, \exp\left(\frac{-(\ln t^s - \mu)^2}{2\,\alpha\,\sigma^2}\right) \mathrm{d}t \\ &= \, s \int_0^\infty t \, \exp\left(\frac{-(\ln t - \mu/s)^2}{2\,(\sigma\,\sqrt{\alpha}/s)^2}\right) \mathrm{d}t \\ &= \, s\,\gamma\,\sqrt{2\,\pi} \int_0^\infty \frac{t^2}{t\,\gamma\,\sqrt{2\,\pi}} \, \exp\left(\frac{-(\ln t - \mu/s)^2}{2\,\gamma^2}\right) \mathrm{d}t. \end{aligned}$$

The last integral is the expected value of the square of a log-normal random variable $\mathbf{X}_{\gamma,\mu/s}$ with the parameter σ replaced by γ and μ replaced by μ/s. Hence

$$\|\omega^{1/\alpha}\|_{L_1(D)}^\alpha = \frac{(s\,\gamma\,\sqrt{2\,\pi})^\alpha}{\sigma\,\sqrt{2\,\pi}} \, \exp\left(2\,\gamma^2\,\alpha + \frac{2\,\mu\,\alpha}{s}\right) \, = \, \frac{(\sigma\,\sqrt{2\,\pi\,\alpha})^\alpha}{\sigma\,\sqrt{2\,\pi}} \, \exp\left(\frac{\sigma^2\,(\alpha-1)^2}{2} + \mu\,(\alpha - 1)\right).$$

This gives us

$$\mathrm{FCTR}(p,q,\omega,\kappa_{c_*}) \;=\; \left(\frac{c_*}{(c_*-\alpha)\,\sigma\,\sqrt{2\,\pi\,\alpha}}\right)^{\alpha} \exp\left(\frac{\sigma^2\left((c_*-1)^2-(\alpha-1)^2\right)}{2}\right)$$

if either $\alpha \geq 2$ or $\alpha < 2$ and $2(2-\alpha) \leq \alpha^2/\sigma^2$, and

$$\mathrm{FCTR}(p,q,\omega,\kappa_2) \;=\; \left(\frac{2}{(2-\alpha)\,\sigma\,\sqrt{2\,\pi\,\alpha}}\right)^{\alpha} \exp\left(\frac{\sigma^2\left(1-(\alpha-1)^2\right)}{2}\right)$$

if $\alpha < 2$ and $2(2-\alpha) > \alpha^2/\sigma^2$.

Rounded values for FCTR for various σ and α are[2]:

σ	1	2	3	
FCTR	1.315	2.948	23.941	$\alpha = 1$
FCTR	2.988	4.615	7.573	$\alpha = 2$

3.3.2 Case of $p > q$

Now

$$\mathcal{E}_p^q(\omega,\kappa_c) \;=\; \frac{1}{\sigma\,\sqrt{2\,\pi}}\,\left(\frac{c-\alpha}{c\,\mathrm{e}^{\mu}}\right)^{\alpha-r}\,(I_1+I_2)^{\alpha-r},$$

where

$$I_1 \;=\; \int_0^{\mathrm{e}^{\mu}} \exp\left(-\frac{1}{\alpha-r}\,\left[\frac{(\ln x-\mu)^2}{2\,\sigma^2}+\ln x\right]\right)\mathrm{d}x$$

and

$$I_2 \;=\; \int_{\mathrm{e}^{\mu}}^{\infty} \exp\left(-\frac{1}{\alpha-r}\,\left[\frac{(\ln x-\mu)^2}{2\,\sigma^2}+\ln x\right]-\frac{r\,c}{\alpha\,(\alpha-r)}\,(\mu-\ln x)\right)\mathrm{d}x.$$

In what follows, for both integrals, we will use first the change of variables $y = \ln x - \mu$. We have

$$\begin{aligned}
I_1 \;&=\; \int_{-\infty}^{0} \exp(y+\mu)\,\exp\left(-\frac{1}{\alpha-r}\,\left[\frac{y^2}{2\,\sigma^2}+y+\mu\right]\right)\mathrm{d}x\\
&=\; \exp\left(\mu\,\frac{\alpha-r-1}{\alpha-r}\right)\int_{-\infty}^{0}\exp\left(-\frac{1}{\alpha-r}\,\left[\frac{y^2}{2\,\sigma^2}+(1+r-\alpha)\,y\right]\right)\mathrm{d}x\\
&=\; \exp\left(\mu\,\frac{\alpha-r-1}{\alpha-r}\right)\int_{-\infty}^{0}\exp\left(-\frac{y^2+2\,y\,\sigma^2\,(1+r-\alpha)}{2\sigma^2\,(\alpha-r)}\right)\mathrm{d}x\\
&=\; \exp\left(\frac{1+r-\alpha}{\alpha-r}\,\left(\frac{\sigma^2\,(1+r-\alpha)}{2}-\mu\right)\right)\int_{-\infty}^{0}\exp\left(-\frac{[y+\sigma^2\,(1+r-\alpha)]^2}{(\alpha-r)\,2\,\sigma^2}\right)\mathrm{d}y\\
&=\; \exp\left(\frac{1+r-\alpha}{\alpha-r}\,\left(\frac{\sigma^2\,(1+r-\alpha)}{2}-\mu\right)\right)\sqrt{\frac{\sigma^2\,(\alpha-r)\,\pi}{2}}\,\left[1+\mathrm{erf}\left(\frac{\sigma\,(1+r-\alpha)}{\sqrt{2\,(\alpha-r)}}\right)\right].
\end{aligned}$$

[2]Computed with MATHEMATICA.

Similarly for I_2 we get

$$\begin{aligned} I_2 &= \exp\left(\mu\,\frac{\alpha-r-1}{\alpha-r}\right)\int_0^\infty \exp\left(-\frac{1}{\alpha-r}\left[\frac{y^2}{2\,\sigma^2}+y-y\,\left(\alpha-r+\frac{r\,c}{\alpha}\right)\right]\right)\mathrm{d}y \\ &= \frac{\exp\left(\frac{\sigma^2\,(1+r-\alpha-r\,c/\alpha)^2}{2\,(\alpha-r)}\right)}{\exp\left(\frac{1+r-\alpha}{\alpha-r}\,\mu\right)}\int_0^\infty \exp\left(-\frac{[y+\sigma^2\,(1+r-\alpha-r\,c/\alpha)]^2}{(\alpha-r)\,2\,\sigma^2}\right)\mathrm{d}y \\ &= \frac{\exp\left(\frac{\sigma^2\,(1+r-\alpha-r\,c/\alpha)^2}{2\,(\alpha-r)}\right)}{\exp\left(\frac{1+r-\alpha}{\alpha-r}\,\mu\right)}\sqrt{\frac{\sigma^2\,(\alpha-r)\pi}{2}}\left[1-\operatorname{erf}\left(\frac{\sigma\,(1+r-\alpha-r\,c/\alpha)}{\sqrt{2\,(\alpha-r)}}\right)\right]. \end{aligned}$$

Hence

$$\begin{aligned} &(I_1+I_2)^{\alpha-r} \\ &\quad= \frac{\exp(\sigma^2\,(1+r-\alpha)^2/2)}{\exp(\mu\,(1+r-\alpha))}\left[\frac{\sigma^2\,(\alpha-r)\,\pi}{2}\right]^{(\alpha-r)/2}\left[1+\operatorname{erf}\left(\frac{\sigma\,(1+r-\alpha)}{\sqrt{2\,(\alpha-r)}}\right)\right. \\ &\qquad\left.+\exp\left(\frac{\sigma^2}{2\,(\alpha-r)}\,\left(-\frac{2\,r\,c}{\alpha}\,(1+r-\alpha)+\left(\frac{r\,c}{\alpha}\right)^2\right)\right)\left[1-\operatorname{erf}\left(\frac{\sigma\,(1+r-\alpha-r\,c/\alpha)}{\sqrt{2\,(\alpha-r)}}\right)\right]\right]^{\alpha-r}. \end{aligned}$$

Since computing $\mathrm{FCTR}(p,q,\omega,\kappa_c)$ for arbitrary parameters $q\le p$ is very challenging, we will do this for $p=\infty$ and $q=1$, which—as already mentioned—corresponds to the integration problem. In this specific case, we have $\alpha=r+1$ and

$$(I_1+I_2)^{\alpha-r} = \sqrt{\frac{\sigma^2\,\pi}{2}}\left[1+\exp\left(\frac{(\sigma\,(\alpha-1)\,c)^2}{2\,\alpha^2}\right)\left[1-\operatorname{erf}\left(-\frac{\sigma\,(\alpha-1)\,c}{\alpha\,\sqrt{2}}\right)\right]\right].$$

This yields

$$\begin{aligned} \mathrm{FCTR}(\infty,1,\omega,\kappa_c) &= \frac{(c-\alpha)\,\sigma\,\sqrt{2\,\pi}}{2\,c}\left(\frac{c}{(c-\alpha)\,\sigma\,\sqrt{2\,\pi\,\alpha}}\right)^{\alpha}\exp\left(-\frac{\sigma^2\,(\alpha-1)^2}{2}-\mu(\alpha-1)\right) \\ &\qquad\times\left[1+\exp\left(\frac{(\sigma\,(\alpha-1)\,c)^2}{2\,\alpha^2}\right)\left[1-\operatorname{erf}\left(\frac{-\sigma\,(\alpha-1)\,c}{\alpha\,\sqrt{2}}\right)\right]\right]. \end{aligned}$$

As a numerical example we consider the case $\mu=0$ and $\sigma=1$. For fixed $\alpha\in\{1.5,2,2.5,3,3.5\}$ we numerically minimize[3] $\mathrm{FCTR}(\infty,1,\omega,\kappa_c)$ as a function in c. The results together with the optimal c_* are presented in the following table:

α	1.5	2	2.5	3	3.5
$\mathrm{FCTR}(\infty,1,\omega,\kappa_{c_*})$	1.058	1.224	1.594	2.314	3.648
c_*	2.555	2.973	3.422	3.899	4.392

3.4 Logistic ϱ and Exponential ψ

Consider $D=\mathbb{R}$,

$$\varrho(x) = \frac{\exp(x/\nu)}{\nu\,(1+\exp(x/\nu))^2} \quad \text{and} \quad \psi(x) = \exp(-b|x|)$$

[3] Using the Mathematica command `FindMinimum`

with parameters $\nu > 0$ and $b > 0$. Then

$$\omega(x) \;=\; \frac{\exp(x/\nu + b\,|x|)}{\nu\,(1+\exp(x/\nu))^2}$$

which is quite complicated, in particular if one considers $\omega^{1/\alpha}$, and is not monotonic. Consider therefore

$$\kappa_a(x) \;=\; \exp(-a|x|) \quad \text{for some } a \;>\; 0.$$

Hence the points $x_{-n}, \ldots, x_n$ satisfying (12) are again given by (14).

To simplify the formulas to come, we use

$$\lambda \;:=\; \frac{1}{\nu}, \quad \text{i.e.,} \quad \omega(x) \;=\; \frac{\lambda\,\exp(\lambda\,x + b\,|x|)}{(1+\exp(\lambda\,x))^2}.$$

For $\|\omega^{1/\alpha}\|_{L_1(D)}^{\alpha}$ and $\|\omega/\kappa_a\|_{L_\infty(D)}$ to be finite, we need to have

$$\lambda \;>\; b \quad \text{and} \quad \lambda \;\geq\; a+b.$$

Since the integral in $\mathcal{E}_p^q(\omega,\kappa_a)$ becomes very complicated for this example we do not distinguish between $p \leq q$ and $p > q$. Instead we use the upper bound (13) here.

We first study $\|\omega/\kappa_a\|_{L_\infty(D)}$. Since ω and κ_a are symmetric, we can restrict the attention to $x \geq 0$. By substituting $z = \exp(\lambda\,x)$, we get that

$$\left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} = \lambda \sup_{z\geq 1} \frac{z^{1+(a+b)/\lambda}}{(1+z)^2}.$$

When $a+b=\lambda$ the supremum is attained at $z=\infty$, otherwise it is attained at $z = (\lambda+a+b)/(\lambda-(a+b))$. Therefore

$$\left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} \;=\; \frac{\lambda}{4}\left(1+\frac{a+b}{\lambda}\right)^{1+(a+b)/\lambda}\left(1-\frac{a+b}{\lambda}\right)^{1-(a+b)/\lambda},$$

with the convention that $0^0 := 1$, i.e., $\|\omega/\kappa_a\|_{L_\infty(D)} = \lambda$ if $a = \lambda - b$.

Indeed, the previous formula for $\|\omega/\kappa_a\|_{L_\infty(D)}$ can be shown by noting that

$$\begin{aligned}
&\lambda\left[\frac{\lambda+a+b}{\lambda-a-b}\right]^{1+\frac{a+b}{\lambda}}\left(1+\frac{\lambda+a+b}{\lambda-a-b}\right)^{-2} = \lambda\left[\frac{\lambda+a+b}{\lambda-a-b}\right]^{1+\frac{a+b}{\lambda}}\left(\frac{\lambda-(a+b)}{2\lambda}\right)^2\\
&= \frac{\lambda}{4}\left[\frac{\lambda+a+b}{\lambda-a-b}\right]^{1+\frac{a+b}{\lambda}}\left(1-\frac{a+b}{\lambda}\right)^2\\
&= \frac{\lambda}{4}\left[\frac{\lambda+a+b}{\lambda-a-b}\right]^{1+\frac{a+b}{\lambda}}\left(1-\frac{a+b}{\lambda}\right)^{1-\frac{a+b}{\lambda}}\left(1-\frac{a+b}{\lambda}\right)^{1+\frac{a+b}{\lambda}}\\
&= \frac{\lambda}{4}\left(1-\frac{a+b}{\lambda}\right)^{1-\frac{a+b}{\lambda}}\left(\frac{\lambda+a+b}{\lambda-a-b}\cdot\frac{\lambda-a-b}{\lambda}\right)^{1+\frac{a+b}{\lambda}}.
\end{aligned}$$

As above,

$$\|\kappa_a^{1/\alpha}\|_{L_1(D)}^{\alpha} \;=\; \left(\frac{2\alpha}{a}\right)^{\alpha}.$$

We also have

$$\begin{aligned}\|\omega^{1/\alpha}\|^{\alpha}_{L_1(D)} &= \lambda\left(2\int_0^\infty \frac{\exp((\lambda+b)\,x/\alpha)}{(1+\exp(\lambda\,x))^{2/\alpha}}\,\mathrm{d}x\right)^{\alpha} \\ &\geq \lambda\left(2\int_0^\infty \frac{\exp(\lambda\,x/\alpha)}{(1+\exp(\lambda\,x/\alpha))^{2}}\,\mathrm{d}x\right)^{\alpha}\end{aligned}$$

due to the fact that $1/(1+A)^{1/\alpha} \geq 1/(1+A^{1/\alpha})$ since $\alpha \geq 1$. Therefore

$$\|\omega^{1/\alpha}\|^{\alpha}_{L_1(D)} \geq \lambda\left(\frac{\alpha}{\lambda}\right)^{\alpha}.$$

This gives

$$\mathrm{FCTR}(p,q,\omega,\kappa_a) \leq \left(\frac{2\,\lambda}{a}\right)^{\alpha}\frac{1}{4}\left(1+\frac{a+b}{\lambda}\right)^{1+(a+b)/\lambda}\left(1-\frac{a+b}{\lambda}\right)^{1-(a+b)/\lambda}.$$

As before the right-hand side above is

$$\left(\frac{2\,\lambda}{\lambda-b}\right)^{\alpha} \quad \text{if } a = \lambda - b.$$

Letting $x = a/\lambda$, the minimum is at $0 < x < 1 - b/\lambda$ that is the root of

$$x\left(\ln\left(1+\frac{b}{\lambda}+x\right)-\ln\left(1-\frac{b}{\lambda}-x\right)\right)-\alpha = 0.$$

Rounded values of the upper bound on FCTR for $\alpha = b = 1$ and various λ's are[4]:

λ	2	5	10	15
Bound on FCTR	3.341	1.710	1.431	1.353

3.5 Student's ϱ and ψ

Consider Student's t-distribution on $D = \mathbb{R}$

$$\varrho(x) = T_\nu\left(1+\frac{x^2}{\nu}\right)^{-(\nu+1)/2} \quad \text{with} \quad T_\nu = \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\,\pi}\,\Gamma(\nu/2)} \quad \text{for} \quad \nu > 0.$$

Here Γ denotes Euler's Gamma function $\Gamma(z) = \int_0^\infty t^{z-1}\mathrm{e}^{-t}\,\mathrm{d}t$. Let

$$\psi(x) = \left(1+\frac{x^2}{\nu}\right)^{-b/2} \quad \text{and} \quad \kappa_a(x) = (1+|x|)^{-a}$$

for $a > 0$ and $b \geq 0$. For $\|\omega^{1/\alpha}\|_{L_1(D)}$, $\|\kappa_a^{1/\alpha}\|_{L_1(D)}$, and $\|\omega/\kappa_a\|_{L_\infty(D)}$ to be finite, we have to assume that

$$\nu + 1 - b \geq a > \alpha.$$

It is easy to see that

$$\|\kappa_a^{1/\alpha}\|^{\alpha}_{L_1(D)} = \left(\frac{2\,\alpha}{a-\alpha}\right)^{\alpha}.$$

[4]Computed with MATHEMATICA.

Hence the points $x_{-n}, \ldots, x_n$ satisfying (12) are given by

$$x_i \;=\; -x_{-i} \;=\; \left(1 - \frac{i}{n}\right)^{-\frac{\alpha}{a-\alpha}} - 1 \quad \text{for } 0 \le i \le n.$$

To compute the norm of $\omega^{1/\alpha}$, make the change of variables $x/\sqrt{\nu} = t/\sqrt{\mu}$, where

$$\mu \;=\; \frac{\nu + 1 - b - \alpha}{\alpha} \quad \text{so that} \quad \frac{\mu+1}{2} \;=\; \frac{\nu+1-b}{2\,\alpha}.$$

Then we get

$$\begin{aligned}
\|\omega^{1/\alpha}\|_{L_1(D)}^{\alpha} &= T_\nu \left(\int_{\mathbb{R}} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1-b)/(2\,\alpha)} \mathrm{d}x\right)^{\alpha} \\
&= T_\nu \left(\frac{\nu}{\mu}\right)^{\alpha/2} T_\mu^{-\alpha} \left(T_\mu \int_{\mathbb{R}} \left(1 + \frac{t^2}{\mu}\right)^{-(\mu+1)/2} \mathrm{d}t\right)^{\alpha} \;=\; T_\nu \left(\frac{\sqrt{\nu}}{T_\mu\,\sqrt{\mu}}\right)^{\alpha}.
\end{aligned}$$

Since

$$\frac{\omega(x)}{\kappa_a(x)} \;=\; T_\nu \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1-b)/2} (1+|x|)^a,$$

we have

$$\left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} \;=\; T_\nu\,(1+\nu)^{(\nu+1-b)/2} \quad \text{for } a \;=\; \nu+1-b,$$

and

$$\left\|\frac{\omega}{\kappa_a}\right\|_{L_\infty(D)} \;=\; \frac{\omega(x_*)}{\kappa(x_*)} \quad \text{for} \quad x_* \;=\; \frac{\sqrt{(\nu+1-b)^2 + 4\,a\,\nu\,(\nu+1-b-a)} - (\nu+1-b)}{2\,(\nu+1-a-b)}$$

for $a < \nu + 1 - b$.

This gives

$$\mathrm{FCTR}(p,q,\omega,\kappa_a) \;\le\; \begin{cases} (1+\nu)^{(\nu+1-b)/2} \left(\frac{2\,T_\mu}{\sqrt{\nu\,\mu}}\right)^{\alpha} & \text{for } a = \nu+1-b, \\ \frac{(1+x_*)^a}{\left(1+\frac{x_*^2}{\nu}\right)^{(\nu+1-b)/2}} \left(T_\mu\,\frac{2\,\alpha}{a-\alpha}\,\sqrt{\frac{\mu}{\nu}}\right)^{\alpha} & \text{for } a \in (\alpha, \nu+1-b), \end{cases}$$

with equality whenever $p \le q$.

In the following numerical experiments for fixed values of α, b and ν, we choose $a \in (\alpha, \nu + 1 - b]$ of the form $a = \alpha + k/10$ such that it gives the smallest value of the above bound on FCTR. For example:

(ν, b, α)	$(3,2,1)$	$(4,2,2)$	$(5,3,2)$	$(6,3,3)$
FCTR	1.427	1.626	1.710	1.861

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