# ON THE OPTIMAL CONSTANTS IN THE TWO-SIDED STECHKIN INEQUALITIES

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ABSTRACT. We address the optimal constants in the strong and the weak Stechkin inequalities, both in their discrete and continuous variants. These inequalities appear in the characterization of approximation spaces which arise from sparse approximation or have applications to interpolation theory. An elementary proof of a constant in the strong discrete Stechkin inequality given by Bennett is provided, and we improve the constants given by Levin and Stechkin and by Copson. Finally, the minimal constants in the weak discrete Stechkin inequalities and both continuous Stechkin inequalities are presented.

#### 1. Introduction

In the present paper, we address the minimal constants  $c_1(q)$ ,  $C_1(q)$ ,  $c_{1,\infty}(q)$ ,  $C_{1,\infty}(q)$ ,  $\bar{c}_1(q)$ ,  $\bar{c}_1(q)$ ,  $\bar{c}_{1,\infty}(q)$ , and  $\bar{C}_{1,\infty}(q) > 0$  in the inequalities

(1.1) 
$$\frac{1}{c_1(q)} \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \le \sum_{n=1}^{\infty} a_n \le C_1(q) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}},$$

$$(1.2) \qquad \frac{1}{c_{1,\infty}(q)} \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \leq \sup_{n \in \mathbb{N}} n a_n \leq C_{1,\infty}(q) \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}},$$

$$\frac{1}{\bar{c}_1(q)} \int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt \le \int_0^\infty f(t) dt \le \bar{C}_1(q) \int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt,$$

and

(1.4)

$$\frac{1}{\bar{c}_{1,\infty}(q)} \sup_{t>0} t \left(\frac{1}{t} \int_t^{\infty} f(s)^q ds\right)^{\frac{1}{q}} \leq \sup_{t>0} t f(t) \leq \bar{C}_{1,\infty}(q) \sup_{t>0} t \left(\frac{1}{t} \int_t^{\infty} f(s)^q ds\right)^{\frac{1}{q}},$$

for sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1 \geq a_2 \geq \ldots \geq 0$ , monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$ , and  $1 < q \leq \infty$ . These inequalities are henceforth referred to as the *strong discrete*, the *weak discrete*, the *strong continuous*, and the *weak continuous Stechkin inequality*, respectively.

The right-hand side inequalities of (1.1), (1.2), and (1.4) also allow for q=1. Note that in case  $q=\infty$ , the expressions  $\left(\frac{1}{n}\sum_{k=n}^{\infty}a_{k}^{q}\right)^{\frac{1}{q}}$  and  $\left(\frac{1}{t}\int_{x}^{\infty}f(s)^{q}\mathrm{d}s\right)^{\frac{1}{q}}$  are replaced by  $\sup\left\{a_{k}\mid k\geq n\right\}=a_{n}$  and  $\sup\left\{f(s)\mid s\geq t\right\}=f(t)$ .

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Copson [3, Theorem 2.3] proves that  $C_1(q) \leq q^{\frac{1}{q}}$ , cf. also Hardy, Littlewood, and Pólya [12, Theorem 345]. Levin and Stechkin [13, Д.61] improve Copson's result [3, Theorem 2.3] when  $1 < q < \infty$ , showing that  $C_1(q) = (q-1)^{\frac{1}{q}}$  when  $3 \leq q < \infty$  and giving upper bounds in the remaining cases. Stechkin revisits (1.1) for q=2 in [17, Лемма 3] and [18, Лемма 1] where it is proved that  $C_1(2) \leq \frac{2}{\sqrt{3}}$ . Gao [9, Theorem 1] provides further improvement by showing that  $C_1(q) = (q-1)^{\frac{1}{q}}$  even for  $q_0 \leq q < \infty$ , where  $q_0 \approx 2.8855$  is a solution of the equation  $2^{\frac{1}{q-1}}\left((q-1)^{\frac{q}{q-1}}-(q-1)\right)-\left(1+\frac{3-q}{2}\right)^{\frac{q}{q-1}}=0$ . De Bruijn [4, p. 174] reports that  $C_1(2)=1.1064957714$  "with an error of at most 9 units at the last decimal place." Stechkin [18, Лемма 1] is first to address the constant  $c_1(2)$ ; he asserts  $c_1(2) \leq 2$  and conjectures an improvement to  $c_1(2) \leq \frac{\pi}{2}$  but proves neither claim; see also [6, Section 7.4] for a historical discussion. The existence of constants validating the inequalities (1.1) and (1.2) is due to Pietsch [15, Example 1 on p. 123], see also [5, Theorem 4]. Bennett [1, Theorem 3] shows that

$$c_1(q) = \frac{\pi}{q \sin(\frac{\pi}{q})}$$

in (1.1), thus confirming Stechkin's conjecture for q = 2, see Figure 1 for an illustration. Hardy, Littlewood, and Pólya [12, Theorem 337] prove

$$\bar{C}_1(q) = (q-1)^{\frac{1}{q}}$$

in (1.3), depicted in Figure 7.

The contribution of the present paper is as follows. First, we give an alternative proof for the optimality of  $c_1(2) = \frac{\pi}{2}$  in (1.1) which uses an elementary insight from convex optimization.

Second, we extend the upper bound for  $C_1(q) \leq 2\left(2\frac{q}{q-1}-1\right)^{-\frac{q-1}{q}}$  proved by Levin and Stechkin for  $\frac{5}{3} \leq q < 3$  to  $1 < q < \infty$  via Proposition 2.3. A more detailed analysis of the same argument leads to  $C_1(2) \leq 1.1086983$  in Corollary 2.6. Third, we improve the upper bounds for  $C_1(q)$  from the literature when  $1 \leq q \leq \frac{2+\ln(2)}{2-\ln(2)}$  and  $q \neq 2$ . Summarizing, the currently best known bounds for the constant  $C_1(q)$  are

$$C_1(q) \begin{cases} \leq \left(\frac{e\ln(2)}{\sqrt{2}}\right)^{1-\frac{1}{q}}, & 1 \leq q \leq \frac{2+\ln(2)}{2-\ln(2)}, \\ \approx 1.1064957714, & q = 2, \\ \leq 2\left(2\frac{q}{q-1}-1\right)^{-\frac{q-1}{q}}, & \frac{2+\ln(2)}{2-\ln(2)} < q < q_0, \\ = (q-1)^{\frac{1}{q}} & q_0 \leq q \leq \infty \end{cases}$$

with  $q_0 \approx 2.8855$ , see Proposition 2.3 and Theorem 2.7 and Figure 4 for an illustration.

Fourth, we determine the optimal constants in (1.2) as

$$c_{1,\infty}(q) = \zeta(q)^{\frac{1}{q}}$$
 and  $C_{1,\infty}(q) = \left(\frac{1}{q}\right)^{-\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{-\left(1 - \frac{1}{q}\right)}$ 

where  $\zeta(\cdot)$  denotes the Riemann zeta-function, see Theorems 3.2 and 3.3 and Figures 5 and 6. Next, we show that

$$\bar{c}_1(q) = c_1(q) = \frac{\pi}{q \sin(\frac{\pi}{q})}$$

in (1.3), see Theorem 4.1 and Figure 7, as well as

$$\bar{c}_{1,\infty}(q) = (q-1)^{-\frac{1}{q}} \quad \text{and} \quad \bar{C}_{1,\infty}(q) = C_{1,\infty}(q) = \left(\frac{1}{q}\right)^{-\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{-\left(1 - \frac{1}{q}\right)},$$

see Theorems 4.3 and 4.4 and Figure 8.

The inequalities discussed in this paper have applications in interpolation theory and nonlinear approximation. On the one hand, (1.3) can be used in the proof of the Marcinkiewicz interpolation theorem, see [2, Theorem 1.3.1]. On the other hand (1.1) and (1.2) play a role in the characterization of the approximation spaces  $\mathcal{A}_r^{\alpha}(\mathcal{H})$ , i.e., the set of elements f of the infinite-dimensional separable Hilbert space  $\mathcal{H}$  for which the quasi-norm

$$||f||_{\mathcal{A}_r^{\alpha}(\mathcal{H})} := \begin{cases} \left( \sum_{n=1}^{\infty} (n^{\alpha} E_n(f)_{\mathcal{H}})^r \frac{1}{n} \right)^{\frac{1}{r}}, & 0 < r < \infty, \\ \sup_{n \in \mathbb{N}} n^{\alpha} E_n(f)_{\mathcal{H}}, & r = \infty \end{cases}$$

is finite. Here,  $\alpha > 0$  and  $E_n(f)$  denote the infimal distance of f to elements of the form  $\sum_{k \in \Lambda} \lambda_k e_k$ , where  $\Lambda \subset \mathbb{N}$  is a set of cardinality n-1 and  $(e_k)_{k=1}^{\infty}$  is an orthonormal basis of  $\mathcal{H}$ . The consequences for the optimal constants in the inequalities stated by DeVore in [5, Theorem 4] on sparse approximation in infinite-dimensional separable real Hilbert spaces are outlined in Section 5.

A recurring technique in this paper is that we prove the inequalities under consideration for finite sequences first, which then yields the general claim through a limiting process. These finite-dimensional versions will be used to gain some geometric insight to (1.1).

# 2. The strong discrete Stechkin inequality

In this section, we are concerned with the minimal constants  $c_1(q)$  and  $C_1(q) > 0$  in the inequality

(2.1) 
$$\frac{1}{c_1(q)} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}} \le \sum_{n=1}^{\infty} a_n \le C_1(q) \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$$

for sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1 \geq a_2 \geq \ldots \geq 0$  and for  $1 < q \leq \infty$  with the appropriate modification for  $q = \infty$ , as indicated in Section 1. The monotonicity assumption on  $(a_n)_{n\in\mathbb{N}}$  gives  $\sup\{a_k \mid k \geq n\} = a_n$ , so  $c_1(\infty) = C_1(\infty) = 1$ . For q = 1, we have  $C_1(1) = 1$  because

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{a_k}{k} \le \sum_{k=1}^{\infty} \sum_{n=1}^{k} \frac{a_k}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} a_k$$

which holds as an equality when  $a_n = 0$  for all  $n \geq 2$ .

2.1. On the optimal lower constant. We give a rather elementary proof of the optimality of  $c_1(2) = \frac{\pi}{2}$  in (2.1), which is due to Bennett [1, Theorem 3]. Our proof uses an elementary insight from convex optimization.

**Theorem 2.1.** The minimal constant  $c_1(2) > 0$  for which

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}} \le c_1(2) \sum_{n=1}^{\infty} a_n$$

holds for all sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq \ldots \geq 0$  is  $c_1(2)=\frac{\pi}{2}$ .

Proof. Step 1. We prove the claim for finite sequences. Consider

$$\sup \left\{ \frac{\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}}}{\sum_{n=1}^{\infty} a_n} \middle| a_1 \ge a_2 \ge \dots \ge 0, a_{N+1} = 0, \sum_{n=1}^{\infty} a_n < \infty \right\}$$

$$= \sup \left\{ \frac{\sum_{n=1}^{N} \left(\frac{1}{n} \sum_{k=n}^{N} a_k^2\right)^{\frac{1}{2}}}{\sum_{n=1}^{N} a_n} \middle| a_1 \ge a_2 \ge \dots \ge 0, a_{N+1} = 0 \right\}$$

$$= \sup \left\{ \sum_{n=1}^{N} \left(\frac{1}{n} \sum_{k=n}^{N} a_k^2\right)^{\frac{1}{2}} \middle| (a_n)_{n \in \mathbb{N}} \in \Delta_N \right\}.$$

The set  $\Delta_N := \left\{ (a_n)_{n \in \mathbb{N}} \mid a_1 \geq a_2 \geq \ldots \geq 0, a_{N+1} = 0, \sum_{n=1}^N a_n = 1 \right\}$  is determined by a single linear equality and n linear inequalities in  $a_1, \ldots, a_n$  and thus is a (n-1)-dimensional simplex in its N-dimensional linear span  $V_N$ . Therefore, the restriction of the convex function

$$V_N \to \mathbb{R},$$

$$(a_n)_{n \in \mathbb{N}} \mapsto \sum_{n=1}^N \left(\frac{1}{n} \sum_{k=n}^N a_k^2\right)^{\frac{1}{2}}$$

to  $\Delta_N$  attains its supremum at one of the vertices of  $\Delta_N$ . Similarly to the discussion in [8, Section 2.1], the vertices of  $\Delta_N$  are precisely those points for which all but one of the defining inequalities are actually equalities. This means that for each of the N vertices of  $\Delta_N$ , there is a number  $k_0 \in \{1, \ldots, N\}$  such that  $a_1 = \ldots = a_{k_0} > a_{k_0+1} = \ldots = a_N = 0$ . Taking  $\sum_{n=1}^N a_n = 1$  into account, we obtain  $a_1 = \ldots = a_{k_0} = \frac{1}{k_0}$  and

$$\sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^2 \right)^{\frac{1}{2}} = \sum_{n=1}^{k_0} \left( \frac{1}{n} \sum_{k=n}^{k_0} \frac{1}{k_0^2} \right)^{\frac{1}{2}} = \sum_{n=1}^{k_0} n^{-\frac{1}{2}} \left( \frac{k_0 - n + 1}{k_0^2} \right)^{\frac{1}{2}}.$$

Therefore

$$\sup \left\{ \frac{\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}}}{\sum_{n=1}^{\infty} a_n} \middle| a_1 \ge a_2 \ge \dots \ge 0, a_{N+1} = 0, \sum_{n=1}^{\infty} a_n < \infty \right\}$$

$$= \sup \left\{ \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^2 \right)^{\frac{1}{2}} \middle| (a_n)_{n \in \mathbb{N}} \in \Delta_N \right\}$$

$$= \sup \left\{ \sum_{n=1}^{k_0} n^{-\frac{1}{2}} \left( \frac{k_0 - n + 1}{k_0^2} \right)^{\frac{1}{2}} \mid k_0 \in \{1, \dots, N\} \right\}.$$

Clearly, this quantity is monotonically increasing in N because not only the set  $\{(a_n)_{n\in\mathbb{N}}\mid a_1\geq a_2\geq\ldots\geq 0, a_{N+1}=0\}$  is, but also  $\Delta_N$  and its vertex set are. We will show that the sequence  $\left(\sum_{n=1}^{k_0}n^{-\frac{1}{2}}\left(\frac{k_0-n+1}{k_0^2}\right)^{\frac{1}{2}}\right)_{k_0\in\mathbb{N}}$  is bounded above and the supremum is  $\frac{\pi}{2}$ . Then we automatically know that

$$\frac{\pi}{2} = \sup_{k_0 \in \mathbb{N}} \sum_{n=1}^{k_0} n^{-\frac{1}{2}} \left( \frac{k_0 - n + 1}{k_0^2} \right)^{\frac{1}{2}}$$

$$= \sup_{N \in \mathbb{N}} \sup \left\{ \sum_{n=1}^{k_0} n^{-\frac{1}{2}} \left( \frac{k_0 - n + 1}{k_0^2} \right)^{\frac{1}{2}} \middle| k_0 \in \{1, \dots, N\} \right\}$$

$$= \sup_{N \in \mathbb{N}} \sup \left\{ \frac{\sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}}}{\sum_{n=1}^{\infty} a_n} \middle| a_1 \ge a_2 \ge \dots \ge 0, a_{N+1} = 0 \right\}.$$

The function  $g:(0,k_0+1)\to [0,\infty),\ g(t):=t^{-\frac{1}{2}}\Big(\frac{k_0-t+1}{k_0^2}\Big)^{\frac{1}{2}}$  is monotonically decreasing, so

(2.2) 
$$f_{k_0} := \int_1^{k_0+1} g(t) dt \le \sum_{n=1}^{k_0} g(n) \le g(1) + \int_1^{k_0} g(t) dt =: h_{k_0}.$$

For the computation of the antiderivative  $\int t^{-\frac{1}{2}}(k_0-t+1)^{\frac{1}{2}}dt$ , the change of variables  $u=\sqrt{\frac{t}{k_0-t+1}}$  or  $t=\frac{u^2(k_0+1)}{u^2+1}$  yields  $\frac{dt}{du}=\frac{2u(k_0+1)}{(u^2+1)^2}$  and

$$\int t^{-\frac{1}{2}} (k_0 - t + 1)^{\frac{1}{2}} dt$$

$$= 2(k_0 + 1) \int \frac{1}{(u^2 + 1)^2} du$$

$$= 2(k_0 + 1) \left( \frac{u}{2(u^2 + 1)} + \frac{1}{2} \arctan(u) \right)$$

$$= t^{\frac{1}{2}} (k_0 - t + 1)^{\frac{1}{2}} + (k_0 + 1) \arctan\left( \left( \frac{t}{k_0 - t + 1} \right)^{\frac{1}{2}} \right).$$

Plugging in the integration bounds, we arrive at

$$\int_{1}^{k_{0}+1} t^{-\frac{1}{2}} (k_{0} - t + 1)^{\frac{1}{2}} dt = (k_{0} + 1) \frac{\pi}{2} - k_{0}^{\frac{1}{2}} - (k_{0} + 1) \arctan\left(k_{0}^{-\frac{1}{2}}\right),$$

$$\int_{1}^{k_{0}} t^{-\frac{1}{2}} (k_{0} - t + 1)^{\frac{1}{2}} dt = \left(\arctan\left(k_{0}^{\frac{1}{2}}\right) - \arctan\left(k_{0}^{-\frac{1}{2}}\right)\right) (k_{0} + 1).$$

It follows that  $\lim_{k_0\to\infty} f_{k_0} = \lim_{k_0\to\infty} h_{k_0} = \frac{\pi}{2}$ . Now, if we can show that the sequences  $(f_{k_0})_{k_0\in\mathbb{N}}$  and  $(h_{k_0})_{k_0\in\mathbb{N}}$  are monotonically increasing, then (2.2) implies

$$\frac{\pi}{2} = \lim_{k_0 \to \infty} f_{k_0} = \sup_{k_0 \in \mathbb{N}} f_{k_0} \le \sup_{k_0 \in \mathbb{N}} \sum_{k_0 \in \mathbb{N}} \sum_{k_0 \in \mathbb{N}} n^{-\frac{1}{2}} \left( \frac{k_0 - n + 1}{k_0^2} \right)^{\frac{1}{2}}$$

$$\leq \sup_{k_0 \in \mathbb{N}} h_{k_0} = \lim_{k_0 \to \infty} h_{k_0} = \frac{\pi}{2}$$

and we are done. Indeed, one has

$$\frac{\partial}{\partial x} \int_{1}^{x+1} \left( \frac{x-t+1}{tx^2} \right)^{\frac{1}{2}} dt = \frac{\partial}{\partial x} \frac{(x+1)\frac{\pi}{2} - x^{\frac{1}{2}} - (x+1)\arctan\left(x^{-\frac{1}{2}}\right)}{x}$$
$$= \frac{x^{\frac{1}{2}} - \arctan\left(x^{\frac{1}{2}}\right)}{x^2} > 0$$

and

$$\frac{\partial}{\partial x} \left( x^{-\frac{1}{2}} + \int_{1}^{x} \left( \frac{x - t + 1}{tx^{2}} \right)^{\frac{1}{2}} dt \right)$$

$$= \frac{\partial}{\partial x} \left( x^{-\frac{1}{2}} + \left( \arctan\left( x^{\frac{1}{2}} \right) - \arctan\left( x^{-\frac{1}{2}} \right) \right) \frac{x + 1}{x} \right)$$

$$= \frac{x^{\frac{1}{2}} - 4\arctan\left( x^{\frac{1}{2}} \right) + \pi}{2x^{2}} > 0$$

for all  $x \ge 1$ . For the latter claim, note that the function  $(0,\infty) \ni x \mapsto \sqrt{x} - 4\arctan(\sqrt{x}) + \pi$  has a global minimizer at x = 3 with minimum  $\sqrt{3} - \frac{\pi}{3} > 0$ , which can be read off the signs of its derivative  $x \mapsto \frac{x-3}{2\sqrt{x}(x+1)}$ .

Step 2. We prove the claim for all sequences. It remains to show that

$$\frac{\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}}}{\sum_{n=1}^{\infty} a_n} \le \frac{\pi}{2}$$

for all sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq \ldots \geq 0$  and  $\sum_{n=1}^{\infty}a_n\leq \infty$ . Let  $\varepsilon>0$ . Then there exists  $N_1,N_2,N_3\in\mathbb{N}$  such that

$$\sum_{n=N+1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}} < \frac{\varepsilon}{2}$$

for all  $N > N_1$ , and

$$\sum_{k=\left|\frac{N}{2}\right|}^{\infty} a_k < \frac{\varepsilon}{8}$$

for all  $N > N_2$ , and

$$\sqrt{N} \left| \frac{N}{2} \right|^{-\frac{1}{2}} \le 2$$

for all  $N > N_3$ . Also, for fixed  $N \in \mathbb{N}$  and  $M \ge N + 1$ , apply [8, Proposition 2.3] with p = 1, q = 2, and  $s = \frac{N}{2}$  to the sequence  $(a_n)_{n \ge \lfloor \frac{N}{2} \rfloor}$  to obtain

$$\left(\sum_{k=N+1}^{\infty} a_k^2\right)^{\frac{1}{2}} \le \left\lfloor \frac{N}{2} \right\rfloor^{-\frac{1}{2}} \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor}^{\infty} a_k.$$

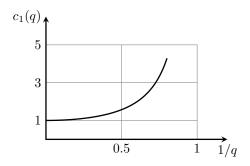


FIGURE 1. The function  $1/q \mapsto \frac{\pi}{q \sin(\frac{\pi}{q})}$ .

For  $N > \max\{N_1, N_2, N_3\}$ , we conclude

$$\sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=N+1}^{\infty} a_k^2 \right)^{\frac{1}{2}} \le \sum_{n=1}^{N} n^{-\frac{1}{2}} \left\lfloor \frac{N}{2} \right\rfloor^{-\frac{1}{2}} \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor}^{\infty} a_k \le \int_0^N x^{-\frac{1}{2}} dx \left\lfloor \frac{N}{2} \right\rfloor^{-\frac{1}{2}} \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor}^{\infty} a_k$$

$$= 2\sqrt{N} \left\lfloor \frac{N}{2} \right\rfloor^{-\frac{1}{2}} \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor}^{\infty} a_k \le 4 \sum_{k=\left\lfloor \frac{N}{2} \right\rfloor}^{\infty} a_k < \frac{\varepsilon}{2}$$

and

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}}$$

$$= \sum_{n=1}^{N} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}} + \sum_{n=N+1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}}$$

$$\leq \sum_{n=1}^{N} \left(\frac{1}{n} \sum_{k=n}^{N} a_k^2\right)^{\frac{1}{2}} + \sum_{n=1}^{N} \left(\frac{1}{n} \sum_{k=N+1}^{\infty} a_k^2\right)^{\frac{1}{2}} + \sum_{n=N+1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^2\right)^{\frac{1}{2}}$$

$$< \frac{\pi}{2} \sum_{n=1}^{N} a_n + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$= \frac{\pi}{2} \sum_{n=1}^{\infty} a_n + \varepsilon.$$

Taking the limit  $\varepsilon \downarrow 0$  proves the assertion.

The precise values for  $c_1(q)$  from [1, Theorem 3] are illustrated in Figure 1.

**Remark 2.2.** Our proof of Theorem 2.1 already emphasizes the importance of finite sequences for our considerations which will be encountered again in the proof of Theorem 3.3. Let us draw a geometric picture. Fix  $N \in \mathbb{N}$  and  $q \in (1, \infty)$ . The quantities

$$||a||_1 = \sum_{n=1}^{N} |a_n|$$

and

$$\gamma_N(a) := \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} |a_k|^q \right)^{\frac{1}{q}}$$

define norms on  $\mathbb{R}^N$ , whose closed unit balls shall be denoted by  $B_1^N$  and  $B_{1,q}^N$ , respectively. Figure 2 illustrates  $B_{1,q}^N$  for several values of q.

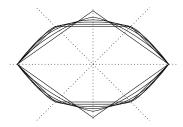


FIGURE 2. The unit balls of the norms  $\gamma_N$  for N=2 and  $q\in\{1,1.4,2,3,6,\infty\}$ .

The inequality

(2.3) 
$$\frac{1}{c_1(q)} \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^q \right)^{\frac{1}{q}} \le \sum_{n=1}^{N} a_n \le C_1(q) \sum_{n=1}^{N} \left( \frac{1}{n} \sum_{k=n}^{N} a_k^q \right)^{\frac{1}{q}}$$

for all  $a = (a_1, \ldots, a_N) \in \mathbb{R}^N$  with  $a_1 \geq \ldots a_N \geq 0$  then translates to the following chain of set inclusions of convex bodies:

(2.4) 
$$\frac{1}{c_1(q)}(B_1^N \cap K_N) \subset B_{1,q}^N \cap K_N \subset C_1(q)(B_1^N \cap K_N).$$

Here  $K_N := \{(a_1, \ldots, a_N) \mid a_1 \geq \ldots a_N \geq 0\}$ . For understanding the shape of the convex bodies  $B_{1,q}^N$ , we note that norm  $\gamma_N$  is the pointwise sum of the functions

 $\gamma_{n,N}:\mathbb{R}^n\to\mathbb{R}$  given by  $\gamma_{n,N}(a)=\left(\frac{1}{n}\sum_{k=n}^N a_k^q\right)^{\frac{1}{q}}$  for  $n\in\{1,\ldots,N\}$ . With ° denoting the polar set with respect to the standard inner product, it follows that  $B_{1,q}^N=\left(B_{1,N}^\circ+\ldots+B_{N,N}^\circ\right)^\circ$ , which is similar to the construction of the harmonic mean of convex bodies introduced by Firey in [7]. For N=q=2, the chain of set inclusions stated in (2.4) with the optimal constants is illustrated in Figure 3. This figure may also be used to convince oneself that for the left-hand side inequality in (2.3), it is relevant to have  $a\in K_N$ , and thus monotonicity is also relevant for the left-hand side inequality in (1.1).

2.2. On the optimal upper constant. In this section, we improve the known upper bounds on  $C_1(q)$  for  $1 < q \le \frac{2+\ln(2)}{2-\ln(2)}$ . The following result adapts Stechkin's proof technique in [17, Лемма 3] and produces upper bounds for  $C_1(q)$  from auxiliary sequences whose entrywise inverses in  $\ell_{q'}$  where q' is the Hölder conjugate of q.

**Proposition 2.3.** Let  $1 < q < \infty$ , set  $q' = \frac{q}{q-1}$ , and assume that  $b = (b_n)_{n \in \mathbb{N}_0}$  is a strictly monotonically increasing sequence with  $b_0 = 0$  and  $\sum_{n=1}^{\infty} \frac{1}{b_n^{q'}} < \infty$ . Then

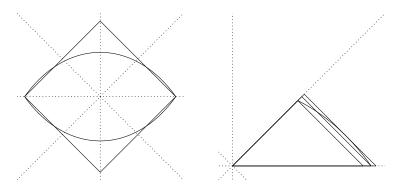


FIGURE 3. The unit balls of the norms  $\gamma_N$  and  $\|\cdot\|_1$  for N=q=2 (left) and an optimally scaled version of their intersections with the cone  $K_N$  (right).

for all  $(a_n)_{n\in\mathbb{N}}\in\ell_q$ , we have

$$\sum_{n=1}^{\infty} a_n \le C_b(q) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$
with  $C_b(q) := \sup \left\{ \left( n^{\frac{q'}{q}} (b_n - b_{n-1})^{q'} \sum_{k=n}^{\infty} \frac{1}{b_k^{q'}} \right)^{\frac{1}{q'}} \, \middle| \, n \in \mathbb{N} \right\}.$ 

Proof. Consider

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a_n}{b_n} b_n = \sum_{n=1}^{\infty} \frac{a_n}{b_n} \sum_{k=1}^{n} (b_k - b_{k-1})$$

$$= \sum_{n=1}^{\infty} (b_n - b_{n-1}) \sum_{k=n}^{\infty} \frac{a_k}{b_k} \le \sum_{n=1}^{\infty} (b_n - b_{n-1}) \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \left( n^{\frac{q'}{q}} \sum_{k=n}^{\infty} \frac{1}{b_k^{q'}} \right)^{\frac{1}{q'}}$$

$$= \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \left( n^{\frac{q'}{q}} (b_n - b_{n-1})^{q'} \sum_{k=n}^{\infty} \frac{1}{b_k^{q'}} \right)^{\frac{1}{q'}} \le C_b(q) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$

where  $C_b(q)$  is chosen as stated in the assertion.

We investigate the choice  $b_k = (k(k+1))^p$  for  $p \in (\frac{1}{2q'}, 1]$ . If we set

$$A_n := n^{\frac{q'}{q}} ((n(n+1))^p - (n(n-1))^p)^{q'} \sum_{k=n}^{\infty} \frac{1}{(k(k+1))^{q'p}}$$

for  $n \in \mathbb{N}$ , then  $C(p,q) := \sup\{A_n^{\frac{1}{q'}} \mid n \in \mathbb{N}\}$  the constant defined in Proposition 2.3 for our particular choice of the sequence  $(b_n)_{n \in \mathbb{N}}$ .

**Lemma 2.4.** Let  $p > \frac{1}{2q'}$ . For all  $n \in \mathbb{N}$ , we have

(2.5) 
$$A_n \le \left(\frac{(n+1)^p - (n-1)^p}{n^{p-1}}\right)^{q'} \frac{1}{2q'p - 1}.$$

*Proof.* The assertion follows from

$$\sum_{k=n}^{\infty} \frac{1}{b_k^{q'}} = \sum_{k=n}^{\infty} \frac{1}{(k(k+1))^{q'p}} = \sum_{k=n}^{\infty} \left( \int_k^{k+1} x^{-2} dx \right)^{q'p}$$

$$\leq \sum_{k=n}^{\infty} \int_k^{k+1} x^{-2q'p} dx = \frac{1}{2q'p-1} n^{-2q'p+1},$$

for which  $p > \frac{1}{2q'}$  is crucial.

We can say even more about the right-hand side of (2.5).

**Lemma 2.5.** Let  $0 . The sequence <math>(A'_n)_{n \in \mathbb{N}}$  defined by

$$A'_n := \frac{(n+1)^p - (n-1)^p}{n^{p-1}}$$

is monotonically decreasing.

*Proof.* For p=1 the claim is trivial. Otherwise, consider the functions  $f_1, g_1: (0, \infty) \to \mathbb{R}$  defined by  $g_1(x) = x^p$  and

$$f_1(x) = \frac{g_1(x+1) - g_1(x-1)}{x^{p-1}}.$$

Note that  $f_1(n) = A'_n$ . We will show that  $f_1$  is monotonically decreasing on  $[2, \infty)$  and that  $A'_1 > A'_2$ . The Taylor expansion of  $g_1$  at  $x \ge 2$  is given by

$$g_1(x+h) = \sum_{k=0}^{n} \frac{h^k}{k!} x^{p-k} \prod_{m=0}^{k-1} (p-m) + R(x,h,n),$$

and the corresponding approximation error is

$$R(x,h,n) := \int_{x}^{x+h} \frac{(x+h-t)^n}{n!} t^{p-n-1} \prod_{m=0}^{n} (p-m) dt.$$

Next, note that

$$|R(x,1,n)| \le \int_{x}^{x+1} \frac{|x+1-t|^{n}}{n!} t^{p-n-1} \prod_{m=0}^{n} |p-m| \, \mathrm{d}t$$

$$\le \int_{x}^{x+1} t^{p-n-1} \, \mathrm{d}t = \frac{1}{p-n} ((x+1)^{p-n} - x^{p-n}) \le \frac{1}{n-p}$$

and

$$|R(x,-1,n)| \le \int_{x-1}^{x} \frac{(x-1-t)^n}{n!} t^{p-n-1} \prod_{m=0}^{n} (p-m) dt$$

$$\le \int_{x-1}^{x} t^{p-n-1} dt = \frac{1}{p-n} (x^{p-n} - (x-1)^{p-n}) \le \frac{1}{n-p}.$$

This follows from  $\frac{1}{n!}\prod_{m=0}^{n}|p-m|=p\prod_{m=1}^{n}\frac{m-p}{m}\leq 1$  and  $|x+h+t|^n\leq 1$  for  $h\in\{-1,1\}$  and t between x and x+h. As  $\lim_{n\to\infty}\frac{-1}{p-n}=0$ , we know that

 $\lim_{n\to\infty} R(x,h,n) = 0$  for  $h\in\{-1,1\}$ , and we may write

$$g_1(x+1) = \sum_{k=0}^{\infty} \frac{1}{k!} x^{p-k} \prod_{m=0}^{k-1} (p-m)$$

and

$$-g_1(x-1) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k!} x^{p-k} \prod_{m=0}^{k-1} (p-m).$$

Setting  $h_n := \frac{1}{(2n-1)!} \prod_{m=0}^{2n-2} (p-m)$  for  $n \in \mathbb{N}$  and noticing  $h_n > 0$ , this gives

$$g_1(x+1) - g_1(x-1) = 2\sum_{n=1}^{\infty} \frac{1}{(2n-1)!} x^{p-(2n-1)} \prod_{m=0}^{2n-2} (p-m)$$

$$= 2px^{p-1} + 2\sum_{n=2}^{\infty} \frac{1}{(2n-1)!} x^{p-(2n-1)} \prod_{m=0}^{2n-2} (p-m)$$

$$= 2px^{p-1} + 2\sum_{n=2}^{\infty} h_n x^{p-(2n-1)}.$$

As a function of x, the expression

$$f_1(x) = \frac{g_1(x+1) - g_1(x-1)}{x^{p-1}} = 2p + \sum_{n=2}^{\infty} \frac{h_n}{x^{2n-2}}$$

is thus monotonically decreasing on  $[2, \infty)$ . In order to show that

(2.6) 
$$A_1' = 2^p \ge \frac{3^p - 1}{2^{p-1}} = A_2'$$

for all  $p \in (0,1]$ , consider the functions  $f_2, g_2 : \mathbb{R} \to \mathbb{R}$  defined by  $f_2(x) = 3^x - 1$  and  $g_2(x) = 2^{2x-1}$ . Then  $f_2'(x) = 3^x \ln(3)$ ,  $g_2'(x) = 4^x \ln(2)$ , f(1) = g(1), and  $f_2, g_2, f_2'$ , and  $g_2'$  are monotonically increasing. Therefore  $f_1'(1) > g_2'(1)$  shows that  $f(x) \leq g(x)$  for all  $x \leq 1$  (with equality only for x = 1). This implies (2.6).  $\square$ 

In Theorem 2.7, we will show how Lemmas 2.4 and 2.5 and an in some sense optimal choice of p in  $b_n = (n(n+1))^p$  yield  $C_1(2) \le \sqrt{\frac{e \ln(2)}{\sqrt{2}}}$  which is already an improvement to Stechkin's  $C_1(2) \le \frac{2}{\sqrt{3}}$ . More detailed analysis for a specific choice of p in this construction yields further improvement when q = 2, coinciding with de Bruijn's result [4, p. 174] in the first two decimal places.

Corollary 2.6. The minimal constant  $C_1(2) > 0$  for which

$$\sum_{n=1}^{\infty} a_n \le C_1(2) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^2 \right)^{\frac{1}{2}}$$

holds for all  $(a_n)_{n\in\mathbb{N}}\in\ell_2$  is at most 1.1086983.

*Proof.* For  $N \in \mathbb{N}$  with  $N \geq 2$ , Lemmas 2.4 and 2.5 give

$$C(p,2) \le \max \left\{ \max_{n=1,\dots,N-1} \sqrt{A_n}, \sup_{n\ge N} \frac{1}{\sqrt{4p-1}} \frac{(n+1)^p - (n-1)^p}{n^{p-1}} \right\}$$

(2.7) 
$$= \max \left\{ \max_{n=1,\dots,N-1} \sqrt{A_n}, \frac{1}{\sqrt{4p-1}} \frac{(N+1)^p - (N-1)^p}{N^{p-1}} \right\}.$$

The last expression evaluated at N=100 and p=0.88 can be bounded above by 1.1086983. For the computation of  $A_1,\ldots,A_{100}$ , the series  $\sum_{k=n}^{\infty}\frac{1}{(k(k+1))^{2p}}$  have been truncated to  $\sum_{k=n}^{M}\frac{1}{(k(k+1))^{2p}}$  with  $M=2\cdot 10^5$ . The proof of Lemma 2.4 then shows  $\sum_{k=M}^{\infty}\frac{1}{(k(k+1))^{2p}}\leq \frac{1}{4p-1}M^{-4p+1}$ . For  $n\in\mathbb{N}$ , we also have

$$n((n(n+1))^p - (n(n-1))^p)^2 = n^{4p-1} \left(\frac{(n+1)^p - (n-1)^p}{n^{p-1}}\right)^2 \le n^{4p-1}$$

by Lemma 2.5. The truncation error is therefore at most  $\frac{1}{4p-1}N^{4p-1}M^{-4p+1} \approx 1.9055 \cdot 10^{-9}$ , which can be neglected in the computation of the maximum in (2.7).

For p=1, Lemma 2.4 gives  $A_n^{\frac{1}{q'}} \leq 2(2q'-1)^{-\frac{1}{q'}}$ , and Proposition 2.3 yields

(2.8) 
$$\sum_{n=1}^{\infty} a_n \le 2(2q'-1)^{-\frac{1}{q'}} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$$

This extends the bound obtained by Levin and Stechkin [13, Д.61] for  $\frac{5}{3} \leq q < 3$  to arbitrary  $1 < q < \infty$ . The q = 2 case  $C_1(2) \leq \frac{2}{\sqrt{3}}$  has been addressed again in [17, Лемма 3]. For  $1 < q < \frac{2 + \ln(2)}{2 - \ln(2)}$ , we can achieve better bounds by choosing the parameter p optimally in  $b_k = (k(k+1))^p$ .

**Theorem 2.7.** Let  $1 < q \le \frac{2 + \ln(2)}{2 - \ln(2)}$ . The minimal constant  $C_1(q) > 0$  for which

$$\sum_{n=1}^{\infty} a_n \le C_1(q) \sum_{n=1}^{\infty} \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$

holds for all  $(a_n)_{n\in\mathbb{N}}\in\ell_q$  is at  $most\left(\frac{e\ln(2)}{\sqrt{2}}\right)^{\frac{1}{q'}}$ .

*Proof.* Choose  $p \in (\frac{1}{2q'}, 1]$  and set  $b_k = (k(k+1))^p$  in Proposition 2.3. Then Lemmas 2.4 and 2.5 show that

$$\sum_{n=1}^{\infty} a_n \le \frac{2^p}{(2q'p-1)^{\frac{1}{q'}}} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=n}^{\infty} a_k^q\right)^{\frac{1}{q}}$$

for all  $(a_n)_{n\in\mathbb{N}}\in\ell_q$ . For fixed  $q\in(1,\frac{2+\ln(2)}{2-\ln(2)}]$ , we find a minimizer of  $p\mapsto\frac{2^p}{(2q'p-1)^{\frac{1}{q'}}}$ .

Through the change of variables  $\lambda := q'p$ , this expression becomes  $\left(\frac{2^{\lambda}}{2\lambda-1}\right)^{\frac{1}{q'}}$ . The latter is minimized at  $\lambda = \frac{2+\ln(2)}{\ln(4)}$ , so  $\frac{1}{2q'} , and the minimum is <math>\frac{e\ln(2)}{\sqrt{2}}$ .

The conclude this section by comparing the bounds from (2.8) and Theorem 2.7 to those from the literature.

Copson [3, Theorem 2.3] shows that  $C_1(q) \leq q^{\frac{1}{q}}$ . This result is also reported by Hardy, Littlewood, and Pólya [12, Theorem 345]. The bound from (2.8) improves

the one from [3, Theorem 2.3] when  $2(2q'-1)^{-\frac{1}{q'}} < q^{\frac{1}{q}}.$  As

$$\lim_{q \to 1} q^{\frac{1}{q}} = \lim_{q \to \infty} q^{\frac{1}{q}} = 1,$$
$$\lim_{q \to 1} 2(2q' - 1)^{-\frac{1}{q'}} = \lim_{q \to \infty} 2(2q' - 1)^{-\frac{1}{q'}} = 2,$$

and  $2(2q'-1)^{-\frac{1}{q'}} < q^{\frac{1}{q}}$  at q=2, there are real numbers  $q_1$  and  $q_2$  such that  $1 < q_1 < q_2$  and  $2(2q'-1)^{-\frac{1}{q'}} < q^{\frac{1}{q}}$  for all  $q \in (q_1,q_2)$ . Furthermore, the function  $q \mapsto q^{\frac{1}{q}}$  is monotonically increasing on (1,e) and monotonically decreasing on  $(e,\infty)$ . Also, for  $q_3 \approx 1.7718$ , the function  $q \mapsto 2(2q'-1)^{-\frac{1}{q'}}$  is monotonically decreasing on  $(1,q_3)$  and monotonically increasing on  $(q_3,\infty)$ . Thus for q sufficiently close to 1 or  $\infty$ , the bound from [3, Theorem 2.3] is smaller than the one from (2.8). It turns out that  $q_1$  and  $q_2$  can be chosen such that  $2(2q'-1)^{-\frac{1}{q'}} < q^{\frac{1}{q}}$  if and only if  $q \in (q_1,q_2)$ . Analytical expressions for  $q_1$  and  $q_2$  are not available through the inequality  $2(2q'-1)^{-\frac{1}{q'}} < q^{\frac{1}{q}}$ . However, we have  $q_1 \approx 1.3229$  and  $q_2 \approx 4.4124$ . An improvement to [3, Theorem 2.3] is reported by Levin and Stechkin in their appendix to the Russian 1948 edition [13] of Hardy, Littlewood, and Pólya's monograph. Levin and Stechkin's bound [13,  $\pi$ .61] translates to our notation as

(2.9) 
$$C_1(q) \begin{cases} \leq 2^{\frac{1}{q}-2} \left(3 - \frac{1}{q}\right) q \left(2 - \frac{1}{q}\right)^{\frac{1}{q}-1}, & 1 < q < \frac{5}{3}, \\ \leq 2 \left(2^{\frac{q}{q-1}} - 1\right)^{-\frac{q-1}{q}}, & \frac{5}{3} \leq q < 3, \\ = (q-1)^{\frac{1}{q}}, & 3 \leq q < \infty. \end{cases}$$

Note that at first glance, (2.9) is not what is stated in [13,  $\Pi$ .61] or its English translation [14, D.61]. The mismatch is  $2^{\frac{1}{q}-2}\left(3-\frac{1}{q}\right)q(2-\frac{1}{q})^{\frac{1}{q}-1}$  versus  $2^{\frac{1}{q}-2}\left(3-\frac{1}{q}\right)q(1-\frac{1}{q})^{\frac{1}{q}-1}$  in the  $1< q<\frac{5}{3}$  case but the latter would not be an improvement over [12, Theorem 345], and it would not fit in as a special case of the two-parameter inequality [13,  $\Pi$ .62] or [14, D.62(v)], regardless of the discrepancies between the formulas in the different versions and regardless of the fact that it is not the r=-p special case as claimed by Levin and Stechkin but actually the r=+p one.

The bound from (2.8) is larger than the one from [13,  $\pi$ .61] when  $3 < q < \infty$ . (In this case the latter result provides the optimal constant.) The bounds coincide for  $\frac{5}{3} \le q \le 3$ . In the remaining case  $1 < q < \frac{5}{3}$ , our bound from (2.8) is monotonically decreasing in q while the Levin and Stechkin's bound [13,  $\pi$ .61] is monotonically increasing, reversing orders at the boundaries. Thus the bounds coincide at most once for  $1 < q < \frac{5}{3}$ , and they do for  $q_4 \approx 1.3725$ , which means that the bound from (2.8) is smaller for  $q_4 < q \le \frac{5}{3}$  and larger for  $1 < q < q_4$  than the one from [13,  $\pi$ .61].

By construction, the bound from Theorem 2.7 outperforms the one from (2.8). Moreover, Theorem 2.7 is an improvement over [13,  $\pi$ .61] for  $1 < q \le \frac{2 + \ln(2)}{2 - \ln(2)} \approx 2.0608$ .

Gao [9, Theorem 1] provides further improvement by showing that  $C_1(q) = (q-1)^{\frac{1}{q}}$  not only for  $3 \le q < \infty$  as stated in [13, Д.61], but even for  $q_0 \le q < \infty$ , where

 $q_0 \approx 2.8855$  is a solution of the equation

$$2^{\frac{1}{q-1}} \left( (q-1)^{\frac{q}{q-1}} - (q-1) \right) - \left( 1 + \frac{3-q}{2} \right)^{\frac{q}{q-1}} = 0.$$

De Bruijn [4, p. 174] reports that  $C_1(2) = 1.1064957714$  "with an error of at most 9 units at the last decimal place." This is an improvement over Levin and Stechkin's bound  $C_1(2) \leq \frac{2}{\sqrt{3}} \approx 1.1547$  established in [13, Д.61], and over our bound  $C_1(2) \leq \sqrt{\frac{e \ln(2)}{\sqrt{2}}} \approx 1.1542$  from Theorem 2.7. A visualization of the various upper bounds on  $C_1(q)$  is given in Figure 4.

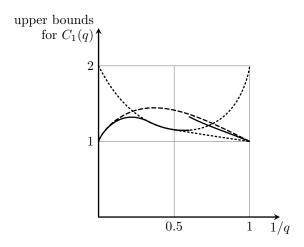


FIGURE 4. The upper bounds on  $C_1(q)$  given by Copson [3, Theorem 2.3] (dashed line), Levin and Stechkin [13,  $\pi$ .61] and Gao [9, Theorem 1] (solid line), and (2.8) and Theorem 2.7 from the paper at hand (dotted line).

## 3. The weak discrete Stechkin inequality

Here we compute the optimal constants  $c_{1,\infty}(q)$  and  $C_{1,\infty}(q) > 0$ , for which the inequality

$$(3.1) \qquad \frac{1}{c_{1,\infty}(q)} \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \leq \sup_{n \in \mathbb{N}} n a_n \leq C_{1,\infty}(q) \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$

holds true for all sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq \ldots \geq 0$ , when  $1< q<\infty$ . With the modification indicated in Section 1, inequality (3.1) holds true also for  $q=\infty$ . The monotonicity assumption on  $(a_n)_{n\in\mathbb{N}}$  gives  $\sup\{a_k\mid k\geq n\}=a_n$ , so  $c_{1,\infty}(\infty)=C_{1,\infty}(\infty)=1$ . For q=1, we have  $C_{1,\infty}(1)=1$  because

$$\sup_{n\in\mathbb{N}} na_n = \sup_{n\in\mathbb{N}} \sum_{k=1}^n a_n \le \sum_{k=1}^\infty a_k = \sup_{n\in\mathbb{N}} \sum_{k=n}^\infty a_k,$$

which holds as an equality when  $a_n = 0$  for all  $n \geq 2$ .

Notable results in the direction of (3.1) are the inequalities

$$n^{1-1/q} \left( \sum_{k=n}^{\infty} a_k^q \right)^{1/q} \le \sum_{n=1}^{\infty} a_n$$

and

$$n^{1-1/q} \left( \sum_{k=n+1}^{\infty} a_k^q \right)^{1/q} \le (q-1)^{-1/q} \sup_{n \in \mathbb{N}} n a_n$$

proved in [19, Лемма IV.2.1] and [8, Proposition 2.11], respectively. The following estimate on the Riemann zeta-function is required in our calculation of  $c_{1,\infty}(q)$ .

**Lemma 3.1.** Let  $1 < q < \infty$ . Then  $\zeta(q) > \frac{1}{q-1} + \frac{1}{2}$ .

Proof. Consider

$$\zeta(q) - \frac{1}{q-1} = \zeta(q) - \int_1^\infty x^{-q} dx > \zeta(q) - \sum_{n=1}^\infty \frac{n^{-q} + (n+1)^{-q}}{2}$$
$$= \zeta(q) - \frac{2\zeta(q) - 1}{2} = \frac{1}{2}.$$

The estimate in Lemma 3.1 is not best possible. In fact, the constant  $\frac{1}{2}$  can be replaced by the Euler–Mascheroni constant, see [20, (2.1.16)]. Nonetheless this estimate enables the computation of the precise constant on the left-hand side of (3.1).

**Theorem 3.2.** Let  $1 < q < \infty$ . The minimal constant  $c_{1,\infty}(q) > 0$  for which

$$\sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \le c_{1,\infty}(q) \sup_{n \in \mathbb{N}} n a_n$$

holds for all  $(a_n)_{n\in\mathbb{N}}$  with  $a_1 \geq a_2 \geq \ldots \geq 0$  is  $c_{1,\infty}(q) = \zeta(q)^{\frac{1}{q}}$ .

*Proof.* The supremum

$$(3.2) \qquad \sup \left\{ \sup_{n \in \mathbb{N}} n^{1 - \frac{1}{q}} \left( \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \middle| a_1 \ge a_2 \ge \dots \ge 0, \sup_{n \in \mathbb{N}} n a_n = 1 \right\}$$

is attained at the sequence  $(a_n)_{n\in\mathbb{N}}$  defined by  $a_n=\frac{1}{n}$  for all  $n\in\mathbb{N}$ . Indeed, for any sequence  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq \ldots \geq 0$  and  $\sup_{n\in\mathbb{N}}na_n=1$ , we have  $0\leq a_n\leq \frac{1}{n}$  for all  $n\in\mathbb{N}$ . Therefore,  $(\sum_{k=n}^\infty a_k^q)^{\frac{1}{q}}\leq (\sum_{k=n}^\infty \frac{1}{k^q})^{\frac{1}{q}}$  for all  $n\in\mathbb{N}$  with equality if and only if  $a_n=\frac{1}{n}$  for all  $n\in\mathbb{N}$ . Also, we have  $\sup_{n\in\mathbb{N}}n\frac{1}{n}=1$ , and (3.2) evaluates to  $\sup_{n\in\mathbb{N}}n^{1-\frac{1}{q}}(\sum_{k=n}^\infty \frac{1}{k^q})^{\frac{1}{q}}=\zeta(q)^{\frac{1}{q}}$ . Note that the supremum  $\sup_{n\in\mathbb{N}}n^{1-\frac{1}{q}}(\sum_{k=n}^\infty \frac{1}{k^q})^{\frac{1}{q}}$  is attained at n=1 because

$$\sum_{k=n}^{\infty} k^{-p} = n^{-p} + \sum_{k=n+1}^{\infty} k^{-p} \le n^{-p} + \int_{n}^{\infty} x^{-p} \mathrm{d}x = n^{-p} + \frac{n^{-p+1}}{p-1}$$

and Lemma 3.1 give

$$n^{1-\frac{1}{q}} \left( \sum_{k=n}^{\infty} k^{-q} \right)^{\frac{1}{q}} \le n^{1-\frac{1}{q}} \left( n^{-q} + \frac{n^{-q+1}}{q-1} \right)^{\frac{1}{q}} = \left( n^{-1} + \frac{1}{q-1} \right)^{\frac{1}{q}} \le \zeta(q)^{\frac{1}{q}}.$$

The result of Theorem 3.2 is depicted in Figure 5.

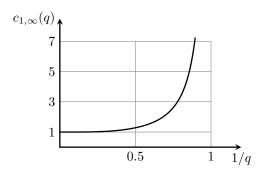


FIGURE 5. The function  $1/q \mapsto \zeta(q)^{\frac{1}{q}}$ .

The optimal constant  $C_{1,\infty}(q)$  in (3.1) turns out to be invariant under taking Hölder conjugates.

**Theorem 3.3.** Let  $1 < q < \infty$ . The minimal constant  $C_{1,\infty}(q) > 0$  for which

$$\sup_{n \in \mathbb{N}} n a_n \le C_{1,\infty}(q) \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}$$

holds for all  $(a_n)_{n \in \mathbb{N}}$  with  $a_1 \ge a_2 \ge ... \ge 0$  is  $C_{1,\infty}(q) = \left(\frac{1}{q}\right)^{-\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{-\left(1 - \frac{1}{q}\right)}$ .

Proof. Step 1. We prove the claim for finite sequences. The infimum

$$\inf \left\{ \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \middle| a_1 \ge a_2 \ge \dots \ge 0, a_{N+1} = 0, \sup_{n \in \mathbb{N}} n a_n = 1 \right\}$$

is attained at the sequence  $(a_n)_{n\in\mathbb{N}}$  given by  $a_n = \frac{1}{N}$  for  $n \in \{1, \dots, N\}$ . Its value is thus equal to

$$\sup_{n=1,\dots,N} n \left( \frac{1}{n} \sum_{k=n}^{N} \frac{1}{N^q} \right)^{\frac{1}{q}} = \sup_{n=1,\dots,N} \frac{1}{N} n^{1-\frac{1}{q}} (N-n+1)^{\frac{1}{q}}.$$

One can readily check that the derivative of the function  $f:(0,N+1)\to\mathbb{R}$ ,  $f(x)=\frac{1}{N}x^{1-\frac{1}{q}}(N-x+1)^{\frac{1}{q}}$  is given by  $f'(x)=\frac{1}{Nq}x^{-\frac{1}{q}}(N-x+1)^{\frac{1}{q}-1}((N+1)(q-1)-qx)$ . Therefore, f is maximized at  $x=(N+1)\left(1-\frac{1}{q}\right)$ , the maximum is

 $\frac{N+1}{N}\left(\frac{1}{q}\right)^{\frac{1}{q}}\left(1-\frac{1}{q}\right)^{1-\frac{1}{q}}$ , and f is monotonically increasing for  $x<(N+1)\left(1-\frac{1}{q}\right)$  and monotonically decreasing for  $x>(N+1)\left(1-\frac{1}{q}\right)$ . Hence we have shown that

$$\inf \left\{ \sup_{n \in \mathbb{N}} n \left( \frac{1}{n} \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \middle| a_1 \ge a_2 \ge \dots \ge 0, a_{N+1} = 0, \sup_{n \in \mathbb{N}} n a_n = 1 \right\}$$

$$= \sup_{n=1,\dots,N} n^{1-\frac{1}{q}} \left( \frac{N-n+1}{N^q} \right)^{\frac{1}{q}}$$

$$\le \sup_{x \in (0,N+1)} x^{1-\frac{1}{q}} \left( \frac{N-x+1}{N^q} \right)^{\frac{1}{q}}$$

$$= \frac{N+1}{N} \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1-\frac{1}{q}}.$$

Taking the infimum over  $N \in \mathbb{N}$ , we see that

$$\inf_{N \in \mathbb{N}} \sup_{n=1,...,N} n^{1-\frac{1}{q}} \left( \frac{N-n+1}{N^q} \right)^{\frac{1}{q}} \le \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1-\frac{1}{q}}$$

for finite sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq\ldots\geq 0$ , which also shows that  $C_{1,\infty}(q)\leq \left(\frac{1}{q}\right)^{-\frac{1}{q}}\left(1-\frac{1}{q}\right)^{-\left(1-\frac{1}{q}\right)}$ . Next we show that also

$$\inf_{N \in \mathbb{N}} \sup_{n=1}^{\infty} n^{1-\frac{1}{q}} \left( \frac{N-n+1}{N^q} \right)^{\frac{1}{q}} \ge \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1-\frac{1}{q}}$$

for finite sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1 \geq a_2 \geq \ldots \geq 0$  and we do it for  $q \geq 2$  first. In this case, we have  $x_N := 1 - \frac{1}{q} - \frac{1}{N+1} \in (0,1)$  for all  $N \in \mathbb{N}$ . The monotonicity properties of the function  $g_N : (0,1) \to \mathbb{R}$ ,  $g_N(x) = f((N+1)x) = \frac{N+1}{N}x^{1-\frac{1}{q}}(1-x)^{\frac{1}{q}}$  yield

$$\sup \left\{ g_N(x) \mid x = \frac{1}{N+1}, \dots, \frac{N}{N+1} \right\} \ge g_N\left(\frac{1}{N+1} \left\lfloor (N+1)\left(1 - \frac{1}{q}\right) \right\rfloor\right)$$

$$\ge g_N(x_N).$$

If we can show that  $(g_N(x_N))_{N\geq 3}$  is monotonically decreasing, then we know that

$$\inf_{N \ge 3} g_N(x_N) = \lim_{N \to \infty} g_N(x_N) = \left(\frac{1}{q}\right)^{\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{1 - \frac{1}{q}}$$

and, in turn,

$$\inf_{N \in \mathbb{N}} \sup_{n=1,\dots,N} n^{1-\frac{1}{q}} \left( \frac{N-n+1}{N^q} \right)^{\frac{1}{q}} \\
= \inf \left\{ 1, \sup \left\{ \frac{1}{2} 2^{\frac{1}{q}}, \frac{1}{2} 2^{1-\frac{1}{q}} \right\}, \inf_{N \ge 3} \sup_{n=1,\dots,N} n^{1-\frac{1}{q}} \left( \frac{N-n+1}{N^q} \right)^{\frac{1}{q}} \right\} \\
\ge \inf \left\{ 1, \sup \left\{ \frac{1}{2} 2^{\frac{1}{q}}, \frac{1}{2} 2^{1-\frac{1}{q}} \right\}, \inf_{N \ge 3} g_N(x_N) \right\}$$

$$\geq \left(\frac{1}{q}\right)^{\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{1 - \frac{1}{q}}.$$

Indeed, let  $a:=\frac{1}{q}\in(0,\frac{1}{2}]$ , and consider the function  $h:(0,1-a)\to\mathbb{R},\ h(t)=\left(\frac{1-a-t}{1-t}\right)^{1-a}\left(\frac{a+t}{1-t}\right)^a$ . Then  $g_N(x_N)=h(\frac{1}{N+1})$ , and it is sufficient to show that h is monotonically increasing on  $(0,\frac{1}{4})$  when  $a\leq\frac{1}{2}$ . The derivative

$$h'(t) = (1-a)\left(\frac{1-a-t}{1-t}\right)^{-a} \frac{-a}{(1-t)^2} \left(\frac{a+t}{1-t}\right)^a$$

$$+ a\left(\frac{a+t}{1-t}\right)^{a-1} \frac{1+a}{(1-t)^2} \left(\frac{1-a-t}{1-t}\right)^{1-a}$$

$$= \frac{1}{(1-t)^2} \left(\frac{a+t}{1-a-t}\right)^{a-1} \left((1-a)\frac{a+t}{1-a-t}(-a) + a(1+a)\right)$$

is  $\geq 0$  if and only if  $(1-a)\frac{a+t}{1-a-t}(-a)+a(1+a)\geq 0$  if and only if  $a(a+t-1)(a+2t-1)\geq 0$ . For  $t\in (0,1-a)$ , we have  $a(a+t-1)\leq 0$ , so  $h'(t)\geq 0$  when  $a+2t-1\leq 0$ . The latter is fulfilled when  $t\leq \frac{1}{4}$ .

For  $q \leq 2$ , we consider the function  $(0,1) \ni x \mapsto g_N(1-x)$  instead of  $g_N$ . This is the same as  $g_N$  for the Hölder conjugate  $q' \geq 2$ , and this is covered by the first case.

Step 2. We prove the claim for all sequences. It remains to show that

$$\frac{\sup_{n \in \mathbb{N}} n^{1 - \frac{1}{q}} \left( \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}}}{\sup_{n \in \mathbb{N}} n a_n} \ge \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1 - \frac{1}{q}}$$

for all sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq\ldots\geq 0$ . Let  $\varepsilon>0$ . Then, for all sequences  $(a_n)_{n\in\mathbb{N}}$  with  $a_1\geq a_2\geq\ldots\geq 0$  and  $\sup\{na_n\mid n\in\mathbb{N}\}=1$ , there exists a number  $N\in\mathbb{N}$  such that  $a_N\geq\frac{1-\varepsilon}{N}$ . It follows that

$$\sup_{n \in \mathbb{N}} n^{1 - \frac{1}{q}} \left( \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \ge \sup_{n=1,\dots,N} n^{1 - \frac{1}{q}} \left( \sum_{k=n}^N \frac{1}{N^q} \right)^{\frac{1}{q}} (1 - \varepsilon)$$

$$= \sup_{n=1,\dots,N} n^{1 - \frac{1}{q}} \left( \frac{N - n + 1}{N^q} \right)^{\frac{1}{q}} (1 - \varepsilon) \ge \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1 - \frac{1}{q}} (1 - \varepsilon).$$

Taking the limit  $\varepsilon \downarrow 0$  yields the desired inequality.

The result of Theorem 3.3 is depicted in Figure 6.

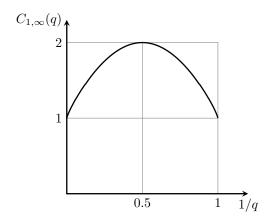


FIGURE 6. The function  $1/q \mapsto \left(\frac{1}{q}\right)^{-\frac{1}{q}} \left(1 - \frac{1}{q}\right)^{-\left(1 - \frac{1}{q}\right)}$ .

## 4. The continuous Stechkin inequalities

In this section, we are concerned with the optimal constants  $\bar{c}_1(q)$ ,  $\bar{C}_1(q)$ ,  $\bar{c}_{1,\infty}(q)$ , and  $\bar{C}_{1,\infty}(q) > 0$  in the inequalities

$$\frac{1}{\bar{c}_1(q)} \int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt \le \int_0^\infty f(t) dt \le \bar{C}_1(q) \int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt$$

$$\frac{1}{\bar{c}_{1,\infty}(q)} \sup_{t>0} t \left(\frac{1}{t} \int_{t}^{\infty} f(s)^{q} \mathrm{d}s\right)^{\frac{1}{q}} \leq \sup_{t>0} t f(t) \leq \bar{C}_{1,\infty}(q) \sup_{t>0} t \left(\frac{1}{t} \int_{t}^{\infty} f(s)^{q} \mathrm{d}s\right)^{\frac{1}{q}}$$

for monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$ . In both cases, precise values for the optimal constants are available, either through the literature or shown here. The proofs in this section are independent of their discrete counterparts, yet there is a strong resemblance in the case of the weak Stechkin inequalities. (The arguments turn out to be less tedious in the continuous inequality, though.)

4.1. The strong continuous Stechkin inequality. Hardy, Littlewood, and Pólya [12, Theorem 337] show that for  $1 < q < \infty$ , the minimal constant  $\bar{C}_1(q) > 0$  for which (4.1)

$$\frac{1}{\bar{c}_1(q)} \int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt \le \int_0^\infty f(t) dt \le \bar{C}_1(q) \int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt$$

holds true for all monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  is  $\bar{C}_1(q)=(q-1)^{\frac{1}{q}}$  when  $1< q<\infty$ . With the modification indicated in Section 1, inequality (4.1) holds true also for  $q=\infty$ . The monotonicity assumption on  $f:(0,\infty)\to [0,\infty)$  gives  $\sup\{f(s)\mid s\geq t\}=f(t),$  so  $\bar{c}_1(\infty)=\bar{C}_1(\infty)=1$ . For q=1, the right-hand side of (4.1) holds in the sense that  $\int_0^\infty \left(\frac{1}{t}\int_t^\infty f(s)^q\mathrm{d}s\right)^{\frac{1}{q}}\mathrm{d}t$  diverges when  $\int_0^\infty f(t)\mathrm{d}t$  is finite. Therefore  $\lim_{q\to 1}\bar{C}_1(q)=0$ . For this, choose a function  $f:(0,\infty)\to[0,\infty)$  with  $\int_0^\infty f(t)\mathrm{d}t\neq0$ . Then  $F(t):=\int_t^\infty f(s)^q\mathrm{d}s$  defines a monotonically decreasing function  $F:(0,\infty)\to[0,\infty)$  for which there exist  $\varepsilon>0$ 

and  $\delta > 0$  such that  $F(t) \geq \delta$  for all  $t \in (0, \varepsilon)$ . It follows that  $\int_0^\infty \frac{1}{t} F(t) dt \geq 0$  $\int_0^{\varepsilon} \frac{1}{t} F(t) dt \ge \int_0^{\varepsilon} \frac{1}{t} \delta dt = \infty.$  We complement these results by computing the minimal constant  $\bar{c}_1(q) > 0$  appear-

ing in (4.1).

**Theorem 4.1.** Let  $1 < q < \infty$ . The minimal constant  $\bar{c}_1(q) > 0$  for which

$$\int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt \le \bar{c}_1(q) \int_0^\infty f(t) dt$$

holds for all monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  is  $\bar{c}_1(q)=$ 

*Proof.* Lower bounds on the constant  $\bar{c}_1(q)$  are given by the quotients

(4.2) 
$$\frac{\int_0^t \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt}{\int_0^\infty f(t) dt}$$

with  $f:(0,\infty)\to[0,\infty)$  a monotonically decreasing function. For  $T\in(0,\infty)$ , take  $f = \frac{1}{T}\chi_{(0,T)}: (0,\infty) \to [0,\infty)$  in (4.2), where  $\chi_{(0,T)}$  denotes the function which is 1 on (0,T) and 0 on  $[T,\infty)$ . Then  $\int_0^\infty f(t) dt = 1$  and

$$\int_0^\infty \left(\frac{1}{t} \int_t^\infty f(s)^q ds\right)^{\frac{1}{q}} dt = \frac{1}{T} \int_0^\infty \left(\frac{1}{t} \int_t^\infty \chi_{(0,T)}(s) ds\right)^{\frac{1}{q}} dt$$

$$= \frac{1}{T} \int_0^T \left(\frac{1}{t} \int_t^\infty \chi_{(0,T)}(s) ds\right)^{\frac{1}{q}} dt = \frac{1}{T} \int_0^T \left(\frac{T-t}{t}\right)^{\frac{1}{q}} dt$$

$$= B\left(1 - \frac{1}{q}, 1 + \frac{1}{q}\right) = \frac{\pi}{q \sin(\frac{\pi}{q})}$$

independently of T. (Here  $B(x,y) := \int_0^1 t^{x-1} (1-t)^{y-1} dt$  denotes the beta function.) This shows  $\bar{c}_1(q) \geq \frac{\pi}{q \sin(\frac{\pi}{q})}$ .

Now fix a function  $f:[0,\infty)\to [0,\infty)$  with  $\int_0^\infty f(x)\mathrm{d}x=1$ . For  $\varepsilon>0$ , let  $N := \left| \frac{1}{\varepsilon} \sup_{t \in [0,\infty)} f(t) \right| \in \mathbb{N} \cup \{\infty\}.$  For  $n \in \mathbb{N}$  with  $n \leq N$ , let

$$T_n := \sup \{T > 0 \mid f(t) \ge \varepsilon n \,\forall \, t \in (0, T)\},$$

 $\lambda_n := \varepsilon T_n$ , and  $g_n := \frac{1}{\lambda_n} \varepsilon \chi_{[0,T_n]}$ . Then  $0 \le h_{\varepsilon}(t) := \sum_{n=1}^N \lambda_n g_n(t) \le f(t)$  for all  $t \in [0, \infty)$  and  $\int_0^\infty g_n(t) dt = 1$  for all n, so  $\sum_{n=1}^N \lambda_n = \int_0^\infty \sum_{n=1}^N \lambda_n g_n(t) dt \le \int_0^\infty f(t) dt = 1$ . From the triangle inequality for integrals, it follows that

$$\int_0^\infty \left(\frac{1}{t} \int_t^\infty h_\varepsilon(s)^q ds\right)^{\frac{1}{q}} dt \le \sum_{n=1}^N \lambda_n \int_0^\infty \left(\frac{1}{t} \int_t^\infty g_n(s)^q ds\right)^{\frac{1}{q}} dt$$
$$= \sum_{n=1}^N \lambda_n \frac{\pi}{q \sin(\frac{\pi}{q})} \le \frac{\pi}{q \sin(\frac{\pi}{q})}.$$

With the abbreviations  $H_{\varepsilon}(t) := \left(\int_t^{\infty} h_{\varepsilon}(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$  and  $F(t) := \left(\int_t^{\infty} f(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$ , we have  $\lim_{k \to \infty} H_{2^{-k}}(t) = F(t)$  and  $0 \le H_{2^{-k}}(t) \le F(t)$  for all  $t \in (0, \infty)$  and  $k \in \mathbb{N}$ .

The dominated convergence theorem then yields

$$\int_0^\infty F(t)\mathrm{d}t = \int_0^\infty \lim_{k \to \infty} H_{2^{-k}}(t)\mathrm{d}t = \lim_{k \to \infty} \int_0^\infty H_{2^{-k}}(t)\mathrm{d}t \le \frac{\pi}{q\sin(\frac{\pi}{q})}.$$

This shows  $\bar{c}_1(q) \leq \frac{\pi}{q \sin(\frac{\pi}{q})}$  and completes the proof.

The precise values for  $\bar{c}_1(q)$  and  $\bar{C}_1(q)$  given in Theorem 4.1 and [12, Theorem 337] are illustrated in Figure 7.

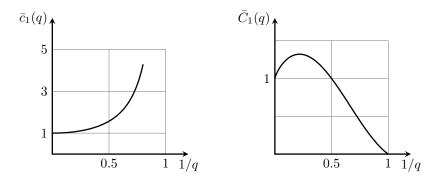


FIGURE 7. The function  $1/q \mapsto \frac{\pi}{q \sin(\frac{\pi}{q})}$  (left) and  $1/q \mapsto (q-1)^{\frac{1}{q}}$  (right).

Theorem 4.1 and [12, Theorem 337] can be transformed into a result on functions  $f: \Omega \to \mathbb{R}$  defined on a measure space  $(\Omega, \mu)$ . For such a function f, the assignment  $f^*(t) := \inf\{s > 0 \mid \mu(\{x \in \Omega \mid |f(x)| > s\}) \le t\}$  defines a function  $f^*: (0, \infty) \to [0, \infty)$ , called the non-increasing rearrangement of f.

**Corollary 4.2.** Let  $(\Omega, \mu)$  be a measure space and  $1 < q < \infty$ . The minimal constants c(q), C(q) > 0 for which

$$\frac{1}{c(q)} \int_0^\infty \left(\frac{1}{t} \int_t^\infty f^*(s)^q ds\right)^{\frac{1}{q}} dt \le \int_0^\infty |f(x)| d\mu(x)$$

$$\le C(q) \int_0^\infty \left(\frac{1}{t} \int_t^\infty f^*(s)^q ds\right)^{\frac{1}{q}} dt$$

holds for all functions  $f: \Omega \to \mathbb{R}$  are  $c(q) = \bar{c}_1(q)$  and  $C(q) = \bar{C}_1(q)$ .

*Proof.* Note that  $\int_{\Omega} |f(x)| d\mu(x) = \int_{0}^{\infty} f^{*}(t) dt$  and apply Theorem 4.1 and [12, Theorem 337] to  $f^{*}$ .

4.2. The weak continuous Stechkin inequality. Here we compute the optimal constants  $\bar{c}_{1,\infty}(q)$  and  $\bar{C}_{1,\infty}(q) > 0$ , for which the inequality (4.3)

$$\frac{1}{\bar{c}_{1,\infty}(q)} \sup_{t>0} t \left(\frac{1}{t} \int_t^\infty f(s)^q \mathrm{d}s\right)^{\frac{1}{q}} \leq \sup_{t>0} t f(t) \leq \bar{C}_{1,\infty}(q) \sup_{t>0} t \left(\frac{1}{t} \int_t^\infty f(s)^q \mathrm{d}s\right)^{\frac{1}{q}}$$

holds true for all monotonically decreasing functions  $f:(0,\infty)\to[0,\infty)$ , when  $1 < q < \infty$ . With the modification indicated in Section 1, inequality (4.3) holds true

also for  $q = \infty$ . The monotonicity assumption on f gives  $\sup \{f(s) \mid s \geq t\} = f(t)$ , so  $\bar{c}_{1,\infty}(\infty) = \bar{C}_{1,\infty}(\infty) = 1$ . For q = 1, we have  $\bar{C}_{1,\infty}(1) = 1$  because

$$\sup_{t>0} tf(t) = \sup_{t>0} \int_0^t f(t) ds \le \int_0^\infty f(s) ds = \sup_{t>0} \int_t^\infty f(s) ds.$$

**Theorem 4.3.** Let  $1 < q < \infty$ . The minimal constant  $\bar{c}_{1,\infty}(q) > 0$  for which

$$\sup_{t>0} t \left(\frac{1}{t} \int_{t}^{\infty} f(s)^{q} ds\right)^{\frac{1}{q}} \leq \bar{c}_{1,\infty}(q) \sup_{t>0} t f(t)$$

holds for all monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  is  $\bar{c}_{1,\infty}(q)=(q-1)^{-\frac{1}{q}}$ .

*Proof.* The supremum of the expression

$$\sup_{t>0} t^{1-\frac{1}{q}} \left( \int_{t}^{\infty} f(s)^{q} ds \right)^{\frac{1}{q}}$$

over the monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  with  $\sup_{t>0} tf(t)=1$  is attained at the function  $f:(0,\infty)\to [0,\infty)$  defined by  $f(t)=\frac{1}{t}$ . Indeed, for any monotonically decreasing function  $f:(0,\infty)\to [0,\infty)$  with  $\sup_{t>0} tf(t)=1$ , we have  $0\le f(t)\le \frac{1}{t}$  for all t>0. Therefore,  $\left(\int_t^\infty f(s)^q\mathrm{d}s\right)^{\frac{1}{q}}\le \left(\int_t^\infty \frac{1}{s^q}\mathrm{d}s\right)^{\frac{1}{q}}$  for all t>0 with equality if and only if  $f(t)=\frac{1}{t}$  for all t>0. Also, we have  $\sup_{t>0} t\frac{1}{t}=1$ , and (4.4) evaluates to  $\sup_{t>0} t^{1-\frac{1}{q}} \left(\int_t^\infty \frac{1}{s^q}\mathrm{d}s\right)^{\frac{1}{q}}=(q-1)^{-\frac{1}{q}}$ .

The optimal constant  $\bar{C}_{1,\infty}(q)$  in (4.3) coincides with its discrete counterpart.

**Theorem 4.4.** Let  $1 < q < \infty$ . The minimal constant  $\bar{C}_{1,\infty}(q) > 0$  for which

$$\sup_{t>0} tf(t) \le \bar{C}_{1,\infty}(q) \sup_{t>0} t\left(\frac{1}{t} \int_{t}^{\infty} f(s)^{q} ds\right)^{\frac{1}{q}}$$

holds for all monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  is  $\bar{C}_{1,\infty}(q)=\left(\frac{1}{q}\right)^{-\frac{1}{q}}\left(1-\frac{1}{q}\right)^{-\left(1-\frac{1}{q}\right)}$ .

Proof. Step 1. We prove the claim for functions supported by an interval (0,T) with T>0. The infimum of the expression

$$\sup_{t>0} t \left(\frac{1}{t} \int_{t}^{\infty} f(s)^{q} ds\right)^{\frac{1}{q}}$$

over the monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  with f(T)=0 and  $\sup_{t>0}tf(t)=1$  is attained at the function  $f=\frac{1}{T}\chi_{(0,T)}:(0,\infty)\to [0,\infty)$ . Its value is thus equal to

$$\sup_{t \in (0,T)} t \left( \frac{1}{t} \int_{t}^{T} \frac{1}{T^{q}} ds \right)^{\frac{1}{q}} = \sup_{t \in (0,T)} \frac{1}{T} t^{1 - \frac{1}{q}} (T - t)^{\frac{1}{q}}.$$

One can readily check that the derivative of the function  $F:(0,T)\to\mathbb{R},\ F(t)=\frac{1}{T}t^{1-\frac{1}{q}}(T-t)^{\frac{1}{q}}$  is given by  $F'(t)=\frac{q(T-t)+T}{q(T-t)T}\left(\frac{T-t}{t}\right)^{\frac{1}{q}}$ . Therefore, F is maximized at  $t=T\left(1-\frac{1}{q}\right)$ , and the maximum is  $\left(\frac{1}{q}\right)^{\frac{1}{q}}\left(1-\frac{1}{q}\right)^{1-\frac{1}{q}}$  independently of T.

Step 2. We prove the claim for all functions. It remains to show that

$$\frac{\sup_{t>0} t^{1-\frac{1}{q}} \left( \int_{t}^{\infty} f(s)^{q} ds \right)^{\frac{1}{q}}}{\sup_{t>0} t f(t)} \ge \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1-\frac{1}{q}}$$

for all monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$ . Let  $\varepsilon>0$ . Then, for all monotonically decreasing functions  $f:(0,\infty)\to [0,\infty)$  with  $\sup\{tf(t)\mid t>0\}=1$ , there exists a number  $T\in\mathbb{N}$  such that  $f(T)\geq \frac{1-\varepsilon}{T}$ . It follows that

$$\sup_{t>0} t^{1-\frac{1}{q}} \left( \int_{t}^{\infty} f(s)^{q} ds \right)^{\frac{1}{q}} \ge \sup_{t \in (0,T)} t^{1-\frac{1}{q}} \left( \int_{t}^{T} \frac{1}{T^{q}} ds \right)^{\frac{1}{q}} (1-\varepsilon)$$

$$= \sup_{t \in (0,T)} t^{1-\frac{1}{q}} \left( \frac{T-t}{T^{q}} \right)^{\frac{1}{q}} (1-\varepsilon) \ge \left( \frac{1}{q} \right)^{\frac{1}{q}} \left( 1 - \frac{1}{q} \right)^{1-\frac{1}{q}} (1-\varepsilon).$$

Taking the limit  $\varepsilon \downarrow 0$  yields the desired inequality.

The precise values for  $\bar{c}_{1,\infty}(q)$  and  $\bar{C}_{1,\infty}(q)$  are illustrated in Figure 8.

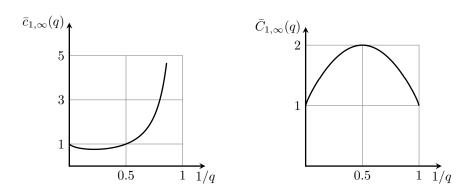


FIGURE 8. The function  $1/q \mapsto (q-1)^{-\frac{1}{q}}$  (left) and  $1/q \mapsto \left(\frac{1}{q}\right)^{-\frac{1}{q}} \left(1-\frac{1}{q}\right)^{-\left(1-\frac{1}{q}\right)}$  (right).

### 5. Applications to sparse approximation

As mentioned in Section 1, the inequalities (1.1) and (1.2) play an important role in nonlinear approximation. More precisely, we will outline the connection of [5, Theorem 4] and our results. Let  $\mathcal{H}$  be an infinite-dimensional separable real Hilbert space with inner product  $\langle \cdot | \cdot \rangle_{\mathcal{H}}$  and norm  $\| \cdot \|_{\mathcal{H}}$ . The choice of an orthonormal basis  $(e_k)_{k \in \mathbb{N}}$  and Parseval's identity give an isometric isomorphism  $\mathcal{H} \to \ell_2$ ,  $f \mapsto (\langle f | e_k \rangle_{\mathcal{H}})_{k \in \mathbb{N}}$ . Sparse approximation in  $\mathcal{H}$  is implemented by defining the approximation error for  $f \in \mathcal{H}$  as

$$E_n(f)_{\mathcal{H}} = \{ \|f - g\|_{\mathcal{H}} \mid g \in \Sigma_{n-1}(\mathcal{H}) \}$$

where  $\Sigma_{n-1}(\mathcal{H}) := \{ \sum_{k \in \Lambda} \lambda_k e_k \mid \lambda_k \in \mathbb{R}, \Lambda \subset \mathbb{N}, \#\Lambda < n \}$ . Then, for  $\alpha > 0$ , the approximation space  $\mathcal{A}_r^{\alpha}(\mathcal{H})$  is defined as the set of elements  $f \in \mathcal{H}$  for which the

quantity

$$||f||_{\mathcal{A}_r^{\alpha}(\mathcal{H})} := \begin{cases} \left( \sum_{n=1}^{\infty} (n^{\alpha} E_n(f)_{\mathcal{H}})^r \frac{1}{n} \right)^{\frac{1}{r}}, & 0 < r < \infty, \\ \sup_{n \in \mathbb{N}} n^{\alpha} E_n(f)_{\mathcal{H}}, & r = \infty \end{cases}$$

is finite. Approximation spaces are then subject to characterizations in terms of Lorentz sequence spaces  $\ell_{p,r}$ , i.e., the sets of bounded sequences for which the quantity

$$\|(f_k)_{k \in \mathbb{N}}\|_{\ell_{p,r}} := \begin{cases} \left( \sum_{n \in \mathbb{N}} (n^{\frac{1}{p}} f_n^*)^r \frac{1}{n} \right)^{\frac{1}{r}}, & 0 < r < \infty, \\ \sup_{n \in \mathbb{N}} n^{\frac{1}{p}} f_n^*, & r = \infty \end{cases}$$

is finite. Here  $0 , and <math>(f_k^*)_{k \in \mathbb{N}}$  is the non-increasing rearrangement of the sequence  $(|f_k|)_{k \in \mathbb{N}}$ . Now, given  $f \in \mathcal{H}$  and  $f_k := \langle f | e_k \rangle_{\mathcal{H}}$  for  $k \in \mathbb{N}$ , DeVore [5, Theorem 4] shows that

$$(5.1) ||f||_{\mathcal{A}_{r}^{\alpha}(\mathcal{H})} \times ||(f_{k})_{k \in \mathbb{N}}||_{\ell_{\tau,r}}.$$

meaning that there exist constants c, C > 0 such that

$$\frac{1}{c} \|f\|_{\mathcal{A}_r^{\alpha}(\mathcal{H})} \le \|(f_k)_{k \in \mathbb{N}}\|_{\ell_{\tau,r}} \le C \|f\|_{\mathcal{A}_r^{\alpha}(\mathcal{H})}.$$

Here  $\tau = (\alpha + \frac{1}{2})^{-1}$ . Notable special cases of (5.1) are re-parameterized by the results from Sections 2 and 3.

**Theorem 5.1.** With the definitions above, the following statements are true for the minimal constants c, C > 0 in the inequality

(5.2) 
$$\frac{1}{c} \|f\|_{\mathcal{A}_{r}^{\alpha}(\mathcal{H})} \leq \|(f_{k})_{k \in \mathbb{N}}\|_{\ell_{\tau,r}} \leq C \|f\|_{\mathcal{A}_{r}^{\alpha}(\mathcal{H})},$$

where  $\tau = (\alpha + \frac{1}{2})^{-1}$ .

(i) If 
$$r = \tau$$
, then  $c = c_1(2\alpha + 1)^{\alpha + \frac{1}{2}}$  and  $C = C_1(2\alpha + 1)^{\alpha + \frac{1}{2}}$ .

(ii) If 
$$r = \infty$$
, then  $c = c_{1,\infty}(2\alpha + 1)^{\alpha + \frac{1}{2}}$  and  $C = C_{1,\infty}(2\alpha + 1)^{\alpha + \frac{1}{2}}$ .

*Proof.* For  $r = \tau$ , inequality (5.1) becomes

$$\frac{1}{C_1} \left( \sum_{n=1}^{\infty} \left( n^{\alpha} \left( \sum_{k=n}^{\infty} (f_k^*)^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{\tau}} \right)^{\frac{1}{\tau}} \le \left( \sum_{k=1}^{\infty} (f_k^*)^{\tau} \right)^{\frac{1}{\tau}} \\
\le C_2 \left( \sum_{n=1}^{\infty} \left( n^{\alpha} \left( \sum_{k=n}^{\infty} (f_k^*)^2 \right)^{\frac{1}{2}} \right)^{\frac{1}{\tau}} \frac{1}{n} \right)^{\frac{1}{\tau}}.$$

Setting  $a_k = (f_k^*)^{\tau}$  and  $q = \frac{2}{\tau} = 2\alpha + 1$  gives

$$\frac{1}{C_1} \left( \sum_{n=1}^{\infty} n^{-\frac{1}{q}} \left( \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \right)^{\frac{1}{\tau}} \le \left( \sum_{k=1}^{\infty} a_k \right)^{\frac{1}{\tau}} \le C_2 \left( \sum_{n=1}^{\infty} n^{-\frac{1}{q}} \left( \sum_{k=n}^{\infty} a_k^q \right)^{\frac{1}{q}} \right)^{\frac{1}{\tau}}.$$

Raising everything to the  $\tau$ th power shows (i). Similarly, for  $r=\infty$ , inequality (5.1) becomes

$$\frac{1}{C_1} \sup_{n \in \mathbb{N}} n^{\alpha} \left( \sum_{k=n}^{\infty} (f_k^*)^2 \right)^{\frac{1}{2}} \le \sup_{k \in \mathbb{N}} k^{\frac{1}{\tau}} f_k^* \le C_2 \sup_{n \in \mathbb{N}} n^{\alpha} \left( \sum_{k=n}^{\infty} (f_k^*)^2 \right)^{\frac{1}{2}}$$

Raising everything to the  $\tau$ th power gives

$$\frac{1}{C_1} \sup_{n \in \mathbb{N}} n^{\alpha \tau} \left( \sum_{k=n}^{\infty} (f_k^*)^2 \right)^{\frac{\tau}{2}} \le \sup_{k \in \mathbb{N}} k (f_k^*)^{\tau} \le C_2 \sup_{n \in \mathbb{N}} n^{\alpha \tau} \left( \sum_{k=n}^{\infty} (f_k^*)^2 \right)^{\frac{\tau}{2}}.$$

Setting  $a_k = (f_k^*)^{\tau}$  and  $q = \frac{2}{\tau} = 2\alpha + 1$  shows (ii).

Stechkin [17] considers the Hilbert space  $\mathcal{H}=L_2(\mathbb{T}^d)$  of square-integrable functions on  $\mathbb{T}^d=[0,2\pi]^d$ . An orthonormal basis in  $L_2(\mathbb{T}^d)$  is given by  $e_k(x):=\frac{1}{\sqrt{2\pi}}\exp(\mathrm{i}kx)$  for  $k\in\mathbb{Z}$  and  $x\in\mathbb{T}^d$ . Then (5.1) for  $r=\tau=1$  shows that the approximation space  $\mathcal{A}_1^{1/2}(L_2(\mathbb{T}^d))$  coincides with the Wiener algebra  $\mathcal{A}(\mathbb{T}^d):=\left\{f\in C(\mathbb{T}) \ \Big| \ \sum_{k\in\mathbb{Z}^d} |\hat{f}(k)| <\infty\right\}$  and, moreover, their quasi-norms are equivalent in the sense of (5.2). Theorem 5.1 yields

$$\frac{2}{\pi} \|f\|_{\mathcal{A}_{1}^{1/2}(L_{2}(\mathbb{T}^{d}))} \le \|f\|_{\mathcal{A}(\mathbb{T}^{d})} \le 1.1064957714 \|f\|_{\mathcal{A}_{1}^{1/2}(L_{2}(\mathbb{T}^{d}))}$$

for  $f \in L_2(\mathbb{T}^d)$ , the constants being independent of d.

Following DeVore [5, Remark 7.4], if the Hilbert space  $\mathcal{H}$  is chosen to be  $L_2(\mathbb{R})$  with a wavelet orthonormal basis  $(\psi_I)_{I\in\mathcal{D}}$ , then the approximation space can be characterized in terms of Besov smoothness. A multivariate version of this result using a tensorized wavelet basis is derived by Sickel and Ullrich in [16, Theorem 2.7]. For further results in this direction, see the papers of Hansen and Sickel [10, 11]. **Acknowledgements.** The authors would like to acknowledge support by the DFG Ul-403/2-1. They would further like to thank Kai Lüttgen, Winfried Sickel, Vladimir Temlyakov, and Gerd Wachsmuth for insightful discussions.

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