# Quantum Approximation I. Embeddings of Finite Dimensional $L_{p}$ Spaces 

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#### Abstract

We study approximation of embeddings between finite dimensional $L_{p}$ spaces in the quantum model of computation. For the quantum query complexity of this problem matching (up to logarithmic factors) upper and lower bounds are obtained. The results show that for certain regions of the parameter domain quantum computation can essentially improve the rate of convergence of classical deterministic or randomized approximation, while there are other regions where the best possible rates coincide for all three settings. These results serve as a crucial building block for analyzing approximation in function spaces in a subsequent paper 11.


## 1 Introduction

In this paper we continue the investigation of numerical problems of analysis in the quantum model of computation. In a number of papers the integration problem and its discretized version, the mean computation, were studied and matching upper and lower bounds (often up to logarithmic factors) were established for various function classes. See the references [2, 5, 16, [19, 8, 14, 10, 25, 15, 12]. It turned out that for these types of problems quantum computing can reach an exponential speedup over deterministic classical computation and a quadratic speedup over randomized classical computation.

All these problems are such that the solution is a single number. Therefore the question arises what happens if we consider problems whose solu-
tion is a family of numbers or, in other words, a function. A particularly typical situation is function approximation, where the solution is just the input function itself (and we are asked to compute an approximation to it in a given norm). A first consideration of an approximation problem in the quantum setting appears in [20], but no matching upper and lower bounds were obtained.

In the present paper we provide the first results for approximation in the quantum model of computation, with matching upper and lower bounds. We start with the very basic situation: the approximation of the embedding $J_{p q}^{N}$ of $L_{p}^{N}$ into $L_{q}^{N}$, or, in other words, the approximation of $N$-sequences with bounded $L_{p}^{N}$ norm in the norm of $L_{q}^{N}$. These embeddings are the elementary building blocks of embeddings of function spaces - in the same way as mean computation is the elementary building block of integration, see 10 and 11 for more on this principle. Our results show that for $p<q$, the quantum model of computation can bring an acceleration up to a factor $N^{-1}$ of the rate of the classical (deterministic or randomized) setting. On the other hand, for $p \geq q$, the optimal rate is the same for all three settings, so in these cases there is no speedup of the rate by quantum computation.

We prove that the following version of Grover's quantum search algorithm is optimal: we find all coordinates of $f \in L_{p}^{N}$ with absolute value not smaller than a suitably chosen threshold and set the other coordinates to zero. The crucial new element in proving lower bounds is a multiplicative inequality for the $n$-th minimal query error, which is - in a wide sense analogous to multiplicativity properties of $s$-numbers, see 21 .

In a subsequent paper [11] we show that, similarly to the analysis in [10, sufficiently precise knowledge about the embeddings $J_{p q}^{N}$ leads to a full understanding of the infinite dimensional problem of approximation of functions from Sobolev spaces.

The paper is organized as follows. In Section 2 we recall notation from the quantum setting of information-based complexity theory as developed in [8]. In Section 3 we derive some new general results which will be needed later on. Section 4 contains the main results on approximation of embeddings of $L_{p}^{N}$ into $L_{q}^{N}$ spaces. Finally, in Section 5 we give some comments on the quantum bit model and a summary including comparisons to the respective results in the classical deterministic and randomized setting.

For more details on the quantum setting of information-based complexity we refer to [8], to the survey [13], and to an introduction [9]. For the classical settings of information-based complexity theory see [18, [26, 7. General background on quantum computing can be found in the surveys [1, 4, 24]
and in the monographs [22, 6] and [17.

## 2 Notation

For nonempty sets $D$ and $K$, we denote by $\mathcal{F}(D, K)$ the set of all functions from $D$ to $K$. For a normed space $G$ we let $\mathcal{B}(G)=\left\{g \in G \mid\|g\|_{G} \leq 1\right\}$ denote the unit ball of $G$. Let $F \subseteq \mathcal{F}(D, K)$ be a nonempty subset. Let $\mathbf{K}$ stand for either $\mathbf{R}$ or $\mathbf{C}$, the field of real or complex numbers, let $G$ be a normed space over $\mathbf{K}$, and let $S: F \rightarrow G$ be a mapping. We seek to approximate $S(f)$ for $f \in F$ by means of quantum computations. Let $H_{1}$ be the two-dimensional complex Hilbert space $\mathbf{C}^{2}$, with its unit vector basis $\left\{e_{0}, e_{1}\right\}$, let

$$
H_{m}=\underbrace{H_{1} \otimes \cdots \otimes H_{1}}_{m},
$$

equipped with the tensor Hilbert space structure. Denote

$$
\mathbf{Z}[0, N):=\{0, \ldots, N-1\}
$$

for $N \in \mathbf{N}$, where we agree to write, as usual, $\mathbf{N}=\{1,2, \ldots\}$ and $\mathbf{N}_{0}=$ $\mathbf{N} \cup\{0\}$. Let $\mathcal{C}_{m}=\left\{|i\rangle: i \in \mathbf{Z}\left[0,2^{m}\right)\right\}$ be the canonical basis of $H_{m}$, where $|i\rangle$ stands for $e_{j_{0}} \otimes \cdots \otimes e_{j_{m-1}}, i=\sum_{k=0}^{m-1} j_{k} 2^{m-1-k}$ is the binary expansion of $i$. Let $\mathcal{U}\left(H_{m}\right)$ denote the set of unitary operators on $H_{m}$.

A quantum query on $F$ is given by a tuple

$$
Q=\left(m, m^{\prime}, m^{\prime \prime}, Z, \tau, \beta\right)
$$

where $m, m^{\prime}, m^{\prime \prime} \in \mathbf{N}, m^{\prime}+m^{\prime \prime} \leq m, Z \subseteq \mathbf{Z}\left[0,2^{m^{\prime}}\right)$ is a nonempty subset, and

$$
\begin{gathered}
\tau: Z \rightarrow D \\
\beta: K \rightarrow \mathbf{Z}\left[0,2^{m^{\prime \prime}}\right)
\end{gathered}
$$

are arbitrary mappings. We let $m(Q):=m$ be the number of qubits of $Q$.
Given a query $Q$, we define for each $f \in F$ the unitary operator $Q_{f}$ by setting for $|i\rangle|x\rangle|y\rangle \in \mathcal{C}_{m}=\mathcal{C}_{m^{\prime}} \otimes \mathcal{C}_{m^{\prime \prime}} \otimes \mathcal{C}_{m-m^{\prime}-m^{\prime \prime}}:$

$$
Q_{f}|i\rangle|x\rangle|y\rangle= \begin{cases}|i\rangle|x \oplus \beta(f(\tau(i)))\rangle|y\rangle & \text { if } \quad i \in Z \\ |i\rangle|x\rangle|y\rangle & \text { otherwise }\end{cases}
$$

where $\oplus$ means addition modulo $2^{m^{\prime \prime}}$.
A quantum algorithm on $F$ with no measurement is a tuple

$$
A=\left(Q,\left(U_{j}\right)_{j=0}^{n}\right)
$$

Here $Q$ is a quantum query on $F, n \in \mathbf{N}_{0}$ and $U_{j} \in \mathcal{U}\left(H_{m}\right)(j=0, \ldots, n)$, with $m=m(Q)$. Given $f \in F$, we define $A_{f} \in \mathcal{U}\left(H_{m}\right)$ as

$$
A_{f}=U_{n} Q_{f} U_{n-1} \ldots U_{1} Q_{f} U_{0}
$$

We denote by $n_{q}(A):=n$ the number of queries and by $m(A)=m=$ $m(Q)$ the number of qubits of $A$. Let $\left(A_{f}(x, y)\right)_{x, y \in \mathbf{Z}\left[0,2^{m}\right)}$ be the matrix of the transformation $A_{f}$ in the canonical basis $\mathcal{C}_{m}$, that is, $A_{f}(x, y)=$ ( $A_{f}|y\rangle,|x\rangle$ ).

A quantum algorithm from $F$ to $G$ with $k$ measurements is a tuple

$$
A=\left(\left(A_{l}\right)_{l=0}^{k-1},\left(b_{l}\right)_{l=0}^{k-1}, \varphi\right)
$$

where $k \in \mathbf{N}, A_{l}(l=0, \ldots, k-1)$ are quantum algorithms on $F$ with no measurement,

$$
\begin{gathered}
b_{0} \in \mathbf{Z}\left[0,2^{m_{0}}\right), \\
b_{l}: \prod_{i=0}^{l-1} \mathbf{Z}\left[0,2^{m_{i}}\right) \rightarrow \mathbf{Z}\left[0,2^{m_{l}}\right) \quad(1 \leq l \leq k-1),
\end{gathered}
$$

where $m_{l}:=m\left(A_{l}\right)$, and

$$
\varphi: \prod_{l=0}^{k-1} \mathbf{Z}\left[0,2^{m_{l}}\right) \rightarrow G
$$

The output of $A$ at input $f \in F$ is a probability measure $A(f)$ on $G$, defined as follows. First put

$$
\begin{aligned}
p_{A, f}\left(x_{0}, \ldots, x_{k-1}\right)= & \left|A_{0, f}\left(x_{0}, b_{0}\right)\right|^{2}\left|A_{1, f}\left(x_{1}, b_{1}\left(x_{0}\right)\right)\right|^{2} \ldots \\
& \ldots\left|A_{k-1, f}\left(x_{k-1}, b_{k-1}\left(x_{0}, \ldots, x_{k-2}\right)\right)\right|^{2} .
\end{aligned}
$$

Then define $A(f)$ by setting for any subset $C \subseteq G$

$$
A(f)(C)=\sum_{\varphi\left(x_{0}, \ldots, x_{k-1}\right) \in C} p_{A, f}\left(x_{0}, \ldots, x_{k-1}\right) .
$$

Let $n_{q}(A):=\sum_{l=0}^{k-1} n_{q}\left(A_{l}\right)$ denote the number of queries used by $A$. For more details and background see [8]. Note that we often use the term 'quantum algorithm' (or just 'algorithm'), meaning a quantum algorithm with measurement(s).

If $A$ is an algorithm with one measurement, the above definition simplifies essentially. Such an algorithm is given by

$$
\begin{equation*}
A=\left(A_{0}, b_{0}, \varphi\right), \quad A_{0}=\left(Q,\left(U_{j}\right)_{j=0}^{n}\right) . \tag{1}
\end{equation*}
$$

The quantum computation is carried out on $m:=m(Q)$ qubits. For $f \in F$ the algorithm starts in the state $\left|b_{0}\right\rangle$ and produces

$$
\begin{equation*}
\left|\psi_{f}\right\rangle=U_{n} Q_{f} U_{n-1} \ldots U_{1} Q_{f} U_{0}\left|b_{0}\right\rangle \tag{2}
\end{equation*}
$$

Let

$$
\begin{equation*}
\left|\psi_{f}\right\rangle=\sum_{i=0}^{2^{m}-1} a_{i, f}|i\rangle \tag{3}
\end{equation*}
$$

(referring to the notation above, we have $a_{i, f}=A_{0, f}\left(i, b_{0}\right)$ ). Then $A$ outputs the element $\varphi(i) \in G$ with probability $\left|a_{i, f}\right|^{2}$. It is shown in [8], Lemma 1, that for each algorithm $A$ with $k$ measurements there is an algorithm $\widetilde{A}$ with one measurement such that $A(f)=\widetilde{A}(f)$ for all $f \in F$ and $\widetilde{A}$ uses just twice the number of queries of $A$, that is, $n_{q}(\widetilde{A})=2 n_{q}(A)$. Hence, as long as we are concerned with studying the minimal query error (see below) up to the order, that is, up to constant factors, we can restrict ourselves to algorithms with one measurement.

Let $\theta \geq 0$. For a quantum algorithm $A$ we define the (probabilistic) error at $f \in F$ as follows. Let $\zeta$ be a random variable with distribution $A(f)$. Then

$$
e(S, A, f, \theta)=\inf \{\varepsilon \geq 0 \mid \mathbf{P}\{\|S(f)-\zeta\|>\varepsilon\} \leq \theta\}
$$

(note that this infimum is always attained). Hence $e(S, A, f, \theta) \leq \varepsilon$ iff the probability that the algorithm $A$ computes $S(f)$ with error at most $\varepsilon$ is at least $1-\theta$. Observe that for algorithms with one measurement ((1))-(3)),

$$
P\{\|S(f)-\zeta\|>\varepsilon\}=\sum_{i:\|S(f)-\varphi(i)\|>\varepsilon}\left|a_{i, f}\right|^{2} .
$$

Trivially, $e(S, A, f, \theta)=0$ for $\theta \geq 1$. We put

$$
e(S, A, F, \theta)=\sup _{f \in F} e(S, A, f, \theta)
$$

(we allow the value $+\infty$ for this quantity). Furthermore, we set

$$
\begin{aligned}
& e_{n}^{\mathrm{q}}(S, F, \theta) \\
& \quad=\inf \left\{e(S, A, F, \theta) \mid A \text { is any quantum algorithm with } n_{q}(A) \leq n\right\} .
\end{aligned}
$$

It is customary to consider these quantities at a fixed error probability level. We denote

$$
e(S, A, f)=e(S, A, f, 1 / 4)
$$

and similarly,

$$
e(S, A, F)=e(S, A, F, 1 / 4), \quad e_{n}^{\mathrm{q}}(S, F)=e_{n}^{\mathrm{q}}(S, F, 1 / 4)
$$

The choice $\theta=1 / 4$ is arbitrary - any fixed $\theta<1 / 2$ would do. The quantity $e_{n}^{\mathrm{q}}(S, F)$ is central for our study - it is the $n$-th minimal query error, that is, the smallest error which can be reached using at most $n$ queries. Note that it essentially suffices to study $e_{n}^{\mathrm{q}}(S, F)$ instead of $e_{n}^{\mathrm{q}}(S, F, \theta)$, since with $\mathcal{O}(\nu)$ repetitions, the error probability can be reduced to $2^{-\nu}$ (see Lemmas 34 and Corollary 1 below).

## 3 Some General Results

Let $G$ and $\widetilde{G}$ be normed spaces. Recall that a mapping $\Phi: G \rightarrow \widetilde{G}$ is said to be Lipschitz, if there is a constant $c \geq 0$ such that

$$
\|\Phi(x)-\Phi(y)\|_{\widetilde{G}} \leq c\|x-y\|_{G} \quad \text { for all } \quad x, y \in G
$$

The Lipschitz constant $\|\Phi\|_{\text {Lip }}$ is the smallest constant $c$ such that the relation above holds.

Given a quantum algorithm $A$ from $F$ to $G$ and a mapping $\Phi$ from $G$ to $\widetilde{G}$, the algorithm $\Phi \circ A$ is defined as the composition, meaning that $\varphi$ in the definition of $A$ is replaced by $\Phi \circ \varphi$ (this is a special case of the definition of the composition given in [8], p. 13). The following direct consequence of the definitions will be needed later.
Lemma 1. Let $S$ be a mapping and $A$ a quantum algorithm, both from $F$ to $G$. Let $\Phi$ be a Lipschitz mapping from $G$ to $\widetilde{G}$. Then for each $f \in F$ and $\theta \geq 0$,

$$
e(\Phi \circ S, \Phi \circ A, f, \theta) \leq\|\Phi\|_{\text {Lip }} e(S, A, f, \theta) .
$$

Consequently, for each $n \in \mathbf{N}$,

$$
e_{n}^{\mathrm{q}}(\Phi \circ S, F, \theta) \leq\|\Phi\|_{\operatorname{Lip}} e_{n}^{\mathrm{q}}(S, F, \theta)
$$

The next result was shown in [10, Lemma 2, for $G=\mathbf{R}$, but the proof of the general case is identical to that one.
Lemma 2. Let $D, K$ and $F \subseteq \mathcal{F}(D, K)$ be nonempty sets, $G$ a normed space, let $k \in \mathbf{N}_{0}$ and let $S_{l}: F \rightarrow G(l=0, \ldots, k)$ be mappings. Define $S: F \rightarrow G$ by $S(f)=\sum_{l=0}^{k} S_{l}(f) \quad(f \in F)$. Let $\theta_{0}, \ldots, \theta_{k} \geq 0, n_{0}, \ldots, n_{k} \in$ $\mathbf{N}_{0}$ and put $n=\sum_{l=0}^{k} n_{l}$. Then

$$
e_{n}^{\mathrm{q}}\left(S, F, \sum_{l=0}^{k} \theta_{l}\right) \leq \sum_{l=0}^{k} e_{n_{l}}^{\mathrm{q}}\left(S_{l}, F, \theta_{l}\right) .
$$

The following results are generalizations of the usual procedure of "boosting the success probability", which decreases the failure probability by repeating the algorithm a number of times and computing the median of the outputs (see, e.g., [8, Lemma 3). This works for algorithms whose outputs are real numbers. Since there is no natural linear order on a normed space, in general, the latter step has to be changed suitably when dealing with outputs in a normed space $G$.

For this purpose, let $\nu \in \mathbf{N}$. Let $\mu: \mathbf{R}^{\nu} \rightarrow \mathbf{R}$ be the mapping given by the median, that is, $\mu\left(a_{0}, \ldots, a_{\nu-1}\right)$ is the value of the $\lceil(\nu+1) / 2\rceil$-th element of the non-decreasing rearrangement of $\left(a_{i}\right)$. First we deal with the case that $G$ is a space of the form $G=l_{\infty}(\mathcal{T})$, where $\mathcal{T}$ is a nonempty set and $l_{\infty}(\mathcal{T})$ denotes the space of all bounded real-valued functions on $\mathcal{T}$, equipped with the supremum norm $\|g\|_{l_{\infty}(\mathcal{T})}=\sup _{t \in \mathcal{T}}|g(t)|$. Define $\bar{\mu}: l_{\infty}(\mathcal{T})^{\nu} \rightarrow l_{\infty}(\mathcal{T})$ as follows:

$$
\bar{\mu}\left(g_{0}, \ldots, g_{\nu-1}\right)=\left(\mu\left(g_{0}(t), \ldots, g_{\nu-1}(t)\right)\right)_{t \in \mathcal{T}}
$$

that is, we apply the median componentwise. For any algorithm $A$ from $F$ to $l_{\infty}(\mathcal{T})$ denote by $\bar{\mu}\left(A^{\nu}\right):=\bar{\mu}(A, \ldots, A)$ the composed algorithm (see again p. 13 of [8])) of repeating $\nu$ times the algorithm $A$ and applying $\bar{\mu}$ to the outputs.

Lemma 3. Let $A$ be any quantum algorithm and $S$ be any mapping, both from $F$ to $l_{\infty}(\mathcal{T})$, and let $\nu \in \mathbf{N}$. Then for each $f \in F$,

$$
e\left(S, \bar{\mu}\left(A^{\nu}\right), f, e^{-\nu / 8}\right) \leq e(S, A, f)
$$

Proof. Fix $f \in F$. Let $\zeta_{0}, \ldots, \zeta_{\nu-1}$ be independent random variables with distribution $A(f)$. Let $\chi_{i}$ be the indicator function of the set

$$
\left\{\left\|S(f)-\zeta_{i}\right\|_{l_{\infty}(\mathcal{T})}>e(S, A, f)\right\}
$$

Then $\mathbf{P}\left\{\chi_{i}=1\right\} \leq 1 / 4$. Hoeffding's inequality, see, e.g., [23], p. 191, yields

$$
\mathbf{P}\left\{\sum_{i=0}^{\nu-1} \chi_{i} \geq \nu / 2\right\} \leq \mathbf{P}\left\{\sum_{i=0}^{\nu-1}\left(\chi_{i}-\mathbf{E} \chi_{i}\right) \geq \nu / 4\right\} \leq e^{-\nu / 8}
$$

Hence, with probability at least $1-e^{-\nu / 8}$,

$$
\begin{equation*}
\left|\left\{i \mid\left\|S(f)-\zeta_{i}\right\|_{l_{\infty}(\mathcal{T})} \leq e(S, A, f)\right\}\right|>\nu / 2 \tag{4}
\end{equation*}
$$

It follows from (4) that for all $t \in \mathcal{T}$

$$
\left|\left\{i\left|\left|S(f)(t)-\zeta_{i}(t)\right| \leq e(S, A, f)\right\} \mid>\nu / 2\right.\right.
$$

Consequently,

$$
\left|S(f)(t)-\mu\left(\zeta_{0}(t), \ldots, \zeta_{\nu-1}(t)\right)\right| \leq e(S, A, f),
$$

which means that

$$
\begin{equation*}
\left\|S(f)-\bar{\mu}\left(\zeta_{0}, \ldots, \zeta_{\nu-1}\right)\right\|_{l_{\infty}(\mathcal{T})} \leq e(S, A, f) \tag{5}
\end{equation*}
$$

Since (4) holds with probability at least $1-e^{-\nu / 8}$, so does (5).
Now let $G$ be a general normed space. For the following construction we consider $G$ as a space over $\mathbf{R}$ (each normed space over $\mathbf{C}$ can also be considered as a normed space over $\mathbf{R}$ ). We define for each $\delta>0$ a suitable mapping $\psi_{\delta}: G^{\nu} \rightarrow G$ as follows. Let $G^{*}$ denote the dual of $G$, that is, the space of all bounded linear functionals on $G$. Let $\mathcal{T} \subseteq \mathcal{B}\left(G^{*}\right)$ be a norming set, i.e., for all $g \in G$

$$
\begin{equation*}
\|g\|=\sup _{t \in \mathcal{T}}|t(g)| \tag{6}
\end{equation*}
$$

(such a $\mathcal{T}$ always exists, one can take, for example, $\mathcal{T}=\mathcal{B}\left(G^{*}\right)$ itself). Then $G$ can be identified with a subspace of $l_{\infty}(\mathcal{T})$ via the embedding map $J: G \rightarrow l_{\infty}(\mathcal{T})$ defined by

$$
J g=(t(g))_{t \in \mathcal{T}}
$$

in such a way that the norm is preserved: $\|J g\|_{l_{\infty}(\mathcal{T})}=\|g\|_{G}$. For $\delta>0$, let finally $\pi_{\delta}: l_{\infty}(\mathcal{T}) \rightarrow G$ be any $\delta$-approximate metric projection, by which we mean a mapping satisfying

$$
\left\|x-\pi_{\delta}(x)\right\|_{l_{\infty}(\mathcal{T})} \leq(1+\delta) \inf _{g \in G}\|x-g\|_{l_{\infty}(\mathcal{T})}
$$

for all $x \in l_{\infty}(\mathcal{T})$. We define $\psi_{\delta}: G^{\nu} \rightarrow G$ by setting

$$
\psi_{\delta}=\pi_{\delta} \circ \bar{\mu} \circ J^{\nu} .
$$

Lemma 4. Let $A$ be any quantum algorithm and $S$ be any mapping, both from $F$ to a normed space $G$, let $\nu \in \mathbf{N}$ and $\delta>0$. Then for each $f \in F$,

$$
e\left(S, \psi_{\delta}\left(A^{\nu}\right), f, e^{-\nu / 8}\right) \leq(2+\delta) e(S, A, f) .
$$

Proof. It follows from (6]) that

$$
e(J S, J A, f)=e(S, A, f)
$$

By Lemma 3 ,

$$
e\left(J S, \bar{\mu}\left((J A)^{\nu}\right), f, e^{-\nu / 8}\right) \leq e(J S, J A, f)=e(S, A, f)
$$

Let $\zeta$ be a random variable with values in $l_{\infty}(\mathcal{T})$ with distribution $\bar{\mu}\left((J A)^{\nu}\right)(f)$. Then with probability at least $1-e^{-\nu / 8}$,

$$
\|J S(f)-\zeta\| \leq e(S, A, f)
$$

Hence

$$
\begin{aligned}
\left\|S(f)-\pi_{\delta}(\zeta)\right\| & \leq\|J S(f)-\zeta\|+\left\|\zeta-\pi_{\delta}(\zeta)\right\| \\
& \leq e(S, A, f)+(1+\delta)\|\zeta-S(f)\| \\
& \leq(2+\delta) e(S, A, f)
\end{aligned}
$$

But $\pi_{\delta}(\zeta)$ is a random variable with distribution

$$
\pi_{\delta} \circ \bar{\mu}\left((J A)^{\nu}\right)(f)=\psi_{\delta}\left(A^{\nu}\right)(f)
$$

and it follows that

$$
e\left(S, \psi_{\delta}\left(A^{\nu}\right), f, e^{-\nu / 8}\right) \leq(2+\delta) e(S, A, f)
$$

Corollary 1. Let $S$ be any mapping from $F \subseteq \mathcal{F}(D, K)$ to a normed space $G$. Then for each $n, \nu \in \mathbf{N}$,

$$
e_{\nu n}^{\mathrm{q}}\left(S, F, e^{-\nu / 8}\right) \leq 2 e_{n}^{\mathrm{q}}(S, F)
$$

If $G$ is a space of the form $l_{\infty}(\mathcal{T})$ for some set $\mathcal{T}$, then the constant 2 above can be replaced by 1.

The definition of $\psi_{\delta}$ is not constructive (and neither is that of $\bar{\mu}$, if $\mathcal{T}$ is infinite). Since we are dealing with the quantum query complexity, the cost of (classically) computing the $\varphi$ part of a quantum algorithm (this is the place where $\bar{\mu}$ and $\psi_{\delta}$ enter) are generally neglected. However, if one looks for a more efficient procedure, here is one which leads to the constant 3 instead of $2+\delta$ of Lemma[4. Define $\varrho: G^{\nu} \rightarrow G$ as follows: $\varrho\left(g_{0}, \ldots, g_{\nu-1}\right)$ is the element $g_{i_{0}}$, where

$$
i_{0}=\arg \min _{i} \mu\left(\left\|g_{0}-g_{i}\right\|, \ldots,\left\|g_{\nu-1}-g_{i}\right\|\right)
$$

(if there is more than one index $i$ at which the minimum is attained, we choose the smallest index, just for definiteness). It can be shown along the lines of the proof of Lemma 3 that for all $f \in F$,

$$
e\left(S, \varrho\left(A^{\nu}\right), f, e^{-\nu / 8}\right) \leq 3 e(S, A, f)
$$

Note that the cost is $\mathcal{O}\left(\nu^{2}\right)$ (which is usually a logarithmic term) times the cost of computing the norm $\left\|g_{i}-g_{j}\right\|$ (which depends on the structure and dimension of $G$, and on the - possible - sparsity of the $g_{i}$, see also the discussion in Section 3 of [11]).

Corollary 2. Let $D, K, F \subseteq \mathcal{F}(D, K), G, k \in \mathbf{N}_{0}$ and $S, S_{l}: F \rightarrow G$ $(l=0, \ldots, k)$ be as in Lemma (2. Assume $\nu_{0}, \ldots, \nu_{k} \in \mathbf{N}$ satisfy

$$
\sum_{l=0}^{k} e^{-\nu_{l} / 8} \leq \frac{1}{4}
$$

Let $n_{0}, \ldots, n_{k} \in \mathbf{N}_{0}$ and put $n=\sum_{l=0}^{k} \nu_{l} n_{l}$. Then

$$
e_{n}^{\mathrm{q}}(S, F) \leq 2 \sum_{l=0}^{k} e_{n_{l}}^{\mathrm{q}}\left(S_{l}, F\right)
$$

If $G=l_{\infty}(\mathcal{T})$, then the relation holds with constant 1.
This is an obvious consequence of Lemma 2 and Corollary In the sequel we need the following mappings. Let $m^{*} \in \mathbf{N}$ and define $\beta: \mathbf{R} \rightarrow \mathbf{Z}\left[0,2^{m^{*}}\right)$ for $z \in \mathbf{R}$ by

$$
\beta(z)= \begin{cases}0 & \text { if } z<-2^{m^{*} / 2-1}  \tag{7}\\ \left\lfloor 2^{m^{*} / 2}\left(z+2^{m^{*} / 2-1}\right)\right\rfloor & \text { if } \quad-2^{m^{*} / 2-1} \leq z<2^{m^{*} / 2-1} \\ 2^{m^{*}}-1 & \text { if } z \geq 2^{m^{*} / 2-1}\end{cases}
$$

Furthermore, let $\gamma: \mathbf{Z}\left[0,2^{m^{*}}\right) \rightarrow \mathbf{R}$ be defined for $y \in \mathbf{Z}\left[0,2^{m^{*}}\right)$ as

$$
\begin{equation*}
\gamma(y)=2^{-m^{*} / 2} y-2^{m^{*} / 2-1} \tag{8}
\end{equation*}
$$

It follows that for $-2^{m^{*} / 2-1} \leq z \leq 2^{m^{*} / 2-1}$,

$$
\begin{equation*}
\gamma(\beta(z)) \leq z \leq \gamma(\beta(z))+2^{-m^{*} / 2} \tag{9}
\end{equation*}
$$

Proposition 1. Let $D$ be a nonempty set and let $\emptyset \neq F \subseteq X \subseteq \mathcal{F}(D, \mathbf{R})$, where $X$ is a linear subspace equipped with a norm $\left\|\|_{X}\right.$, such that (i) $\sup _{f \in F}|f(t)|<\infty$ for each $t \in D$, and
(ii) $X$ separates the points of $D$ in the following sense: Given $t_{0} \in D$ and a finite subset $D_{0} \subseteq D \backslash\left\{t_{0}\right\}$, there is an $g \in X$ with $g\left(t_{0}\right) \neq 0$ and $g(t)=0$ for all $t \in D_{0}$.

Let $J: F \rightarrow X$ be the embedding map, let $G$ be a normed space and $S: X \rightarrow G$ a bounded linear operator. Then for all $\tilde{n}, n \in \mathbf{N}, 0 \leq \theta_{1}, \theta_{2} \leq 1$,

$$
e_{\tilde{n}+2 n}^{\mathrm{q}}\left(S J, F, \theta_{1}+\theta_{2}-\theta_{1} \theta_{2}\right) \leq e_{\tilde{n}}^{\mathrm{q}}\left(J, F, \theta_{1}\right) e_{n}^{\mathrm{q}}\left(S, \mathcal{B}(X), \theta_{2}\right)
$$

Proof. Let $\delta>0$, let $\widetilde{A}$ be a quantum algorithm from $F$ to $X$ with $q(\widetilde{A}) \leq \tilde{n}$ and

$$
\begin{equation*}
e\left(J, \widetilde{A}, F, \theta_{1}\right) \leq e_{\widetilde{n}}^{\mathrm{q}}\left(J, F, \theta_{1}\right)+\delta:=\sigma_{1} \tag{10}
\end{equation*}
$$

Put

$$
\begin{equation*}
\sigma=\sigma_{1}+\delta \tag{11}
\end{equation*}
$$

Let $A$ be a quantum algorithm from $\mathcal{B}(X)$ to $G$ with $q(A) \leq n$ and

$$
\begin{equation*}
e\left(S, A, \mathcal{B}(X), \theta_{2}\right) \leq e_{n}^{\mathrm{q}}\left(S, \mathcal{B}(X), \theta_{2}\right)+\delta:=\sigma_{2} \tag{12}
\end{equation*}
$$

Let

$$
\widetilde{A}=\left(\left(\widetilde{A}_{l}\right)_{l=0}^{\tilde{k}-1},\left(\widetilde{( }_{l}\right)_{l=0}^{\tilde{k}-1}, \widetilde{\varphi}\right)
$$

with

$$
\widetilde{A}_{l}=\left(\widetilde{Q}_{l},\left(\widetilde{U}_{l j}\right)_{j=0}^{\tilde{n}_{l}}\right),
$$

and

$$
\widetilde{Q}_{l}=\left(\widetilde{m}_{l}, \widetilde{m}_{l}^{\prime}, \widetilde{m}_{l}^{\prime \prime}, \widetilde{Z}_{l}, \widetilde{\tau}_{l}, \widetilde{\beta}_{l}\right)
$$

for $l=0, \ldots, \tilde{k}-1$. Furthermore, let

$$
A=\left(\left(A_{l}\right)_{l=0}^{k-1},\left(b_{l}\right)_{l=0}^{k-1}, \varphi\right)
$$

and for $l=0, \ldots, k-1$,

$$
A_{l}=\left(Q_{l},\left(U_{l j}\right)_{j=0}^{n_{l}}\right),
$$

and

$$
Q_{l}=\left(m_{l}, m_{l}^{\prime}, m_{l}^{\prime \prime}, Z_{l}, \tau_{l}, \beta_{l}\right)
$$

We need some auxiliary functions and relations. Let

$$
D_{A}=\left\{\tau_{l}(i) \mid l=0, \ldots, k-1, i \in Z_{l}\right\}
$$

(the set of all points at which the quantum algorithm $A$ queries the function). By assumption (ii), for each $t \in D_{A}$ there is a $g_{t} \in X$ such that $g(t)=1$ and $g(s)=0$ for all $s \in D_{A} \backslash\{t\}$. Let $M_{1}=\max _{t \in D_{A}}\left\|g_{t}\right\|_{X}$. By asumption (i),

$$
M_{2}:=\max _{f \in F, t \in D_{A}}|f(t)|<\infty
$$

Now choose the $m^{*}$ in the definition of the mappings $\beta, \gamma$ in (7) and (8) in such a way that for $a \in \mathbf{R}$ with $|a| \leq M_{2}$,

$$
|a-\gamma \circ \beta(a)| \leq M_{1}^{-1}\left|D_{A}\right|^{-1} \delta
$$

Define for $f \in F$ and $x=\left(x_{0}, \ldots, x_{\tilde{k}-1}\right) \in \prod_{l=0}^{\tilde{k}-1} \mathbf{Z}\left[0,2^{\widetilde{m}_{l}}\right)$,

$$
h_{f, x}=\sigma^{-1}\left(f-\widetilde{\varphi}(x)+\sum_{t \in D_{A}}(\gamma \circ \beta \circ f(t)-f(t)) g_{t}\right)
$$

(recall that $\widetilde{\varphi}(x) \in X)$. Then $h_{f, x} \in X$,

$$
\begin{equation*}
h_{f, x}(s)=\sigma^{-1}(\gamma \circ \beta \circ f(s)-\widetilde{\varphi}(x)(s)) \quad\left(s \in D_{A}\right) \tag{13}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|f-\widetilde{\varphi}(x)-\sigma h_{f, x}\right\|_{X} \leq M_{1}\left|D_{A}\right| M_{1}^{-1}\left|D_{A}\right|^{-1} \delta=\delta \tag{14}
\end{equation*}
$$

Moreover,

$$
\begin{align*}
\left\|h_{f, x}\right\|_{X} & =\sigma^{-1}\left\|(f-\widetilde{\varphi}(x))-\left(f-\widetilde{\varphi}(x)-\sigma h_{f, x}\right)\right\|_{X} \\
& \leq \sigma^{-1}\left(\|\left(f-\widetilde{\varphi}(x) \|_{X}+\delta\right)\right. \tag{15}
\end{align*}
$$

We build an algorithm as follows. It has $\tilde{\sim}+k$ cycles. The first $\tilde{k}$ cycles are exactly those of $\widetilde{A}$. After the $\tilde{k}$ measurements we have the result, say

$$
x=\left(x_{0}, x_{1}, \ldots, x_{\tilde{k}-1}\right)
$$

(from which $\widetilde{\varphi}\left(x_{0}, \ldots, x_{\tilde{k}-1}\right)$ would be computed - but we don't do that yet). Next the $k$ cycles of $A$ follow, with certain modifications. In each cycle we add $\widetilde{m}=\sum_{l=0}^{\tilde{k}-1} \widetilde{m}_{l}$ qubits which are initialized in the state

$$
|x\rangle=\left|x_{0}\right\rangle\left|x_{1}\right\rangle \ldots\left|x_{\tilde{k}-1}\right\rangle
$$

and remain there all the way. We add $m^{*}$ further auxiliary qubits, initially set to zero (and being zero again at the end of each cycle). We also want
to modify the queries $Q_{l}$ of $A$. For $0 \leq l<k$ introduce the following new query:

$$
\bar{Q}_{l}=\left(m_{l}+\widetilde{m}+m^{*}, m_{l}^{\prime}, m^{*}, Z_{l}, \tau_{l}, \beta\right),
$$

where $\beta$ was defined in (77). Define a unitary operator $V_{l}$ on

$$
H_{m_{l}^{\prime}} \otimes H_{m_{l}^{\prime \prime}} \otimes H_{m_{l}-m_{l}^{\prime}-m_{l}^{\prime \prime}} \otimes H_{\tilde{m}} \otimes H_{m^{*}}
$$

by setting for

$$
|i\rangle|z\rangle|u\rangle|x\rangle|v\rangle \in \mathcal{C}_{m_{l}^{\prime}} \otimes \mathcal{C}_{m_{l}^{\prime \prime}} \otimes \mathcal{C}_{m_{l}-m_{l}^{\prime}-m_{l}^{\prime \prime}} \otimes \mathcal{C}_{\widetilde{m}} \otimes \mathcal{C}_{m^{*}}
$$

$$
V_{l}|i\rangle|z\rangle|u\rangle|x\rangle|v\rangle=|i\rangle\left|z \oplus \beta_{l}\left(\sigma^{-1}\left(\gamma(v)-\widetilde{\varphi}(x)\left(\tau_{l}(i)\right)\right)\right)\right\rangle|u\rangle|x\rangle|v\rangle
$$

if $i \in Z_{l}$, and

$$
V_{l}|i\rangle|z\rangle|u\rangle|x\rangle|v\rangle=|i\rangle|z\rangle|u\rangle|x\rangle|v\rangle
$$

otherwise (recall that $\widetilde{\varphi}(x) \in X \subseteq \mathcal{F}(D, \mathbf{R})$ and $\tau_{l}(i) \in D$, so the respective expression above is well-defined). We also need

$$
W_{l}|i\rangle|z\rangle|u\rangle|x\rangle|v\rangle=|i\rangle|z\rangle|u\rangle|x\rangle|\ominus v\rangle
$$

with $\ominus v=\left(2^{m^{*}}-v\right) \bmod 2^{m^{*}}$. Now we consider the following composition

$$
\begin{equation*}
P_{l} \bar{Q}_{l, f} P_{l} W_{l} V_{l} P_{l} \bar{Q}_{l, f} P_{l}, \tag{16}
\end{equation*}
$$

where $P_{l}$ exchanges the $|z\rangle$ with the $|v\rangle$ component. Let us look how the combination (16) acts, when the last $m^{*}$ qubits are in the state $|0\rangle$. Assume $i \in Z_{l}$. Then

$$
|i\rangle|z\rangle|u\rangle|x\rangle|0\rangle
$$

is mapped by $P_{l} \bar{Q}_{l, f} P_{l}$ to

$$
|i\rangle|z\rangle|u\rangle|x\rangle\left|\beta \circ f \circ \tau_{l}(i)\right\rangle .
$$

Using (13), we see that $V_{l}$ produces

$$
\begin{aligned}
& |i\rangle\left|z \oplus \beta_{l}\left(\sigma^{-1}\left(\gamma \circ \beta \circ f \circ \tau_{l}(i)-\widetilde{\varphi}(x)\left(\tau_{l}(i)\right)\right)\right)\right\rangle|u\rangle|x\rangle\left|\beta \circ f \circ \tau_{l}(i)\right\rangle \\
= & |i\rangle\left|z \oplus \beta_{l} \circ h_{f, x} \circ \tau_{l}(i)\right\rangle|u\rangle|x\rangle\left|\beta \circ f \circ \tau_{l}(i)\right\rangle .
\end{aligned}
$$

Finally, the application of $P_{l} \bar{Q}_{l, f} P_{l} W_{l}$ leads to

$$
\begin{equation*}
|i\rangle\left|z \oplus \beta_{l} \circ h_{f, x} \circ \tau_{l}(i)\right\rangle|u\rangle|x\rangle|0\rangle=\left(Q_{l, h_{f, x}}|i\rangle|z\rangle|u\rangle\right)|x\rangle|0\rangle . \tag{17}
\end{equation*}
$$

It can be checked analogously that if $i \notin Z_{l}$, we also end in the state given by the right-hand side of (17). That means, the combination (16) acts as if we apply the original query $Q_{l}$, but with $f$ replaced by $h_{f, x}$. Now we replace each occurrence of $Q_{l, f}$ by this string (16), while the $U_{l j}$ are replaced by $\bar{U}_{l j}$, which are the $U_{l j}$, extended to $H_{m} \otimes H_{\tilde{m}} \otimes H_{m^{*}}$ by tensoring with the identity on $H_{\widetilde{m}} \otimes H_{m^{*}}$. The respective $\bar{b}_{l}$ are defined in such a way that

$$
\bar{b}_{l}\left(x_{0}, \ldots, x_{\tilde{k}-1},\left(y_{0}, x, 0\right), \ldots,\left(y_{l-1}, x, 0\right)\right)=\left(b_{l}\left(y_{0}, \ldots, y_{l-1}\right), x, 0\right)
$$

for all $x_{0}, \ldots, x_{\tilde{k}-1}, y_{0}, \ldots, y_{l-1}, l=0, \ldots, k-1$.
After the completion of the $\tilde{k}+k$ cycles, let the measurement results be

$$
x_{0}, \ldots, x_{\tilde{k}-1},\left(y_{0}, x, 0\right), \ldots,\left(y_{k-1}, x, 0\right)
$$

where, as before, $x=\left(x_{0}, \ldots, x_{\tilde{k}-1}\right)$. Then we apply the mapping $\bar{\varphi}$ defined by

$$
\bar{\varphi}\left(x,\left(y_{0}, x, 0\right), \ldots,\left(y_{k-1}, x, 0\right)\right):=S \widetilde{\varphi}(x)+\sigma \varphi\left(y_{0}, \ldots, y_{k-1}\right) .
$$

Denote the resulting quantum algorithm from $F$ to $G$ by $B$. Clearly,

$$
\begin{equation*}
q(B)=\tilde{n}+2 n \tag{18}
\end{equation*}
$$

By (17), the modified $A$-part, applied to $f \in F$, acts like algorithm A, applied to $h_{f, x}$. More precisely, in algorithm $B$, applied to $f$, given $x$ as the outcome of the measurements of the first part of $B$, the probability of measuring

$$
\left(y_{0}, x, 0\right), \ldots,\left(y_{k-1}, x, 0\right)
$$

in the second part of $B$ is the same as that of measuring

$$
y_{0}, \ldots, y_{k-1}
$$

in algorithm $A$, applied to $h_{f, x}$. For a fixed $f \in F$, we have, by (10), with probability at least $1-\theta_{1}$,

$$
\|J f-\widetilde{\varphi}(x)\|_{X} \leq \sigma_{1}
$$

thus, by (15) (recalling also $J f=f$ ),

$$
\left\|h_{f, x}\right\|_{X} \leq \sigma^{-1}\left(\sigma_{1}+\delta\right)=1
$$

Thus, for fixed $f \in F$, with probability at least $1-\theta_{1}$,

$$
\begin{equation*}
h_{f, x} \in \mathcal{B}(X) . \tag{19}
\end{equation*}
$$

But for each $x$ satisfying (19), we have by (12), with probability at least $1-\theta_{2}$,

$$
\begin{equation*}
\left\|S h_{f, x}-\varphi\left(y_{0}, \ldots, y_{k-1}\right)\right\|_{G} \leq \sigma_{2} \tag{20}
\end{equation*}
$$

Summarizing, we see that (19) and (20) together hold with probability at least $\left(1-\theta_{1}\right)\left(1-\theta_{2}\right)$. We have

$$
\begin{aligned}
& \left\|S J f-\bar{\varphi}\left(x,\left(y_{0}, x, 0\right), \ldots,\left(y_{k-1}, x, 0\right)\right)\right\|_{G} \\
= & \left\|S J f-S \widetilde{\varphi}(x)-\sigma \varphi\left(y_{0}, \ldots, y_{k-1}\right)\right\|_{G} \\
= & \left\|S J f-S \widetilde{\varphi}(x)-\sigma S h_{f, x}+\sigma S h_{f, x}-\sigma \varphi\left(y_{0}, \ldots, y_{k-1}\right)\right\|_{G} \\
\leq & \left\|S\left(f-\widetilde{\varphi}(x)-\sigma h_{f, x}\right)\right\|_{G}+\sigma\left\|S h_{f, x}-\varphi\left(y_{0}, \ldots, y_{k-1}\right)\right\|_{G} \\
\leq & \|S\| \delta+\sigma \sigma_{2},
\end{aligned}
$$

by (14) and (20), with probability at least $\left(1-\theta_{1}\right)\left(1-\theta_{2}\right)$. Thus, using (11),

$$
e\left(S J, B, F, \theta_{1}+\theta_{2}-\theta_{1} \theta_{2}\right) \leq\|S\| \delta+\left(\sigma_{1}+\delta\right) \sigma_{2}
$$

hence, by (18), (10), and (12),

$$
\begin{aligned}
& e_{\tilde{n}+2 n}^{\mathrm{q}}\left(S J, F, \theta_{1}+\theta_{2}-\theta_{1} \theta_{2}\right) \\
& \quad \leq\|S\| \delta+\left(e_{\tilde{n}}^{\mathrm{q}}\left(J, F, \theta_{1}\right)+2 \delta\right)\left(e_{n}^{\mathrm{q}}\left(S, \mathcal{B}(X), \theta_{2}\right)+\delta\right)
\end{aligned}
$$

Since $\delta>0$ was arbitrary, the result follows.
Corollary 3. Let $\nu_{1}, \nu_{2} \in \mathbf{N}$ with

$$
e^{-\nu_{1} / 8}+e^{-\nu_{2} / 8}-e^{-\left(\nu_{1}+\nu_{2}\right) / 8} \leq 1 / 4 .
$$

Then under the same assumptions as in Proposition 1 .

$$
e_{\nu_{1} \tilde{n}+2 \nu_{2} n}^{\mathrm{q}}(S J, F) \leq 4 e_{\tilde{n}}^{\mathrm{q}}(J, F) e_{n}^{\mathrm{q}}(S, \mathcal{B}(X))
$$

Proof. By Proposition 1 and Corollary 1 .

$$
\begin{aligned}
e_{\nu_{1} \tilde{n}+2 \nu_{2} n}^{\mathrm{q}}(S J, F) & \leq e_{\nu_{1} \tilde{n}}^{\mathrm{q}}\left(J, F, e^{-\nu_{1} / 8}\right) e_{\nu_{2 n}}^{\mathrm{q}}\left(S, \mathcal{B}(X), e^{-\nu_{2} / 8}\right) \\
& \leq 4 e_{\tilde{n}}^{\mathrm{q}}(J, F) e_{n}^{\mathrm{q}}(S, \mathcal{B}(X)) .
\end{aligned}
$$

## 4 Approximation of Finite Dimensional Embeddings

For $N \in \mathbf{N}$ and $1 \leq p \leq \infty$, let $L_{p}^{N}$ denote the space of all functions $f: \mathbf{Z}[0, N) \rightarrow \mathbf{R}$, equipped with the norm

$$
\|f\|_{L_{p}^{N}}=\left(\frac{1}{N} \sum_{i=0}^{N-1}|f(i)|^{p}\right)^{1 / p}
$$

if $p<\infty$ and

$$
\|f\|_{L_{\infty}^{N}}=\max _{0 \leq i \leq N-1}|f(i)| .
$$

Define $J_{p q}^{N}: L_{p}^{N} \rightarrow L_{q}^{N}$ to be the identity operator $J_{p q}^{N} f=f\left(f \in L_{p}^{N}\right)$. Furthermore, for a real $M \geq 0$ define the operator $C_{p q}^{N, M}: L_{p}^{N} \rightarrow L_{q}^{N}$ for $f=(f(i))_{i=0}^{N-1}$ as

$$
\left(C_{p q}^{N, M} f\right)(i)= \begin{cases}f(i) & \text { if }|f(i)| \geq M \\ 0 & \text { otherwise }\end{cases}
$$

Lemma 5. Let $1 \leq p, q \leq \infty$. There is a constant $c>0$ sucht that for all $n, N \in \mathbf{N}$, and $M \in \mathbf{R}$ with $M \geq 0$,

$$
e_{n}^{\mathrm{q}}\left(C_{p q}^{N, M}, \mathcal{B}\left(L_{p}^{N}\right)\right)=0
$$

whenever

$$
M \geq c(N / n)^{2 / p} \max (\log (n / \sqrt{N}), 1)^{2 / p}
$$

Remark. Throughout the paper $\log$ means $\log _{2}$. Furthermore, we often use the same symbol $c, c_{1}, \ldots$ for possibly different positive constants (also when they appear in a sequence of relations). These constants are either absolute or may depend only on $p$ and $q$ - in all statements of lemmas, propositions, etc. this is precisely described anyway by the order of the quantifiers.

Proof. This is an immediate consequence of the proof of Proposition 1 and Corollary 3 of [14. Namely, it contains an algorithm with $n$ queries that produces, with probability $\geq 3 / 4$, all indices $i$ with $|f(i)| \geq M$ and for each such $i$ an (arbitrarily precise) approximation $y_{i}$ to $f(i)$, where $M$ is any number satisfying

$$
M \geq c(N / n)^{2 / p} \max (\log (n / \sqrt{N}), 1)^{2 / p}
$$

Lemma 6. Let $1 \leq p \leq q \leq \infty$. For all $N \in \mathbf{N}$, and $M \in \mathbf{R}$ with $M \geq 0$,

$$
\sup _{f \in \mathcal{B}\left(L_{p}^{N}\right)}\left\|f-C_{p q}^{N, M} f\right\|_{L_{q}^{N}} \leq M^{1-p / q}
$$

Proof. We have for $f \in \mathcal{B}\left(L_{p}^{N}\right)$

$$
\begin{aligned}
& \frac{1}{N} \sum_{i=0}^{N-1}\left|f(i)-\left(C_{p q}^{N, M} f\right)(i)\right|^{q} \\
& \quad \leq \frac{1}{N} \max _{j}\left|f(j)-\left(C_{p q}^{N, M} f\right)(j)\right|^{q-p} \sum_{i=0}^{N-1}\left|f(i)-\left(C_{p q}^{N, M} f\right)(i)\right|^{p} \\
& \quad \leq \frac{M^{q-p}}{N} \sum_{i=0}^{N-1}|f(i)|^{p} \leq M^{q-p}
\end{aligned}
$$

Next we give an upper bound.
Proposition 2. Let $1 \leq p, q \leq \infty$. In the case $p<q$ there is a constant $c>0$ such that for all $n, N \in \mathbf{N}$

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \leq c \min \left(\left(\frac{N}{n} \log (n / \sqrt{N}+2)\right)^{2 / p-2 / q}, N^{1 / p-1 / q}\right)
$$

In the case $p \geq q$, we have

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \leq 1
$$

Proof. For $p<q$ the estimate involving the first term of the minimum follows from the previous two lemmas, since by Lemma 6 (i) of [8,

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \leq e_{n}^{\mathrm{q}}\left(C_{p q}^{N, M}, \mathcal{B}\left(L_{p}^{N}\right)\right)+\sup _{f \in \mathcal{B}\left(L_{p}^{N}\right)}\left\|f-C_{p q}^{N, M} f\right\|_{L_{q}^{N}} .
$$

The estimate involving the second term is a trivial consequence of $\left\|J_{p q}^{N}\right\|=$ $N^{1 / p-1 / q}$. The case $p \geq q$ follows from $\left\|J_{p q}^{N}\right\|=1$.

Before we derive lower bounds we recall some tools from [8]. Let $D$ and $K$ be nonempty sets, let $L \in \mathbf{N}$, and let to each $u=\left(u_{0}, \ldots, u_{L-1}\right) \in\{0,1\}^{L}$ an $f_{u} \in \mathcal{F}(D, K)$ be assigned such that the following is satisfied:

Condition (I): For each $t \in D$ there is an $l, 0 \leq l \leq L-1$, such that $f_{u}(t)$ depends only on $u_{l}$, in other words, for $u, u^{\prime} \in\{0,1\}^{L}, u_{l}=u_{l}^{\prime}$ implies $f_{u}(t)=f_{u^{\prime}}(t)$.

For $u \in\{0,1\}^{L}$ let $|u|$ denote the number of 1 's in $u$. Define the function $\varrho\left(L, l, l^{\prime}\right)$ for $L \in \mathbf{N}, 0 \leq l \neq l^{\prime} \leq L$ by

$$
\begin{equation*}
\varrho\left(L, l, l^{\prime}\right)=\sqrt{\frac{L}{\left|l-l^{\prime}\right|}}+\frac{\min _{j=l, l^{\prime}} \sqrt{j(L-j)}}{\left|l-l^{\prime}\right|} . \tag{21}
\end{equation*}
$$

The following was proved in [8, using the polynomial method [3] and based on a result from [16]:

Lemma 7. There is a constant $c_{0}>0$ such that the following holds: Let $D, K$ be nonempty sets, let $F \subseteq \mathcal{F}(D, K)$ be a set of functions, $G$ a normed space, $S: F \rightarrow G$ a mapping, and $L \in \mathbf{N}$. Suppose $\left(f_{u}\right)_{u \in\{0,1\}^{L}} \subseteq \mathcal{F}(D, K)$ is a system of functions satisfying condition (I). Let finally $0 \leq l \neq l^{\prime} \leq L$ and assume that

$$
\begin{equation*}
f_{u} \in F \quad \text { whenever } \quad|u| \in\left\{l, l^{\prime}\right\} . \tag{22}
\end{equation*}
$$

Then

$$
\begin{equation*}
e_{n}^{\mathrm{q}}(S, F) \geq \frac{1}{2} \min \left\{\left\|S\left(f_{u}\right)-S\left(f_{u^{\prime}}\right)\right\|| | u\left|=l,\left|u^{\prime}\right|=l^{\prime}\right\}\right. \tag{23}
\end{equation*}
$$

for all $n$ with

$$
\begin{equation*}
n \leq c_{0} \varrho\left(L, l, l^{\prime}\right) . \tag{24}
\end{equation*}
$$

For the case $q=\infty$ we give the following lower bound.
Proposition 3. Let $1 \leq p \leq \infty$. There are constants $c_{1}, c_{2}>0$ such that for all $n, N \in \mathbf{N}$ with $n \leq c_{1} N$

$$
e_{n}^{\mathrm{q}}\left(J_{p, \infty}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq c_{2} \min \left(\left(\frac{N}{n}\right)^{2 / p}, N^{1 / p}\right)
$$

Proof. First we assume

$$
\begin{equation*}
n \leq c_{0} \sqrt{N} \tag{25}
\end{equation*}
$$

where $c_{0}$ is the constant from Lemma 7 Put

$$
L=N, \quad l=0, \quad l^{\prime}=1 .
$$

Then

$$
\begin{equation*}
n \leq c_{0} \sqrt{L}=c_{0} \varrho\left(L, l, l^{\prime}\right) . \tag{26}
\end{equation*}
$$

Define $\psi_{j} \quad(j=0, \ldots, L-1)$ by

$$
\psi_{j}(i)= \begin{cases}N^{1 / p} & \text { if } \quad i=j \\ 0 & \text { otherwise } .\end{cases}
$$

Note that $\psi_{j} \in \mathcal{B}\left(L_{p}^{N}\right)$ and

$$
\begin{equation*}
\left\|J_{p, \infty}^{N} \psi_{j}\right\|_{L_{\infty}^{N}}=\left\|\psi_{j}\right\|_{L_{\infty}^{N}}=N^{1 / p} \tag{27}
\end{equation*}
$$

For each $u=\left(u_{0}, \ldots, u_{L-1}\right) \in\{0,1\}^{L}$ define

$$
\begin{equation*}
f_{u}=\sum_{j=0}^{L-1} u_{j} \psi_{j} \tag{28}
\end{equation*}
$$

Since the functions $\psi_{j}$ have disjoint supports, the system $\left(f_{u}\right)_{u \in\{0,1\}^{L}}$ satisfies condition (I). Lemma 7 and relations (26) and (27) give

$$
\begin{aligned}
e_{n}^{\mathrm{q}}\left(J_{p, \infty}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) & \geq \frac{1}{2} \min \left\{\left\|J_{p, \infty}^{N} f_{u}-J_{p, \infty}^{N} f_{u^{\prime}}\right\|_{L_{\infty}^{N}}| | u\left|=0,\left|u^{\prime}\right|=1\right\}\right. \\
& =\frac{1}{2} N^{1 / p} .
\end{aligned}
$$

This proves the statement in the first case. Let

$$
\begin{equation*}
c_{1}=c_{0} / \sqrt{12} . \tag{29}
\end{equation*}
$$

Now we assume

$$
\begin{equation*}
c_{0} \sqrt{N}<n \leq c_{1} N . \tag{30}
\end{equation*}
$$

We set

$$
\begin{equation*}
L=N, \quad l=\left\lceil 2 c_{0}^{-2} n^{2} N^{-1}\right\rceil, \quad l^{\prime}=l+1 . \tag{31}
\end{equation*}
$$

It follows from (30) that $l>2$. Moreover, from (31),

$$
\begin{equation*}
n \leq c_{0} \sqrt{l N / 2} \tag{32}
\end{equation*}
$$

and, taking into account that $l>2$,

$$
l / 2<l-1<2 c_{0}^{-2} n^{2} N^{-1}
$$

hence, by (29) and (30),

$$
\begin{equation*}
l+1 \leq 3 l / 2<6 c_{0}^{-2} n^{2} N^{-1} \leq 6 c_{0}^{-2} c_{1}^{2} N=N / 2 \tag{33}
\end{equation*}
$$

We have, by (32), (33) and (31),

$$
\begin{equation*}
n \leq c_{0} \sqrt{l N / 2} \leq c_{0} \min _{j=l, l+1} \sqrt{j(N-j)} \leq c_{0} \varrho\left(L, l, l^{\prime}\right) \tag{34}
\end{equation*}
$$

Now we define $\psi_{j} \in L_{p}^{N} \quad(j=0, \ldots, L-1)$ as

$$
\psi_{j}(i)= \begin{cases}(l+1)^{-1 / p} N^{1 / p} & \text { if } \quad i=j \\ 0 & \text { otherwise } .\end{cases}
$$

Then

$$
\begin{equation*}
\left\|J_{p, \infty}^{N} \psi_{j}\right\|_{L_{\infty}^{N}}=\left\|\psi_{j}\right\|_{L_{\infty}^{N}}=(l+1)^{-1 / p} N^{1 / p} . \tag{35}
\end{equation*}
$$

Defining the system $\left(f_{u}\right)_{u \in\{0,1\}^{L}}$ as above, it satisfies condition (I), and $f_{u} \in \mathcal{B}\left(L_{p}^{N}\right)$ whenever $|u|=l, l+1$. Lemma 7] relations (34), (35), and the left and middle part of (33) give

$$
\begin{aligned}
e_{n}^{\mathrm{q}}\left(J_{p, \infty}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) & \geq \frac{1}{2} \min \left\{\left\|J_{p, \infty}^{N} f_{u}-J_{p, \infty}^{N} f_{u^{\prime}}\right\|_{L_{\infty}^{N}}| | u\left|=l,\left|u^{\prime}\right|=l+1\right\}\right. \\
& =\frac{1}{2}(l+1)^{-1 / p} N^{1 / p} \geq \frac{1}{2}\left(6 c_{0}^{-2} n^{2} N^{-1}\right)^{-1 / p} N^{1 / p} \\
& =\frac{c_{0}^{2 / p}}{2 \cdot 6^{1 / p}} n^{-2 / p} N^{2 / p} .
\end{aligned}
$$

The previous results together with Corollary 3 give lower bounds also for arbitrary $q$.

Proposition 4. Let $1 \leq p, q \leq \infty$. There are constants $c_{0}, c_{1}>0$ such that for all $n, N \in \mathbf{N}$, with $n \leq c_{0} N$ the following hold: If $p \leq q$, then
$e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq c_{1} \min \left(\left(\frac{N}{n}\right)^{2 / p-2 / q}(\log (n / \sqrt{N}+2))^{-2 / q}, N^{1 / p-1 / q}\right)$,
and if $p>q$, then

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq c_{1}(\log N+1)^{-2 / q} .
$$

Proof. Fix any $\nu_{1}, \nu_{2} \in \mathbf{N}$ with

$$
e^{-\nu_{1} / 8}+e^{-\nu_{2} / 8}-e^{-\left(\nu_{1}+\nu_{2}\right) / 8} \leq 1 / 4 .
$$

By Corollary 3

$$
e_{\left(\nu_{1}+2 \nu_{2}\right) n}^{\mathrm{q}}\left(J_{p, \infty}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \leq 4 e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) e_{n}^{\mathrm{q}}\left(J_{q, \infty}^{N}, \mathcal{B}\left(L_{q}^{N}\right)\right),
$$

therefore,

$$
\begin{equation*}
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq 4^{-1} e_{\left(\nu_{1}+2 \nu_{2}\right) n}^{\mathrm{q}}\left(J_{p, \infty}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) e_{n}^{\mathrm{q}}\left(J_{q, \infty}^{N}, \mathcal{B}\left(L_{q}^{N}\right)\right)^{-1} . \tag{36}
\end{equation*}
$$

By Proposition 2

$$
\begin{equation*}
e_{n}^{\mathrm{q}}\left(J_{q, \infty}^{N}, \mathcal{B}\left(L_{q}^{N}\right)\right) \leq c \min \left(\left(\frac{N}{n} \log (n / \sqrt{N}+2)\right)^{2 / q}, N^{1 / q}\right) \tag{37}
\end{equation*}
$$

while by Proposition 3 for $n$ such that $\left(\nu_{1}+2 \nu_{2}\right) n \leq c_{1} N\left(c_{1}\right.$ the constant from Proposition (3)

$$
\begin{align*}
e_{\left(\nu_{1}+2 \nu_{2}\right) n}^{\mathrm{q}}\left(J_{p, \infty}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) & \geq c_{2} \min \left(\left(\frac{N}{\left(\nu_{1}+2 \nu_{2}\right) n}\right)^{2 / p}, N^{1 / p}\right) \\
& \geq c \min \left(\left(\frac{N}{n}\right)^{2 / p}, N^{1 / p}\right) . \tag{38}
\end{align*}
$$

We first consider the case $n \leq \min \left(\sqrt{N}, c_{1} N /\left(\nu_{1}+2 \nu_{2}\right)\right)$. Then (361), (37), and (38) give

$$
\begin{equation*}
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq c N^{1 / p-1 / q} \tag{39}
\end{equation*}
$$

In the case $\sqrt{N}<n \leq c_{1} N /\left(\nu_{1}+2 \nu_{2}\right)$ we obtain similarly,

$$
\begin{equation*}
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq c\left(\frac{N}{n}\right)^{2 / p}\left(\frac{N}{n} \log (n / \sqrt{N}+2)\right)^{-2 / q} \tag{40}
\end{equation*}
$$

For $p \leq q$ relations (39) and (40) give the lower bound. In the case $p>q$ we note that it suffices to prove the lower bound for $n=\left\lfloor c_{1} N /\left(\nu_{1}+2 \nu_{2}\right)\right\rfloor$. But in this case (40) gives

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq c(\log N+1)^{-2 / q} .
$$

To present the results in a form which emphasizes the main, polynomial parts of the estimates and suppresses the logarithmic factors, we introduce the following notation: For functions $a, b: \mathbf{N}^{2} \rightarrow[0, \infty)$ we write $a(n, N) \asymp_{\log } b(n, N)$ if there are constants $c_{0}, c_{1}, c_{2}>0, n_{0} \in \mathbf{N}, \alpha_{1}, \alpha_{2} \in \mathbf{R}$ such that

$$
c_{1}(\log (N+n))^{\alpha_{1}} b(n, N) \leq a(n, N) \leq c_{2}(\log (N+n))^{\alpha_{2}} b(n, N)
$$

for all $n, N \in \mathbf{N}$ with $n_{0} \leq n \leq c_{0} N$. In this notation, we summarize the results of Propositions 2 and 4 in

Theorem 1. Let $1 \leq p, q \leq \infty$. If $p<q$ then

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \asymp_{\log } \min \left(\left(\frac{N}{n}\right)^{2 / p-2 / q}, N^{1 / p-1 / q}\right)
$$

and if $p \geq q$, then

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \asymp_{\log } 1 .
$$

Although the polynomial part of the order has been determined, there remains some room for improvements of the logarithmic factors. In the sequel we present some improvements of the lower bounds for particular situations. They involve a different way of applying Corollary 3 which is interesting in itself: we combine it with known results about summation. Furthermore, we will use these bounds for the Sobolev case 11. We need to recall some previous results about summation. Define $S_{N}: L_{p}^{N} \rightarrow \mathbf{R}$ by

$$
S_{N} f=\frac{1}{N} \sum_{i=0}^{N-1} f(i)
$$

Let us recall what is known about the minimal query errors of the $S_{N}$ :
Proposition 5. Let $1 \leq p \leq \infty$. There are constants $c_{0}, c_{1}, c_{2}>0$ such that for all $n, N \in \mathbf{N}$ with $2<n \leq c_{0} N$,

$$
\begin{align*}
& c_{1} n^{-1} \leq e_{n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \leq c_{2} n^{-1} \quad \text { if } 2<p \leq \infty,  \tag{41}\\
& c_{1} n^{-1} \leq e_{n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{2}^{N}\right)\right) \leq c_{2} n^{-1} \log ^{3 / 2} n \log \log n, \tag{42}
\end{align*}
$$

and

$$
\begin{align*}
& c_{1} \min \left(n^{-2(1-1 / p)}, n^{-2 / p} N^{2 / p-1}\right) \leq e_{n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \\
& \quad \leq c_{2} \min \left(n^{-2(1-1 / p)}, n^{-2 / p} N^{2 / p-1}\right)(\log (n / \sqrt{N}+2))^{2 / p-1} \tag{43}
\end{align*}
$$

if $1 \leq p<2$.
The upper bound in the case $p=\infty$ is contained in [5], 2], the lower bound for $p=\infty$ is from (16, while (41) for $2<p<\infty$ and (42) were obtained in [8], and (43) is from [14]. For the case $p=2$ we need another upper estimate, which is more precise than (42) for $n$ close to $N$. It can be found in [10], Lemma 6.

Lemma 8. There is a constant $c>0$ such that for all $n, N \in \mathbf{N}$ with $n \leq N$,

$$
e_{n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{2}^{N}\right)\right) \leq c n^{-1} l(n, N)^{3 / 2} \log l(n, N),
$$

where

$$
l(n, N)=\log (N / n)+\log \log (n+1)+2 .
$$

Proposition 6. Let $1 \leq p, q \leq \infty$. There are constants $c_{0}, c_{1}>0$ such that for all $n, N \in \mathbf{N}$ with $N>4$ and $n \leq c_{0} N$

$$
\begin{aligned}
e_{n}^{\mathrm{q}}\left(J_{p, q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) & \geq c_{1} \quad \text { if } \quad 2<q \leq \infty \\
e_{n}^{\mathrm{q}}\left(J_{p, 2}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) & \geq c_{1}(\log \log N)^{-3 / 2}(\log \log \log N)^{-1} \\
e_{n}^{\mathrm{q}}\left(J_{p, q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) & \geq c_{1}(\log N)^{-2 / q+1} \quad \text { if } \quad 1 \leq q<2 .
\end{aligned}
$$

Proof. The proof is similar to that of Proposition (4) Fix $\nu_{1}, \nu_{2} \in \mathbf{N}$ with

$$
e^{-\nu_{1} / 8}+e^{-\nu_{2} / 8}-e^{-\left(\nu_{1}+\nu_{2}\right) / 8} \leq 1 / 4 .
$$

Since $J_{p q}^{N}$ is the identity, we have $S_{N}=S_{N} J_{p q}^{N}$, and we deduce from Corollary 3

$$
e_{\left(\nu_{1}+2 \nu_{2}\right) n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \leq 4 e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) e_{n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{q}^{N}\right)\right)
$$

which gives

$$
e_{n}^{\mathrm{q}}\left(J_{p q}^{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) \geq 4^{-1} e_{\left(\nu_{1}+2 \nu_{2}\right) n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{p}^{N}\right)\right) e_{n}^{\mathrm{q}}\left(S_{N}, \mathcal{B}\left(L_{q}^{N}\right)\right)^{-1} .
$$

It remains to apply Proposition 5 and Lemma 8 with $n=\left\lfloor c_{0}\left(\nu_{1}+2 \nu_{2}\right)^{-1} N\right\rfloor$.

## 5 Comments

First we discuss the cost of the presented (optimal with respect to the number of queries) algorithms in the bit model of computation. For this purpose we assume that $n$ and $N$ are powers of 2 . The algorithm behind Proposition 2 for approximating $J_{p q}^{N}$ is nontrivial only for $n \geq \sqrt{N}$. In this case it uses $n$ quantum queries to finds all coordinates of $f \in \mathcal{B}\left(L_{p}^{N}\right)$ with absolute value not smaller than

$$
M=c(N / n)^{2 / p} \max (\log (n / \sqrt{N}), 1)^{2 / p}
$$

(with $c$ a concrete constant, see [14]). It follows from $\|f\|_{L_{p}^{N}} \leq 1$ that there are at most

$$
N M^{-p}=\mathcal{O}\left(n^{2} N^{-1} \max (\log (n / \sqrt{N}), 1)^{-2}\right)
$$

of them. The algorithm of finding them, which is described and analyzed in [14], needs $\mathcal{O}(n \log N)$ quantum gates, $\mathcal{O}(\log N)$ qubits,

$$
\mathcal{O}\left(n^{2} N^{-1} \max (\log (n / \sqrt{N}), 1)^{-1}\right)
$$

measurements, and $\mathcal{O}\left(n^{2} N^{-1} \log N\right)$ classical bit operations. So the total bit cost is $\mathcal{O}(n \log N)=\mathcal{O}(n \log n)$, hence, up to a logarithm, the same amount as the number of queries.

In the following table we summarize the results of this paper on the quantum approximation of the identical embeddings $J_{p q}^{N}: \mathcal{B}\left(L_{p}^{N}\right) \rightarrow L_{q}^{N}$ and compare them with the respective known quantities in the classical deterministic and randomized settings. We refer to $[7$ and the bibliography therein for more information on the classical settings. The respective entries of the table give the minimal errors, constants and logarithmic factors are suppressed. We always assume $n \leq c N$.

| $J_{p q}^{N}: \mathcal{B}\left(L_{p}^{N}\right) \rightarrow L_{q}^{N}$ | deterministic | random | quantum |
| :--- | :--- | :--- | :--- |
| $1 \leq p<q \leq \infty, n \leq \sqrt{N}$ | $N^{1 / p-1 / q}$ | $N^{1 / p-1 / q}$ | $N^{1 / p-1 / q}$ |
| $1 \leq p<q \leq \infty, n>\sqrt{N}$ | $N^{1 / p-1 / q}$ | $N^{1 / p-1 / q}$ | $\left(\frac{N}{n}\right)^{2 / p-2 / q}$ |
| $1 \leq q \leq p \leq \infty$ | 1 | 1 | 1 |

We see that the quantum rate can improve the classical deterministic and randomized rates by a factor of order $N^{-1}$ (for $p=1, q=\infty$, and $n$ of the order of $N$ ). It is this case which will lead to a speedup for Sobolev embeddings by an exponent 1, see [11. We observe that there are also regions where the speedup is smaller or there is no speedup at all.

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