# Sharp Error Bounds on Quantum Boolean Summation in Various Settings* 

Marek Kwas and Henryk Woźniakowski<br>Department of Computer Science<br>Columbia University<br>and<br>Institute of Applied Mathematics<br>University of Warsaw

November 3, 2018


#### Abstract

We study the quantum summation (QS) algorithm of Brassard, Høyer, Mosca and Tapp, see [1, that approximates the arithmetic mean of a Boolean function defined on $N$ elements. We improve error bounds presented in [1] in the worst-probabilistic setting, and present new error bounds in the average-probabilistic setting.

In particular, in the worst-probabilistic setting, we prove that the error of the QS algorithm using $M-1$ quantum queries is $\frac{3}{4} \pi M^{-1}$ with probability $\frac{8}{\pi^{2}}$, which improves the error bound $\pi M^{-1}+\pi^{2} M^{-2}$ of [1]. We also present error bounds with probabilities $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$, and show that they are sharp for large $M$ and $N M^{-1}$.

In the average-probabilistic setting, we prove that the QS algorithm has error of order $\min \left\{M^{-1}, N^{-1 / 2}\right\}$ iff $M$ is divisible by 4 . This bound is optimal, as recently shown in [10. For $M$ not divisible by 4, the QS algorithm is far from being optimal if $M \ll N^{1 / 2}$ since its error is proportional to $M^{-1}$.


## 1 Introduction

The quantum summation (QS) algorithm (also known as the amplitude estimation algorithm) of Brassard, Høyer, Mosca and Tapp, see [1], computes an approximation to the arithmetic

[^0]mean of all values of a Boolean function defined on a set of $N=2^{n}$ elements. Information regarding the Boolean function is supplied by quantum queries. The quantum queries play a role similar to the use of function values in the worst case and randomized settings. Suppose that we use $M-1$ quantum queries. Obviously, the only case of interest is when $M$ is much smaller than $N$. It was proven in [1] that the error of the QS algorithm is at most
\[

$$
\begin{equation*}
\frac{\pi}{M}+\frac{\pi^{2}}{M^{2}} \quad \text { with probability } \frac{8}{\pi^{2}}=0.81 \ldots \tag{1}
\end{equation*}
$$

\]

Nayak and Wu, see [7], showed that for any $p \in\left(\frac{1}{2}, 1\right]$ the error of any quantum algorithm that uses no more than $M-1$ quantum queries must be proportional to $M^{-1}$ with probability $p$. Therefore, the QS algorithm enjoys the smallest possible error modulo a factor multiplying $M^{-1}$.

The minimal error estimate of order $M^{-1}$ in the quantum setting should be compared to the minimal error estimates in the worst case and randomized settings of algorithms using $M-1$ function values. It is known, see [8], that in the worst case setting, the error bound is roughly $\frac{1}{2}(1-M / N)$. This means that as long as $M$ is much less than $N$ the error is almost $\frac{1}{2}$, and is therefore of order $M$ times larger than in the quantum setting. In the randomized setting, the classical Monte Carlo is almost optimal, and the error bound is roughly $1 /(2 \sqrt{M})$, see again [8]. Hence, it is of order $\sqrt{M}$ larger than in the quantum setting.

The QS algorithm has many applications. In particular, it can be used for approximation of the arithmetic mean of a real function, which is the basic step for approximation of many continuous problems such as multivariate integration, multivariate approximation and path integration, see [4, 5, 6, 9, 11].

Since the QS algorithm has so many applications, it seems reasonable to check whether the estimate (11) is sharp and how the error decreases if we lower the probability $p=\frac{8}{\pi^{2}}$ to $p>\frac{1}{2}$. It also seems reasonable to study the error of the $\mathbf{Q S}$ algorithm in various settings. The estimate (1) corresponds to the worst-probabilistic setting, which is most frequently used in the quantum setting. The essence of this setting is that it holds for all Boolean functions. It is also interesting to study the average performance of the QS algorithm with respect to some measure on Boolean functions. This is the average-probabilistic setting. In the worst-average and average-average settings, we study the worst or average performance with respect to Boolean functions and the average performance with respect to all outcomes of a quantum algorithm. We add in passing that the worst-average setting is usually used for the study of the classical Monte Carlo algorithm.

Sharp error bounds in the worst- and average-probabilistic settings are addressed in this paper whereas the worst- and average-average settings will be studied in a future paper. We study error bounds with probabilities $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. If we want to obtain error bounds with higher probability, it is known that it is enough to run the QS algorithm several times and take the median as the final result, see e.g., 4].

In the worst-probabilistic setting, we show that (11) can be slightly improved. Namely, the error of the QS algorithm is at most

$$
\frac{3}{4} \frac{\pi}{M} \quad \text { with probability } \frac{8}{\pi^{2}}
$$

Furthermore, for large $M$ and $N / M$ we prove that the last estimate is sharp. More generally, for $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$ we prove that the error of the $\mathbf{Q S}$ algorithm is at most

$$
\frac{\left(1-v^{-1}(p)\right) \pi}{M} \quad \text { with probability } p
$$

where $v^{-1}$ is the inverse of the function $v(\Delta)=\sin ^{2}(\pi \Delta) /(\pi \Delta)^{2}$. We prove that the last estimate is sharp for large $M$ and $N / M$. We have $1-v^{-1}(p) \in\left(\frac{1}{2}, \frac{3}{4}\right]$ and it is well approximated by $\frac{1}{16} \pi^{2} p+\frac{1}{4}$. In particular, for the most commonly used values of $p$ we have

$$
\left(1-v^{-1}\left(\frac{1}{2}+\right)\right) \pi=1.75 \ldots,\left(1-v^{-1}\left(\frac{3}{4}\right)\right) \pi=2.23 \ldots,\left(1-v^{-1}\left(\frac{8}{\pi^{2}}\right)\right)=\frac{3}{4} \pi=2.35 \ldots
$$

In the average-probabilistic setting, we consider two measures on the set of Boolean functions. The first measure is uniform on Boolean functions, while the second measure is uniform on arithmetic means of Boolean functions. The results for these two measures are quite different. The mean element of the arithmetic means is $\frac{1}{2}$ for both measures. However, the first moment is of order $N^{-1 / 2}$ for the first measure, and about $\frac{1}{4}$ for the second. The first moment is exactly equal to the error of the constant algorithm that always outputs $\frac{1}{2}$. This explains why we can obtain the error of order $N^{-1 / 2}$ without any quantum queries for the first measure. This provides the motivation for us to check whether the error of the QS algorithm enjoys a similar property. It turns out that this is indeed the case iff $M$ is divisible by 4 . That is, for $M$ divisible by 4 , the average-probabilistic error of the QS algorithm is of order $\min \left\{M^{-1}, N^{-1 / 2}\right\}$, and if $M$ is not divisible by 4 , then the error is of order $M^{-1}$. For the second measure, since the first moment is not small, the average-probabilistic error of the QS algorithm is of order $M^{-1}$ for all $M$. For both measures, the upper bounds presented in this paper match lower bounds that were recently obtained by Papageorgiou, see [10]. Hence, the QS algorithm enjoys minimal error bounds also in the average-probabilistic setting if we choose $M$ divisible by 4 for the first measure and with no restriction on $M$ for the second measure.

The quantum setting, and in particular the QS algorithm, is relatively new and probably not well known, especially for people interested in continuous complexity. Hence we present all details of this algorithm, emphasizing its quantum parts. Since we wanted also to find sharp error bounds, we needed a very detailed analysis of the outcome probabilities of the QS algorithm.

We outline the contents of this paper. In Section 2 we define the QS algorithm. Section 3 deals with the performance analysis of the QS algorithm in the worst-probabilistic setting, see Section 3.1, and in the average-probabilistic setting, see Section 3.2.

## 2 Quantum Summation Algorithm

We consider the most basic form of the summation problem, i.e., the summation of Boolean functions. Let $\mathbb{B}_{N}$ denote the set of Boolean functions $f:\{0, \ldots, N-1\} \rightarrow\{0,1\}$. Let

$$
a_{f}=\frac{1}{N} \sum_{i=0}^{N-1} f(i)
$$

denote the arithmetic mean of all values of $f$. Clearly, $a_{f} \in[0,1]$.
Problem: For $f \in \mathbb{B}_{N}$, compute an $\varepsilon$-approximation $\bar{a}_{f}$ of the sum $a_{f}$ such that

$$
\begin{equation*}
\left|\bar{a}_{f}-a_{f}\right| \leq \varepsilon \tag{2}
\end{equation*}
$$

We are interested in the minimal number of evaluations of the function $f$ that are needed to compute $\bar{a}_{f}$ satisfying (2). It is known that in the worst case setting, we need roughly $N(1-$ $\varepsilon)$ evaluations of the function $f$. In the randomized setting, we assume that $\bar{a}_{f}$ is a random variable and require that (2) holds for the expected value of $\left|\bar{a}_{f}-a_{f}\right|$ for any function $f$. It is known, see e.g., [8], that in the randomized setting we need roughly $\min \left\{N, \varepsilon^{-1 / 2}\right\}$ function evaluations. In the quantum setting, we want to compute a random variable $\bar{a}_{f}$ such that (21) holds with a high probability (greater than $\frac{1}{2}$ ) either for all Boolean functions or on the average with respect to a probability measure defined on the set $\mathbb{B}_{N}$. These two error criteria in the quantum setting will be precisely defined in Section 3

In this section we describe the quantum summation algorithm, which is also called the quantum amplitude estimation algorithm. This algorithm was discovered by Brassard, Høyer, Mosca and Tapp [1], and uses Grover's iterate operator as its basic component, see [2]. We use standard notation of quantum computation, see e.g., 3].

For simplicity we assume that $N=2^{n}$. Let $\mathcal{H}_{n}$ denote the tensor product $\mathbb{C}^{2} \otimes \cdots \otimes \mathbb{C}^{2}$ of $n$ copies of $\mathbb{C}^{2}$, with $\mathbb{C}^{2}$ the 2-dimensional complex vector space. Unit vectors from $\mathbb{C}^{2}$ are called one qubit quantum states (or qubits). Let $|0\rangle$ and $|1\rangle$ be an orthonormal basis of $\mathbb{C}^{2}$. Then any qubit $|\psi\rangle$ can be represented as

$$
|\psi\rangle=\psi_{0}|0\rangle+\psi_{1}|1\rangle \quad \text { with } \quad \psi_{k} \in \mathbb{C} \quad \text { and } \quad\left|\psi_{0}\right|^{2}+\left|\psi_{1}\right|^{2}=1
$$

For $j=0,1, \ldots, N-1$, we have $j=\sum_{k=0}^{n-1} 2^{n-1-k} j_{k}$, with $j_{k} \in\{0,1\}$. Let

$$
|j\rangle=\bigotimes_{k=0}^{n-1}\left|j_{k}\right\rangle
$$

The set $\{|j\rangle: j=0, \ldots, N-1\}$ forms an orthonormal basis of $\mathcal{H}_{n}$ and any unit vector $|\psi\rangle \in \mathcal{H}_{n}$ can be represented as

$$
|\psi\rangle=\sum_{j=0}^{N-1} \psi_{j}|j\rangle \quad \text { with } \quad \psi_{j} \in \mathbb{C} \quad \text { and } \quad \sum_{j=0}^{N-1}\left|\psi_{j}\right|^{2}=1
$$

Unit vectors from $\mathcal{H}_{n}$ are called $n$ qubit quantum states (or quantum states or just states, whenever $n$ is clear from the context).

The only transforms that can be performed on quantum states are defined by certain unitary operators on $\mathcal{H}_{n}$. We now define the six unitary operators that are basic components of the summation algorithm. Since unitary operators are linear, it is enough to define them on the basis states $|j\rangle$.

1. Let $S_{0}: \mathcal{H}_{n} \rightarrow \mathcal{H}_{n}$ denote the inversion about zero transform

$$
S_{0}|j\rangle=(-1)^{\delta_{j, 0}}|j\rangle,
$$

where $\delta_{j, 0}$ is the Kronecker delta. Hence, $S_{0}|0\rangle=-|0\rangle$ and $S_{0}|j\rangle=|j\rangle$ for all $j \neq 0$. This corresponds to the diagonal matrix with one element equal to -1 , and the rest equal to 1. The operator $S_{0}$ can be also written as the Householder operator

$$
S_{0}=I-2|0\rangle\langle 0| .
$$

Here, for a state $|\psi\rangle$, we let $|\psi\rangle\langle\psi|$ denote the projection onto the space span $\{|\psi\rangle\}$ given by

$$
(|\psi\rangle\langle\psi|)|x\rangle=\langle\psi \mid x\rangle|\psi\rangle,
$$

where $\langle\psi \mid x\rangle$ is the inner product ${ }^{1}$ in $\mathcal{H}_{n},\langle\psi \mid x\rangle=\sum_{k=0}^{N-1} \overline{\psi_{k}} x_{k}$. The matrix form of the projector $|\psi\rangle\langle\psi|$ in the basis $\{|j\rangle\}$ is $\left(\overline{\psi_{k}} \psi_{j}\right)_{j, k=0}^{N-1}$. One can also view the matrix form of the projector $|\psi\rangle\langle\psi|$ as the matrix product of the $N \times 1$ column vector $|\psi\rangle$ and the $N \times 1$ row vector $\langle\psi|$, which is the Hermitian conjugate of $|\psi\rangle,\langle\psi|=|\psi\rangle^{\dagger}$. For any $|x\rangle \in \mathcal{H}_{n}$ we have

$$
\langle k|(I-2|0\rangle\langle 0|)|x\rangle=\langle k \mid x\rangle-2\langle 0 \mid x\rangle\langle k \mid 0\rangle= \begin{cases}x_{k}-2 x_{k}=-x_{k} & \text { for } k=0 \\ x_{k}-0=x_{k} & \text { for } k \neq 0\end{cases}
$$

Hence $I-2|0\rangle\langle 0|=S_{0}$, as claimed.
2. Let $W_{N}: \mathcal{H}_{n} \rightarrow \mathcal{H}_{n}$ denote the Walsh-Hadamard transform

$$
W_{N}|j\rangle=\frac{1}{\sqrt{N}} \bigotimes_{k=0}^{n-1}\left(|0\rangle+(-1)^{j_{k}}|1\rangle\right)
$$

That is, the Walsh-Hadamard transform corresponds to the matrix with entries

$$
\langle i| W_{N}|j\rangle=\frac{1}{\sqrt{N}} \prod_{k=0}^{n-1}\left\langle i_{k}\right|\left(|0\rangle+(-1)^{j_{k}}|1\rangle\right)=\frac{1}{\sqrt{N}} \prod_{k=0}^{n-1}(-1)^{i_{k} j_{k}}=\frac{1}{\sqrt{N}}(-1)^{\sum_{k=0}^{n-1} i_{k} j_{k}}
$$

The matrix $\left(\langle i| W_{N}|j\rangle\right)_{i, j=0}^{N-1}$ is symmetric. Furthermore,

$$
\begin{aligned}
W_{N}^{2}|j\rangle & =\frac{1}{\sqrt{N}} W_{n} \bigotimes_{k=0}^{n-1}\left(|0\rangle+(-1)^{j_{k}}|1\rangle\right) \\
& =\frac{1}{\sqrt{N}} \bigotimes_{k=0}^{n-1}\left(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)+\frac{(-1)^{j_{k}}}{\sqrt{2}}(|0\rangle-|1\rangle)\right) \\
& =\frac{1}{\sqrt{N}} \bigotimes_{k=0}^{n-1} \sqrt{2}\left|j_{k}\right\rangle=|j\rangle .
\end{aligned}
$$

Thus, $W_{N}^{2}=I$ and $W_{N}^{-1}=W_{N}$ is orthogonal. This means that the operator $W_{N}$ is symmetric and unitary.

[^1]3. For $K=1,2, \ldots, 2^{n}$, let $F_{K, n}: \mathcal{H}_{n} \rightarrow \mathcal{H}_{n}$ denote the quantum Fourier transform
\[

F_{K, n}|j\rangle= $$
\begin{cases}\frac{1}{\sqrt{K}} \sum_{k=0}^{K-1} e^{2 \pi i j k / K}|k\rangle, & \text { for } \quad j=0,1, \ldots, K-1, \quad(i=\sqrt{-1}) \\ |j\rangle & \text { for } \quad j=K, \ldots, 2^{n}-1 .\end{cases}
$$
\]

Hence, $F_{K, n}$ corresponds to the unitary block-diagonal matrix

$$
\left[\begin{array}{cc}
F_{K} & 0 \\
0 & I
\end{array}\right],
$$

where $F_{K}=\left(K^{-1 / 2} e^{2 \pi i j k / k}\right)_{j, k=0}^{K-1}$ is the matrix of the inverse quantum Fourier transform. For $K=2^{n}=N$ we have

$$
F_{N, n}|\psi\rangle=\sum_{j=0}^{N-1} \psi_{j} F_{N, n}|j\rangle=\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1}\left(\sum_{j=0}^{N-1} \psi_{j} e^{2 \pi i j k / N}\right)|k\rangle .
$$

The coefficients of $F_{N, n}|\psi\rangle$ in the basis $\{|j\rangle\}$ are the inverse quantum Fourier transforms of the coefficients of the state $|\psi\rangle$. Note that $W_{N}$ and $F_{N, n}$ coincide for the state $|0\rangle$, i.e.,

$$
W_{N}|0\rangle=F_{N, n}|0\rangle=\frac{1}{\sqrt{N}} \sum_{j=0}^{N-1}|j\rangle .
$$

4. Let $S_{f}: \mathcal{H}_{n} \rightarrow \mathcal{H}_{n}$ denote the quantum query operator

$$
S_{f}|j\rangle=(-1)^{f(j)}|j\rangle
$$

This again corresponds to the diagonal matrix with elements $\pm 1$ depending on the values of the Boolean function $f$. This operator is the only one that provides information about the Boolean function $f$. This is analogous to the concept of an oracle or a black-box which is used in classical computation and which supplies information about the function $f$ through its values.
The standard definition of the quantum query $\bar{S}_{f}$ is

$$
\bar{S}_{f}: \mathcal{H}_{n} \otimes \mathbb{C}^{2} \rightarrow \mathcal{H}_{n} \otimes \mathbb{C}^{2}, \quad \bar{S}_{f}|j\rangle|i\rangle=|j\rangle|i \oplus f(j)\rangle,
$$

where $\oplus$ means addition modulo 2 . We can simulate $S_{f}$ by $\bar{S}_{f}$ if we use an auxiliary qubit $(1 / \sqrt{2})(|1\rangle-|0\rangle)$, namely,

$$
\begin{aligned}
\bar{S}_{f}\left(|j\rangle \frac{|1\rangle-|0\rangle}{\sqrt{2}}\right)=|j\rangle \frac{|1 \oplus f(j)\rangle-|f(j)\rangle}{\sqrt{2}} & \\
& =(-1)^{f(j)}|j\rangle \frac{|1\rangle-|0\rangle}{\sqrt{2}}=\left(S_{f}|j\rangle\right) \frac{|1\rangle-|0\rangle}{\sqrt{2}} .
\end{aligned}
$$

5. Let $Q_{f}: \mathcal{H}_{n} \rightarrow \mathcal{H}_{n}$ denote the Grover operator

$$
Q_{f}=-W_{N} S_{0} W_{N}^{-1} S_{f}
$$

This is the basic component of Grover's search algorithm, see [2]. As we shall see, $Q_{f}$ also plays the major role for the summation algorithm. The eigenvectors and eigenvalues of $Q_{f}$ will be useful in further considerations. Let

$$
|\psi\rangle=W_{N}|0\rangle=\frac{1}{\sqrt{N}} \sum_{k=0}^{N-1}|k\rangle
$$

and $\left|\psi_{0}\right\rangle,\left|\psi_{1}\right\rangle$ denote the orthogonal projections of $|\psi\rangle$ onto the subspaces $\operatorname{span}\{|j\rangle: f(j)=0\}$ and $\operatorname{span}\{|j\rangle: f(j)=1\}$, respectively. That is,

$$
\left|\psi_{j}\right\rangle=\frac{1}{\sqrt{N}} \sum_{k: f(k)=j}|k\rangle \quad j=0,1
$$

Then $|\psi\rangle=\left|\psi_{0}\right\rangle+\left|\psi_{1}\right\rangle$ and $\left\langle\psi_{0} \mid \psi_{1}\right\rangle=0$. Furthermore, $\left\langle\psi_{j} \mid \psi_{j}\right\rangle=N^{-1} \sum_{k: f(k)=j} 1$, for $j=0,1$, so that $\left\langle\psi_{1} \mid \psi_{1}\right\rangle=a$ and $\left\langle\psi_{0} \mid \psi_{0}\right\rangle=1-a$, where $a=a_{f}$ is the sum we want to approximate.
From [1], we know that

$$
\begin{align*}
Q_{f}\left|\psi_{0}\right\rangle & =(1-2 a)\left|\psi_{0}\right\rangle+2(1-a)\left|\psi_{1}\right\rangle \\
Q_{f}\left|\psi_{1}\right\rangle & =-2 a\left|\psi_{0}\right\rangle+(1-2 a)\left|\psi_{1}\right\rangle \tag{3}
\end{align*}
$$

For the sake of completeness, we provide a short proof of (3). By the definition of the operator $S_{f}$ we have

$$
S_{f}\left|\psi_{j}\right\rangle=(-1)^{j}\left|\psi_{j}\right\rangle, \quad j=0,1,
$$

and

$$
W_{N} S_{0} W_{N}^{-1}=W_{N}(I-2|0\rangle\langle 0|) W_{N}^{-1}=I-2\left(W_{N}|0\rangle\langle 0| W_{N}\right)
$$

Since $\langle 0| W_{N}=\left(W_{N}|0\rangle\right)^{\dagger}=(|\psi\rangle)^{\dagger}=\langle\psi|$, we obtain for $j=0,1$,

$$
W_{N} S_{0} W_{N}^{-1}\left|\psi_{j}\right\rangle=\left|\psi_{j}\right\rangle-2(|\psi\rangle\langle\psi|)\left|\psi_{j}\right\rangle=\left|\psi_{j}\right\rangle-2\left\langle\psi \mid \psi_{j}\right\rangle|\psi\rangle=\left|\psi_{j}\right\rangle-2\left\langle\psi_{j} \mid \psi_{j}\right\rangle|\psi\rangle .
$$

From this we calculate for $j=0,1$,

$$
\begin{aligned}
& Q_{f}\left|\psi_{j}\right\rangle=(-1)^{1+j} W_{N} S_{0} W_{N}^{-1}\left|\psi_{j}\right\rangle \\
&=(-1)^{\delta j, 0}\left(\left|\psi_{j}\right\rangle-2\left(\delta_{j, 1} a+\delta_{j, 0}(1-a)\right)\left(\left|\psi_{0}\right\rangle+\left|\psi_{1}\right\rangle\right)\right)
\end{aligned}
$$

which is equivalent to (3).
Thus, the space $\operatorname{span}\left\{\left|\psi_{0}\right\rangle,\left|\psi_{1}\right\rangle\right\}$ is an invariant space of $Q_{f}$ and its eigenvectors and corresponding eigenvalues can be computed by solving the eigenproblem for the $2 \times 2$ matrix

$$
\left[\begin{array}{cc}
1-2 a & -2 a \\
2(1-a) & 1-2 a
\end{array}\right]
$$

For $a \in(0,1)$, the eigenvectors and the corresponding orthonormalized eigenvalues of $Q_{f}$ are

$$
\left|\psi_{ \pm}\right\rangle=\frac{1}{\sqrt{2}}\left( \pm \frac{i}{\sqrt{1-a}}\left|\psi_{0}\right\rangle+\frac{1}{\sqrt{a}}\left|\psi_{1}\right\rangle\right) \quad \text { and } \quad \lambda_{ \pm}=1-2 a \pm 2 i \sqrt{a(1-a)}=e^{ \pm 2 i \theta_{a}}
$$

where $\theta_{a}=\arcsin \sqrt{a}$. Moreover, it is easy to check that

$$
\begin{equation*}
|\psi\rangle=\frac{-i}{\sqrt{2}}\left(e^{i \theta_{a}}\left|\psi_{+}\right\rangle-e^{-i \theta_{a}}\left|\psi_{-}\right\rangle\right) \tag{4}
\end{equation*}
$$

For $a \in\{0,1\}$, we have $\operatorname{span}\left\{\left|\psi_{0}\right\rangle,\left|\psi_{1}\right\rangle\right\}=\operatorname{span}\{|\psi\rangle\}$ and $|\psi\rangle$ is the eigenvector of $Q_{f}$ with eigenvalues $\pm 1$, respectively. For $a \in\{0,1\}$, we define

$$
\left|\psi_{+}\right\rangle=i^{1-a} \sqrt{2}|\psi\rangle \quad \text { and } \quad\left|\psi_{-}\right\rangle=0 .
$$

Then it is easy to check that (4) is valid, and $\lambda_{ \pm}=e^{ \pm 2 i \theta_{a}}=(-1)^{a}$ is an eigenvalue of $Q_{f}$ for all $a \in[0,1]$.
6. The next unitary transform, called the Grover iterate operator, is defined on the tensor product of $\mathcal{H}_{m} \otimes \mathcal{H}_{n}$ and uses $m+n$ qubits. The first space $\mathcal{H}_{m}$ and $m$ qubits will be related to the accuracy of the quantum summation algorithm. The Grover iterate operator $\Lambda_{m}\left(Q_{f}\right): \mathcal{H}_{m} \otimes \mathcal{H}_{n} \rightarrow \mathcal{H}_{m} \otimes \mathcal{H}_{n}$ is defined by

$$
\Lambda_{m}\left(Q_{f}\right)|j\rangle|y\rangle=|j\rangle Q_{f}^{j}|y\rangle \quad \text { for } \quad|j\rangle|y\rangle \in \mathcal{H}_{m} \otimes \mathcal{H}_{n}
$$

Hence, the power of $Q_{f}$ applied to the second component depends on the first one. Note that $j$ may vary from 0 to $2^{m}-1$. Therefore $\Lambda_{m}\left(Q_{f}\right)$ may use the powers of $Q_{f}$ up to the $\left(2^{m}-1\right)$ st.

We need one more concept of quantum computation, that of measurement. Suppose $s$ is a positive integer and consider the space $\mathcal{H}_{s}$. Given the state

$$
|\psi\rangle=\sum_{k=0}^{2^{s}-1} \psi_{k}|k\rangle \in \mathcal{H}_{s}
$$

we cannot, in general, recover all the coefficients $\psi_{k}$. We can only measure the state $|\psi\rangle$ with respect to a finite collection of linear operators $\left\{M_{j}\right\}_{j=0}^{p}$, where the $M_{j}: \mathcal{H}_{s} \rightarrow \mathcal{H}_{s}$ satisfy the completeness relation

$$
\sum_{j=0}^{p} M_{j}^{\dagger} M_{j}=I
$$

After performing the measurement, we obtain the outcome $j$ and the state $|\psi\rangle$ collapses into the state

$$
\frac{1}{\sqrt{\langle\psi| M_{j}^{\dagger} M_{j}|\psi\rangle}} M_{j}|\psi\rangle
$$

both these events occur with probability $\langle\psi| M_{j}^{\dagger} M_{j}|\psi\rangle$. Note that for $M_{j}|\psi\rangle=0$ the outcome $j$ cannot happen with positive probability. Hence, with probability 1 the outcome $j$ corresponds to $M_{j}|\psi\rangle \neq 0$ for $j=0,1, \ldots, p$.

The most important example of such a collection of operators is $\{|j\rangle\langle j|\}_{j=0}^{2^{s}-1}$. Then, the measurement of the state $|\psi\rangle$ with respect to this collection of operators gives us the outcome $j$ and the collapse into the state

$$
\frac{\langle j \mid \psi\rangle}{|\langle j \mid \psi\rangle|}|j\rangle
$$

with probability $\left|\psi_{j}\right|^{2}, j=0,1, \ldots, 2^{s}-1$.
Another example is a variation of the previous example and will be used in the quantum summation algorithm. We now let $s=m+n$, as for the Grover iterate operator, and define $M_{j}: \mathcal{H}_{m} \otimes \mathcal{H}_{n} \rightarrow \mathcal{H}_{m} \otimes \mathcal{H}_{n}$ by

$$
M_{j}=|j\rangle\langle j| \otimes I
$$

for $j=0,1, \ldots, 2^{m}-1$, with $I$ denoting the identity operator on $\mathcal{H}_{n}$. That is,

$$
(|j\rangle\langle j| \otimes I)|x\rangle|y\rangle=\langle j \mid x\rangle|j\rangle|y\rangle
$$

for $|x\rangle \in \mathcal{H}_{m}$ and $|y\rangle \in \mathcal{H}_{n}$.
Since $\sum_{j=0}^{2^{m}-1}(|j\rangle\langle j| \otimes I)|x\rangle|y\rangle=|x\rangle|y\rangle$ for all basis states $|x\rangle$ of $\mathcal{H}_{m}$ and $|y\rangle$ of $\mathcal{H}_{n}$, the completeness relation is satisfied. Consider now the probability of the outcome $j$ for a special state $|\psi\rangle$ of the form $|\psi\rangle=\left|\psi_{1}\right\rangle\left|\psi_{2}\right\rangle$ with $\left|\psi_{1}\right\rangle \in \mathcal{H}_{m},\left|\psi_{2}\right\rangle \in \mathcal{H}_{n}$ and $\left\langle\psi_{k} \mid \psi_{k}\right\rangle=1$ for $k=1,2$. Since $|j\rangle\langle j| \otimes I$ is self-adjoint, the outcome $j$ and the collapse of the state $|\psi\rangle$ to the state

$$
\frac{\left\langle j \mid \psi_{1}\right\rangle}{\left|\left\langle j \mid \psi_{1}\right\rangle\right|}|j\rangle\left|\psi_{2}\right\rangle
$$

occur with probability $\left|\left\langle j \mid \psi_{1}\right\rangle\right|^{2}$. Hence, this collection of operators measures the components of the so-called first register $\left|\psi_{1}\right\rangle$ of the quantum state $|\psi\rangle$.

Following [1], we are ready to describe the quantum summation (QS) algorithm for solving our problem. The QS algorithm depends on a Boolean function $f$ and on an integer parameter $M$ that controls the number of quantum queries of $f$ used by the algorithm. We perform computations in the space $\mathcal{H}_{m} \otimes \mathcal{H}_{n}$, with $m=\left\lceil\log _{2} M\right\rceil$, so we use $n+m$ qubits. As we will see later, the accuracy of the algorithm is related to the dimension of the space $\mathcal{H}_{m}$.

## Algorithm QS $(f, M)$

Input state: $|0\rangle|0\rangle \in \mathcal{H}_{m} \otimes \mathcal{H}_{n}$ with $m=\left\lceil\log _{2} M\right\rceil$ and $n=\log _{2} N$.

## Computation:

$$
\begin{aligned}
& \text { 1. }\left|\eta_{1}\right\rangle=F_{M, m} \otimes W_{N}|0\rangle|0\rangle, \\
& \text { 2. }\left|\eta_{2}\right\rangle=\Lambda_{m}\left(Q_{f}\right)\left|\eta_{1}\right\rangle, \\
& \text { 3. }\left|\eta_{3}\right\rangle=\left(F_{M, m}^{-1} \otimes I\right)\left|\eta_{2}\right\rangle .
\end{aligned}
$$

## Measurement:

Perform the measurement of the state $\left|\eta_{3}\right\rangle$ with respect to the collection $\{(|j\rangle\langle j|) \otimes I\}_{j=0}^{2^{m}-1}$. Denote the outcome by $j$.

Output: $\bar{a}_{f}(j)=\sin ^{2}(\pi j / M)$.

We briefly comment on the QS algorithm. The input state is always the same and does not depend on $f$. Step 1 computes $\left|\eta_{1}\right\rangle=(N M)^{-1 / 2} \sum_{j=0}^{M-1} \sum_{k=0}^{N-1}|j\rangle|k\rangle$, which is the equally weighted superposition of the basis states. Step 2 computes $\left|\eta_{2}\right\rangle$ by using the Grover iterate
operator. During this step we use the successive powers of the Grover operator $Q_{f}$, and this is the only step where information about the Boolean function $f$ is used. We shall see that the QS algorithm uses $M-1$ quantum queries. Step 3 computes $\left|\eta_{3}\right\rangle$ by performing the inverse quantum Fourier transform on the first $m$ qubits, and prepares the system for measurement. After Step 3, we perform the measurement, obtain the outcome $j$ and compute the output $\bar{a}_{f}(j)$ on a classical computer. We stress that the distribution of the outcomes $j$ depends on the Boolean function $f$, and this is the only dependence of the output $\bar{a}_{f}(j)$ on $f$.

## 3 Performance Analysis

In this section we analyze the error of the QS algorithm. As we have seen in Section 2, the output $\bar{a}_{f}(j)$ of the QS algorithm is a random value chosen according to a certain distribution dependent on the input function $f$. In this way, the $\mathbf{Q S}$ algorithm is a randomized algorithm. Various ways of measuring the performance of randomized algorithms are commonly used in the analysis of algorithms and computational complexity. They correspond to various error criteria. In this paper we consider two error criteria: worst-probabilistic and average-probabilistic. In a future paper we consider other two error criteria: worst-average and average-average, which correspond to the worst or average performance with respect to Boolean functions and the average performance with respect to all outcomes.

## Worst-Probabilistic Error

We start with the error criterion that is used in most papers dealing with quantum computations. We are interested in the worst case error of the QS algorithm that holds with a given probability $p$. Here $p \in[0,1]$ and $1-p$ measures the probability of QS algorithm's failure and usually $p$ is set to be $\frac{3}{4}$. In our analysis, however, we will allow an arbitrary $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. The choice of the upper bound $\frac{8}{\pi^{2}}=0.81 \ldots$ will be clear from the analysis of the QS algorithm. The QS algorithm outputs $\bar{a}_{f}(j)$ with probability $p_{f}(j)$ for $j=0,1, \ldots, M-1$, see Theorem 2 where the $p_{j}(f)$ 's are given. Its worst-probabilistic error is formally defined as the smallest error bound that holds for all Boolean function with probability at least $p$, i.e.,

$$
e^{\text {wor }-\operatorname{pro}}(M, p)=\inf \left\{\alpha: \sum_{j:\left|a_{f}-\bar{a}_{f}(j)\right| \leq \alpha} p_{f}(j) \geq p \quad \forall f \in \mathbb{B}_{N}\right\}
$$

It is easy to see that $e^{\text {wor-pro }}(M, p)$ can be rewritten as follows. Let $A \subset\{0,1, \ldots, M-1\}$. For $f \in \mathbb{B}_{N}$ define the measure of $A$ as

$$
\mu(A, f)=\sum_{j \in A} p_{f}(j)
$$

Then

$$
\begin{equation*}
e^{\mathrm{wor}-\mathrm{pro}}(M, p)=\max _{f \in \mathbb{B}_{N}} \min _{A: \mu(A, f) \geq p} \max _{j \in A}\left|a_{f}-\bar{a}_{f}(j)\right| . \tag{5}
\end{equation*}
$$

## Average-Probabilistic Error

The worst-probabilistic error $e^{\text {wor-pro }}(M, p)$ of the $\mathbf{Q S}$ algorithm is defined by the worst performance with respect to Boolean functions. It is also natural to consider the average performance of the QS algorithm with respect to Boolean functions. Let $\mathbf{p}$ be a probability measure on the set $\mathbb{B}_{N}$. That is, any Boolean function $f \in \mathbb{B}_{N}$ occurs with probability $\mathbf{p}(f)$. Obviously, $\mathbf{p}(f) \geq 0$ and $\sum_{f \in \mathbb{B}_{N}} \mathbf{p}(f)=1$. The average-probabilistic error is defined by replacing the first max in (5) by the expectation, i.e.,

$$
e^{\operatorname{avg}-\operatorname{pro}}(M, p)=\sum_{f \in \mathbb{B}_{N}} \mathbf{p}(f) \min _{A: \mu(A, f) \geq p} \max _{j \in A}\left|a_{f}-\bar{a}_{f}(j)\right|,
$$

Hence, we are interested in the average error that holds with a certain fixed probability.

### 3.1 Worst-Probabilistic Error

We begin by citing a theorem from [1] for which we will propose a number of improvements.
Theorem 1. [1] For any Boolean function $f \in \mathbb{B}_{N}$, the $\boldsymbol{Q S}$ algorithm uses exactly $M-1$ quantum queries and outputs $\bar{a}$ that approximates $a=a_{f}$ such that

$$
|\bar{a}-a| \leq \frac{2 \pi}{M} \sqrt{a(1-a)}+\frac{\pi^{2}}{M^{2}} \leq \frac{\pi}{M}+\frac{\pi^{2}}{M^{2}}
$$

with probability at least $\frac{8}{\pi^{2}}=0.81 \ldots$. Hence,

$$
\sum_{j:\left|\bar{a}_{f}(j)-a_{f}\right| \leq(2 \pi / M) \sqrt{a_{f}\left(1-a_{f}\right)}+\pi^{2} / M^{2}} p_{f}(j) \geq \frac{8}{\pi^{2}} \quad \forall f \in \mathbb{B}_{N}
$$

and, therefore,

$$
e^{\mathrm{wor}-\mathrm{pro}}\left(M, \frac{8}{\pi^{2}}\right) \leq \frac{\pi}{M}+\frac{\pi^{2}}{M^{2}}
$$

Using proof ideas of Theorem 1 from [1] we present the following theorem and the subsequent corollaries.

Theorem 2. For any Boolean function $f \in \mathbb{B}_{N}$, denote

$$
\sigma_{a}=\sigma_{a_{f}}=\frac{M}{\pi} \arcsin \sqrt{a} \in\left[0, \frac{1}{2} M\right]
$$

1. The $\boldsymbol{Q S}$ algorithm uses exactly $M-1$ quantum queries, and $\log _{2} N+\left\lceil\log _{2} M\right\rceil$ qubits.
2. For $j=0,1, \ldots, M-1$, the outcome $j$ of the $\boldsymbol{Q S}$ algorithm occurs with probability

$$
\begin{equation*}
p_{f}(j)=\frac{\sin ^{2}\left(\pi\left(j-\sigma_{a_{f}}\right)\right)}{2 M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(j-\sigma_{a_{f}}\right)\right)}\left(1+\frac{\sin ^{2}\left(\pi\left(j-\sigma_{a_{f}}\right) / M\right)}{\sin ^{2}\left(\pi\left(j+\sigma_{a_{f}}\right) / M\right)}\right) \tag{6}
\end{equation*}
$$

(If $\sin \left(\pi\left(j \pm \sigma_{a_{f}}\right) / M\right)=0$ we need to apply the limiting value of the formula above.) For $j=M, M+1, \ldots, 2^{\left\lceil\log _{2} M\right\rceil}-1$, the outcome $j$ occurs with probability 0 .
3. If $\sigma_{a_{f}}$ is an integer then the $\boldsymbol{Q S}$ algorithm outputs the exact value of $a_{f}$ with probability 1. This holds iff $a_{f}=\sin ^{2}(k \pi / M)$ for some integer $k \in\left[0, \frac{1}{2} M\right]$. In particular, this holds for $a_{f}=0$, for $a_{f}=1$ and even $M$, and for $a_{f}=\frac{1}{2}$ and $M$ divisible by 4 .
4. Let $\bar{x}=\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right) / M$ and $\underline{x}=\pi\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right) / M$. If $\sigma_{a_{f}}$ is not an integer then the $\boldsymbol{Q S}$ algorithm outputs the same value $\bar{a}=\bar{a}_{f}\left(\left\lceil\sigma_{a}\right\rceil\right)=\bar{a}_{f}\left(M-\left\lceil\sigma_{a}\right\rceil\right)$ for the outcomes $\left\lceil\sigma_{a}\right\rceil$ and $M-\left\lceil\sigma_{a}\right\rceil$ such that

$$
\begin{align*}
|\bar{a}-a| & =|\sin (\bar{x})(2 \sqrt{a(1-a)} \cos (\bar{x})+(1-2 a) \sin (\bar{x}))| \\
& \leq \frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right) \tag{7}
\end{align*}
$$

with probability

$$
\begin{align*}
& \frac{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}\left(1+\left(1-\delta_{\left\lceil\sigma_{a}\right\rceil, M / 2}\right) \frac{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right) / M\right)}{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil+\sigma_{a}\right) / M\right)}\right) \\
\geq & \frac{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}{\pi^{2}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)^{2}}=1-\frac{\pi^{2}}{3}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)^{2}+O\left(\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)^{4}\right), \tag{8}
\end{align*}
$$

and outputs the same value $\bar{a}=\bar{a}_{f}\left(\left\lfloor\sigma_{a}\right\rfloor\right)=\bar{a}_{f}\left(\left(1-\delta_{\left\lfloor\sigma_{a}\right\rfloor, 0}\right) M-\left\lfloor\sigma_{a}\right\rfloor\right)$ for the outcomes $\left\lfloor\sigma_{a}\right\rfloor$ and $\left(1-\delta_{\left\lfloor\sigma_{a}\right\rfloor, 0}\right) M-\left\lfloor\sigma_{a}\right\rfloor$ such that

$$
\begin{align*}
|\bar{a}-a| & =|\sin (\underline{x})(2 \sqrt{a(1-a)} \cos (\underline{x})+(1-2 a) \sin (\underline{x}))| \\
& \leq \frac{\pi}{M}\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right) \tag{9}
\end{align*}
$$

with probability

$$
\begin{align*}
& \frac{\sin ^{2}\left(\pi\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)}\left(1+\left(1-\delta_{\left\lfloor\sigma_{a}\right\rfloor, 0}\right) \frac{\sin ^{2}\left(\pi\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right) / M\right)}{\sin ^{2}\left(\pi\left(\sigma_{a}+\left\lfloor\sigma_{a}\right\rfloor\right) / M\right)}\right) \\
\geq & \frac{\sin ^{2}\left(\pi\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)}{\pi^{2}\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)^{2}}=1-\frac{\pi^{2}}{3}\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)^{2}+O\left(\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)^{4}\right) . \tag{10}
\end{align*}
$$

Proof. As before, let $\theta_{a}=\arcsin \sqrt{a}$ and

$$
\left|S_{M}(\omega)\right\rangle=\frac{1}{\sqrt{M}} \sum_{k=0}^{M-1} e^{2 \pi i \omega k}|k\rangle, \quad i=\sqrt{-1}
$$

for arbitrary $\omega \in \mathbb{R}$. Note that

$$
F_{M, m}|j\rangle= \begin{cases}\left|S_{M}(j / M)\right\rangle & \text { for } j=0,1, \ldots, M-1, \\ |j\rangle & \text { for } j=M, M+1, \ldots, 2^{m}-1\end{cases}
$$

The steps $1-3$ of the $\mathbf{Q S}$ algorithm are equivalent to the application of the operator $\left(F_{M, m}^{-1} \otimes\right.$ I) $\Lambda_{m}\left(Q_{f}\right) F_{M, m} \otimes W_{N}$ to the state $|0\rangle|0\rangle \in \mathcal{H}_{m} \otimes \mathcal{H}_{n}$. Then $\left|\eta_{1}\right\rangle$ can be written as $M^{-1 / 2} \sum_{j=0}^{M-1}|j\rangle|\psi\rangle$, and $|\psi\rangle=W_{N}|0\rangle$ is given by (4). Hence

$$
\left|\eta_{1}\right\rangle=\frac{-i}{\sqrt{2 M}} \sum_{j=0}^{M-1}|j\rangle\left(e^{i \theta_{a}}\left|\psi_{+}\right\rangle-e^{-i \theta_{a}}\left|\psi_{-}\right\rangle\right)
$$

Applying $\Lambda_{m}\left(Q_{f}\right)$ in Step 2 and remembering that $Q_{f}^{j}\left|\psi_{ \pm}\right\rangle=\lambda_{ \pm}^{j}\left|\psi_{ \pm}\right\rangle$, we obtain

$$
\begin{aligned}
\left|\eta_{2}\right\rangle & =\Lambda_{m}\left(Q_{f}\right)\left|\eta_{1}\right\rangle=\frac{-i}{\sqrt{2 M}} \sum_{j=0}^{M-1}|j\rangle\left(e^{2 i j \theta_{a}} e^{i \theta_{a}}\left|\psi_{+}\right\rangle-e^{-2 i j \theta_{a}} e^{-i \theta_{a}}\left|\psi_{-}\right\rangle\right) \\
& =\frac{-i}{\sqrt{2}}\left(e^{i \theta_{a}}\left|S_{M}\left(\sigma_{a} / M\right)\right\rangle\left|\psi_{+}\right\rangle-e^{-i \theta_{a}}\left|S_{M}\left(-\sigma_{a} / M\right)\right\rangle\left|\psi_{-}\right\rangle\right)
\end{aligned}
$$

Since $j=0,1, \ldots, M-1$, the largest power of $Q_{f}$ is $M-1$. Hence, we use exactly $M-1$ quantum queries to compute $\left|\eta_{2}\right\rangle$. The remaining steps of the QS algorithm do not use quantum queries. This means that the total number of quantum queries used by the QS algorithm is $M-1$, and obviously we are using $n+m$ qubits. This proves the first part of Theorem 2,

Step 3 yields the state

$$
\left|\eta_{3}\right\rangle=\left(F_{M, m}^{-1} \otimes I\right)\left|\eta_{2}\right\rangle=\frac{-i}{\sqrt{2}}\left(e^{i \theta_{a}} F_{M, m}^{-1}\left|S_{M}\left(\sigma_{a} / M\right)\right\rangle\left|\psi_{+}\right\rangle-e^{-i \theta_{a}} F_{M, m}^{-1}\left|S_{M}\left(-\sigma_{a} / M\right)\right\rangle\left|\psi_{-}\right\rangle\right) .
$$

We are ready to analyze the probability of the outcome $j$ of the $\mathbf{Q S}$ algorithm. Observe that

$$
\begin{aligned}
\left|\alpha_{ \pm}\right\rangle & :=(|j\rangle\langle j| \otimes I) F_{M, m}^{-1}\left|S_{M}\left( \pm \sigma_{a} / M\right)\right\rangle\left|\psi_{ \pm}\right\rangle \\
& =\langle j| F_{M, m}^{-1}\left|S_{M}\left( \pm \sigma_{a} / M\right)\right\rangle|j\rangle\left|\psi_{ \pm}\right\rangle \\
& = \begin{cases}\left\langle S_{M}(j / M) \mid S_{M}\left( \pm \sigma_{a} / M\right)\right\rangle|j\rangle\left|\psi_{ \pm}\right\rangle & \text {for } j=0,1, \ldots, M-1, \\
0 & \text { for } j=M, M+1, \ldots, 2^{m}-1,\end{cases}
\end{aligned}
$$

and therefore

$$
\left\langle\alpha_{ \pm} \mid \alpha_{ \pm}\right\rangle= \begin{cases}\left|\left\langle S_{M}(j / M) \mid S_{M}\left( \pm \sigma_{a} / M\right)\right\rangle\right|^{2}\left\langle\psi_{ \pm} \mid \psi_{ \pm}\right\rangle & \text {for } j=0,1, \ldots, M-1 \\ 0 & \text { for } j=M, M+1, \ldots, 2^{m}-1\end{cases}
$$

Observe that for $a \in(0,1)$, we have $\left\langle\psi_{ \pm} \mid \psi_{ \pm}\right\rangle=1$, whereas for $a \in\{0,1\}$, we have $\left\langle\psi_{+} \mid \psi_{+}\right\rangle=2$ and $\left\langle\psi_{-} \mid \psi_{-}\right\rangle=0$.

For $\omega_{1}, \omega_{2} \in \mathbb{R}$ we have

$$
\begin{aligned}
\left|\left\langle S_{M}\left(\omega_{1}\right) \mid S_{M}\left(\omega_{2}\right)\right\rangle\right|^{2} & =\left|\left(\frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{-2 \pi i \omega_{1} j}\langle j|\right)\left(\frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} e^{2 \pi i \omega_{2} j}|j\rangle\right)\right|^{2} \\
& =\frac{1}{M^{2}}\left|\sum_{j=0}^{M-1} e^{-2 \pi i\left(\omega_{1}-\omega_{2}\right) j}\right|^{2}
\end{aligned}
$$

If $\omega_{1}-\omega_{2}$ is an integer then the last sum is clearly $M$, and the whole expression is 1 . If $\omega_{1}-\omega_{2}$ is not an integer then

$$
\frac{1}{M} \sum_{j=0}^{M-1} e^{-2 \pi i\left(\omega_{1}-\omega_{2}\right) j}=\frac{e^{-2 \pi i M\left(\omega_{1}-\omega_{2}\right)}-1}{M\left(e^{-2 \pi i\left(\omega_{1}-\omega_{2}\right)}-1\right)},
$$

which holds for all $\omega_{1}, \omega_{2} \in \mathbb{R}$ if we take $0 / 0$ as 1 . Therefore

$$
\left|\frac{1}{M} \sum_{j=0}^{M-1} e^{-2 \pi i\left(\omega_{1}-\omega_{2}\right) j}\right|^{2}=\frac{1-\cos \left(2 \pi M\left(\omega_{1}-\omega_{2}\right)\right)}{M^{2}\left(1-\cos \left(2 \pi\left(\omega_{1}-\omega_{2}\right)\right)\right)}=\frac{\sin ^{2}\left(\pi M\left(\omega_{1}-\omega_{2}\right)\right)}{M^{2}\left(\sin ^{2}\left(\pi\left(\omega_{1}-\omega_{2}\right)\right)\right)}
$$

Hence

$$
\begin{equation*}
\left|\left\langle S_{M}\left(\omega_{1}\right) \mid S_{M}\left(\omega_{2}\right)\right\rangle\right|^{2}=\frac{\sin ^{2}\left(M \pi\left(\omega_{1}-\omega_{2}\right)\right)}{M^{2} \sin ^{2}\left(\pi\left(\omega_{1}-\omega_{2}\right)\right)} \tag{11}
\end{equation*}
$$

which holds for all $\omega_{1}, \omega_{2} \in \mathbb{R}$ if we take $0 / 0$ as 1 . Applying this we conclude that

$$
\left\langle\alpha_{ \pm} \mid \alpha_{ \pm}\right\rangle= \begin{cases}\frac{\sin ^{2}\left(\pi\left(j \neq \sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\pi\left(j \mp \sigma_{a}\right) / M\right)}\left\langle\psi_{ \pm} \mid \psi_{ \pm}\right\rangle & \text {for } j=0,1, \ldots, M-1, \\ 0 & \text { for } j=M, M+1, \ldots, 2^{m}-1\end{cases}
$$

The outcome $j$ occurs after the measurement and the state $\left|\eta_{3}\right\rangle$ collapses to the state $\left(\left\langle\eta_{3}\right| M_{j}^{\dagger} M_{j}\left|\eta_{3}\right\rangle\right)^{-1} M_{j}\left|\eta_{3}\right\rangle$, where $M_{j}=|j\rangle\langle j| \otimes I$. For $j=0,1, \ldots, M-1$, we have

$$
\begin{aligned}
M_{j}\left|\eta_{3}\right\rangle=|j\rangle\left(\frac { - i } { \sqrt { 2 } } \left(e^{i \theta_{a}}\left\langle S_{M}(j / M) \mid S_{M}\left(\sigma_{a} / M\right)\right\rangle\left|\psi_{+}\right\rangle\right.\right. & \\
& \left.\left.-e^{-i \theta_{a}}\left\langle S_{M}(j / M) \mid S_{M}\left(-\sigma_{a} / M\right)\right\rangle\left|\psi_{-}\right\rangle\right)\right),
\end{aligned}
$$

whereas $M_{j}\left|\eta_{3}\right\rangle=0$ for $j=M, M+1, \ldots, 2^{m}-1$. Since $\left|\psi_{+}\right\rangle$and $\left|\psi_{-}\right\rangle$are orthogonal we have

$$
\left\langle\eta_{3}\right| M_{j}^{\dagger} M_{j}\left|\eta_{3}\right\rangle=\frac{1}{2}\left(\left\langle\alpha_{+} \mid \alpha_{+}\right\rangle+\left\langle\alpha_{-} \mid \alpha_{-}\right\rangle\right) .
$$

Hence, the outcome $j, j=0,1, \ldots, M-1$, occurs with probability

$$
\begin{equation*}
p_{f}(j)=\frac{1}{2}\left(\frac{\sin ^{2}\left(\pi\left(j-\sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\pi\left(j-\sigma_{a}\right) / M\right)}+\frac{\sin ^{2}\left(\pi\left(j+\sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\pi\left(j+\sigma_{a}\right) / M\right)}\right) \tag{12}
\end{equation*}
$$

Indeed, for $a \in(0,1)$, we have $\left\langle\psi_{ \pm} \mid \psi_{ \pm}\right\rangle=1$ and (12) follows from the form of $\left\langle\alpha_{ \pm} \mid \alpha_{ \pm}\right\rangle$. For $a \in\{0,1\}$, we have $\left\langle\psi_{+} \mid \psi_{+}\right\rangle=2$ and $\left\langle\psi_{-} \mid \psi_{-}\right\rangle=0$. Since the two terms in (12) are now the same, the formula for $\left\langle\alpha_{+} \mid \alpha_{+}\right\rangle$again yields (12).

Since $\sin ^{2}\left(\pi\left(j-\sigma_{a}\right)\right)=\sin ^{2}\left(\pi\left(j+\sigma_{a}\right)\right)$, the last formula is equivalent to (6). Obviously for $j=M, M+1, \ldots, 2^{m}-1$, the probability of the outcome $j$ is zero. This proves the second part of Theorem 2.

Assume now that $\sigma_{a} \in \mathbb{Z}$. If $\sigma_{a}=0$ or $\sigma_{a}=\frac{1}{2} M$ (if $M$ is even) then the probability $p_{f}\left(\sigma_{a}\right)$ of the outcome $\sigma_{a}$ is 1 . For $\sigma_{a}=0$ we have $a=0$ and the output is $\bar{a}_{f}(0)=0$. For $\sigma_{a}=\frac{1}{2} M$ we have $a=1$ and the output is $\bar{a}_{f}\left(\frac{1}{2} M\right)=1$. Hence, in both cases the QS algorithm outputs the exact value with probability 1.

If $\sigma_{a} \in \mathbb{Z}$ and $\sigma_{a} \notin\left\{0, \frac{1}{2} M\right\}$ then the probability of the distinct outcomes $\sigma_{a}$ and $M-\sigma_{a}$ is $\frac{1}{2}$. These two values of the outcomes yield the same output

$$
\sin ^{2}\left(\pi \sigma_{a} / M\right)=\sin ^{2}\left(\pi\left(M-\sigma_{a}\right) / M\right)=a
$$

Hence, the QS algorithm outputs the exact value with probability 1. This proves the third part of Theorem 2.

We now turn to the case when $\sigma_{a} \notin \mathbb{Z}$. It is easy to check that the third part of Theorem 2 holds for $M=1$. Assume then that $M \geq 2$ which implies that $\left\lceil\frac{1}{2} M\right\rceil \leq M-1$. Since $\sigma_{a}$ is not an integer, we have $\left\lceil\sigma_{a}\right\rceil \geq 1,\left\lceil\sigma_{a}\right\rceil \leq\left\lceil\frac{1}{2} M\right\rceil \leq M-1$ and $M-\left\lceil\sigma_{a}\right\rceil \leq M-1$. This means that both $\left\lceil\sigma_{a}\right\rceil$ and $M-\left\lceil\sigma_{a}\right\rceil$ may be the outcomes of the QS algorithm. Obviously, these two outcomes are different iff $\left\lceil\sigma_{a}\right\rceil \neq \frac{1}{2} M$. Similarly, both $\left\lfloor\sigma_{a}\right\rfloor$ and $\left(1-\delta_{\left\lfloor\sigma_{a}\right\rfloor, 0}\right) M-\left\lfloor\sigma_{a}\right\rfloor$ may be also the outcomes. They are different iff $\left\lfloor\sigma_{a}\right\rfloor \neq 0$.

We show that the outputs for the outcomes $\left\lceil\sigma_{a}\right\rceil$ and $\left\lfloor\sigma_{a}\right\rfloor$ satisfy (7) and (19) with probabilities (8) and (10), respectively. We focus on the output for the outcome $\left\lceil\sigma_{a}\right\rceil$ and its probability. The proof for the outcome $\left\lfloor\sigma_{a}\right\rfloor$ is similar.

We estimate the error of the QS algorithm for the output $\bar{a}=\sin ^{2}\left(\pi\left\lceil\sigma_{a}\right\rceil / M\right)$. Recall that $\bar{x}=\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right) / M$. We have

$$
\begin{aligned}
|\bar{a}-a| & =\left|\sin ^{2}\left(\pi\left\lceil\sigma_{a}\right\rceil / M\right)-\sin ^{2}\left(\pi \sigma_{a} / M\right)\right|=\left|\sin (\bar{x}) \sin \left(\bar{x}+2 \pi \sigma_{a} / M\right)\right| \\
& =\left|\sin (\bar{x})\left(\sin \left(2 \pi \sigma_{a} / M\right) \cos (\bar{x})+\cos \left(2 \pi \sigma_{a} / M\right) \sin (\bar{x})\right)\right| \\
& \leq \pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right) / M .
\end{aligned}
$$

Since $\sin \left(2 \pi \sigma_{a} / M\right)=2 \sqrt{a(1-a)}$ and $\cos \left(2 \pi \sigma_{a} / M\right)=1-2 a$, this proves the estimate of the error of the QS algorithm in the fourth part of Theorem [2,

We find the probability of the output $\bar{a}$. Since $\sin ^{2}(\pi t / M)$ is injective for $t \in\left[0, \frac{1}{2} M\right]$, the output $\bar{a}$ occurs only for the outcomes $\left\lceil\sigma_{a}\right\rceil$ and $M-\left\lceil\sigma_{a}\right\rceil$. If $\left\lceil\sigma_{a}\right\rceil=\frac{1}{2} M$ then these two outcomes are the same and $\bar{a}$ occurs with probability $p_{f}\left(\frac{1}{2} M\right)$. Due to (12)

$$
p_{f}\left(\frac{1}{2} M\right)=\frac{\sin ^{2}\left(\pi\left(\frac{1}{2} M-\sigma_{a_{f}}\right)\right)}{M^{2} \sin ^{2}\left(\pi\left(\frac{1}{2} M-\sigma_{a_{f}}\right) / M\right)}
$$

which agrees with the claim in Theorem 2,
If $\left\lceil\sigma_{a}\right\rceil \neq \frac{1}{2} M$ then $\left\lceil\sigma_{a}\right\rceil \neq M-\left\lceil\sigma_{a}\right\rceil$ and $\bar{a}$ occurs for exactly two distinct outcomes. The probability of $\bar{a}$ is now equal to the sum of the probabilities $p_{f}\left(\left\lceil\sigma_{a}\right\rceil\right)+p_{f}\left(M-\left\lceil\sigma_{a}\right\rceil\right)$ with $p_{f}$ 's given by (12). Since both terms are equal, the probability of $\bar{a}$ is $2 p_{f}\left(\left\lceil\sigma_{a}\right\rceil\right)$ which also agrees with the claim in Theorem [2. Since $\sin \left(\frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right) \leq \frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)$ we have

$$
\frac{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)} \geq \frac{\sin ^{2}\left(\pi\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)}{\pi^{2}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)^{2}}
$$

We finish proving (8) using the standard expansion of the sine. This completes the proof.

Based on Theorem [2 we present simplified estimates of the error of the QS algorithm and of the corresponding probability.

Corollary 1. The $\boldsymbol{Q S}$ algorithm outputs $\bar{a}$ such that

$$
\begin{equation*}
|\bar{a}-a| \leq \frac{\pi}{M} \max \left\{\left\lceil\sigma_{a}\right\rceil-\sigma_{a}, \sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right\} \tag{13}
\end{equation*}
$$

with probability at least

$$
\begin{equation*}
\frac{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}+\frac{\sin ^{2}\left(\pi\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)} \geq \frac{8}{\pi^{2}} . \tag{14}
\end{equation*}
$$

Proof. It is enough to prove Corollary $\mathbb{T}^{\text {if }} \sigma_{a}$ is not an integer. The estimate of the error of the QS algorithm by the maximum of the estimates (7) and (9) holds with probability that is the sum of the probabilities (8) and (10). Moreover, $\left\lceil\sigma_{a}\right\rceil-\sigma_{a}=1-\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)$. It now suffices to observe that the function

$$
g(\Delta)=\frac{\sin ^{2}(\pi \Delta)}{\pi^{2} \Delta^{2}}+\frac{\sin ^{2}(\pi(1-\Delta))}{\pi^{2}(1-\Delta)^{2}}
$$

is a lower bound of the left hand side of (14) with $\Delta=\left\lceil\sigma_{a}\right\rceil-\sigma_{a}$, and attains the minimum $\frac{8}{\pi^{2}}$ on the interval $[0,1]$ for $\Delta=\frac{1}{2}$, see also [1].

Corollary $\mathbb{1}$ guarantees high probability of the estimate (13). Unfortunately this estimate does not preserve the continuity of the estimates (7) and (9) with respect to $\left\lceil\sigma_{a}\right\rceil-\sigma_{a}$ and $\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor$. The continuity of the estimates will be present in the next corollary at the expense of the probability of the outcome. This corollary will also play an essential role in the study of the average-probabilistic error of the QS algorithm.

Corollary 2. The $\boldsymbol{Q S}$ algorithm outputs $\bar{a}$ such that

$$
\begin{equation*}
|\bar{a}-a| \leq \frac{\pi}{M} \min \left\{\left\lceil\sigma_{a}\right\rceil-\sigma_{a}, \sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right\} \tag{15}
\end{equation*}
$$

with probability at least

$$
\begin{equation*}
\max \left\{\frac{\sin ^{2}\left(\pi\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)}, \frac{\sin ^{2}\left(\pi\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)}{M^{2} \sin ^{2}\left(\frac{\pi}{M}\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right)}\right\} \geq \frac{4}{\pi^{2}} \tag{16}
\end{equation*}
$$

Proof. We may again assume that $\sigma_{a}$ is not an integer. Let us define

$$
w(\Delta)=\frac{\sin ^{2}(\pi \Delta)}{M^{2} \sin ^{2}\left(\frac{\pi}{M} \Delta\right)} \quad \text { for } \quad \Delta \in[0,1]
$$

Then $w\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)$ is the probability of (7) and $w\left(1-\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)$ is the probability of (91). For $\Delta \in\left[0, \frac{1}{2}\right]$, note that $w(\cdot)$ is decreasing, and $w(1-\cdot)$ is increasing. Therefore

$$
w(\Delta) \geq w\left(\frac{1}{2}\right) \geq w(1-\Delta) \quad \text { for } \quad \Delta \in\left[0, \frac{1}{2}\right]
$$

Let $\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \leq \sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor$. Then $\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \leq \frac{1}{2}$. In this case (15) is equivalent to (7) and holds with probability at least $w\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)$, which corresponds to (16). Analogously, if
$\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \geq \sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor$ then $\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \geq \frac{1}{2}$. In this case (15) is equivalent to (9) and holds with probability at least $w\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)$, which also corresponds to (16). Finally, note that

$$
\max \left\{w\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right), w\left(\sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right)\right\}
$$

is minimal for $\left\lceil\sigma_{a}\right\rceil-\sigma_{a}=\frac{1}{2}$ and is equal to

$$
\frac{1}{M^{2}} \sin ^{-2} \frac{\pi}{2 M} \geq \frac{4}{\pi^{2}}
$$

Unfortunately for $\left\lceil\sigma_{a}\right\rceil-\sigma_{a}$ close to $\frac{1}{2}$ the probability of the estimate (15) is too small. However, in this case we may use Corollary

We now turn to global error estimates, that is, estimates independent of $a$. Theorem 1 of [1] states, in particular, that $|\bar{a}-a| \leq \pi / M+\pi^{2} / M^{2}$ with probability at least $\frac{8}{\pi^{2}}$. We now improve this estimate by combining the estimates (13) and (15).

Corollary 3. The $\boldsymbol{Q S}$ algorithm outputs $\bar{a}$ such that

$$
\begin{equation*}
|\bar{a}-a| \leq \frac{3}{4} \frac{\pi}{M} \tag{17}
\end{equation*}
$$

with probability at least $\frac{8}{\pi^{2}}$. That is,

$$
e^{\text {wor }-\mathrm{pro}}\left(M, \frac{8}{\pi^{2}}\right) \leq \frac{3}{4} \frac{\pi}{M} .
$$

Proof. Let us define

$$
\begin{equation*}
h(\Delta)=\max \left\{\frac{\sin ^{2}(\pi \Delta)}{\pi^{2} \Delta^{2}}, \frac{\sin ^{2}(\pi(1-\Delta))}{\pi^{2}(1-\Delta)^{2}}\right\} . \tag{18}
\end{equation*}
$$

Clearly, $h\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)$ is a lower bound of the $\max \left\{w\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right), w\left(1-\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)\right)\right\}$ and therefore $h\left(\left\lceil\sigma_{a}\right\rceil-\sigma_{a}\right)$ is a lower bound of the probability of the output satisfying (15). We consider two cases.

Assume first that $\Delta=\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \in\left[0, \frac{1}{4}\right] \cup\left[\frac{3}{4}, 1\right]$. It is easy to see that then $h(\Delta) \geq \frac{8}{\pi^{2}}$ and the estimate (15) yields

$$
|\bar{a}-a| \leq \frac{\pi}{M} \min \left\{\left\lceil\sigma_{a}\right\rceil-\sigma_{a}, \sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right\} \leq \frac{1}{4} \frac{\pi}{M}
$$

with probability at least $\frac{8}{\pi^{2}}$.
Assume now that $\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \in\left(\frac{1}{4}, \frac{3}{4}\right)$. Then we can use the estimate (13), which holds unconditionally with probability at least $\frac{8}{\pi^{2}}$. In this case, we have

$$
|\bar{a}-a| \leq \frac{\pi}{M} \max \left\{\left\lceil\sigma_{a}\right\rceil-\sigma_{a}, \sigma_{a}-\left\lfloor\sigma_{a}\right\rfloor\right\} \leq \frac{3}{4} \frac{\pi}{M} .
$$

These two estimates combined together yield (17).

The obvious consequence of Corollary 3 is that for $M$ large enough we can compute the value of $a$ exactly by rounding the output.

Corollary 4. Assume that

$$
M>\frac{3 \pi}{2} N
$$

Then the rounding of the $\boldsymbol{Q S}$ algorithm output to the nearest number of the form $k / N$ yields the exact value of the sum a with probability at least $\frac{8}{\pi^{2}}$.

The proof of Corollary 3 may suggest that the constant $\frac{3}{4}$ in (17) can be decreased. Furthermore one may want to decrease the constant $\frac{3}{4}$ at the expense of decreasing the probability $\frac{8}{\pi^{2}}$. These points are addressed in the next corollary. We shall see that the constant $\frac{3}{4}$ may be lowered only by decreasing the probability.

Corollary 5. Define

$$
C(p)=\inf \left\{C:\left|\bar{a}_{f}-a_{f}\right| \leq C \frac{\pi}{M} \quad \forall f \in \mathbb{B}_{N} \text { with probability at least } p\right\}
$$

and

$$
v(\Delta)=\frac{\sin ^{2}(\pi \Delta)}{\pi^{2} \Delta^{2}} \quad \text { for } \Delta \in\left[\frac{1}{4}, \frac{1}{2}\right]
$$

Then

$$
C(p) \leq\left\{\begin{array}{lll}
\frac{1}{2} & \text { for } & p \in\left[0, \frac{4}{\pi^{2}}\right)  \tag{19}\\
1-v^{-1}(p) & \text { for } & p \in\left[\frac{4}{\pi^{2}}, \frac{8}{\pi^{2}}\right] \\
M / \pi & \text { for } & p \in\left(8 / \pi^{2}, 1\right]
\end{array}\right.
$$

Moreover, $1-v^{-1}(p) \in\left[\frac{1}{2}, \frac{3}{4}\right]$ and

$$
\begin{equation*}
\left|\frac{\pi^{2}}{16} p+\frac{1}{4}-\left(1-v^{-1}(p)\right)\right| \leq 0.0085 \quad \text { for } p \in\left[\frac{4}{\pi^{2}}, \frac{8}{\pi^{2}}\right] \tag{20}
\end{equation*}
$$

Proof. For $p \in\left[0, \frac{4}{\pi^{2}}\right)$, Corollary 5 is a consequence of Corollary 2 For $p \in\left(\frac{8}{\pi^{2}}, 1\right]$, Corollary 5 trivially holds since $|\bar{a}-a| \leq 1=(M / \pi) \pi / M$. For the remaining $p$ 's we use a proof technique similar to that of Corollary 3.

Let $p \in\left[\frac{4}{\pi^{2}}, \frac{8}{\pi^{2}}\right]$. It is easy to check that $v$ is decreasing and, therefore, $v^{-1}(p)$ is well defined and $v^{-1}(p) \in\left[\frac{1}{4}, \frac{1}{2}\right]$. We have to show that the estimate

$$
\begin{equation*}
|\bar{a}-a| \leq\left(1-v^{-1}(p)\right) \frac{\pi}{M} \tag{21}
\end{equation*}
$$

holds with probability at least $p$. We consider two cases.
Assume first that $\Delta=\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \in\left[0, v^{-1}(p)\right] \cup\left[1-v^{-1}(p), 1\right]$. Observe that the function $h$ defined in (18) can be rewritten as

$$
h(\Delta)=\max \{v(\Delta), v(1-\Delta)\}
$$

It is easy to see that in this case, $h(\Delta) \geq p$ and the estimate (15) yields

$$
|\bar{a}-a| \leq \frac{\pi}{M} \min \{\Delta, 1-\Delta\} \leq v^{-1}(p) \frac{\pi}{M} \leq\left(1-v^{-1}(p)\right) \frac{\pi}{M}
$$

with probability at least $p$.
Assume now that $\Delta=\left\lceil\sigma_{a}\right\rceil-\sigma_{a} \in\left(v^{-1}(p), 1-v^{-1}(p)\right)$. Then we can use the estimate (13), which holds unconditionally with probability at least $\frac{8}{\pi^{2}}>p$. In this case, we have

$$
|\bar{a}-a| \leq \frac{\pi}{M} \max \{\Delta, 1-\Delta\} \leq\left(1-v^{-1}(p)\right) \frac{\pi}{M}
$$

This proves (21).
We found the estimate (20) by numerical computations.


Figure 1: The estimate (19) of $C(p)$ for $p \in\left[0, \frac{8}{\pi^{2}}\right]$

From Figure 1 we see that the estimate (19) is almost linear on the interval $\left[\frac{4}{\pi^{2}}, \frac{8}{\pi^{2}}\right]$, which explains why the right hand side of the estimate (20) is small.


Figure 2: The function $v$ on $[0,1]$. The two horizontal lines show $\frac{4}{\pi^{2}}$ and $\frac{8}{\pi^{2}}$ levels. The part of the graph between the arrows shows that $v$ is almost linear.

We now find a sharp bound on the worst-probabilistic error of the QS algorithm. We show that for large $M$ and $N / M$ the bound obtained in Corollary 5 is optimal for $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. Theorem 3. For large $M$ and $N / M$, the worst-probabilistic error of the $\boldsymbol{Q} \boldsymbol{S}$ algorithm is given by

$$
e^{\mathrm{wor}-\mathrm{pro}}(M, p)=\left(1-v^{-1}(p)\right) \frac{\pi}{M}\left(1+O\left(M^{-1}\right)+O\left(M N^{-1}\right)\right) \quad \text { for } p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]
$$

Here, $v$ is as in Corollary , and $1-v^{-1}(p) \approx\left(\pi^{2} / 16\right) p+\frac{1}{4}$ by (20).
Proof. From Corollary [ it is enough to show a lower bound on the error. Define

$$
s_{1}=\sin ^{2}\left(\frac{\pi\left\lceil\frac{1}{4} M\right\rceil}{M}\right) \quad \text { and } \quad s_{2}=\sin ^{2}\left(\frac{\pi\left(1+\left\lceil\frac{1}{4} M\right\rceil\right)}{M}\right) .
$$

For large $M$, we have

$$
s_{i}=\frac{1}{2}+O\left(M^{-1}\right) \quad \text { and } \quad s_{2}-s_{1}=\sin \left(\frac{\pi}{M}\right) \sin \left(\frac{\pi\left(1+2\left\lceil\frac{1}{4} M\right\rceil\right)}{M}\right)=\frac{\left(1+O\left(M^{-1}\right)\right) \pi}{M}
$$

There exist two Boolean functions $f_{1}$ and $f_{2}$ with sums $a_{1}=a_{f_{1}}$ and $a_{2}=a_{f_{2}}$ such that

$$
\left|a_{i}-s_{i}\right| \leq N^{-1} \quad \text { for } \quad i=1,2
$$

Since $\sigma_{s_{i}}=\left\lceil\frac{1}{4} M\right\rceil+(i-1)$ and the derivative of $\sigma_{a}$ for $a=\frac{1}{2}$ is $M / \pi$, we have

$$
\sigma_{a_{1}}=\left\lceil\frac{1}{4} M\right\rceil+O\left(M N^{-1}\right) \quad \text { and } \quad \sigma_{a_{2}}=\left\lceil\frac{1}{4} M\right\rceil+1+O\left(M N^{-1}\right)
$$

Obviously, $a_{i}=k_{i} / N$ for some integers $k_{i}$ with $k_{1}<k_{2}$. Consider $\sigma_{x / N}$ for $x \in\left\{k_{1}, k_{1}+\right.$ $\left.1, \ldots, k_{2}\right\}$. Then $\sigma_{x / N}$ varies from $\sigma_{a_{1}}$ for $x=k_{1}$ to $\sigma_{a_{2}}$ for $x=k_{2}$. Since $v^{-1}(p) \in\left[\frac{1}{4}, \frac{1}{2}\right)$, for a positive and small $\eta$ (we finally let $\eta$ go to zero), we can choose $x=x_{\eta}$ such that for $a^{*}=x_{\eta} / N$ we have

$$
\sigma_{a^{*}}:=\left\lceil\frac{1}{4} M\right\rceil+v^{-1}(p)+\eta+O\left(M N^{-1}\right)
$$

For large $N / M$, we then have

$$
\begin{array}{rll}
\left\lfloor\sigma_{a^{*}}\right\rfloor=\left\lceil\frac{1}{4} M\right\rceil & \text { and } & \left\lceil\sigma_{\left.a^{*}\right\rceil}\right\rceil\left\lceil\frac{1}{4} M\right\rceil+1, \\
\sigma_{a^{*}}-\left\lfloor\sigma_{a^{*}}\right\rfloor=v^{-1}(p)+\eta+O(M / N) & \text { and } & \left\lceil\sigma_{a^{*}}\right\rceil-\sigma_{a^{*}}=1-v^{-1}(p)-\eta+O(M / N) .
\end{array}
$$

Let $\bar{a}_{1}^{*}$ denote the output for the outcome $\left\lceil\sigma_{a^{*}}\right\rceil$, and $\bar{a}_{2}^{*}$ for $\left\lfloor\sigma_{a^{*}}\right\rfloor$.
Due to (7) and (9) of Theorem [2 we have

$$
\begin{aligned}
\left|a^{*}-\bar{a}_{1}^{*}\right| & =\frac{\pi}{M}\left(1-v^{-1}(p)-\eta\right)\left(1+O\left(M^{-1}+M N^{-1}\right)\right) \\
\left|a^{*}-\bar{a}_{2}^{*}\right| & =\frac{\pi}{M}\left(v^{-1}(p)+\eta\right)\left(1+O\left(M^{-1}+M N^{-1}\right)\right)
\end{aligned}
$$

Let us write $1+o(1)$ for $1+O\left(\eta^{2}+M^{-1}+M N^{-1}\right)$. The probability of $\bar{a}_{2}^{*}$ is given by (101) and is now equal to

$$
\frac{\sin ^{2}\left(\pi\left(v^{-1}(p)+\eta\right)\right)}{\left(\pi\left(v^{-1}(p)+\eta\right)\right)^{2}}(1+o(1))=\frac{\sin ^{2}\left(\pi v^{-1}(p)\right)+\pi \eta \sin \left(2 \pi v^{-1}(p)\right)}{\pi^{2} v^{-1}(p)^{2}\left(1+2 \eta / v^{-1}(p)\right)}(1+o(1)) .
$$

Since $p=v\left(v^{-1}(p)\right)=\sin ^{2}\left(\pi v^{-1}(p)\right) /\left(\pi v^{-1}(p)\right)^{2}$, the probability of $\bar{a}_{2}^{*}$ is

$$
p\left(1-2 \eta\left(\frac{1}{v^{-1}(p)}-\pi \cot \left(\pi v^{-1}(p)\right)\right)\right)(1+o(1)) .
$$

Since $\cot (t)<1 / t$ for $t \in\left[\frac{1}{4} \pi, \frac{1}{2} \pi\right]$, we see that the probability of $\bar{a}_{2}^{*}$ is slightly less than $p$ for small $\eta$.

We are ready to find a lower bound on the worst-probabilistic error

$$
e^{\text {wor-pro }}(M, p)=\max _{f \in \mathbb{B}_{N}} \min _{A: \mu(A, f) \geq p} \max _{j \in A}\left|a_{f}-\bar{a}_{f}(j)\right|
$$

of the QS algorithm. Take the function $f$ that corresponds to $a^{*}$. We claimed that the error is minimized if $A=\left\{\left\lfloor\sigma_{a^{*}}\right\rfloor,\left\lceil\sigma_{a^{*}}\right\rceil\right\}$. Indeed, $\left\lfloor\sigma_{a^{*}}\right\rfloor$ must belong to $A$ since otherwise $\mu(A, f) \leq 1-p_{f}\left(\left\lfloor\sigma_{a^{*}}\right\rfloor\right)=1-p+o(1)<p$ for $p>\frac{1}{2}$. The probability of $\left\lfloor\sigma_{a^{*}}\right\rfloor$ is slightly less than $p$, and so the set $A$ must also contain some other outcome $j$. If $j=\left\lceil\sigma_{a^{*}}\right\rceil$ then the error bound is roughly $\left(1-v^{-1}(p)-\eta\right) \pi / M$, and the sum of the probabilities of the outputs for the outcome $\left\lfloor\sigma_{a^{*}}\right\rfloor$ and $\left\lceil\sigma_{a^{*}}\right\rceil$ is always at least $\frac{8}{\pi^{2}} \geq p$. On the other hand, if $\left\lceil\sigma_{a^{*}}\right\rceil$ does not belong to the set $A$ then any other outcome $j$ yields the output $\sin ^{2}(\pi j / M)$. Since $\sin ^{2}(\pi(j+1) / M)-\sin ^{2}(\pi j / M)=\sin (\pi / M) \sin (\pi(2 j+1) / M)$, the distribution of the outcomes around $\frac{1}{2}$ is a mesh with step size roughly $\pi / M$. Hence, if $j \neq\left\lceil\sigma_{a^{*}}\right\rceil$, the error is at least roughly $\left(1+v^{-1}(p)\right) \pi / M>\pi\left(1-v^{-1}(p)\right) / M$. Thus the choice $j=\left\lceil\sigma_{a^{*}}\right\rceil$ minimizes the error and for $\eta$ tending to zero, the error is roughly $\left(1-v^{-1}(p)\right) \pi / M$. This completes the proof.

From these results, it is obvious how to guarantee that the error of the QS algorithm is at most $\varepsilon$ with probability at least $p$. Since $|\bar{a}-a| \leq\left(1-v^{-1}(p)\right) \pi / M$ holds with probability $p$, it is enough to take $M \geq\left(1-v^{-1}(p)\right) \pi / \varepsilon$. Due to Theorem 3 this bound is sharp for small $\varepsilon$ and large $\varepsilon N$. We have

Corollary 6. For $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$, the algorithm $\boldsymbol{Q S}\left(f,\left\lceil\left(1-v^{-1}(p)\right) \pi / \varepsilon\right\rceil\right)$ computes $\bar{a}$ with the error $\varepsilon$ and probability at least $p$ with $\left\lceil\left(1-v^{-1}(p)\right) \pi / \varepsilon\right\rceil-1$ quantum queries. For small $\varepsilon$ and large $\varepsilon N$, the estimate of the number of quantum queries is sharp.

### 3.2 Average-Probabilistic Error

In this section we study the average performance of the QS algorithm with respect to some measure on the set $\mathbb{B}_{N}$ of all Boolean functions defined on the set $\{0, \ldots, N-1\}$. We consider two such measures. The first measure $\mathbf{p}_{1}$ is uniformly distributed on the set $\mathbb{B}_{N}$, i.e.,

$$
\mathbf{p}_{1}(f)=2^{-N} \quad \forall f \in \mathbb{B}_{N}
$$

The second measure $\mathbf{p}_{2}$ is uniformly distributed on the set of results, i.e.,

$$
\mathbf{p}_{2}(f)=\frac{1}{\binom{N}{k}(N+1)} \quad \text { if } \quad a_{f}=\frac{k}{N}
$$

For the average-probabilistic error we want to estimate

$$
e_{\mathbf{p}_{i}}^{\text {avg-pro }}(M, p)=\sum_{f \in \mathbb{B}_{N}} \mathbf{p}_{i}(f) \min _{A: \mu(A, f) \geq p} \max _{j \in A}\left|a_{f}-a_{f}(j)\right| \quad \text { for } i=1,2 .
$$

For the measures $\mathbf{p}_{i}$, the mean of the random variable $a_{f}$ is clearly $\frac{1}{2}$. However, their first (central) moments are very different. As we shall see, the moment for the measure $\mathbf{p}_{1}$ is small
since it is of order $N^{-1 / 2}$ whereas the moment for measure $\mathbf{p}_{2}$ is roughly $\frac{1}{4}$. Since the first moments are the same as the error of the constant algorithm $\bar{a}_{f}(j)=\frac{1}{2}$, we can achieve small error of order $N^{-1 / 2}$ for the measure $\mathbf{p}_{1}$ without any quantum queries, while this property is not true for the measure $\mathbf{p}_{2}$.

We now consider the measure $\mathbf{p}_{1}$. It is interesting to ask if the QS algorithm has the same property as the constant algorithm. We shall prove that this is indeed the case iff $M$ is divisible by 4 .

We compute the first moment or the error of the constant algorithm, which is

$$
\sum_{k=0}^{N} 2^{-N}\binom{N}{k}\left|\frac{1}{2}-\frac{k}{N}\right|
$$

We do it only for odd $N$ since the case of even $N$ is analogous. We have

$$
\begin{aligned}
\sum_{k=0}^{N} 2^{-N}\binom{N}{k}\left|\frac{1}{2}-\frac{k}{N}\right| & =2 \sum_{k=0}^{\lfloor N / 2\rfloor}\binom{N}{k}\left(\frac{1}{2}-\frac{k}{N}\right)=\sum_{k=0}^{\lfloor N / 2\rfloor}\binom{N}{k}-2 \sum_{k=0}^{\lfloor N / 2\rfloor-1}\binom{N-1}{k} \\
& =2^{N-1}-2 \frac{1}{2}\left(2^{N-1}-\binom{N-1}{(N-1) / 2}\right)=\binom{N-1}{(N-1) / 2} .
\end{aligned}
$$

Thus

$$
e_{\mathbf{p}_{1}}^{\text {avg }-\operatorname{pro}}(0,1)=\sum_{k=0}^{N} 2^{-N}\binom{N}{k}\left|\frac{1}{2}-\frac{k}{N}\right|= \begin{cases}2^{-N}\binom{N-1}{(N-1) / 2} & \text { if } N \text { is odd },  \tag{22}\\ 2^{-(N+1)}\binom{N}{N / 2} & \text { if } N \text { is even } .\end{cases}
$$

By Stirling's formula

$$
k!=\sqrt{2 \pi k}\left(\frac{k}{e}\right)^{k} e^{\theta_{k} / 12 k} \quad \text { for certain } \theta_{k} \in[0,1]
$$

we estimate the both binomial quantities in (22) by

$$
\frac{1}{\sqrt{2 \pi}} \frac{1}{\sqrt{N-1}} e^{1 /(12(N-1))}
$$

proving that

$$
\begin{equation*}
e_{\mathbf{p}_{1}}^{\operatorname{avg}-\operatorname{pro}}(0,1)=\sum_{k=0}^{N} 2^{-N}\binom{N}{k}\left|\frac{1}{2}-\frac{k}{N}\right|=\frac{1}{\sqrt{2 \pi}} \frac{1}{\sqrt{N}}(1+o(1)) \leq \frac{1}{\sqrt{2 \pi}} \frac{1}{\sqrt{N-1}} e^{1 /(12(N-1))} \tag{23}
\end{equation*}
$$

We are ready to analyze the average-probabilistic error of the QS algorithm.
Theorem 4. Assume that $M$ is divisible by 4 and let $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. Then the averageprobabilistic error of the $\boldsymbol{Q S}$ algorithms with respect to the measure $\mathbf{p}_{1}$ satisfies

$$
\begin{aligned}
e_{\mathbf{p}_{1}}^{\text {avg-pro }}(M, p) & \leq \min \left\{\frac{3}{4} \frac{\pi}{M}, \sqrt{\frac{3}{2 \pi}} \sqrt{1+\frac{\pi^{2}}{4 M^{2}}} \frac{1}{\sqrt{N-1}} e^{1 /(12(N-1))}\right\} \\
& \leq \frac{3}{4} \pi(1+o(1)) \min \left\{\frac{1}{M}, \frac{1}{\sqrt{N}}\right\}
\end{aligned}
$$

Proof. The estimate $e_{\mathbf{p}_{1}}^{\text {avg-pro }}(M, p) \leq e_{\mathbf{p}_{1}}^{\text {avg-pro }}\left(M, \frac{8}{\pi^{2}}\right) \leq e_{\mathbf{p}_{1}}^{\text {wor-pro }}\left(M, \frac{8}{\pi^{2}}\right)$ is obvious and, applying Corollary 3, we get

$$
e_{\mathbf{p}_{1}}^{\text {avg }-\operatorname{pro}}\left(M, \frac{8}{\pi^{2}}\right) \leq \frac{3}{4} \frac{\pi}{M} .
$$

As before denote $\sigma_{a}=\frac{M}{\pi} \arcsin \sqrt{a}$. Let $a=\frac{1}{2}+x$. We are interested in the behavior of $\sigma_{\frac{1}{2}+x}$ for $|x|<\frac{1}{2}$. Clearly $\sigma_{\frac{1}{2}}^{\pi}=\frac{1}{4} M$. Let $|x|<\frac{1}{2}$, By Taylor's theorem, we have

$$
\sigma_{\frac{1}{2}+x}=\frac{M}{4}+\frac{M}{\pi} \frac{x}{2 \sqrt{\left(1-\xi_{x}\right) \xi_{x}}} \quad \text { for } \quad \xi_{x} \in\left(\frac{1}{2}, \frac{1}{2}+x\right)
$$

and $2 \sqrt{\left(1-\xi_{x}\right) \xi_{x}} \geq \sqrt{1-4 x^{2}}$. Assume additionally that

$$
\frac{M}{\pi} \frac{|x|}{\sqrt{1-4 x^{2}}} \leq \frac{1}{4}
$$

which is equivalent to assuming that

$$
|x| \leq \frac{\pi}{\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}}
$$

Since $M$ is divisible by 4 then $\left\lfloor\sigma_{\frac{1}{2}+x}\right\rfloor=\frac{1}{4} M$ for $x \geq 0$, and $\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil=\frac{1}{4} M$ for $x \leq 0$. This yields

$$
\begin{equation*}
\min \left\{\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil-\sigma_{\frac{1}{2}+x}, \sigma_{\frac{1}{2}+x}-\left\lfloor\sigma_{\frac{1}{2}+x}\right\rfloor\right\} \leq \frac{M}{\pi} \frac{|x|}{\sqrt{1-4 x^{2}}} \tag{24}
\end{equation*}
$$

Observe that

$$
\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil-\sigma_{\frac{1}{2}+x} \in\left[0, \frac{1}{4}\right] \cup\left[\frac{3}{4}, 1\right] .
$$

Indeed, for $x \leq 0$ we have

$$
\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil-\sigma_{\frac{1}{2}+x}=\frac{M}{\pi} \frac{|x|}{2 \sqrt{\left(1-\xi_{x}\right) \xi_{x}}} \in\left[0, \frac{1}{4}\right]
$$

and for $x \geq 0$ we have

$$
\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil-\sigma_{\frac{1}{2}+x}=1-\frac{M}{\pi} \frac{|x|}{2 \sqrt{\left(1-\xi_{x}\right) \xi_{x}}} \in\left[1-\frac{1}{4}, 1\right]=\left[\frac{3}{4}, 1\right]
$$

as claimed.
Let $a=\frac{1}{2}+x$. By the proof of Corollary 3, the error of the QS algorithm satisfies

$$
|\bar{a}-a| \leq \frac{\pi}{M} \min \left\{\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil-\sigma_{\frac{1}{2}+x}, \sigma_{\frac{1}{2}+x}-\left\lfloor\sigma_{\frac{1}{2}+x}\right\rfloor\right\}
$$

and by (24), we have

$$
|\bar{a}-a| \leq \frac{|x|}{\sqrt{1-4 x^{2}}}
$$

We split the sum that defines $e_{\mathbf{p}_{1}}^{\text {avg-pro }}\left(M, \frac{8}{\pi^{2}}\right)$ into two sums. The first sum is for $f \in \mathbb{B}_{N}$ for which $a=a_{f}=\frac{1}{2}+x$ with $|x| \leq \pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}$ and the second sum is for $f$ for which $a=a_{f}=\frac{1}{2}+x$ with $|x|>\pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}$. For the first sum we estimate the error of the QS algorithm by $|x| / \sqrt{1-4 x^{2}}$ and for the second sum, by the worst-case error $3 \pi /(4 M)$. Hence we have

$$
\begin{aligned}
e_{\mathbf{p}_{1}}^{\text {avg-pro }}\left(M, \frac{8}{\pi^{2}}\right) \leq & \sum_{f:\left|a_{f}-\frac{1}{2}\right| \leq \pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} \mathbf{p}_{1}(f) \frac{\left|a_{f}-\frac{1}{2}\right|}{\sqrt{1-4\left(a_{f}-\frac{1}{2}\right)^{2}}} \\
& +\sum_{f:\left|a_{f}-\frac{1}{2}\right|>\pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} \mathbf{p}_{1}(f) \frac{3}{4} \frac{\pi}{M} .
\end{aligned}
$$

Since $a_{f}=k / N$ for some integer $k \in[0, N]$,

$$
\begin{aligned}
e_{\mathbf{p}_{1}}^{\operatorname{avg}-\operatorname{pro}}\left(M, \frac{8}{\pi^{2}}\right) & \leq \sum_{k:\left|k / N-\frac{1}{2}\right| \leq \pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k} \frac{\left|k / N-\frac{1}{2}\right|}{\sqrt{1-4\left(k / N-\frac{1}{2}\right)^{2}}} \\
& +\frac{3}{4} \frac{\pi}{M} \sum_{k:\left|k / N-\frac{1}{2}\right|>\pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k} .
\end{aligned}
$$

Since $1-4\left(k / N-\frac{1}{2}\right)^{2} \geq 1-\pi^{2} /\left(4 M^{2}\right) \geq \frac{3}{4}$, the first sum can be estimated as $\sum_{k:\left|k / N-\frac{1}{2}\right| \leq \pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k} \frac{\left|k / N-\frac{1}{2}\right|}{\sqrt{1-4\left(k / N-\frac{1}{2}\right)^{2}}}$

$$
\leq \frac{2}{\sqrt{3}} \sum_{k:\left|k / N-\frac{1}{2}\right| \leq \pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k}\left|\frac{1}{2}-\frac{k}{N}\right| .
$$

The second sum can be estimated by

$$
\begin{aligned}
\frac{3}{4} \frac{\pi}{M} \sum_{k:\left|k / N-\frac{1}{2}\right|>\pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k} \frac{\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}}{\pi}\left|\frac{k}{N}-\frac{1}{2}\right| \\
\leq 3 \sqrt{1+\frac{\pi^{2}}{4 M^{2}}} \sum_{k:\left|k / N-\frac{1}{2}\right|>\pi /\left(16 M^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k}\left|\frac{k}{N}-\frac{1}{2}\right|
\end{aligned}
$$

Adding the estimates of these two sums we obtain

$$
e_{\mathbf{p}_{1}}^{\text {avg-pro }}\left(M, \frac{8}{\pi^{2}}\right) \leq 3 \sqrt{1+\frac{\pi^{2}}{4 M^{2}}} \sum_{k=0}^{N} 2^{-N}\binom{N}{k}\left|\frac{1}{2}-\frac{k}{N}\right| .
$$

The last sum is given by (22) and estimated by (23). Hence

$$
e_{\mathbf{p}_{1}}^{\text {avg-pro }}\left(M, \frac{8}{\pi^{2}}\right) \leq \sqrt{\frac{3}{2 \pi}} \sqrt{1+\frac{\pi^{2}}{4 M^{2}}} \frac{1}{\sqrt{N-1}} e^{1 /(12(N-1))}
$$

which completes the proof.

In the next theorem we consider the case when $M$ is not divisible by 4 .

Theorem 5. Assume that $M>4$ is not divisible by 4 , and let $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. Then the averageprobabilistic error of the $\boldsymbol{Q S}$ algorithm with respect to the measure $\mathbf{p}_{1}$ satisfies

$$
e_{\mathbf{p}_{1}}^{\text {avg-pro }}(M, p) \geq \frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right)\left(1-2 \exp \left(-\frac{N \pi^{2}}{(8 \beta M)^{2}}\right)\right) \quad \forall \beta>1
$$

Proof. Let $M=4 M^{\prime}+\tau$ for $\tau \in\{1,2,3\}$. Let, as before, $\sigma_{a}=(M / \pi) \arcsin \sqrt{a}$. As in the proof of Theorem 4, for $|x|<\frac{1}{2}$ we have

$$
\sigma_{\frac{1}{2}+x}=\frac{M}{4}+\frac{M}{\pi} \frac{x}{2 \sqrt{\left(1-\xi_{x}\right) \xi_{x}}} \quad \text { for } \quad \xi_{x} \in\left(\frac{1}{2}, \frac{1}{2}+x\right)
$$

and $2 \sqrt{\left(1-\xi_{x}\right) \xi_{x}} \geq \sqrt{1-4 x^{2}}$. Assume additionally that

$$
\frac{M}{\pi} \frac{|x|}{\sqrt{1-4 x^{2}}} \leq \frac{1}{4 \beta}
$$

which is equivalent to assuming that

$$
\begin{equation*}
|x| \leq \frac{\pi}{\left(16 M^{2} \beta^{2}+4 \pi^{2}\right)^{1 / 2}} \tag{25}
\end{equation*}
$$

Thus for $x$ satisfying (25) we have

$$
\sigma_{\frac{1}{2}+x}=M^{\prime}+\frac{\tau}{4}+\frac{M}{\pi} \frac{x \theta(x)}{\sqrt{1-4 x^{2}}} \text { with } \theta(x) \in[0,1]
$$

and $\left\lfloor\sigma_{\frac{1}{2}+x}\right\rfloor=M^{\prime}$ and $\left\lceil\sigma_{\frac{1}{2}+x}\right\rceil=M^{\prime}+1$.
From the proof of Corollary 1 we have $\mu\left(\left\{\left\lceil\sigma_{a_{f}}\right\rceil,\left\lfloor\sigma_{a_{f}}\right\rfloor\right\}, f\right) \geq \frac{8}{\pi^{2}}$. Since $\mu(A, f) \geq p>\frac{1}{2}$ then either $\left\lceil\sigma_{a_{f}}\right\rceil \in A$ or $\left\lfloor\sigma_{a_{f}}\right\rfloor \in A$. We then estimate

$$
\begin{equation*}
e_{\mathbf{p}_{1}}^{\operatorname{avg}-\operatorname{pro}}\left(M, \frac{8}{\pi^{2}}\right) \geq \sum_{f \in \mathbb{B}_{N}} \mathbf{p}_{1}(f) \min \left\{\left|a_{f}-\bar{a}_{f}\left(\left\lfloor\sigma_{a_{f}}\right\rfloor\right)\right|,\left|a_{f}-\bar{a}_{f}\left(\left\lceil\sigma_{a_{f}}\right\rceil\right)\right|\right\} \tag{26}
\end{equation*}
$$

We now estimate the error of the QS algorithm for $f \in \mathbb{B}_{N}$ such that $a_{f}=\frac{1}{2}+x$ for $x$ satisfying (25) and the outcome $j=\left\lceil\sigma_{a_{f}}\right\rceil=M^{\prime}$ or $j=\left\lfloor\sigma_{a_{f}}\right\rfloor=M^{\prime}+1$. Denote the outcome by $M^{\prime}+\kappa$ for $\kappa \in\{0,1\}$. By Taylor's theorem we have

$$
\begin{aligned}
\sin ^{2}\left(\frac{M^{\prime}+\kappa}{M} \pi\right)= & \sin ^{2}\left(\frac{\pi}{4}+\frac{\pi}{M}\left(\kappa-\frac{1}{4} \tau\right)\right) \\
& =\frac{1}{2}+\sin \left(2 \xi_{\kappa, \tau}\right) \frac{\pi}{M}\left(\kappa-\frac{1}{4} \tau\right) \quad \text { for } \xi_{\kappa, \tau} \in\left[\frac{1}{4} \pi, \frac{1}{4} \pi+(\pi / M)\left(\kappa-\frac{1}{4} \tau\right)\right]
\end{aligned}
$$

Since $\sin (t) \geq 2 t / \pi$ for $t \in[0, \pi / 2]$, we have $\left|\sin \left(2 \xi_{\kappa, \tau}\right)\right| \geq 1-|4 \kappa-\tau| / M$. Consider the error for the outcome $M^{\prime}+\kappa$ and $x$ satisfying (25). Then $|x| \leq \pi /(4 \beta M)$ and the error can be estimated by

$$
\begin{aligned}
\left|\frac{1}{2}+x-\sin ^{2}\left(\frac{M^{\prime}+\kappa}{M} \pi\right)\right| & =\left|x-\sin \left(2 \xi_{\kappa, \tau}\right) \frac{\pi}{M}\left(\kappa-\frac{1}{4} \tau\right)\right| \\
& \geq \frac{\pi}{M}\left|\kappa-\frac{1}{4} \tau\right|\left|\sin \left(2 \xi_{\kappa, \tau}\right)\right|-|x| \\
& \geq \frac{\pi}{4} \frac{|4 \kappa-\tau|}{M}\left(1-\frac{|4 \kappa-\tau|}{M}\right)-\frac{\pi}{4 \beta M}
\end{aligned}
$$

Clearly, $|4 \kappa-\tau| \in\{1,2,3\}$ and $|4 \kappa-\tau| / M \in[1 / M, 3 / M]$. Then $|4 \kappa-\tau|(1-|4 \kappa-\tau| / M) / M \geq$ $(1-1 / M) / M$. Therefore

$$
\left|\frac{1}{2}+x-\sin ^{2}\left(\frac{M^{\prime}+\kappa}{M} \pi\right)\right| \geq \frac{\pi}{M}\left(\frac{1}{4}\left(1-\frac{1}{M}\right)-\frac{1}{4 \beta}\right)=\frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right) .
$$

Hence, for $f$ such that $a_{f}=\frac{1}{2}+x$ with $x$ satisfying (25) we have

$$
\begin{equation*}
\min \left\{\left|a_{f}-\bar{a}_{f}\left(\left\lfloor\sigma_{a_{f}}\right\rfloor\right)\right|,\left|a_{f}-\bar{a}_{f}\left(\left\lceil\sigma_{a_{f}}\right\rceil\right)\right|\right\} \geq \frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right) . \tag{27}
\end{equation*}
$$

We are now ready to estimate $e_{\mathbf{p}_{1}}^{\text {avg-pro }}(M, p)$. First, by (26), we have

$$
e_{\mathbf{p}_{1}}^{\operatorname{avg}-\operatorname{pro}}(M, p) \geq \sum_{f:\left|a_{f}-\frac{1}{2}\right| \leq\left(16 M^{2} \beta^{2}+4 \pi^{2}\right)^{1 / 2}} \mathbf{p}_{1}(f) \min \left\{\left|a_{f}-\bar{a}_{f}\left(\left\lfloor\sigma_{a_{f}}\right\rfloor\right)\right|,\left|a_{f}-\bar{a}_{f}\left(\left\lceil\sigma_{a_{f}}\right\rceil\right)\right|\right\} .
$$

This, (27) and the Bernstein inequality, $\sum_{k:\left|k / N-\frac{1}{2}\right|>\varepsilon} 2^{-N}\binom{N}{k} \leq 2 e^{-N \varepsilon^{2} / 4}$, yields

$$
\begin{aligned}
e_{\mathbf{p}_{1}}^{\text {avg-pro }}(M, p) & \geq \sum_{f:\left|a_{f}-\frac{1}{2}\right| \leq \pi\left(16 M^{2} \beta^{2}+4 \pi^{2}\right)^{1 / 2}} \mathbf{p}_{1}(f) \frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right) \\
& =\frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right) \sum_{k:\left|k / N-\frac{1}{2}\right| \leq \pi\left(16 M^{2} \beta^{2}+4 \pi^{2}\right)^{1 / 2}} 2^{-N}\binom{N}{k} \\
& \geq \frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right)\left(1-2 \exp \left(-\frac{N \pi^{2}}{4\left(16 M^{2} \beta^{2}-4\right)}\right)\right) \\
& \geq \frac{\pi}{4 M}\left(1-\frac{1}{M}-\frac{1}{\beta}\right)\left(1-2 \exp \left(-\frac{N \pi^{2}}{(8 \beta M)^{2}}\right)\right)
\end{aligned}
$$

which completes the proof.

Obviously, in the average-probabilistic setting, we should use the QS algorithm with $M$ divisible by 4. Then Theorem 4 states that the error is of order $\min \left\{M^{-1}, N^{-1 / 2}\right\}$. Recently, Papageorgiou [10] proved that for any quantum algorithm that uses $M$ quantum queries the
error is bounded from below by of $c \min \left\{M^{-1}, N^{-1 / 2}\right\}$ with probability $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. Here, $c$ is a positive number independent of $M$ and $N$. Hence, the QS algorithm enjoys an optimality property also in the average-probabilistic setting for the measure $\mathbf{p}_{1}$ as long as we use it with $M$ divisible by 4.

We now turn to the measure $\mathbf{p}_{2}$. Clearly, the average-probabilistic error of the QS algorithm is bounded by its worst-probabilistic error, which is of order $M^{-1}$ with probability $p \in\left(\frac{1}{2}, \frac{8}{\pi^{2}}\right]$. It turns out, again due to a recent result of Papageorgiou [10] that this bound is the best possible, since any quantum algorithm that uses $M$ quantum queries must have an error proportional at least to $M^{-1}$. Hence, the factor $N^{-1 / 2}$ that is present for the measure $\mathbf{p}_{1}$ does not appear for the measure $\mathbf{p}_{2}$, and the behavior of the QS algorithm is roughly the same in the worst- and average-probabilistic settings.

## Acknowledgment

We are grateful for many valuable comments on this paper from J. Creutzig, A. Papageorgiou, J. F. Traub, and A.G. Werschulz.

## References

[1] G. Brassard, P. Høyer, M. Mosca, A. Tapp, Quantum Amplitude Estimation and Amplification, http://arXiv.org/quant-ph/0005055, 2000.
[2] L. Grover, A fast quantum mechanical algorithm for database search, Proceedings of the 28th Annual ACM Symposium on the Theory of Computing (STOC), 212-219, 1996, http://arXiv.org/quant-ph/9605043, 1996.
[3] M. A. Nielsen, I. L. Chuang Quantum Computation and Quantum Information, Cambridge University Press, 2001,
[4] S. Heinrich, Quantum Summation with an Application to Integration, J. Complexity, 18, 1-50, 2002, http://arXiv.org/quant-ph/0105116, 2001.
[5] S. Heinrich, From Monte Carlo to Quantum Computation, http://arXiv.org/quant-ph/0112152, 2001.
[6] S. Heinrich and E. Novak, On a Problem in Quantum Summation, to appear in $J$. Complexity, 19, 2003, http://arXiv.org/quant-ph/0109038, 2001.
[7] A. Nayak and F. Wu, The quantum query complexity of approximating the median and related statistics, Proceedings of the 31th Annual ACM Symposium on the Theory of Computing (STOC), 384-393, 1999, http://arXiv.org/quant-ph/9804066, 1998.
[8] E. Novak, Quantum Complexity of Integration, J. Complexity, 17, 2-16, 2001, http://arXiv.org/quant-ph/0008124, 2000.
[9] E. Novak, I. H. Sloan and H. Woźniakowski, Tractability of Approximation for Weighted Korobov Spaces on classical and Quantum Computers, http://arXiv.org/quant-ph/0206023, 2002.
[10] A. Papageorgiou, Average case quantum lower bounds for computing the Boolean means, in progress.
[11] J. F. Traub and H. Woźniakowski, Path Integration on a Quantum Computer, to appear in Quantum Information Processing, http://arXiv.org/quant-ph/0109113, 2001.


[^0]:    *This research was supported in part by the National Science Foundation (NSF) and by the Defence Advanced Research Agency (DARPA) and Air Force Research Laboratory under agreement F30602-01-20523.

[^1]:    ${ }^{1}$ We follow the quantum mechanics notation in which the first argument is conjugated in the inner product, whereas in the standard mathematical notation the second argument is usually conjugated.

