# Solving Linear Programs with Finite Precision: III. Sharp Expectation bounds * 

Dennis Cheung<br>Division of Continuing Professional Education<br>The Hong Kong Institute of Education<br>HONG KONG<br>e-mail: chinwing@ied.edu.hk<br>Felipe Cucker<br>Department of Mathematics<br>City University of Hong Kong<br>HONG KONG<br>e-mail: macucker@cityu.edu.hk


#### Abstract

We give an $\mathcal{O}(\log n)$ bound for the expectation of the logarithm of the condition number $\mathcal{K}(A, b, c)$ introduced in "Solving linear programs with finite precision: I. Condition numbers and random programs." Math. Programm., 99:175-196, 2004. This bound improves the previously existing bound, which was of $\mathcal{O}(n)$.


## 1 Introduction

Consider the following linear programming problem (in standard form),

$$
\begin{align*}
\min c^{\mathrm{T}} x & \\
\text { s.t. } A x & =b  \tag{P}\\
x & \geq 0 .
\end{align*}
$$

Here $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^{m}, c \in \mathbb{R}^{n}$, and $n \geq m \geq 1$.
Assuming this problem is feasible (i.e., the set given by $A x=b, x \geq 0$, is not empty) and bounded (i.e., the function $x \mapsto c^{T} x$ is bounded below on the feasible set), algorithms solving ( P ) may return an optimizer $x^{*} \in \mathbb{R}^{n}$ and/or the

[^0]optimal value $c^{\mathrm{T}} x^{*}$. Whereas these two computations are essentially equivalent in the presence of infinite precision, obtaining an optimizer appears to be more difficult if only finite precision is available. Accuracy analyses of interior-point algorithms for these problems have been done in [13] - for the computation of the optimal value - and in [6] - for the computation of an optimizer. In both cases, accuracy bounds (as well as complexity bounds) are given in terms of the dimensions $m$ and $n$, as well as of the logarithm of a condition number. The bounds in both analyses are similar. What turns out to be different is their relevant condition numbers.

In [13] this is Renegar's condition number $C(A, b, c)$ which, roughly speaking, is the relativized inverse of the size of the smallest perturbation needed to make ( P ) either infeasible or unbounded. In [6] it is the condition number $\mathcal{K}(A, b, c)$ which, following the same idea, is the relativized inverse of the size of the smallest perturbation needed to change the optimal basis of (P) (a detailed definition is in Section 2 below).

A characteristic of these (and practicality all other) condition numbers is that they cannot be easily computed from the data at hand. Their computation appears to be at least as difficult as that of the solution for the problem whose condition they are measuring (see [10] for a discussion on this) and requires at least the same amount of precision (see [5]). A way out of this dilemma going back to the very beginning of condition numbers is to randomize the data and to estimate the expectation of its condition. Indeed, the first papers on condition are published independently by Turing [12] and by Goldstine and von Neumann [14], both for the condition of linear equation solving and in a sequel [15] to the latter the matix $A$ of the input linear system was considered to be random and some probabilistic estimates on its condition number were derived. This approach was subsequently championed by Demmel [8] and Smale [11].

A number of probabilistic estimates for Renegar's condition number (or for a close relative introduced in [3]) have been obtained in the last decade [7, 2, 9]. The overall picture is that the contribution of the $\log$ of this condition number to complexity and accuracy bounds is, on the average, $\mathcal{O}(\log n)$. In contrast with this satisfactory state of affairs, little is known for the condition number $\mathcal{K}$ on random triples $(A, b, c)$. In [4] it was shown that for these triples, conditioned to (P) being feasible and bounded, $\log \mathcal{K}(A, b, c)$ is $\mathcal{O}(n)$ on the average but this estimate appears to be poor. In the present paper we improve this result and show a $\mathcal{O}(\log n)$ bound (see Theorem 1 below for a precise statement).

## 2 Statement of the Main Result

In this section we fix notations, recall the definition of $\mathcal{K}(A, b, c)$, and state our main result.

For any subset $B$ of $\{1,2, \ldots, n\}$, denote by $A_{B}$ the submatrix of $A$ obtained by removing from $A$ all the columns with index not in $B$. If $x \in \mathbb{R}^{n}, x_{B}$ is defined
analogously. A set $B \subset\{1,2, \ldots, n\}$ such that $|B|=m$ and $A_{B}$ is invertible is said to be a basis for $A$.

Let $B$ be a basis. Then we may uniquely solve $A_{B} x^{\prime}=b$. Consider the point $x^{*} \in \mathbb{R}^{n}$ defined by $x_{j}^{*}=0$ for $j \notin B$ and $x_{B}^{*}=x^{\prime}$. Clearly, $A x^{*}=b$. We say that $x^{*}$ is a primal basic solution. If, in addition, $x^{*} \geq 0$, which is equivalent to $x_{B}^{*} \geq 0$, then we say $x^{*}$ is a primal basic feasible solution.

The dual of (P), which in the sequel we denote by (D), is the following problem,

$$
\begin{align*}
& \max b^{\mathrm{T}} y  \tag{D}\\
& \text { s.t. } A^{\mathrm{T}} y \leq c .
\end{align*}
$$

For any basis $B$, we may now uniquely solve $A_{B}^{\mathrm{T}} y^{*}=c_{B}$. The point $y^{*}$ thus obtained is said to be a dual basic solution. If, in addition, $A^{\mathrm{T}} y \leq c, y^{*}$ is said to be a dual basic feasible solution.

Let $B$ be a basis. We say that $B$ is an optimal basis (for the pair ( $\mathrm{P}-\mathrm{D})$ ) if both the primal and dual basic solutions are feasible. In this case the points $x^{*}$ and $y^{*}$ above are the optimizers of $(\mathrm{P})$ and (D), respectively.

We denote by $d$ the input data ( $A, b, c$ ). We say that $d$ is feasible when there exist $x \in \mathbb{R}^{n}, x \geq 0$, and $y \in \mathbb{R}^{m}$ such that $A x=b$ and $A^{\mathrm{T}} y \leq c$. Let

$$
\mathcal{U}=\{d=(A, b, c) \mid d \text { has a unique optimal basis }\} .
$$

By definition, triples in $\mathcal{U}$ are feasible.
To define conditioning, we need a norm in the space of data triples. To do so, we associate to each triple $d=(A, b, c) \in \mathbb{R}^{m n+m+n}$ the matrix

$$
M_{d}=\left(\begin{array}{cc}
c^{\mathrm{T}} & 0 \\
A & b
\end{array}\right)
$$

and we define $\|d\|$ to be the operator norm $\left\|M_{d}\right\|_{r s}$ of $M_{d}$ considered as a linear map from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{m+1}$. Note that this requires norms $\left\|\|_{r}\right.$ and $\| \|_{s}$ in $\mathbb{R}^{n+1}$ and $\mathbb{R}^{m+1}$, respectively.

Let $\Sigma_{U}$ be the boundary of $\mathcal{U}$ in $\mathbb{R}^{m n+m+n}$. For any data input $d \in \mathcal{U}$, we define the distance to ill-posedness and the condition number for $d$, respectively, as follows,

$$
\varrho(d)=\min \left\{\|\delta d\|: d+\delta d \in \Sigma_{U}\right\} \quad \text { and } \quad \mathcal{K}(d)=\frac{\|d\|}{\varrho(d)}
$$

We next state our main result, after making precise the underlying probability model.

Definition 1 We say that $d=(A, b, c)$ is Gaussian, and we write $d \sim N(0$, Id $)$, when all entries of $A, b$ and $c$ are i.i.d. with standard normal distribution.

Theorem 1 For the \|\| $\|_{12}$ norm we have

$$
\underset{d \sim N(0, \mathrm{Id})}{\mathbb{E}}(\ln \mathcal{K}(d) \mid d \in \mathcal{U}) \leq \frac{5}{4} \ln (m+1)+\frac{3}{2} \ln (n+1)+\ln (12)
$$

Remark 1 The use of the $\left\|\|_{12}\right.$ norm in Theorem 1 is convenient but inessential. Well known norm equivalences yield $\mathcal{O}(\log n)$ bounds for any of the usually considered matrix norms.

## 3 Proof of the Main Result

### 3.1 A useful characterization

Write $\mathcal{D}=\mathbb{R}^{m n+m+n}$ for the space of data inputs, and

$$
\mathcal{B}=\{B \subset\{1,2, \ldots, n\}| | B \mid=m\}
$$

for the family of possible bases.
For any $B \in \mathcal{B}$ and any triple $d \in \mathcal{D}$, let $\mathcal{S}_{1}$ be the set of all $m$ by $m$ submatrices of $\left[A_{B}, b\right], \mathcal{S}_{2}$ the set of all $m+1$ by $m+1$ submatrices of $\left(A^{\mathrm{T}}, c\right)^{\mathrm{T}}$ containing $A_{B}$, and $\mathcal{S}_{B}(d)=\mathcal{S}_{1} \bigcup \mathcal{S}_{2}$. Note that $\left|\mathcal{S}_{1}\right|=m+1$ and $\left|\mathcal{S}_{2}\right|=n-m$, so $\mathcal{S}_{B}(d)$ has $n+1$ elements.

Let Sing be the set of singular matrices. For any square matrix $S$, we define the distance to singularity as follows.

$$
\rho_{\text {Sing }}(S):=\min \{\|\delta S\|:(S+\delta S) \in \text { Sing }\}
$$

For any $B \in \mathcal{B}$ consider the function

$$
\begin{aligned}
h_{B}: \mathcal{D} & \rightarrow[0,+\infty) \\
d & \mapsto \min _{S \in \mathcal{S}_{B}(d)} \rho_{\mathrm{Sing}}(S)
\end{aligned}
$$

The following characterization of $\varrho(d)$ is Theorem 2 in [4].

Theorem 2 For any $d \in \mathcal{U}$,

$$
\varrho(d)=h_{B}(d)
$$

where $B$ is the optimal basis of $d$.

### 3.2 The group action

We consider the group (with respect to componentwise multiplication) $\mathfrak{G}_{n}=$ $\{-1,1\}^{n}$. This group acts on $\mathcal{D}$ as follows. For $u \in \mathfrak{G}_{n}$ let $D_{\mathrm{u}}$ be the diagonal matrix having $\mathrm{u}_{j}$ as its $j$ th diagonal entry, and

$$
\begin{aligned}
\mathrm{u}(A) & :=A D_{\mathrm{u}}=\left(\mathrm{u}_{1} a_{1}, \mathrm{u}_{2} a_{2}, \ldots, \mathrm{u}_{n} a_{n}\right), \\
\mathrm{u}(c) & :=D_{\mathrm{u}} c=\left(\mathrm{u}_{1} c_{1}, \mathrm{u}_{2} c_{2}, \ldots, \mathrm{u}_{n} c_{n}\right),
\end{aligned}
$$

where $a_{i}$ denotes the $i$ th column of $A$. We define $\mathbf{u}(d):=(\mathbf{u}(A), b, \mathbf{u}(c))$. The group $\mathfrak{G}_{n}$ also acts on $\mathbb{R}^{n}$ by $\mathbf{u}(x):=\left(\mathrm{u}_{1} x_{1}, \ldots, \mathrm{u}_{n} x_{n}\right)$. It is immediate to verify that for all $A \in \mathbb{R}^{m \times n}$, all $x \in \mathbb{R}^{n}$, and all $\mathrm{u} \in \mathfrak{G}_{n}$ we have $\mathbf{u}(A) \mathbf{u}(x)=A x$.

Lemma 1 The functions $h_{B}$ are $\mathfrak{G}_{n}$-invariant. That is, for any $d \in \mathcal{D}, B \in \mathcal{B}$ and $\mathrm{u} \in \mathfrak{G}_{n}$,

$$
h_{B}(d)=h_{B}(\mathbf{u}(d)) .
$$

Proof. Let $S^{*}$ be any matrix in $\mathcal{S}_{B}(d)$ such that

$$
\begin{equation*}
\rho_{\text {Sing }}\left(S^{*}\right)=\min _{S \in \mathcal{S}_{B}(d)} \rho_{\text {Sing }}(S) . \tag{1}
\end{equation*}
$$

Let $k$ be the number of rows (or columns) of $S^{*}$ and $E$ be any matrix in $\mathbb{R}^{k \times k}$ such that $S^{*}+E \in$ Sing and

$$
\begin{equation*}
\|E\|=\rho_{\operatorname{Sing}}\left(S^{*}\right) \tag{2}
\end{equation*}
$$

Then, there exists $z \in \mathbb{R}^{k}$ such that

$$
\begin{equation*}
\left(S^{*}+E\right) z=0 \tag{3}
\end{equation*}
$$

Suppose $S^{*}$ consists of the $j_{1}, j_{2}, \ldots, j_{k}$ columns of $M_{d}$ and let $\bar{u}=$ $\left(\mathbf{u}_{j_{1}}, \mathbf{u}_{j_{2}}, \ldots, \mathbf{u}_{j_{k}}\right) \in \mathfrak{G}_{k}$. Then, by the definition of $\mathcal{S}_{B}(d)$ and $\mathcal{S}_{B}(\mathbf{u}(d))$, we have $\overline{\mathrm{u}}\left(S^{*}\right) \in \mathcal{S}_{B}(\mathrm{u}(d))$. Furthermore,

$$
\left(\overline{\mathrm{u}}\left(S^{*}\right)+\overline{\mathrm{u}}(E)\right) \overline{\mathrm{u}}(z)=\overline{\mathrm{u}}\left(S^{*}+E\right) \overline{\mathrm{u}}(z)=\left(S^{*}+E\right)(z)=0,
$$

the last by Equation (3). That is, $\left(\overline{\mathrm{u}}\left(S^{*}\right)+\overline{\mathrm{u}}(E)\right)$ is also singular. By the definition of $\rho_{\text {Sing }}$,

$$
\begin{equation*}
\rho_{\text {Sing }}\left(\overline{\mathrm{u}}\left(S^{*}\right)\right) \leq\|\overline{\mathrm{u}}(E)\| . \tag{4}
\end{equation*}
$$

Since operator norms are invariant under multiplication of arbitrary matrix columns by -1 we have $\|E\|=\|\overline{\mathrm{u}}(E)\|$. Combining this equality with Equations (1), (2), and (4) we obtain

$$
\rho_{\text {Sing }}\left(\overline{\mathrm{u}}\left(S^{*}\right)\right) \leq \min _{S \in \mathcal{S}_{B}(d)} \rho_{\text {Sing }}(S) .
$$

Since $\overline{\mathrm{u}}\left(S^{*}\right) \in \mathcal{S}_{B}(\mathrm{u}(d))$ we obtain

$$
\min _{S \in \mathcal{S}_{B}(\mathrm{u}(d))} \rho_{\text {Sing }}(S) \leq \min _{S \in \mathcal{S}_{B}(d)} \rho_{\text {Sing }}(S) .
$$

The reversed inequality follows by exchanging the roles of $S(\mathrm{u})$ and $S$.
For any $B \in \mathcal{B}$, let

$$
\mathcal{U}_{B}=\{d \in \mathcal{D} \mid B \text { is the only optimal basis for } d\}
$$

The set $\mathcal{U}$ of well-posed feasible triples is thus partitioned by the sets $\left\{\mathcal{U}_{B} \mid B \in \mathcal{B}\right\}$.
Lemma 2 Let $d \in \mathcal{D}$ and $B \in \mathcal{B}$. If $h_{B}(d)>0$, then there exists a unique $\mathbf{u} \in \mathfrak{G}_{n}$ such that $\mathbf{u}(d) \in \mathcal{U}_{B}$.

Proof. First observe that, since $\min _{S \in \mathcal{S}_{B}(d)} \rho_{\text {Sing }}(S)>0$, we have $A_{B}$ invertible and therefore $B$ is a basis for $A$. Let $y^{*}$ and $x^{*}$ be the dual and primal basic solutions of $d$ for the basis $B$, i.e.

$$
\begin{equation*}
y^{*}=A_{B}^{-\mathrm{T}} c_{B}, \quad x_{B}^{*}=A_{B}^{-1} b, \quad x_{j}^{*}=0, \forall j \notin B \tag{5}
\end{equation*}
$$

Similarly, let $y^{u}$ and $x^{u}$ be the dual and primal basic solutions of $\mathbf{u}(d)$ for the same basis. Then, using that $\mathrm{u}(A)=A D_{\mathrm{u}}$ and $\mathrm{u}(c)=D_{\mathrm{u}} c$,

$$
\begin{equation*}
y^{\mathrm{u}}=\mathbf{u}(A)_{B}^{-\mathrm{T}} \mathbf{u}(c)_{B}=A_{B}^{-\mathrm{T}}\left(D_{\mathrm{u}}\right)_{B}^{-\mathrm{T}}\left(D_{\mathrm{u}}\right)_{B} c_{B}=A_{B}^{-\mathrm{T}} c_{B}=y^{*} \tag{6}
\end{equation*}
$$

the third equality by the definition of $\left(D_{\mathrm{u}}\right)_{B}$. Similarly,

$$
\begin{equation*}
x_{B}^{\mathrm{u}}=\mathrm{u}(A)_{B}^{-1} b=\left(D_{\mathrm{u}}\right)_{B}^{-1} A_{B}^{-1} b=\left(D_{\mathrm{u}}\right)_{B} A_{B}^{-1} b=\left(D_{\mathrm{u}}\right)_{B} x_{B}^{*} \tag{7}
\end{equation*}
$$

and $x_{j}^{\mathrm{u}}=0$ for all $j \notin B$. Therefore,

$$
\begin{align*}
B \text { is optimal for } \mathbf{u}(d) & \Leftrightarrow x^{\mathbf{u}} \text { and } y^{\mathbf{u}} \text { are both feasible } \\
& \Leftrightarrow\left\{\begin{array}{l}
x_{B}^{\mathbf{u}} \geq 0 \\
\mathbf{u}(A)_{j}^{\mathrm{T}} y^{\mathbf{u}} \leq \mathbf{u}(c)_{j}, \text { for } j \notin B
\end{array}\right. \\
& \Leftrightarrow\left\{\begin{array}{l}
\left(D_{\mathbf{u}}\right)_{B} x_{B}^{*} \geq 0 \\
\mathbf{u}_{j}\left(a_{j}\right)^{\mathrm{T}} y \leq \mathbf{u}(c)_{j}, \text { for } j \notin B
\end{array} \quad\right. \text { (by (6) and (7)) } \\
& \Leftrightarrow\left\{\begin{array}{l}
\mathbf{u}_{j} x_{j}^{*} \geq 0, \text { for } j \in B \\
\mathbf{u}_{j}\left(c_{j}-a_{j}^{\mathrm{T}} y\right) \geq 0, \text { for } j \notin B .
\end{array}\right. \tag{8}
\end{align*}
$$

Since by hypothesis $\min _{S \in \mathcal{S}_{B}(d)} \rho_{\text {Sing }}(S)>0$,

$$
\begin{equation*}
x_{j}^{*} \neq 0, \forall j \in B \quad \text { and } \quad a_{j}^{\mathrm{T}} y \neq c_{j}, \forall j \notin B . \tag{9}
\end{equation*}
$$

Combining Equations (8) and (9), the statement follows for $u \in \mathfrak{G}_{n}$ given by $\mathrm{u}_{j}=\operatorname{sign}\left(x_{j}^{*}\right)$ if $j \in B$ and $\mathrm{u}_{j}=\operatorname{sign}\left(c_{j}-a_{j}^{\mathrm{T}} y\right)$ otherwise. Clearly, this u is unique.

For $B \in \mathcal{B}$ let

$$
\Sigma_{B}:=\left\{d \in \mathcal{D} \mid h_{B}(d)=0\right\}
$$

and $\mathcal{D}_{B}:=\mathcal{D} \backslash \Sigma_{B}$. Lemma 1 implies that, for all $B \in \mathcal{B}, \Sigma_{B}$ and $\mathcal{D}_{B}$ are $\mathfrak{G}_{n^{-}}$ invariant. Lemma 2 immediately implies the following corollary.

Corollary 1 For all $B \in \mathcal{B}$ the sets

$$
\mathcal{D}_{\mathrm{u}}:=\left\{d \in \mathcal{D}_{B} \mid \mathbf{u}(d) \in \mathcal{U}_{B}\right\}, \quad \text { for } \mathrm{u} \in \mathfrak{G}_{n}
$$

are a partition of $\mathcal{D}_{B}$.

### 3.3 Probabilities

Definition 2 We say that a distribution $\mathscr{D}$ on the set of triples $d=(A, b, c)$ is $\mathfrak{G}_{n}$-invariant when
(i) if $d \sim \mathscr{D}$ then $\mathbf{u}(d) \sim \mathscr{D}$ for all $\mathbf{u} \in \mathfrak{G}_{n}$.
(ii) for all $B \in \mathcal{B}, \underset{d \sim \mathscr{D}}{\operatorname{Prob}}\left\{h_{B}(d)=0\right\}=0$.

Note that Gaussianity is a special case of $\mathfrak{G}_{n}$-invariance. Consequently, all results true for a $\mathfrak{G}_{n}$-invariant distribution also hold for Gaussian data.
Note: For a time to come we fix a $\mathfrak{G}_{n}$-invariant distribution $\mathscr{D}$ with density function $f$.

Lemma 3 For any $\mathrm{u} \in \mathfrak{G}_{n}$ and $B \in \mathcal{B}$,

$$
\underset{d \sim \mathscr{D}}{\operatorname{Prob}}\left\{\mathbf{u}(d) \in \mathcal{U}_{B}\right\}=\underset{d \sim \mathscr{D}}{\operatorname{Prob}}\left\{d \in \mathcal{U}_{B}\right\}=\frac{1}{2^{n}} .
$$

Proof. The equality between probabilities follows from (i) in Definition 2. Therefore, by Corollary 1 and Definition 2(ii), the probability of each of them is $2^{-n}$.

The following lemma tells us that, for all $B \in \mathcal{B}$, the random variable $h_{B}(d)$ is independent of the event " $d \in \mathcal{U}_{B}$."

Lemma 4 For all measurable $g: \mathbb{R} \rightarrow \mathbb{R}$ and $B \in \mathcal{B}$,

$$
\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(g\left(h_{B}(d)\right) \mid d \in \mathcal{U}_{B}\right)=\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(g\left(h_{B}(d)\right)\right) .
$$

Proof. From the definition of conditional expectation and Lemma 3 we have
where $\mathbb{1}_{B}$ denotes the indicator function of $\mathcal{U}_{B}$. Now, for any $u \in \mathfrak{G}_{n}$, the map $d \mapsto \mathrm{u}(d)$ is a linear isometry on $\mathcal{D}$. Therefore

$$
\int_{d \in \mathcal{D}} \mathbb{1}_{B}(d) g\left(h_{B}(d)\right) f(d)=\int_{d \in \mathcal{D}} \mathbb{1}_{B}(\mathrm{u}(d)) g\left(h_{B}(\mathrm{u}(d))\right) f(\mathrm{u}(d)) .
$$

Using that $h_{B}(d)=h_{B}\left(\mathbf{u}(d)\right.$ ) (by Lemma 1) and $f(d)=f(\mathbf{u}(d))$ (by the $\mathfrak{G}_{n^{-}}$ invariance of $\mathscr{D})$, it follows that

$$
\begin{aligned}
\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(g\left(h_{B}(d)\right)\right. & \left.\mid d \in \mathcal{U}_{B}\right)=2^{n} \int_{d \in \mathcal{D}} \mathbb{1}_{B}(d) g\left(h_{B}(d)\right) f(d) \\
= & \sum_{\mathbf{u} \in \mathfrak{G}_{n}} \int_{d \in \mathcal{D}} \mathbb{1}_{B}(\mathbf{u}(d)) g\left(h_{B}(\mathbf{u}(d))\right) f(\mathbf{u}(d)) \\
= & \sum_{\mathbf{u} \in \mathfrak{G}_{n}} \int_{d \in \mathcal{D}} \mathbb{1}_{B}(\mathbf{u}(d)) g\left(h_{B}(d)\right) f(d) \\
= & \int_{d \in \mathcal{D}} g\left(h_{B}(d)\right) f(d)=\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(g\left(h_{B}(d)\right)\right),
\end{aligned}
$$

the last line by Corollary 1.
Let $B^{*}=\{1,2, \ldots, m\}$.
Lemma 5 For all measurable $g: \mathbb{R} \rightarrow \mathbb{R}$

$$
\underset{d \sim N(0, \mathrm{Id})}{\mathbb{E}}(g(\varrho(d)) \mid d \in \mathcal{U})=\underset{d \sim N(0, \mathrm{ld})}{\mathbb{E}}\left(g\left(h_{B^{*}}(d)\right)\right)
$$

Proof. Let $\varphi$ be the probability density function of $N(0, \mathrm{Id})$.

$$
\begin{equation*}
\underset{d \sim N(0, \mathrm{Id})}{\mathbb{E}}(g(\varrho(d)) \mid d \in \mathcal{U})=\frac{\int_{d \in \mathcal{U}} g(\varrho(d)) \varphi(d) \mathrm{d}(d)}{\underset{d \sim N(0, \mathrm{Id})}{\operatorname{Prob}}\{d \in \mathcal{U}\}} \tag{11}
\end{equation*}
$$

Since $d$ is Gaussian, the probability that $d$ has two optimal bases is 0 . Using this and Lemma 3 we see that

$$
\begin{equation*}
\underset{d \sim N(0, \mathrm{ld})}{\operatorname{Prob}}\{d \in \mathcal{U}\}=\sum_{B \in \mathcal{B}} \operatorname{Prob}_{d \sim N(0, \mathrm{Id})}\left\{d \in \mathcal{U}_{B}\right\}=\sum_{B \in \mathcal{B}} \frac{1}{2^{n}}=\binom{n}{m}\left(\frac{1}{2^{n}}\right) . \tag{12}
\end{equation*}
$$

Combining Equations (11) and (12), we have

$$
\begin{aligned}
\binom{n}{m}\left(\frac{1}{2^{n}}\right) \underset{d \sim N(0, \mathrm{Id})}{\mathbb{E}}(g(\varrho(d)) \mid d \in \mathcal{U}) & =\int_{d \in \mathcal{U}} g(\varrho(d)) \varphi(d) \mathrm{d}(d) \\
& =\sum_{B \in \mathcal{B}} \int_{d \in \mathcal{U}_{B}} g(\varrho(d)) \varphi(d) \mathrm{d}(d)
\end{aligned}
$$

the last since the probability that $d$ has two optimal bases is 0 . Using now that the entries of $d$ are i.i.d. and Theorem 2 we obtain

$$
\begin{aligned}
\left(\frac{1}{2^{n}}\right) \underset{d \sim N(0, \mathrm{ld})}{\mathbb{E}}(g(\varrho(d)) \mid d \in \mathcal{U}) & =\int_{d \in \mathcal{U}_{B^{*}}} g(\varrho(d)) \varphi(d) \mathrm{d}(d) \\
& =\int_{d \in \mathcal{U}_{B^{*}}} g\left(h_{B^{*}}(d)\right) \varphi(d) \mathrm{d}(d)
\end{aligned}
$$

Therefore, by Lemma 3 with $B=B^{*}$,

$$
\underset{d \sim N(0, \mathrm{ld})}{\operatorname{Prob}}\left\{d \in \mathcal{U}_{B^{*}}\right\} \underset{d \sim N(0, \mathrm{Id})}{\mathbb{E}}(g(\varrho(d)) \mid d \in \mathcal{U})=\int_{d \in \mathcal{U}_{B^{*}}} g\left(h_{B^{*}}(d)\right) \varphi(d) \mathrm{d}(d) .
$$

We conclude since, by the definition of conditional expectation and Lemma 4,

$$
\begin{aligned}
\underset{d \sim N(0, \mathrm{ld})}{\mathbb{E}}(g(\varrho(d)) \mid d \in \mathcal{U}) & =\underset{d \sim N(0, \text { ld })}{\mathbb{E}}\left(g\left(h_{B^{*}}(d)\right) \mid d \in \mathcal{U}_{B^{*}}\right) \\
& =\underset{d \sim N(0, \text { ld })}{\mathbb{E}}\left(g\left(h_{B^{*}}(d)\right)\right) .
\end{aligned}
$$

The following is Lemma 11 in [4].
Lemma 6 For the $\left\|\|_{12}\right.$ in the definition of $\rho_{\text {Sing }}$ we have

$$
\underset{S \sim N(0, \text { Id })}{\mathbb{E}}\left(\sqrt{\frac{1}{\rho_{\text {Sing }}(S)}}\right) \leq 2 m^{5 / 4}
$$

where $N(0, \mathrm{ld})$ is the Gaussian distribution in the set of $m \times m$ real matrices.
Lemma 7 Let $B \in \mathcal{B}$ fixed. Then, for the $\left\|\|_{12}\right.$ in the definition of $\rho_{\text {sing }}$ we have

$$
\underset{d \sim N(0, \mathrm{ld})}{\mathbb{E}}\left(\sqrt{\frac{1}{h_{B}(d)}}\right) \leq 2(m+1)^{5 / 4}(n+1) .
$$

Proof. For any fixed $d \in \mathcal{D}$,

$$
\sum_{S \in \mathcal{S}_{B}} \sqrt{\frac{1}{\rho_{\mathrm{Sing}}(S)}} \geq \max _{S \in \mathcal{S}_{B}} \sqrt{\frac{1}{\rho_{\mathrm{Sing}}(S)}}=\sqrt{\frac{1}{h_{B}(d)}}
$$

Take average on both sides,

$$
\begin{aligned}
\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(\sqrt{\frac{1}{h_{B}(d)}}\right) & \leq \underset{d \sim \mathscr{D}}{\mathbb{E}}\left(\sum_{S \in \mathcal{S}_{B}} \sqrt{\frac{1}{\rho_{\text {Sing }}(S)}}\right) \leq \sum_{S \in \mathcal{S}_{B}} \underset{d \sim \mathscr{D}}{\mathbb{E}}\left(\sqrt{\frac{1}{\rho_{\text {Sing }}(S)}}\right) \\
& \leq \sum_{S \in \mathcal{S}_{B}} 2(m+1)^{5 / 4} \quad \text { by Lemma } 6 \\
& \leq 2(m+1)^{5 / 4}(n+1) .
\end{aligned}
$$

The following lemma is proved as Lemma 4.
Lemma 8 For all $r, s \geq 1$ we have

$$
\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(\|d\|_{r s} \mid d \in \mathcal{U}\right)=\underset{d \sim \mathscr{D}}{\mathbb{E}}\left(\|d\|_{r s}\right) .
$$

Lemma 9 We have

$$
\underset{d \sim N(0, \text { ld })}{\mathbb{E}}\left(\|d\|_{12}\right) \leq 6 \sqrt{n+1} .
$$

Proof. Recall that $\|d\|_{12}=\left\|M_{d}\right\|_{12}$. It is well known that $\left\|M_{d}\right\|_{12} \leq\left\|M_{d}\right\|$ where the latter is spectral norm. The statement now follows from the fact that, for a random Gaussian $A \in \mathbb{R}^{(m+1) \times(n+1)}$ we have $\mathbb{E}(\|A\|) \leq 6 \sqrt{n+1}$ [1, Lemma 2.4].

Proof of Theorem 1. By Jensen's inequality and Lemma 9,

$$
\begin{equation*}
\underset{d}{\mathbb{E}}\left(\ln \|d\|_{12}\right) \leq \ln \underset{d}{\mathbb{E}}\left(\|d\|_{12}\right) \leq \frac{1}{2} \ln (n+1)+\ln 6 \tag{13}
\end{equation*}
$$

In addition, using now Lemma 7,

$$
\begin{align*}
\underset{d}{\mathbb{E}}\left(\ln \left(h_{B^{*}}(d)\right)\right) & =-2 \underset{d}{\mathbb{E}}\left(\ln \sqrt{\frac{1}{h_{B^{*}}(d)}}\right) \geq-2 \ln \underset{d}{\mathbb{E}}\left(\sqrt{\frac{1}{h_{B^{*}}(d)}}\right) \\
& \geq-\ln \left(2(m+1)^{\frac{5}{4}}(n+1)\right) . \tag{14}
\end{align*}
$$

By the definition of $\mathcal{K}(d)$ and Lemmas 8 and 5 ,

$$
\begin{align*}
\underset{d}{\mathbb{E}}(\ln \mathcal{K}(d) \mid d \in \mathcal{U}) & =\underset{d}{\mathbb{E}}\left(\ln \|d\|_{12} \mid d \in \mathcal{U}\right)-\underset{d}{\mathbb{E}}(\ln \varrho(d) \mid d \in \mathcal{U}) \\
& =\underset{d}{\mathbb{E}}\left(\ln \|d\|_{12}\right)-\underset{d}{\mathbb{E}}\left(\ln \left(h_{B^{*}}(d)\right)\right) . \tag{15}
\end{align*}
$$

Combining Equations (13), (14), and (15), the proof is done.

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