# Representing Probability Measures using Probabilistic Processes

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**Abstract**

In the Type-2 Theory of Effectivity, one considers representations of topological spaces in which infinite words are used as "names" for the elements they represent. Given such a representation, we show that *probabilistic processes* on infinite words, under which each successive symbol is determined by a finite probabilistic choice, generate Borel probability measures on the represented space. Conversely, for several well-behaved types of space, every Borel probability measure is represented by a corresponding probabilistic process. Accordingly, we consider probabilistic processes as providing "probabilistic names" for Borel probability measures. We show that integration is computable with respect to the induced representation of measures.

*Key words:* Borel measures, probabilistic processes, Type-2 Theory of Effectivity

## 1 Introduction

Different notions of "measure" are used in mathematics and computer science, with the choice of definition depending on the situation at hand. For example, topologists and analysts generally consider *(regular) Borel measures* [6], whereas domain theorists instead use the more general notion of *continuous valuation* [3]. These definitions have proven themselves through the development of useful and powerful mathematical theories of integration based upon them. Furthermore, in well-behaved cases, the two notions coincide, in the sense that every continuous valuation extends to a unique Borel measure, see e.g. [1], a fact that lends an apparent canonicity to the notions. Nevertheless, the definitions themselves merely consist of a collection of intuitive consistency conditions for assigning weights to

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sets, and it is hard to see, *prima facie*, reasons that the asserted conditions are exactly the right ones.

In this paper, we provide conceptual justification for these notions, in the special case of probability measures over spaces that arise in computable analysis. To achieve this, we simultaneously address the more practical goal of obtaining a treatment of probability appropriate for use in computable mathematics.

One approach to this would be to borrow the standard definitions from analysis, giving them suitable computational representations. In fact, such an approach was previously taken by Weihrauch in [12], where he defined a representation for Borel probability measures over the closed interval $[0, 1]$, showed its admissibility with respect to the weak topology, and established a number of computability results including the computability of integration.

Our approach is different. Fundamental to computable analysis is the idea that a topological space $X$ should come with a *representation* given as a surjective partial function $\delta :\subseteq \Sigma^\omega \rightarrow X$, where $\Sigma$ is some finite (sometimes countable) alphabet. Thus each element $x \in X$ has an associated nonempty set of *names*, infinite words over $\Sigma$ acting as representatives for $x$. For a space $X$ with such a representation, there is a very intuitive notion of "probability distribution" over $X$. The idea is simple: to probabilistically generate an element of $X$ is to probabilistically generate a name for the element. This idea leads to a conceptual simplification, because there is an evident natural computational mechanism for probabilistically generating names, i.e. infinite sequences, which is easily formalized as a notion of *probabilistic process* over $\Sigma^\omega$. As long as such a probabilistic process is certain to return a name for an element of $X$, i.e. a word in the domain of $\delta$, the process can be seen as inducing a probability measure over $X$. We consider the probability measures generated in this way as the computationally interesting ones, and we think of the probabilistic process itself as a *probabilistic name* representing the measure it determines.

After technical preliminaries in Section 2, we give formal definitions of probabilistic process and name in Section 3, and we prove that probabilistic names induce Borel measures on represented spaces. In Section 4, we use probabilistic names to construct a representation for the Borel measures they induce. In Section 5, we show, for several well-behaved classes of spaces, that probabilistic names over a space $X$ generate exactly the Borel probability measures over $X$. This provides the conceptual justification for Borel measures mentioned above. Finally, in Section 6, we show that integration is a computable operation with respect to probabilistic names.

## 2  Preliminaries

Throughout the paper, we assume $\Sigma$ to be a finite alphabet containing the symbols $0, 1$. The set of finite words over $\Sigma$ is denoted by $\Sigma^*$ and the set $\{p \,|\, p : \mathbb{N} \rightarrow \Sigma\}$ of $\omega$-words by $\Sigma^\omega$. For $p \in \Sigma^\omega$, $n \in \mathbb{N}$, a word $w \in \Sigma^{\omega*}$ and a subset

$W \subseteq \Sigma^*$ we denote by $p^{<n}$ the prefix $p(0) \ldots p(n-1)$ of length $n$, by $w\Sigma^\omega$ the set $\{p \in \Sigma^\omega \,|\, (\exists n \in \mathbb{N}) \, p^{<n} = w\}$, by $W\Sigma^\omega$ the set $\bigcup_{w \in W} w\Sigma^\omega$ and by $\lg(w)$ the length of $w$. Moreover, $\sqsubseteq$ and $\sqsubsetneq$ denote, respectively, the reflexive and the irreflexive prefix relation on $\Sigma^* \cup \Sigma^\omega$. For a $\sqsubseteq$-monotone function $h : (\Sigma^*)^k \to \Sigma^*$ we define $h^\omega :\subseteq (\Sigma^\omega)^k \to \Sigma^\omega$ by

$$h^\omega(p_1, \ldots, p_k) = q \;:\Longleftrightarrow\; q = \sup_{\sqsubseteq}\{h(p_1^{<n}, \ldots, p_k^{<n}) \,|\, n \in \mathbb{N}\} \,.$$

We denote the topology of a topological space $X$ by $\mathcal{O}(X)$ and its underlying set by the symbol $X$ as well. The set of continuous functions from $X$ to another topological space $Y$ is denoted by $\mathcal{C}(X, Y)$. On $\Sigma^\omega$ we consider the Cantor topology $\mathcal{O}(\Sigma^\omega) := \{W\Sigma^\omega \,|\, W \subseteq \Sigma^*\}$. For a set $S \subseteq \Sigma^\omega$, $\mathcal{O}(S)$ is the subspace topology induced on $S$.

## 2.1 Background from Type-2 Theory

We recall some notions and facts from Type-2 Theory of Effectivity ([13]). Its basic idea is to represent infinite objects like real numbers, functions or sets by infinite words over some alphabet $\Sigma$. The corresponding partial surjective function $\delta :\subseteq \Sigma^\omega \to X$ is called a *representation* of set $X$.

Given two representations $\delta :\subseteq \Sigma^\omega \to X$ and $\gamma :\subseteq \Sigma^\omega \to Y$, a total function $f : X \to Y$ is called $(\delta, \gamma)$-*computable* iff there is a Type-2-computable function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ *realising* $g$, i.e. $\gamma(g(p)) = f(\delta(p))$ for all $p \in \mathrm{dom}(\delta)$, where $\mathrm{dom}(\delta)$ denotes the domain of $\delta$. If there are ambient representations of $X$ and $Y$, then we simply say that $f$ is *computable* rather than $f$ is $(\delta, \gamma)$-*computable*. A function $g :\subseteq \Sigma^\omega \to \Sigma^\omega$ is *Type-2-computable* iff there is a $\sqsubseteq$-monotone computable function $h : \Sigma^* \to \Sigma^*$ satisfying $h^\omega = g$. Every Type-2 computable function is continuous w.r.t. the Cantor topology $\mathcal{O}(\Sigma^\omega)$ and has a $G_\delta$-domain. The function $f$ is called $(\delta, \gamma)$-*continuous* (or *relatively continuous* when $\delta, \gamma$ are understood) iff there is a continuous function $g$ realising $f$ w.r.t. $\delta$ and $\gamma$. For multivariate functions the above notions are modified in the obvious way.

The category $\mathsf{Rep_t}$ whose objects are the representations over $\Sigma$ and whose morphisms are the relatively continuous functions is cartesian closed. There is a canonical way to construct a representation $[\delta, \gamma]$ of $X \times Y$ and a representation $[\delta \to \gamma]$ of the set $\mathcal{C}(\delta, \gamma)$ of $(\delta, \gamma)$-continuous total functions (cf. [13]). The representations $[\delta, \gamma]$ and $[\delta \to \gamma]$ form, respectively, the product and the exponential of the objects $\delta$ and $\gamma$ in $\mathsf{Rep_t}$.

Given a further representation $\delta'$ of $X$, we write $\delta \leq_t \delta'$ iff the identity function is $(\delta, \delta')$-continuous. We say that $\delta$ and $\delta'$ are *topologically equivalent*, in symbols $\delta \equiv_t \delta'$, iff $\delta \leq_t \delta' \leq_t \delta$. Computable equivalence is defined analogously and denoted by $\delta \equiv_{cp} \delta'$. Note that computably equivalent representations induce the same class of relatively computable functions.

The property of admissibility is defined to reconcile relative continuity with mathematical continuity. We call $\gamma :\subseteq \Sigma^\omega \to Y$ an *admissible* representation of a sequential space $Y$ iff $\gamma$ is continuous and every continuous representation

$\phi :\subseteq \Sigma^\omega \to Y$ satisfies $\phi \leq_t \gamma$. If $\gamma$ is admissible, then $\mathcal{C}(\delta, \gamma)$ is exactly the set of functions which are continuous w.r.t. to the quotient topologies $\mathcal{O}(\delta)$ and $\mathcal{O}(\gamma)$ (cf. [8]). The *quotient topolopy* $\mathcal{O}(\gamma)$ induced by $\gamma$ is the family $\{U \subseteq Y \,|\, (\exists O \in \mathcal{O}(\Sigma^\omega))\, O \cap \mathrm{dom}(\gamma) = \gamma^{-1}[U]\}$. If $\mathcal{O}(\gamma)$ is equal to $\mathcal{O}(Y)$, then $\gamma$ is called a *quotient representation* of $Y$.

We equip the unit interval $\mathbb{I} = [0, 1]$ with two representations $\varrho_=$ and $\varrho_<$. They are the restriction to $\mathbb{I}$ of the respective representations of $\mathbb{R}$ from [13, Definition 4.1.3]. The first one is admissible w.r.t. the Euclidean topology $\mathcal{O}(\mathbb{I}_=)$ and the second one is admissible w.r.t. the lower topology $\mathcal{O}(\mathbb{I}_<) := \{\emptyset, (x, 1], [0, 1] \,|\, x \in [0, 1)\}$ on $\mathbb{I}$. As the ambient representation of $\Sigma^*$, we will use $\varrho_{\Sigma^*} :\subseteq \Sigma^\omega \to \Sigma^*$ defined by

$$\varrho_{\Sigma^*}(0a_1 0 \ldots 0 a_k 11 \ldots) := a_1 \ldots a_k \,,$$

which is admissible w.r.t. the discrete topology on $\Sigma^*$.

## 2.2   Background from Measure Theory

Let $X$ be a set. A *lattice* over $X$ is a collection of subsets of $X$ which contains the emptyset, $X$ and is closed under finite intersections and finite unions. An algebra $\mathcal{A}$ over $X$ is a lattice over $X$ which is closed under complement. It is well-known that the smallest algebra $\mathcal{A}(\mathcal{L})$ containing a given lattice $\mathcal{L}$ consists of the sets of the form $\biguplus_{i=1}^k U_i \setminus V_i$, where $U_i, V_i \in \mathcal{L}$ and the *crescent* sets $U_i \setminus V_i$ are pairwise disjoint (cf. [5,3]). A *$\sigma$-algebra* over $X$ is an algebra that is closed under countable unions (and thus under countable intersections).

A *(probabilistic) valuation* $\nu$ on a lattice $\mathcal{L}$ is a function from $\mathcal{L}$ into the unit interval $\mathbb{I} = [0, 1]$ which is *strict* (i.e. $\nu(\emptyset) = 0$), monotone, *modular* (i.e. $\nu(U) + \nu(V) = \nu(U \cup V) + \nu(U \cap V)$) and *probabilistic* [1] (i.e. $\nu(X) = 1$). By the Smiley-Horn-Tarski Theorem (cf. [3, Prop. IV-9.3]), any valuation $\nu : \mathcal{L} \to \mathbb{I}$ extends uniquely to a valuation $\overline{\nu}$ on $\mathcal{A}(\mathcal{L})$. A *measure* on an algebra $\mathcal{A}$ is a valuation $\mu$ on $\mathcal{A}$ that is *$\sigma$-additive*, i.e. $\mu(\biguplus_{i\in\mathbb{N}} U_i) = \sum_{i\in\mathbb{N}} \mu(U_i)$ for every pairwise disjoint sequence $(U_i)_i$ in $\mathcal{A}$ such that $\biguplus_{i\in\mathbb{N}} U_i \in \mathcal{A}$.

Given a topological space $X$, we are mainly interested in continuous valuations on the lattice of opens and in Borel measures. A valuation $\nu : \mathcal{O}(X) \to \mathbb{I}$ is called *continuous* iff $\nu(\bigcup F) = \sup\{\nu(U) \,|\, U \in F\}$ holds for every directed family $F$ of opens. This continuity notion is equivalent to topological continuity with respect to the Scott-topology on the lattice of opens (cf. [11]) and the lower topology $\mathcal{O}(\mathbb{I}_<) = \{\emptyset, (x, 1], [0, 1] \,|\, x \in [0, 1)\}$ on the unit interval. Note that if $X$ has a continuous representation, then a valuation on $X$ is continuous if and only if every increasing sequence $(U_i)_i$ of opens satisfies $\nu(\bigcup_{i\in\mathbb{N}} U_i) = \sup_{i\in\mathbb{N}} \nu(U_i)$, because $X$ is a hereditarily Lindelöf space. A *Borel measure* is a measure defined

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[1] Here we are only interested in probabilistic valuations and measures. Therefore we will omit the adjective "probabilistic" in the following. Usually, one allows $\nu(X)$ to be any number in $[0, \infty]$.

on the smallest $\sigma$-algebra $\mathcal{B}(X)$ containing $\mathcal{O}(X)$. The elements of $\mathcal{B}(X)$ are the *Borel sets* of $X$. A Borel measure $\mu$ is called *outer regular* iff

$$\mu(B) = \inf\{\mu(U) \,|\, U \in \mathcal{O}(X), \; U \supseteq B\}$$

holds for all Borel sets $B \in \mathcal{B}(X)$. Any Borel measure on a hereditarily Lindelöf space is continuous by being $\sigma$-additive. Given a Borel measure $\mu : \mathcal{B}(X) \to \mathbb{I}$, the sets $M$ satisfying $\mu(X) = \mu^*(M) + \mu^*(X \setminus M)$ are called $\mu$-measurable, where the *outer measure* $\mu^* : 2^X \to \mathbb{I}$ is defined by

$$\mu^*(Y) := \inf\{\mu(B) \,|\, B \in \mathcal{B}(X), \; B \supseteq Y\}.$$

Clearly, $M$ is $\mu$-measurable iff there are Borels sets $A, B$ satisfying $A \subseteq M \subseteq B$ and $\mu(A) = \mu(B)$. The collection of $\mu$-measurable sets is a $\sigma$-algebra and contains $\mathcal{B}(X)$. The restriction of $\mu^*$ to the $\mu$-measurable sets is a measure (cf. [6]). A $\mu$-null-set is a set $N \subseteq X$ with $\mu^*(N) = 0$. If $\mu$ is outer regular, then

$$\mu^*(Y) = \inf\{\mu(U) \,|\, U \in \mathcal{O}(X), \; U \supseteq Y\}. \tag{1}$$

One easily shows the following lemma.

**Lemma 1** *Let $X$ and $Y$ be topological spaces, $f : X \to Y$ be a continuous function and $\nu : \mathcal{O}(X) \to \mathbb{I}$ be a continuous valuation on $X$. Then the function $\mathrm{Tr}(\nu, f) : \mathcal{O}(Y) \to \mathbb{I}$ defined by $\mathrm{Tr}(\nu, f)(V) := \nu \circ f^{-1}[V]$ is a continuous valuation on $Y$. If $\hat{\nu}$ is a Borel measure on $X$ extending $\nu$, then $\mathrm{Tr}(\hat{\nu}, f) : \mathcal{B}(Y) \to \mathbb{I}$ is a Borel measure extending $\mathrm{Tr}(\nu, f)$.*

We mention the following extension results from [1]:

**Theorem 2** *Let $X$ be a regular space or a locally compact sober space. Then every continuous valuation $\nu : \mathcal{O}(X) \to \mathbb{I}$ on $X$ extends uniquely to a Borel measure $\mu : \mathcal{B}(X) \to \mathbb{I}$. If $X$ is a regular space, then this measure is outer regular.*

## 3 Probabilistic Names

As discussed in the introduction, we wish to consider probabilistic processes over $\Sigma^\omega$ as inducing probability measures on a space $X$ with representation $\delta :\subseteq \Sigma^\omega \to X$. The notion of probabilistic process encapsulates the natural way of generating an infinite word by making a sequence of probabilistic choices, each dependent upon the outcomes of the previous choices.

**Definition 3 (Probabilistic process)** A *probabilistic process* on $\Sigma^\omega$ is a function $\pi : \Sigma^* \to \mathbb{I}$ satisfying

$$\pi(\epsilon) = 1 \;\text{ and }\; \pi(w) = \sum_{a \in \Sigma} \pi(wa) \text{ for all } w \in \Sigma^*. \tag{2}$$

A probabilistic process $\pi$ can be considered as a valuation on the base sets $w\Sigma^\omega$ of the Cantor space assigning $\pi(w)$ to $w\Sigma^\omega$ as its mass. It turns out that this assignment extends to a Borel measure on the Cantor space, which we will denote by $\hat{\pi}$.

**Lemma 4**
*Let $\pi : \Sigma^* \to \mathbb{I}$ be a probabilistic process. Then the function $\hat{\pi} : \mathcal{O}(\Sigma^\omega) \to \mathbb{I}$ defined*

*by*

$$\hat{\pi}(W\Sigma^\omega) := \sum_{w \in W} \pi(w) \quad \text{for all prefix-free sets } W \subseteq \Sigma^* \tag{3}$$

*is a continuous valuation on $\Sigma^\omega$. It extends to an outer regular Borel measure $\hat{\pi} : \mathcal{B}(\Sigma^\omega) \to \mathbb{I}$.*

**Sketch of Proof:** At first one shows that all $U \in \mathcal{O}(\Sigma^\omega)$ and all prefix-free sets $W \subseteq \Sigma^*$ with $U = W\Sigma^\omega$ satisfy $\sum_{w \in W} \pi(w) = \sum_{u \in A_U} \pi(u)$, where $A_U := \{u \in \Sigma^* \,|\, u\Sigma^\omega \subseteq U, \ (\forall w \sqsubsetneq u) \, w\Sigma^\omega \nsubseteq U\}$. For every $u \in A_U$, the set $F_u := \{w \in W \,|\, u \sqsubseteq w\}$ is finite, because $\{w\Sigma^\omega \,|\, w \in F_u\}$ is a disjoint open cover of the compact set $u\Sigma^\omega$. By induction one can show $\pi(u) = \sum_{w \in F_u} \pi(w)$. Hence $\sum_{w \in W} \pi(w) = \sum_{u \in A_U} \sum_{w \in F_u} \pi(w) = \sum_{u \in A_U} \pi(u) = \hat{\pi}(U)$. Therefore Equation (3) defines $\hat{\pi}$ unambiguously. With the help of this observation one can easily derive that $\hat{\pi}$ is a continuous valuation. Since $\mathcal{O}(\Sigma^\omega)$ is a regular topology, $\hat{\pi}$ extends to an outer regular Borel measure $\hat{\pi} : \mathcal{B}(\Sigma^\omega) \to \mathbb{I}$ by Theorem 2, which is given by $\hat{\pi}(B) = \inf\{\hat{\pi}(U) \,|\, U \in \mathcal{O}(\Sigma^\omega), \ B \subseteq U\}$ for any Borel set $B$. $\quad\square$

Let $X$ be a topological space with continuous representation $\delta :\subseteq \Sigma^\omega \to X$. One can think of an open subset of $U \subseteq X$ as being "observable" by checking whether a name $r \in \Sigma^\omega$ is contained in a given open subset $V \subseteq \Sigma^\omega$ with $V \cap \text{dom}(\delta) = \delta^{-1}[U]$, where the latter test is observable in the sense that $V$ is a countable union of basic opens $v\Sigma^\omega$, and to check $r \in v\Sigma^\omega$ requires only a finite test. In the case that $\delta$ is a quotient representation, the open subsets of $X$ are the only subsets that are observable in this sense.

The next definition implements a natural notion of when a probabilistic process on $\Sigma^\omega$ can be seen as implementing a notion of probabilistic choice over the represented space $X$. The basic idea is that every observable property of $X$ should have a uniquely determined probability of being satisfied by the probabilistic process. Thus, for any open subset $U \subseteq X$, there should be a uniquely determined probability that the process satisfies any test $V \subseteq \Sigma^\omega$ for $U$, independent of the choice of such a $V$.

**Definition 5 (Probabilistic name)** A *probabilistic name* for a continuous representation $\delta :\subseteq \Sigma^\omega \to X$ is a probabilistic process $\pi$ on $\Sigma^\omega$ satisfying, for all open $U \in \mathcal{O}(X)$ and $V_1, V_2 \in \mathcal{O}(\Sigma^\omega)$,

$$V_1 \cap \text{dom}(\delta) = \delta^{-1}[U] = V_2 \cap \text{dom}(\delta) \quad \text{implies} \quad \hat{\pi}(V_1) = \hat{\pi}(V_2). \tag{4}$$

If $\pi$ is a probabilistic process for $\delta$, then it follows in particular that $\hat{\pi}^*(\text{dom}(\delta)) = 1$. This is a weak way of saying that the process $\pi$ lands in $\text{dom}(\delta)$ with probability 1. A stronger requirement would be to ask for $\Sigma^\omega \backslash \text{dom}(\delta)$ to be a $\hat{\pi}$-null-set, which is equivalent to requiring that $\text{dom}(\delta)$ be $\hat{\pi}$-measurable. We call a probabilistic name satisfying this additional requirement a *strong probabilistic name.*

We now work towards establishing that every probabilistic name gives rise to a continuous valuation and a Borel measure on the represented space $X$ (Theorem 8 below). Given a Borel measure $\mu : \mathcal{B}(\Sigma^\omega) \to \mathbb{I}$ and a subset $S \subseteq \Sigma^\omega$, we define

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the restriction $\mu\!\downarrow_S : \mathcal{B}(S) \to \mathbb{I}$ by $\mu\!\downarrow_S(B) := \mu^*(B)$ for $B \in \mathcal{B}(S)$. In general, $\mu\!\downarrow_S$ is neither a Borel measure nor a valuation on the subspace $S$. However, when $\mu^*(S) = 1$ it is.

**Lemma 6** *Let $\pi : \Sigma^* \to \mathbb{I}$ be a probabilistic process and let $S \subseteq \Sigma^\omega$ be a subset satisfying $\hat{\pi}^*(S) = 1$.*

(1) *The function $\hat{\pi}\!\downarrow_S$ is a Borel measure on the subspace $S$.*

(2) *For all $O \in \mathcal{O}(\Sigma^\omega)$, we have $\hat{\pi}\!\downarrow_S(O \cap S) = \hat{\pi}(O)$.*

(3) *If $S$ is a Borel set, then $\hat{\pi}\!\downarrow_S(B) = \hat{\pi}(B)$ for all Borel sets $B \in \mathcal{B}(S)$.*

**PROOF.**

(2) Let $O \in \mathcal{O}(\Sigma^\omega)$. Choose a prefix-free set $W \subseteq \Sigma^*$ with $W\Sigma^\omega = O$. Let $V \in \mathcal{O}(\Sigma^\omega)$ with $V \cap S = O \cap S$. For every $w \in W$, we have $S \subseteq (w\Sigma^\omega \cap S) \uplus (\Sigma^\omega \setminus w\Sigma^\omega) \subseteq (w\Sigma^\omega \cap V) \uplus (\Sigma^\omega \setminus w\Sigma^\omega)$ and thus $\hat{\pi}(w\Sigma^\omega \cap V) + \hat{\pi}(\Sigma^\omega \setminus w\Sigma^\omega) \geq \hat{\pi}^*(S) = 1 = \hat{\pi}(w\Sigma^\omega) + \hat{\pi}(\Sigma^\omega \setminus w\Sigma^\omega)$, implying $\hat{\pi}(w\Sigma^\omega \cap V) = \hat{\pi}(w\Sigma^\omega)$. It follows $\hat{\pi}(V) \geq \hat{\pi}(\uplus_{w \in W} w\Sigma^\omega \cap V) = \sum_{w \in W} \hat{\pi}(w\Sigma^\omega \cap V) = \sum_{w \in W} \hat{\pi}(w\Sigma^\omega) = \hat{\pi}(O)$. Hence

$$\hat{\pi}\!\downarrow_S(O \cap S) = \hat{\pi}^*(O \cap S) = \inf\left\{\hat{\pi}(V) \,\middle|\, V \in \mathcal{O}(\Sigma^\omega),\, O \cap S \subseteq V\right\}$$
$$= \inf\left\{\hat{\pi}(V) \,\middle|\, V \in \mathcal{O}(\Sigma^\omega),\, V \cap S = O \cap S\right\} = \hat{\pi}(O).$$

(1) From (2) and the fact that $\hat{\pi}$ is a continuous valuation, it can easily be deduced that $\hat{\pi}\!\downarrow_S : \mathcal{O}(S) \to \mathbb{I}$ is a continuous valuation. By Theorem 2 this continuous valuation extends to an outer regular Borel measure $\mu$. For all $B \in \mathcal{B}(S)$ we have

$$\mu(B) = \inf\left\{\hat{\pi}\!\downarrow_S(U) \,\middle|\, U \in \mathcal{O}(S),\, B \subseteq U\right\}$$
$$= \inf\left\{\inf\{\hat{\pi}(O) \,|\, O \in \mathcal{O}(\Sigma^\omega),\, U \subseteq O\} \,\middle|\, U \in \mathcal{O}(S),\, B \subseteq U\right\}$$
$$= \inf\left\{\hat{\pi}(O) \,\middle|\, O \in \mathcal{O}(\Sigma^\omega),\, B \subseteq O\right\}$$
$$= \hat{\pi}^*(B) = \hat{\pi}\!\downarrow_S(B).$$

Therefore $\hat{\pi}\!\downarrow_S$ is a Borel measure.

(3) If $S$ is a Borel set, then $\mathcal{B}(S) \subseteq \mathcal{B}(\Sigma^\omega)$. Hence $\hat{\pi}\!\downarrow_S(B) = \hat{\pi}^*(B) = \hat{\pi}(B)$. $\square$

It follows from statement (2) in Lemma 6 that property (4) in Definition 5 can be replaced with the apparently weaker requirement that $\hat{\pi}^*(\mathrm{dom}(\delta)) = 1$.

**Proposition 7** *A probabilistic process $\pi$ is a probabilistic name for a representation $\delta$ if and only if $\hat{\pi}^*(\mathrm{dom}(\delta)) = 1$.*

Statement (1) of Lemma 6 implies that a probabilistic name induces a Borel measure on the represented topological spaces.

**Theorem 8** *Let $X$ be a topological space, $\delta$ be a continuous representation of $X$ and $\pi$ be a probabilistic name for $\delta$. Then:*

(1) *$\nu := \hat{\pi}\!\downarrow_{\mathrm{dom}(\delta)} \circ \delta^{-1} : \mathcal{O}(X) \to \mathbb{I}$ is a continuous valuation, and*

(2) $\mu := \hat{\pi}\downarrow_{\mathrm{dom}(\delta)} \circ \delta^{-1} : \mathcal{B}(X) \to \mathbb{I}$ *is a Borel measure.*
*Moreover, if* $\mathrm{dom}(\delta)$ *is a Borel set of* $\Sigma^{\omega}$, *then* $\pi$ *is a strong probabilistic name.*

**PROOF.** From Lemmas 1 and 6 it follows that $\mu$ is a Borel measure. Since $X$ is hereditarily Lindelöf by having a continuous representation, this implies that $\nu$ is a continuous valuation. If $\mathrm{dom}(\delta)$ is a Borel set, then $\hat{\pi}^*(\Sigma^{\omega} \setminus \mathrm{dom}(\delta)) = \hat{\pi}(\Sigma^{\omega} \setminus \mathrm{dom}(\delta)) = 1 - \hat{\pi}(\mathrm{dom}(\delta)) = 0$. $\square$

## 4 Representing Valuations and Borel Measures

We have seen that a probabilistic name gives rise to a Borel measure on a represented space. In this section, we consider probabilistic names as inducing a natural representation for the set of measures so determined. Further, we prove the fundamental fact that the sets of representable (by a probabilistic name) Borel measures over a space $X$ induced by two equivalent representations of $X$ coincide (Corollary 12). In particular, any two admissible quotient representations of $X$ give rise to the same set of measures, and so this set of measures is a topological invariant of the space $X$.

Throughout this section we work, for convenience, with continuous valuations rather than Borel measures. This makes no difference since the representable (by a probabilistic name) valuations all extend (uniquely) to a Borel measure, by Theorem 8.

Let $\delta :\subseteq \Sigma^{\omega} \to X$ be a representation. Let $\nu : \mathcal{O}(X) \to \mathbb{I}$ be a valuation and $\pi$ a probabilistic process. Then we say that $\pi$ is a *($\delta$–) probabilistic name of* $\nu$ if $\nu(U) = \hat{\pi}\downarrow_{\mathrm{dom}(\delta)}(\delta^{-1}[U])$ for all $U \in \mathcal{O}(X)$. It is clear that $\pi$ is indeed a probabilistic name. Moreover, $\nu$ is called *representable* if $\nu$ has a probabilistic name. We denote by $\mathcal{V}(\delta)$ (by $\mathcal{V}_S(\delta)$) the set of valuations $\nu : \mathcal{O}(\delta) \to \mathbb{I}$ that have an ordinary (respectively strong) $\delta$-probabilistic name. Analogously, we define the sets of probabilistic Borel measures that have an ordinary (a strong) $\delta$-probabilistic name. We denote these sets by $\mathcal{M}(\delta)$ and $\mathcal{M}_S(\delta)$, respectively.

There are two straightforward ways to equip $\mathcal{V}(\delta)$ and $\mathcal{V}_S(\delta)$ with representations, namely by using either $[\varrho_{\Sigma^*} \to \varrho_<]$ or $[\varrho_{\Sigma^*} \to \varrho_=]$ as a representation of $\mathcal{C}(\Sigma^*, \mathbb{I})$ (cf. Section 2.1). We define $\delta^{\mathcal{V}<}, \delta^{\mathcal{V}} :\subseteq \Sigma^{\omega} \to \mathcal{V}(\delta)$ by

$$\delta^{\mathcal{V}<}(p) = \nu \; :\Longleftrightarrow \; [\varrho_{\Sigma^*} \to \varrho_<](p) \text{ is a probabilistic name of } \nu \text{ under } \delta,$$
$$\delta^{\mathcal{V}}(p) = \nu \; :\Longleftrightarrow \; [\varrho_{\Sigma^*} \to \varrho_=](p) \text{ is a probabilistic name of } \nu \text{ under } \delta.$$

By $\delta^{\mathcal{V}_S<}$, $\delta^{\mathcal{V}_S}$, $\delta^{\mathcal{M}<}$, $\delta^{\mathcal{M}}$, $\delta^{\mathcal{M}_S<}$ and $\delta^{\mathcal{M}_S}$, we denote the corresponding representations of $\mathcal{V}_S(\delta)$, $\mathcal{M}(\delta)$ and $\mathcal{M}_S(\delta)$. It turns out that both constructions lead to computably equivalent representations. So we can equivalently use either of them to construct our ambient representations of $\mathcal{V}(\delta)$, $\mathcal{V}_S(\delta)$, $\mathcal{M}(\delta)$ and $\mathcal{M}_S(\delta)$.

**Lemma 9** *For any representation* $\delta$, *we have* $\delta^{\mathcal{V}<} \equiv_{cp} \delta^{\mathcal{V}}$, $\delta^{\mathcal{V}_S<} \equiv_{cp} \delta^{\mathcal{V}_S}$, $\delta^{\mathcal{M}<} \equiv_{cp}$ $\delta^{\mathcal{M}}$ *and* $\delta^{\mathcal{M}_S<} \equiv_{cp} \delta^{\mathcal{M}_S}$.

**PROOF.** Since $\varrho_= \leq_{cp} \varrho_<$, we have $\delta^{\mathcal{V}} \leq_{cp} \delta^{\mathcal{V}<}$.

Conversely, let $p \in \mathrm{dom}(\delta^{\mathcal{V}<})$ and $\pi = [\varrho_{\Sigma^*} \to \varrho_<](p)$. The representation $\varrho_<$ allows the approximation of $\pi(w)$ from below. Since $\pi(w) = 1 - \sum\{\pi(u) \mid \lg(u) = \lg(w) \wedge u \neq w\}$, we can also compute $\pi(w)$ from above. By [13, Lemma 4.1.9] this implies that we can compute a $\varrho_=$-name of $\pi(w)$ out of $p$ and $w$. It follows $\delta^{\mathcal{V}<} \leq_{cp} \delta^{\mathcal{V}}$.

The proofs for the other computable equivalences are similar. $\quad\square$

Note that this lemma only holds because of our restriction to probabilistic valuations.

Given two topological spaces $X$ and $Y$, a continuous function $f : X \to Y$ and a valuation (or Borel measure) $\nu$ on $X$, we know from Lemma 1 that the function $\mathrm{Tr}(v, f) : \mathcal{O}(Y) \to \mathbb{I}$ defined by $\mathrm{Tr}(\nu, f)(V) = \nu(f^{-1}[V])$ is a valuation (Borel measure) on $Y$. We show some computability properties of $\mathrm{Tr}$.

**Proposition 10** *Let* $\delta :\subseteq \Sigma^\omega \to X$ *and* $\gamma :\subseteq \Sigma^\omega \to Y$ *be representations.*
 (1) *The function* $\mathrm{Tr} : \mathcal{V}(\delta) \times \mathcal{C}(\delta, \gamma) \to \mathcal{V}(\gamma)$ *is computable* [2].
 (2) *The function* $\mathrm{Tr} : \mathcal{V}_S(\delta) \times \mathcal{C}(\delta, \gamma) \to \mathcal{V}_S(\gamma)$ *is computable* [2].
 (3) *The function* $\mathrm{Tr} : \mathcal{M}(\delta) \times \mathcal{C}(\delta, \gamma) \to \mathcal{M}(\gamma)$ *is computable* [3].
 (4) *The function* $\mathrm{Tr} : \mathcal{M}_S(\delta) \times \mathcal{C}(\delta, \gamma) \to \mathcal{M}_S(\gamma)$ *is computable* [3].

For the proof, we need the following lemma.

**Lemma 11** *Let* $\pi : \Sigma^* \to \mathbb{I}$ *be a probabilistic process and let* $h : \Sigma^* \to \Sigma^*$ *be a monotone function such that* $\hat{\pi}(\mathrm{dom}(h^\omega)) = 1$. *Define* $\xi : \Sigma^* \to \mathbb{I}$ *by*

$$\xi(v) := \sum \left\{ \pi(u) \,\middle|\, v \sqsubseteq h(u) ,\ (\forall w \sqsubsetneq u)\, v \not\sqsubseteq h(w) \right\}$$

 (1) *The function* $\xi$ *is a probabilistic process satisfying* $\hat{\xi} = \mathrm{Tr}(\hat{\pi}{\downarrow}_{\mathrm{dom}(h^\omega)}, h^\omega)$.
 (2) *A subset* $N \subseteq \Sigma^\omega$ *is a* $\hat{\xi}$*-null-set if and only if* $(h^\omega)^{-1}[N]$ *is a* $\hat{\pi}$*-null-set.*

**PROOF.**
 (1) By Lemmas 1 and 6, $\lambda := \mathrm{Tr}(\hat{\pi}{\downarrow}_{\mathrm{dom}(h^\omega)}, h^\omega)$ is a Borel measure. Let $v \in \Sigma^\omega$. Define $W := \{u \in \Sigma^* \mid v \sqsubseteq h(u) ,\ (\forall w \sqsubsetneq u)\, v \not\sqsubseteq h(w)\}$. Then $W$ is prefix-free. For every $r \in \mathrm{dom}(h^\omega)$ we have

$$h^\omega(r) \in v\Sigma^\omega \Leftrightarrow (\exists u \sqsubseteq r)\, v \sqsubseteq h(u) \Leftrightarrow (\exists u \in W)\, r \in u\Sigma^\omega \Leftrightarrow r \in W\Sigma^\omega,$$

hence $(h^\omega)^{-1}[v\Sigma^\omega] = W\Sigma^\omega \cap \mathrm{dom}(h^\omega)$. By Lemma 6, it follows $\lambda(v\Sigma^\omega) = \hat{\pi}{\downarrow}_{\mathrm{dom}(h^\omega)}((h^\omega)^{-1}[v\Sigma^\omega]) = \hat{\pi}(W\Sigma^\omega) = \xi(v)$. In particular, this means that $\xi$ is a probabilistic process. Structural induction shows $\hat{\xi}(B) = \lambda(B)$ for all Borel sets $B \in \mathcal{B}(\Sigma^\omega)$.
 (2) *If-part:* Let $(h^\omega)^{-1}[N]$ be a $\hat{\pi}$-null-set. Then there is a Borel set $B$ satisfying $(h^\omega)^{-1}[N] \subseteq B$ and $\hat{\pi}(B) = 0$. Since the set $A := h^\omega[\Sigma^\omega \setminus B]$ is the image of

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[2] The corresp. representations are $\delta^{\mathcal{V}}, [\delta \to \gamma], \gamma^{\mathcal{V}}$ and $\delta^{\mathcal{V}_S}, [\delta \to \gamma], \gamma^{\mathcal{V}_S}$, respectively.
[3] The corresp. representations are $\delta^{\mathcal{M}}, [\delta \to \gamma], \gamma^{\mathcal{M}}$ and $\delta^{\mathcal{M}_S}, [\delta \to \gamma], \gamma^{\mathcal{M}_S}$, respectively.

a Borel set under a partial function whose graph $\{(p, h^\omega(p)) \mid p \in \mathrm{dom}(h^\omega)\}$ is a Borel set, $A$ is the projection of a Borel set in $\Sigma^\omega \times \Sigma^\omega$ and thus analytic (i.e. the continuous image of $\mathbb{N}^\mathbb{N}$) by [4, Lemma 11.6]. By [2, Theorem 8.4.1], any analytic subset of the Polish space $\Sigma^\omega$ is measurable with respect to any Borel measure on $\Sigma^\omega$. Hence $A$ is $\hat\xi$-measurable, implying that there are Borels sets $C, D$ with $C \subseteq h^\omega[\Sigma^\omega \setminus B] \subseteq D$ and $\hat\xi(C) = \hat\xi(D)$. Since $N \subseteq \Sigma^\omega \setminus h^\omega[\Sigma^\omega \setminus B] \subseteq \Sigma^\omega \setminus C$ and $(h^\omega)^{-1}[\Sigma^\omega \setminus D]$ is a Borel set contained in $B$, we obtain by Lemma 6

$$\hat\xi^*(N) \le \hat\xi(\Sigma^\omega \setminus C) = \hat\xi(\Sigma^\omega \setminus D) = \hat\pi\!\downarrow_{\mathrm{dom}(h^\omega)}((h^\omega)^{-1}[\Sigma^\omega \setminus D])$$
$$= \hat\pi((h^\omega)^{-1}[\Sigma^\omega \setminus D]) \le \hat\pi(B) = 0\,.$$

Hence $N$ is a $\hat\xi$-null-set.

*Only-if-part:* Let $N$ be a $\hat\xi$-null-set and $\varepsilon > 0$. Since $\hat\xi$ is regular, there is some open set $V \supseteq N$ with $\hat\xi(V) < \varepsilon$. As $(h^\omega)^{-1}[N] \subseteq (h^\omega)^{-1}[V]$, we have $\hat\pi^*((h^\omega)^{-1}[N]) \le \hat\pi((h^\omega)^{-1}[V]) = \hat\xi(V) < \varepsilon$. Thus $(h^\omega)^{-1}[N]$ is a $\hat\pi$-null-set. $\square$

Now we are ready to prove Proposition 10.

**Proof of Proposition 10:** Using the computable function $\eta :\subseteq (\Sigma^\omega)^2 \to \Sigma^\omega$ from [13, Theorem 2.3.10], the function space representation $[\delta \to \gamma]$ is defined by $[\delta \to \gamma](q) = f$ iff $\eta_q$ is a $(\delta, \gamma)$-realiser of $f$. Let $h : (\Sigma^*)^2 \to \Sigma^*$ be a $\sqsubseteq$-monotone computable word function satisfying $h^\omega = \eta$.

Let $p \in \mathrm{dom}(\delta^\mathcal{V})$ and $q \in \mathrm{dom}([\delta \to \gamma])$. Set $\nu := \delta^\mathcal{V}(p)$, $\pi := [\varrho_{\Sigma^*} \to \varrho_=](p)$ and $f := [\delta \to \gamma](q)$. Define $h_q : \Sigma^* \to \Sigma^*$ and $\xi_{p,q} : \Sigma^* \to \mathbb{I}$ by $h_q(u) := h(q^{<\mathrm{lg}(u)}, u)$ and

$$\xi_{p,q}(v) := \sum \Big\{ \pi(u) \,\Big|\, v \sqsubseteq h_q(u)\,, \ (\forall w \sqsubsetneq u)\, v \not\sqsubseteq h_q(w) \Big\}\,.$$

Let $V \in \mathcal{O}(Y)$ and let $O \in \mathcal{O}(\Sigma^\omega)$ with $\gamma^{-1}[V] = O \cap \mathrm{dom}(\gamma)$. Then $\delta^{-1}[f^{-1}[V]] = (h_q^\omega)^{-1}[O] \cap \mathrm{dom}(\delta)$, because $h_q^\omega$ is a $(\delta, \gamma)$-realiser of $f$. Since $\pi$ is a $\delta$-probabilistic name of $\nu$ and $\mathrm{dom}(h_q^\omega)$ contains $\mathrm{dom}(\delta)$, we have $\hat\pi^*(\mathrm{dom}(h_q^\omega)) = 1$. From Lemmas 6 and 11 we obtain

$$\mathrm{Tr}(\nu, f)(V) = \nu(f^{-1}[V]) = \hat\pi\!\downarrow_{\mathrm{dom}(\delta)}\Big(\delta^{-1}[f^{-1}[V]]\Big)$$
$$= \hat\pi\!\downarrow_{\mathrm{dom}(\delta)}\Big((h_q^\omega)^{-1}[O] \cap \mathrm{dom}(\delta)\Big) = \hat\pi^*\Big((h_q^\omega)^{-1}[O]\Big)$$
$$= \hat\pi\!\downarrow_{\mathrm{dom}(h_q^\omega)}\Big((h_q^\omega)^{-1}[O]\Big) = \widehat{\xi_{p,q}}(O)\,.$$

Therefore $\widehat{\xi_{p,q}}$ is a $\gamma$-probabilistic name of $\mathrm{Tr}(\nu, f)$.

Since any name $r$ provided by $\varrho_<$ encodes in an effective way all rationals below $\varrho_<(r)$, one can show by standard methods of Type-2 Theory that there is a computable function $g :\subseteq (\Sigma^\omega)^2 \to \Sigma^\omega$ satisfying $\xi_{p,q} = [\varrho_{\Sigma^*} \to \varrho_<](g(p, q))$. Hence $\gamma^{\mathcal{V}<}(g(p, q)) = \mathrm{Tr}(\nu, f)$, i.e. $g$ realises $\mathrm{Tr}$ with respect to $\delta^\mathcal{V}$, $[\delta \to \gamma]$ and

$\gamma^{\mathcal{V}<}$. This implies by Lemma 9 that $\mathrm{Tr} : \mathcal{V}(\delta) \times \mathcal{C}(\delta, \gamma) \to \mathcal{V}(\gamma)$ is computable with respect to $\delta^{\mathcal{V}}$, $[\delta \to \gamma]$ and $\gamma^{\mathcal{V}}$.

Now we assume additionally $\nu = \delta^{\mathcal{V}_S}(p)$, hence $\pi = [\varrho_{\Sigma^*} \to \varrho_=](p)$ is a $\delta$-probabilistic name of $\nu$ and $\hat{\pi}^*(\Sigma^\omega \setminus \mathrm{dom}(\delta)) = 0$. Then $(h_q^\omega)^{-1}(\Sigma^\omega \setminus \mathrm{dom}(\gamma))$ is a $\hat{\pi}$-nullset by being a subset of $\Sigma^\omega \setminus \mathrm{dom}(\delta)$. Lemma 11 implies that $\Sigma^\omega \setminus \mathrm{dom}(\gamma)$ is $\widehat{\xi_{p,q}}$-null-set. Thus $\xi_{p,q}$ is a strong $\gamma$-probabilistic name of $\mathrm{Tr}(\nu, f)$ and $\gamma^{\mathcal{V}_S<}(g(p,q)) = \mathrm{Tr}(\nu, f)$. Therefore $\mathrm{Tr} : \mathcal{V}_S(\delta) \times \mathcal{C}(\delta, \gamma) \to \mathcal{V}_S(\gamma)$ is computable with respect to $\delta^{\mathcal{V}_S}$, $[\delta \to \gamma]$ and $\gamma^{\mathcal{V}_S}$ by Lemma 9.
The proofs of Statements (3) and (4) are similar. $\quad \square$

As a corollary we obtain that topologically equivalent representations $\delta$ induce the same class of valuations that have ordinary (or strong) $\delta$-probabilistic names.

**Corollary 12** *Let $\delta$ and $\gamma$ be representations of a set $X$.*
(1) *If $\delta \leq_t \gamma$, then $\mathcal{V}(\delta) \subseteq \mathcal{V}(\gamma)$, $\mathcal{V}_S(\delta) \subseteq \mathcal{V}_S(\gamma)$, $\mathcal{M}(\delta) \subseteq \mathcal{M}(\gamma)$, $\mathcal{M}_S(\delta) \subseteq \mathcal{M}_S(\gamma)$.*
(2) *If $\delta \equiv_t \gamma$, then $\mathcal{V}(\delta) = \mathcal{V}(\gamma)$, $\mathcal{V}_S(\delta) = \mathcal{V}_S(\gamma)$, $\mathcal{M}(\delta) = \mathcal{M}(\gamma)$, $\mathcal{M}_S(\delta) = \mathcal{M}_S(\gamma)$.*

## 5 Lifting Valuations to Probabilistic Names

In this section we show that for well-behaved classes of topological spaces, every continuous valuation (and hence Borel measure) has a probabilistic name under an admissible representation.

A *point-pseudobase* of a topological space $X$ is a countable family $\mathcal{P}$ of subsets of $X$ such that any open subset of $X$ is the union of a subfamily of $\mathcal{P}$.

**Proposition 13** *Let $\delta$ be an admissible representation of a topological space $X$. Let $\mu$ be a (probabilistic) Borel measure on $X$, and let $\mathcal{P}$ be a countable point-pseudobase of $X$ consisting of $\mu$-measurable sets. Then $\mu$ has a $\delta$-probabilistic name.*

**PROOF.** Let $\mathcal{P} = \{B_0, B_1, \dots\}$. We define an injective representation $\phi :\subseteq \Sigma^* \to X$ by

$$\phi(p) = x :\Longleftrightarrow \left(\forall i \in \mathbb{N}\right) p(i) = \begin{cases} 1 \text{ if } x \in B_i \\ 0 \text{ otherwise.} \end{cases}$$

Then $\phi$ is well-defined, since $X$ is a $T_0$-space by having an admissible representation (cf. [8]). Moreover, $\phi$ is continuous, because for every $p \in \mathrm{dom}(\phi)$ and every open $U$ containing $\phi(p)$ there is some $i \in \mathbb{N}$ such that $\phi(p) \in B_i \subseteq U$, hence $p \in \{q \in \mathrm{dom}(\phi) \,|\, q(i) = 1\} \subseteq \phi^{-1}[B_i] \subseteq \phi^{-1}[U]$. For every word $w \in \{0, 1\}^*$, $\phi[w\Sigma^\omega]$ is $\mu$-measurable by being an intersection of $\mu$-measurable sets, thus $\mu^*(\phi[w\Sigma^\omega]) = \mu^*(\phi[w1\Sigma^\omega]) + \mu^*(\phi[w0\Sigma^\omega])$. Hence $\pi : \Sigma^* \to \mathbb{I}$ defined by $\pi(w) := \mu^*(\phi[w\Sigma^\omega])$ is a probabilistic process.
Let $U \in \mathcal{O}(X)$ and let $W$ be prefix-free with $\phi^{-1}[U] = W\Sigma^\omega \cap \mathrm{dom}(\phi)$. Then $U$ is the disjoint union of the sets in $\{\phi[w\Sigma^\omega] \,|\, w \in W\}$. By $\sigma$-additivity of the

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restriction of $\mu^*$ to the $\mu$-measurable sets it follows $\mu(U) = \sum_{w \in W} \mu^*(\phi[w\Sigma^\omega]) = \sum_{w \in W} \pi(w)$. Hence $\pi$ is a $\phi$-probabilistic name of $\mu$. Since $\phi$ is continuous and $\delta$ is admissible, we have $\phi \leq_t \delta$. By Corollary 12, $\mu$ has a $\delta$-probabilistic name as well. $\square$

Note that this proposition does not hold if we only require $\nu$ to be a continuous valuation rather than a Borel measure. As a counterexample, we consider $\mathbb{N}$ equipped with the Alexandroff topology $\tau := \{\emptyset, \{n, n+1, \dots\} \mid n \in \mathbb{N}\}$. The continuous valuation $\nu$ defined by $\nu(U) = 1 :\Longleftrightarrow U \neq \emptyset$ cannot be extended to a Borel measure $\mu$, because $\mu$ would have to assign 0 to the closed sets $A_n := \{0, \dots, n\}$, but the countable union of the sets $A_n$ has weight 1. Thus $\nu$ does not have a probabilistic name under any admissible representation of the (locally compact non-sober) space $(\mathbb{N}, \tau)$.

We present three classes of spaces such that every continuous valuation has a strong probabilistic name w.r.t. any admissible representation. A topological space $X$ is *locally compact* iff for every point $x$ and every open neighbourhood $U$ of $x$ there is a compact neighbourhood $K$ of $x$ with $K \subseteq U$. It is *sober* iff for every completely prime [4] filter $\mathcal{F} \subseteq \mathcal{O}(X)$ there is some point $x$ with $\mathcal{F} = \{U \in \mathcal{O}(X) \mid x \in U\}$ (cf. [11]). Finally, a topological space is *co-countably-based* iff the Scott topology on the opens is countably-based (cf. [9]).

**Theorem 14** *Let $X$ be a sequential space that is (a) separable and completely metrisable or (b) countably-based $T_0$, locally compact and sober or (c) co-countably-based and regular. Then every continuous valuation on $X$ has a strong probabilistic name under every admissible representation of $X$.*

**PROOF.** By Theorem 2, Proposition 13 and Corollary 12, it suffices to show that, in each case, $X$ has an admissible representation whose domain is a Borel set.

(a) Let $d$ be a complete metric on $X$ inducing $\mathcal{O}(X)$, and let $\{\alpha_0, \alpha_1, \dots\}$ be a dense subset of $(X, d)$. We define the *Cauchy representation* $\varrho :\subseteq \Sigma^\omega \to X$ by

$$\varrho(p) = x :\Longleftrightarrow \left(\exists k_0, k_1, \dots \in \mathbb{N}\right)\left(p = 0^{k_0} 1 0^{k_1} 1 \dots, \right.$$
$$\left. (\forall i < j)\, d(\alpha_{k_i}, \alpha_{k_j}) \leq 2^{-i} \text{ and } \lim_{n \to \infty} \alpha_{k_n} = x\right).$$

Similar to [13, Theorem 8.1.4], one can prove admissibility of $\varrho$. Since $d$ is a complete metric, we have

$$\mathrm{dom}(\varrho) = \Big\{p \in \Sigma^\omega \,\Big|\, \left(\exists k_0, k_1, \dots \in \mathbb{N}\right)$$
$$\left(p = 0^{k_0} 1 0^{k_1} 1 \dots \text{ and } (\forall i < j)\, d(\alpha_{k_i}, \alpha_{k_j}) \leq 2^{-i}\right)\Big\}.$$

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[4] A filter $\mathcal{F} \subseteq \mathcal{O}(X)$ is *completely prime* iff, for any family $\mathcal{U}$ of opens, $\bigcup \mathcal{U} \in \mathcal{F}$ implies $\mathcal{U} \cap \mathcal{F} \neq \emptyset$.

Thus the domain of $\varrho$ is a $G_\delta$-set and hence a Borel set.

(b) Let $\{B_0, B_1, \dots\}$ be a countable base of $X$. Define the representation $\varrho :\subseteq \Sigma^\omega \to X$ by

$$\varrho(p) = x \;:\Longleftrightarrow\; \begin{cases} \mathrm{En}(p) \subseteq \{i \in \mathbb{N} \mid x \in B_i\} \text{ and} \\[2mm] (\forall U \in \mathcal{O}(X) : x \in U)(\exists i \in \mathrm{En}(p))B_i \subseteq U \,, \end{cases}$$

where $\mathrm{En} :\subseteq \Sigma^\omega \to 2^\mathbb{N}$ is an open and admissible representation of $2^\mathbb{N}$ equipped with the Scott-topology (cf. [13, Definition 3.1.2]). Similar to [8, Theorem 12] one can show that $\varrho$ is admissible. Define

$$D_1 := \left\{ p \in \Sigma^\omega \,\middle|\, \mathrm{En}(p) \neq \emptyset, \; (\forall E \subseteq \mathrm{En}(p) \text{ finite}) B_E^\cap \neq \emptyset \right\}$$
$$D_2 := \left\{ p \in \Sigma^\omega \,\middle|\, (\forall a, b \in \mathrm{En}(p))(\exists c \in \mathrm{En}(p)) \, B_c \ll B_a \cap B_b \right\}$$
$$D_3 := \left\{ p \in \Sigma^\omega \,\middle|\, (\forall a \in \mathrm{En}(p))(\forall F \text{ finite} : B_a \subseteq B_F^\cup) \right.$$
$$\left. (\exists b \in F)(\exists c \in \mathrm{En}(p)) \, B_c \ll B_a \cap B_b \right\},$$

where $B_F^\cap := \bigcap_{i \in F} B_i$ and $B_F^\cup := \bigcup_{i \in F} B_i$ for a finite set $F \subseteq \mathbb{N}$ and $\ll$ denotes the way-below relation on the open sets given by $U \ll V :\Longleftrightarrow (\exists K \text{ compact}) \, U \subseteq K \subseteq V$. We prove $p \in \mathrm{dom}(\varrho) \iff p \in D_1 \cap D_2 \cap D_3$.
*If-part:* As $p \in D_1 \cap D_2$, the family $\mathcal{F} := \{U \in \mathcal{O}(X) \mid (\exists a \in \mathrm{En}(p))B_a \ll U\}$ is a filter. Now let $U \cup V \in \mathcal{F}$ and $a \in \mathrm{En}(p)$ with $B_a \ll U \cup V$. Let $I := \{i \in \mathbb{N} \mid B_i \ll U\}$ and $J := \{j \in \mathbb{N} \mid B_j \ll V\}$. By local compactness of $X$, we have $U \cup V = \bigcup_{i \in I \cup J} B_i$. Hence there is a finite subset $F$ of $I \cup J$ with $B_a \ll B_F^\cup$. As $p \in D_3$, there are $b \in F$ and $c \in \mathrm{En}(p)$ with $B_c \ll B_a \cap B_b$. Depending on whether $b \in I$ or $b \in J$, we have $U \in \mathcal{F}$ or $V \in \mathcal{F}$, thus the filter $\mathcal{F}$ is prime. Since $\mathcal{F}$ is Scott-open by being the union of sets of the form $\{U \in \mathcal{O}(X) \mid C \subseteq U\}$ for some compact set $C \subseteq X$, $\mathcal{F}$ is even completely prime.

By sobriety, there is some $x \in X$ such that $\{U \in \mathcal{O}(X) \mid x \in U\} = \mathcal{F}$. One easily verifies $\varrho(p) = x$.

*Only-if-part:* Let $p \in \mathrm{dom}(\varrho)$. Clearly, $B_E^\cap \neq \emptyset$ for every finite subset $E \subseteq \mathrm{En}(p)$. By local compactness, for every $a, b \in \mathrm{En}(p)$ there are $i \in \mathbb{N}$ and $c \in \mathrm{En}(p)$ with $\varrho(p) \in B_i \ll B_a \cap B_b$ and $B_c \subseteq B_i$, hence $B_c \ll B_a \cap B_b$. Finally, if $a \in \mathrm{En}(p)$ and $F$ is finite with $B_a \subseteq B_F^\cup$, then there is some $b \in F$ with $x \in B_b$. Again by local compactness, there is some $c \in \mathrm{En}(p)$ such that $x \in B_c \ll B_a \cap B_b$. Hence $p \in D_1 \cap D_2 \cap D_3$.

For every finite set $F \subseteq \mathbb{N}$, the set $P_F := \{p \in \Sigma^\omega \mid F \subseteq \mathrm{En}(p)\}$ is open, because $\mathrm{En}$ is open and admissible. Since

$$D_1 = \Sigma^\omega \setminus \bigcup \{P_F \mid F \subseteq \mathbb{N} \text{ finite}, \; B_F^\cap = \emptyset\} \,,$$
$$D_2 = \bigcap \left\{ (\Sigma^\omega \setminus P_{\{a,b\}}) \cup \bigcup \{P_{\{a,b,c\}} \mid B_c \ll B_a \cap B_b\} \,\middle|\, a, b \in \mathbb{N} \right\},$$

$$D_3 = \bigcap \Big\{ (\Sigma^\omega \setminus P_{\{a\}}) \cup \bigcup \{ P_{F \cup \{a,c\}} \mid c \in \mathbb{N}, \ (\exists b \in F) \ B_c \ll B_a \cap B_b \} \Big|$$
$$a \in \mathbb{N}, \ F \subseteq \mathbb{N} \text{ finite}, \ B_a \subseteq B_F^\cup \Big\},$$

the domain of $\varrho$ is a Borel set.

(c) By [10, Theorem 7.3], $X$ has countable pseudobase consisting of compact and hence closed sets, thus Proposition 13 can be applied. Moreover, there is a sequence of compact Hausdorff spaces $X_i$ such that $X_i$ is a closed subspace of $X_{i+1}$ and $X$ is the inductive limit of $(X_i)_i$. Statement (b) yields an admissible representation $\varrho_i$ of $X_i$ such that $\mathrm{dom}(\varrho_i)$ is a Borel set. By [8, Theorem 19], the representation $\varrho :\subseteq \Sigma^\omega \to X$ defined by
$$\mathrm{dom}(\varrho) := \{ 0^n 1 p \mid n \in \mathbb{N}, \ p \in \mathrm{dom}(\varrho_n) \} \quad \text{and} \quad \varrho(0^n 1 p) := \varrho_n(p)$$
is an admissible representation of $X$. Obviously, $\mathrm{dom}(\varrho)$ is a Borel set. $\quad\square$

## 6 Integration

In this section we show that integration with respect to representable valuations is computable with respect to the representation on valuations defined in Section 4.

Let $X$ be a topological space and $\nu$ be a continuous valuation on $X$. For any lower semicontinuous function $f : X \to \mathbb{I}_<$, the mapping $t \mapsto \nu\{x \in X \mid f(x) > t\}$ is decreasing and right-continuous and therefore Riemann-integrable (cf. [2]). Thus integration of $f$ with respect to $\nu$ can be defined as

$$\int f \, d\nu := \int_0^1 \nu\{x \in X \mid f(x) > t\} \, dt \tag{5}$$
$$= \sup \Big\{ \sum_{i=1}^k (a_i - a_{i-1}) \cdot \nu\{x \in X \mid f(x) > a_i\} \,\Big|\, 0 = a_0 < a_1 < \ldots < a_k \leq 1 \Big\}.$$

This integral is known as the *horizontal integral*. It is monotone and satisfies $\int f \, d\nu + \int g \, d\nu = 2 \int (f+g)/2 \, d\nu$ (cf. [7]).

Now let $\pi$ be a $\delta$-probabilistic name. Then we define the integration of $f$ with respect to $\pi$ by [5]

$$\int f \, d\pi := \sup \Big\{ \sum_{w \in W} \inf(f \delta[w \Sigma^\omega]) \cdot \pi(w) \,\Big|\, W \subseteq \Sigma^* \text{ finite, prefix-free} \Big\}. \tag{6}$$

We prove that this integral is equivalent to the horizontal integral.

**Proposition 15** *Let $\delta$ be a quotient representation of a topological space $X$, let $\pi$ be a $\delta$-probabilistic name of a valuation $\nu$, and let $f : X \to \mathbb{I}_<$ be a lower semicontinuous function. Then $\int f \, d\pi = \int f \, d\nu$.*

**PROOF.** "$\geq$": Let $\varepsilon > 0$. Then there is a finite sequence $0 = a_0 < a_1 < \ldots < a_k$ such that $\sum_{i=1}^k (a_i - a_{i-1}) \cdot \nu(f^{-1}(a_i, 1]) > \int f \, d\nu - \varepsilon/2$. By continuity of $f$ and $\delta$,

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[5] We set $\inf(\emptyset) = 0$.

there are open sets $O_1 \supseteq \ldots \supseteq O_k$ in $\mathcal{O}(\Sigma^\omega)$ such that $O_i \cap \mathrm{dom}(\delta) = \delta^{-1} f^{-1}(a_i, 1]$. Hence $\hat{\pi}(O_i) = \nu(f^{-1}(a_i, 1])$ and $\inf(f\delta[O_i]) \geq a_i$. Since $\hat{\pi}$ is continuous, there are finite prefix-free sets $F_1 \supseteq \ldots \supseteq F_k$ such that $F_i \Sigma^\omega \subseteq O_i$ and $\hat{\pi}(F_i \Sigma^\omega) > \nu(f^{-1}(a_i, 1]) - \frac{\varepsilon}{2k(a_i - a_{i-1})}$ for all $i$. We set $F_{k+1} := \emptyset$ and conclude

$$\int f \, d\pi \geq \sum_{w \in F_1} \inf(f\delta[w\Sigma^\omega]) \cdot \pi(w) = \sum_{i=1}^{k} \sum_{w \in F_i \setminus F_{i+1}} \inf(f\delta[w\Sigma^\omega]) \cdot \pi(w)$$

$$\geq \sum_{i=1}^{k} \sum_{w \in F_i \setminus F_{i+1}} a_i \cdot \pi(w) = \sum_{i=1}^{k} \sum_{w \in F_i} (a_i - a_{i-1}) \cdot \pi(w)$$

$$= \sum_{i=1}^{k} (a_i - a_{i-1}) \cdot \hat{\pi}(F_i \Sigma^\omega) \geq \sum_{i=1}^{k} (a_i - a_{i-1}) \cdot \nu(f^{-1}(a_i, 1]) - \frac{\varepsilon}{2}$$

$$> \int f \, d\nu - \varepsilon.$$

This implies $\int f \, d\pi \geq \int f \, d\nu$.

"$\leq$": Let $\varepsilon > 0$. There exists a finite prefix-free set $\{w_1, \ldots, w_k\} \subseteq \Sigma^*$ such that $\sum_{i=1}^{k} \inf(f\delta[w_i \Sigma^\omega]) \cdot \pi(w_i) \geq \int f \, d\pi - \varepsilon/2$. Define $b_i := \max\{0, \inf(f\delta[w_i \Sigma^\omega]) - \frac{\varepsilon}{2}\}$ and $b_0 := 0$. Wlog. we can assume $b_1 \leq \ldots \leq b_k$. Then

$$\int f \, d\pi - \varepsilon \leq \sum_{i=1}^{k} (b_i + \tfrac{\varepsilon}{2}) \cdot \pi(w_i) - \frac{\varepsilon}{2} \leq \sum_{i=1}^{k} b_i \cdot \pi(w_i) = \sum_{i=1}^{k} (b_i - b_{i-1}) \cdot \sum_{j=i}^{k} \pi(w_j)$$

$$\leq \sum_{i=1}^{k} (b_i - b_{i-1}) \cdot \hat{\pi}\Big( \bigcup \{w\Sigma^\omega \mid w\Sigma^\omega \subseteq \delta^{-1} f^{-1}(b_i, 1]\} \Big)$$

$$\leq \sum_{i=1}^{k} (b_i - b_{i-1}) \cdot \nu(f^{-1}(b_i, 1]) \leq \int f \, d\nu.$$

Therefore $\int f \, d\pi = \int f \, d\nu$. $\quad \square$

With the help of Proposition 10 and 15, we show that horizontal integration is computable. For the space $X := \mathbb{I}_=$, this has been shown in [12] using a different representation.

**Theorem 16 (Computability of horizontal integration)**
*Let $\delta$ be a quotient representation of a topological space $X$.*
(1) *The function $\int : \mathcal{C}(X, \mathbb{I}_<) \times \mathcal{V}(\delta) \to \mathbb{I}_<$ is computable*[6] *.*
(2) *The function $\int : \mathcal{C}(X, \mathbb{I}_=) \times \mathcal{V}(\delta) \to \mathbb{I}_=$ is computable*[6] *.*

**PROOF.**
(1) We prove at first that the operator $I : \mathcal{V}(\rho_<) \to \mathbb{I}_<, \ \mu \mapsto \int \mathrm{id}_{\mathbb{I}_<} \, d\mu$, is computable with respect to $(\varrho_<)^{\mathcal{V}}$ and $\varrho_<$. Let $p \in \mathrm{dom}((\varrho_<)^{\mathcal{V}})$ and $\mu =$

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[6] The associated representations are $[\delta \to \varrho_<], \delta^{\mathcal{V}}, \varrho_<$ and $[\delta \to \varrho_=], \delta^{\mathcal{V}}, \varrho_=$, respectively.

$(\varrho_<)^{\mathcal{V}}(p)$. Since $\pi := [\varrho_{\Sigma^*} \to \varrho_=](p)$ is a $\delta$-probabilistic name of $\mu$, we obtain by Proposition 15

$$I(\mu) = \sup \left\{ \sum_{w \in W} \inf(\varrho_<[w\Sigma^\omega]) \cdot \pi(w) \,\Big|\, W \subseteq \Sigma^* \text{ finite, prefix-free} \right\}.$$

The representation $\varrho_<$ has the property that $\inf(\varrho_<[w\Sigma^\omega] \cup \{0\})$ is a rational number and that $w \mapsto \inf(\varrho_<[w\Sigma^\omega] \cup \{0\})$ is computable. Thus we can compute the finite sum $\sum_{w \in W} \inf(\varrho_<[w\Sigma^\omega]) \cdot \pi(w)$ from a finite prefix-free set $W \subseteq \Sigma^*$. This means that we can effectively approximate the supremum $I(\mu)$ from below. Therefore $I$ is $((\varrho_<)^{\mathcal{V}}, \varrho_<)$-computable.

For any $f \in \mathcal{C}(X, \mathbb{I}_<)$ and any $\nu \in \mathcal{V}(\delta)$, we have $\int f \, d\nu = \int \mathrm{id}_{\mathbb{I}_<} \, d\mathrm{Tr}(\nu, f) = I(\mathrm{Tr}(\nu, f))$. Thus $\int$ is computable w.r.t. $[\delta \to \varrho_<]$, $\delta^{\mathcal{V}}$, and $\varrho_<$ by Proposition 10 and by computability of $I$.

(2) For any $f \in \mathcal{C}(X, \mathbb{I}_=)$, we have $\int f \, d\nu = 1 - \int(1 - f) \, d\nu$, as $\int f \, d\nu + \int(1 - f) \, d\nu = 2 \int(f + 1 - f)/2 \, d\nu = 1$. With some standard methods from Type-2 Theory, one can prove that $f \mapsto 1 - f$ is $([\delta \to \varrho_=], [\delta \to \varrho_<])$-computable. By (1) we can compute $\int f \, d\nu$ and $\int(1 - f) \, d\nu$ from below and hence $1 - \int(1 - f) \, d\nu$ from above. This means that we can produce a $\varrho_=$-name of $\int f \, d\nu$ (cf. [13, Lemma 4.1.9]). Therefore $\int$ is computable with respect to $[\delta \to \varrho_=]$, $\delta^{\mathcal{V}}$ and $\varrho_=$. □

In particular $\mathbb{I}_<$-valued integration is relatively continuous, so the final topology on $\mathcal{V}(\delta)$, induced by its representation, refines the weak topology.

## 7  Discussion

The results of this paper establish probabilistic names as determining a natural class of Borel measures on a space $X$ which, in good cases, coincides with the set of all Borel measures. Moreover, we have argued that probabilistic names induce a natural representation $\delta^{\mathcal{M}}$ on representable measures, and the computability of integration gives some justification of the utility of the notion. Further justification for our representation will appear in a subsequent paper, where it will be shown that, for many spaces, the representation $\delta^{\mathcal{M}}$ is admissible with respect to the weak topology on measures. This holds, in particular, for complete separable metric spaces and for $\omega$-continuous dcpos. As one special case, it follows that our representation for the Borel measures of the unit interval is equivalent to the one considered by K. Weihrauch in [12].

## References

[1]  M. Alvarez-Manilla, Extension of valuations on locally compact sober spaces, *Topology and its Applications 124* (http://www.elsevier.com/locate/topol,

2002) 397-433.

[2] D.L. Cohn, *Measure Theory* (Birkhäuser, Boston, 1993).

[3] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, D.S. Scott, *Continuous Lattices and Domains* (Cambridge University Press, 2003).

[4] T. Jech, *Set Theory*, 3rd millenium edition (Springer, 2003).

[5] C. Jones, *Probabilistic Non-Determinism*, PhD-Thesis (University of Edinburgh, 1989).

[6] J.C.K. Kingman, S.J. Taylor, *Introduction to Measure and Probability* (Cambridge University Press, 1966).

[7] J.D. Lawson, Domains, integration and 'positive analysis', *Math. Struct. in Comp. Science* **14** (2004) 815-832.

[8] M.Schröder, Extended Admissibility, *Theoretical Computer Science* **284** (2002) 519-538.

[9] M. Schröder, Admissible Representations for Continuous Computations, PhD-thesis, *Informatik–Berichte* **29** (FernUniversität Hagen 2003).

[10] M. Schröder, Spaces allowing Type-2 complexity Theory revisited, *Mathematical Logic Quarterly* **50** (2004) 443-459.

[11] M.B. Smyth, Topology, in: *Handbook of Logic in Computer Science* **1** (Oxford science publications 1992) 641-761.

[12] K. Weihrauch, Computability on the probability measures on the Borel sets of the unit interval, *Theoretical Computer Science* **219** (1999) 421-437.

[13] K. Weihrauch: *Computable Analysis* (Springer, Berlin 2000).