Expected dispersion of uniformly distributed points

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March 27, 2020

Abstract

The dispersion of a point set in $[0, 1]^d$ is the volume of the largest axis parallel box inside the unit cube that does not intersect with the point set. We study the expected dispersion with respect to a random set of npoints determined by an i.i.d. sequence of uniformly distributed random variables. Depending on the number of points n and the dimension d we provide an upper and lower bound of the expected dispersion. In particular, we show that the minimal number of points required to achieve an expected dispersion less than $\varepsilon \in (0, 1)$ depends linearly on the dimension d.

Keywords: expected dispersion, dispersion, delta cover Classification. Primary: 62D05; Secondary: 52B55, 65Y20, 68Q25.

1 Introduction and main result

Given n points $\{x_1, \ldots, x_n\} \subset [0, 1]^d$, the dispersion is the volume of the largest axis parallel box that does not contain a point. It is defined by

$$\operatorname{disp}(x_1,\ldots,x_n) := \sup_{B \cap \{x_1,\ldots,x_n\} = \emptyset} \lambda_d(B), \tag{1}$$

where λ_d denotes the *d*-dimensional Lebesgue measure and the supremum is taken over all boxes $B = I_1 \times \cdots \times I_d$ with intervals $I_k \subseteq [0, 1]$. In this note we study the expected dispersion of random points based on an i.i.d. sequence of uniformly

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distributed random variables $(X_i)_{i \in \mathbb{N}}$, where each X_i maps from a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $[0, 1]^d$. For simplicity we write $X_1, X_2, \ldots \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$. We ask for the behavior of

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n))$$

in terms of n and d.

In recent years the proof of existence and the construction of point sets with small dispersion attracted considerable attention, see [1, 6, 13, 15, 20, 21]. In order to describe optimality of such point sets of cardinality n in the d-dimensional setting, let us define the minimal dispersion

$$\operatorname{disp}(n,d) := \inf_{\{x_1,\dots,x_n\} \subset [0,1]^d} \operatorname{disp}(x_1,\dots,x_n),$$

and its inverse

$$n(\varepsilon, d) := \min\{n \in \mathbb{N} \mid \operatorname{disp}(n, d) \le \varepsilon\},\$$

where $\varepsilon \in (0, 1)$. A lower bound for the minimal dispersion growing with the dimension d is provided in [1, Theorem 1]. Moreover, [1, Section 4] contains an upper bound due to Gerhard Larcher, based on constructions of digital nets, which give explicitly constructable point sets. For $\varepsilon \in (0, 1/8)$ the bounds are

$$2^{-3}\varepsilon^{-1}\log_2 d \le n(\varepsilon, d) \le 2^{7d+1}\varepsilon^{-1}.$$
(2)

Clearly, the dependence on ε^{-1} in (2) cannot be improved. However, the upper bound grows exponentially in d, while the lower bound only grows logarithmically. For large dimensions the upper bound can be improved significantly. It is shown in [15, 20] that for fixed ε the quantity $n(\varepsilon, d)$ increases at most logarithmically in d. This means that also the d-dependence of the lower bound in (2) is optimal.

The results of [15, 20] are based on probabilistic arguments. Namely, points are drawn uniformly at random from a regular grid whose parameters depend on ε and d, and it is shown that these points are suitable with positive probability. By the use of a derandomization technique, [21] provides a deterministic algorithm for the construction of point sets with cardinality $c_{\varepsilon} \log_2(d)$ and dispersion at most ε , where $c_{\varepsilon} > 0$ depends only polynomially on ε . Comparable results can be obtained via a careful translation of statements on the so-called hitting set problem, see [10]. In Table 1 we survey explicit bounds for $n(\varepsilon, d)$, in particular, it contains the ones of [15, 20] and their dependence on ε .

In one way or another, most of these upper bounds rely on randomly drawn points and probabilistic arguments. In particular, the estimate of [13, Corollary 1] is based on an i.i.d. sequence of random variables uniformly distributed on $[0, 1]^d$. Maybe this is the most canonical randomly chosen point set and one might ask how good it is compared to deterministic point sets. Here, the measure of goodness is the expected dispersion and our main result reads as follows:

Theorem 1.1. For any n > d we have

$$\max\left\{\frac{\log(n)}{9n}, \frac{d}{2en}\right\} \le \mathbb{E}(\operatorname{disp}(X_1, \dots, X_n)) \le \frac{9d}{n}\log\left(\frac{en}{d}\right).$$

Reference	Upper bound of $n(\varepsilon, d)$	Remarks
[1] (due to Larcher)	$\lceil 2^{7d+1} \varepsilon^{-1} \rceil$	optimal in ε^{-1} digital net construction
[6]	$\min\{(2d)^{\lceil \log_2(\varepsilon^{-1})-1\rceil}, \varepsilon^{-1}\lceil \log_2(\varepsilon^{-1})\rceil^{d-1}\}$	sparse grid construction
[13]	$8d\varepsilon^{-1}\log(33\varepsilon^{-1})$	existence of point set
[15]	$\log_2(d) \left[\varepsilon^{-1}\right]^{\left(\left[\varepsilon^{-1}\right]^2+2\right)} \left(4\log\left[\varepsilon^{-1}\right]+1\right)$	optimal in d existence of point set
[20]	$2^7 \log_2(d) \varepsilon^{-2} (1 + \log_2(\varepsilon^{-1}))^2$	optimal in d existence of point set

Table 1: The table contains several upper bounds on $n(\varepsilon, d)$ based on existence results of "good" points as well as explicit constructions.

Let us also state our result in terms of the inverse of the expected dispersion. For $\varepsilon \in (0, 1)$ and $d \in \mathbb{N}$, the inverse of the expected dispersion is defined as

$$N(\varepsilon, d) := \min\{n \in \mathbb{N} \mid \mathbb{E}\operatorname{disp}(X_1, \dots, X_n) \le \varepsilon\}.$$

Corollary 1.2. For all $\varepsilon \in (0, \frac{1}{9e})$ and $d \in \mathbb{N}$ we have

$$\max\left\{\frac{1}{9\varepsilon}\log\left(\frac{1}{9\varepsilon}\right), \frac{d}{2e\varepsilon}\right\} \le N(\varepsilon, d) \le \left\lceil 9(1+e^{-1})\frac{d}{\varepsilon}\log\left(\frac{9(e+1)}{\varepsilon}\right) \right\rceil.$$

These estimates show that $N(\varepsilon, d)$ for fixed ε behaves linearly w.r.t. the dimension, and for fixed d behaves like $\varepsilon^{-1} \log(\varepsilon^{-1})$. It is interesting to note that the linear behavior w.r.t. d is in contrast to the $\log_2(d)$ dependence of the inverse of the minimal dispersion.

The upper bound of Theorem 1.1 follows by exploiting a δ -cover approximation and a concentration inequality stated in [13]. The proof of the lower bound is separated into two parts. First, we derive the bound $\log(n)/(9n)$ from well known results on the coupon collector's problem. After that the *d*-dependent lower bound d/(2en) is proven by a reduction to the expected dispersion of *d* points and, eventually, a constant lower bound for this quantity.

The proof of Theorem 1.1, along with the necessary notation, is given in Section 2. Further discussions and extensions of the results are provided in Section 3.

2 Proof of Theorem 1.1

2.1 The upper bound

Before we start with the proof of the upper bound let us provide some further notation. Let \mathcal{B} be the set of boxes given as follows,

$$\mathcal{B} := \left\{ \prod_{k=1}^{d} [a^{(k)}, b^{(k)}) \subseteq [0, 1]^{d} \mid a^{(k)}, b^{(k)} \in \mathbb{Q} \cap [0, 1], k = 1, \dots, d \right\}.$$

Then, obviously, we have

$$\operatorname{disp}(x_1,\ldots,x_n) = \sup_{\substack{B \in \mathcal{B} \\ B \cap \{x_1,\ldots,x_n\} = \emptyset}} \lambda_d(B).$$

Note that with this we can restrict ourself to boxes determined by half-open intervals with rational boundary values. Thus, the supremum within the dispersion is only taken over a countable set, which leads to the measurability of the mapping $(x_1, \ldots, x_n) \mapsto \operatorname{disp}(x_1, \ldots, x_n)$. Occasionally, we also call \mathcal{B} the set of test sets. Let $\delta \in (0, 1]$, then a δ -cover of the set of test sets \mathcal{B} is given by a finite set $\Gamma_{\delta} \subset \mathcal{B}$ that satisfies

$$\forall B \in \mathcal{B} \quad \exists L_B, U_B \in \Gamma_{\delta} \quad \text{with} \quad L_B \subseteq B \subseteq U_B \quad \text{and} \quad \lambda(U_B \setminus L_B) \leq \delta.$$

Furthermore, for $x_1, \ldots, x_n \in [0, 1]^d$ and a δ -cover Γ_{δ} for \mathcal{B} define

$$\operatorname{disp}_{\delta}(x_1,\ldots,x_n) := \sup_{\substack{A \in \Gamma_{\delta} \\ A \cap \{x_1,\ldots,x_n\} = \emptyset}} \lambda_d(A).$$

Having introduced those quantities we state two results from [13]. From Γ_{δ} being a δ -cover it follows that

$$\operatorname{disp}(x_1, \dots, x_n) \le \delta + \operatorname{disp}_{\delta}(x_1, \dots, x_n), \tag{3}$$

and, via a union bound, it follows that for any $s \in (0, 1)$ we have

$$\mathbb{P}\left(\operatorname{disp}_{\delta}(X_1,\ldots,X_n) > s\right) \le |\Gamma_{\delta}|(1-s)^n.$$
(4)

We refer to [13, Lemma 1] and the proof of [13, Theorem 1] for details. These results lead to the following lemma.

Lemma 2.1. Let $\delta \in (0, 1]$ and assume that the set Γ_{δ} is a δ -cover of \mathcal{B} . Then, for any $n \geq \log |\Gamma_{\delta}|$ we have

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) \le \delta + \frac{\log|\Gamma_{\delta}|}{n} + \frac{1}{n+1}$$

Proof. From (3) we have

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) \leq \delta + \mathbb{E}(\operatorname{disp}_{\delta}(X_1,\ldots,X_n)).$$

Furthermore, by using (4) we obtain

$$\mathbb{E}(\operatorname{disp}_{\delta}(X_{1},\ldots,X_{n})) = \int_{0}^{1} \mathbb{P}\left(\operatorname{disp}_{\delta}(X_{1},\ldots,X_{n}) > s\right) \,\mathrm{d}s$$

$$\leq \frac{\log|\Gamma_{\delta}|}{n} + \int_{(\log|\Gamma_{\delta}|)/n}^{1} \mathbb{P}\left(\operatorname{disp}_{\delta}(X_{1},\ldots,X_{n}) > s\right) \,\mathrm{d}s$$

$$\leq \frac{\log|\Gamma_{\delta}|}{n} + |\Gamma_{\delta}| \int_{(\log|\Gamma_{\delta}|)/n}^{1} (1-s)^{n} \,\mathrm{d}s \leq \frac{\log|\Gamma_{\delta}|}{n} + \frac{|\Gamma_{\delta}|}{n+1} \left(1 - \frac{\log|\Gamma_{\delta}|}{n}\right)^{n+1}$$

Note that for any $0 \le a \le n$ we have $(1 - a/n)^n \le \exp(-a)$, hence,

$$|\Gamma_{\delta}| \left(1 - \frac{\log |\Gamma_{\delta}|}{n}\right)^{n+1} \le 1,$$

which finishes the proof.

Remark 2.2. Except for the assumption that we have a δ -cover, we did not use any property of the set of test sets \mathcal{B} .

Now, the upper bound of Theorem 1.1 is deduced by the results on δ -covers for \mathcal{B} from Gnewuch, see [4]. Namely, from [4, Formula (1), Theorem 1.15 with $d! \geq (d/e)^d$, and Lemma 1.18] one obtains that there is a δ -cover for \mathcal{B} with $|\Gamma_{\delta}| \leq (6e \delta^{-1})^{2d}$. By setting $\delta = 6d/n$, the upper bound of Theorem 1.1 follows with Lemma 2.1 and $n \geq d$.

Finally, this upper estimate implies the upper bound of Corollary 1.2. For the convenience of the reader, we add a few arguments. We are looking for preferably small integers $n \ge d$ such that

$$\frac{9d}{n}\log\left(\frac{\mathrm{e}\,n}{d}\right) \le \varepsilon\,.$$

Note that the term on the left-hand side of this inequality is monotonically decreasing for $n \ge d$. Let $c \ge 1$ be a constant, then, any integer

$$n \ge c \frac{d}{\varepsilon} \log\left(\frac{c \, \mathrm{e}}{\varepsilon}\right)$$

will also comply with $n \ge d$, and hence satisfies the estimate

$$\frac{9d}{n}\log\left(\frac{\mathrm{e}\,n}{d}\right) \le \frac{9}{c}\,\varepsilon\cdot\left(1 + \frac{\log\log\left(\frac{c\mathrm{e}}{\varepsilon}\right)}{\log\left(\frac{c\mathrm{e}}{\varepsilon}\right)}\right) \le \frac{9(1 + \mathrm{e}^{-1})}{c}\,\varepsilon\,,$$

where we used that $(\log x)/x$ attains its maximum for x = e. Choosing the constant $c = 9(1 + e^{-1}) = 12.31...$ we obtain the desired guarantee.

2.2The lower bound

In Section 2.2.1 we show that $\mathbb{E}(\operatorname{disp}(X_1, \ldots, X_n)) \geq \frac{\log(n)}{9n}$, and in Section 2.2.2 we prove that $\mathbb{E}(\operatorname{disp}(X_1, \ldots, X_n)) \geq \frac{d}{2en}$ for n > d. Both lower bounds together yield the corresponding statement of Theorem 1.1. By convention, all random variables are defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

2.2.1Lower bound without dimension dependence

We start with an auxiliary tool, using results on the *coupon collector's problem*.

Lemma 2.3. For $\ell \in \mathbb{N}$ let $(Y_i)_{i \in \mathbb{N}}$ be an *i.i.d.* sequence of uniformly distributed random variables in $\{1, \ldots, \ell\}$. Define $H_{\ell} := \sum_{j=1}^{\ell} j^{-1}$ and

$$\tau_{\ell} := \min\{k \in \mathbb{N} \mid \{Y_1, \dots, Y_k\} = \{1, \dots, \ell\}\}.$$

Then, for any integer $n \leq (H_{\ell} - 2)\ell$ we have $\mathbb{P}(\tau_{\ell} > n) > 1/2$.

Proof. It is well known that the mean and the variance of τ_{ℓ} satisfy

$$\mathbb{E} \tau_{\ell} = \ell H_{\ell}$$
 and $\operatorname{Var} \tau_{\ell} \leq \ell^2 \sum_{j=1}^{\ell} j^{-2} \leq \frac{\pi^2}{6} \ell^2$.

For details concerning these estimates, see for example [9] or [7, Proposition 4.7]. Then, for $n \leq (H_{\ell} - 2)\ell$, by Chebyshev's inequality we have

$$\mathbb{P}(\tau_{\ell} \le n) \le \mathbb{P}(\tau_{\ell} \le (H_{\ell} - 2)\ell) = \mathbb{P}(\ell H_{\ell} - \tau_{\ell} \ge 2\ell) \le \frac{\operatorname{Var}(\tau_{\ell})}{4\ell^2} \le \frac{\pi^2}{24} < \frac{1}{2},$$

ch finishes the proof.

which finishes the proof.

By means of the previous result we are able to prove the desired lower bound in the following lemma.

Lemma 2.4. For any integer $n \geq 3$ we have $\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) > \frac{\log(n)}{9n}$.

Proof. Fix some $\ell \in \mathbb{N}$ and split $[0,1]^d$ into ℓ disjoint boxes B_1, \ldots, B_ℓ of equal volume $1/\ell$ (e.g. split along the first coordinate). For i = 1, ..., n define the random variable $Y_i: \Omega \to \{1, \ldots, \ell\}$ that indicates the box the point X_i lies in, i.e. $X_i(\omega) \in B_{Y_i(\omega)}$. Note that Y_1, \ldots, Y_n are i.i.d. and each uniformly distributed in $\{1, \ldots, \ell\}$. Furthermore, for $\omega \in \Omega$ satisfying

$$\{Y_1(\omega),\ldots,Y_n(\omega)\}\neq\{1,\ldots,\ell\},\$$

there is an index $r \in \{1, \ldots, \ell\}$ such that $\{X_1(\omega), \ldots, X_n(\omega)\} \cap B_r = \emptyset$. Thus, for such an ω we obtain

$$\operatorname{disp}(X_1(\omega),\ldots,X_n(\omega)) \ge 1/\ell.$$

This yields

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) = \int_{\Omega} \operatorname{disp}(X_1(\omega),\ldots,X_n(\omega))\mathbb{P}(\operatorname{d}\omega)$$
$$\geq \frac{1}{\ell} \mathbb{P}(\{Y_1,\ldots,Y_n\} \neq \{1,\ldots,\ell\}).$$

Observe that with τ_{ℓ} defined in Lemma 2.3 we have

$$\mathbb{P}(\{Y_1,\ldots,Y_n\}\neq\{1,\ldots,\ell\})=\mathbb{P}(\tau_\ell>n).$$

Choosing $\ell := \left\lceil \frac{(1+e)n}{\log(n)} \right\rceil$, we get $\frac{n}{\ell} \le \frac{\log(n)}{1+e} \le \log\left(\frac{(1+e)n}{\log(n)}\right) - 2 \le \log(\ell) - 2 < H_{\ell} - 2,$

where we used the inequality $\log\left(\frac{(1+e)x}{\log(x)}\right) - 2 - \frac{\log(x)}{1+e} \ge 0$ for x > 1 (attaining equality in $x = \exp(1+1/e)$), as well as $H_{\ell} = \sum_{j=1}^{\ell} j^{-1} > \log(\ell+1)$. This asserts $n \le (H_{\ell}-2)\ell$, and by Lemma 2.3 we obtain $\mathbb{P}(\tau_{\ell} > n) > 1/2$. Taking everything together yields

$$\mathbb{E}(\operatorname{disp}(X_1, \dots, X_n)) > \frac{1}{2\ell} \ge \frac{1}{2} \cdot \frac{\log(n)}{(1+\mathrm{e})n + \log(n)} > \frac{\log(n)}{9n}$$

which completes the proof. Our derivation holds for integers $n \ge 2$, but the bound starts decaying for $n \ge 3$, in the first place.

Having the result of the previous lemma, the first part within the maximum of the lower bound in Corollary 1.2 follows. For the convenience of the reader we add a few arguments. If the expected dispersion shall be smaller than a given $\varepsilon > 0$, the number of points, n, must satisfy $\frac{\log n}{9n} \leq \varepsilon$. Note that the left-hand side is monotonically decreasing only for $n \geq e$, but the expected dispersion for $n \in \{1, 2\}$ should be larger or equal the expected dispersion for n = 3. Restricting to $\varepsilon \in (0, \frac{1}{9e})$, for $e \leq n < \frac{1}{9\varepsilon} \log \left(\frac{1}{9\varepsilon}\right)$ we would have

$$\frac{\log n}{9n} > \varepsilon \cdot \left(1 + \frac{\log \log \left(\frac{1}{9\varepsilon}\right)}{\log \left(\frac{1}{9\varepsilon}\right)} \right) > \varepsilon \,.$$

Hence, $n \ge \frac{1}{9\varepsilon} \log \left(\frac{1}{9\varepsilon}\right)$ is necessary for the expected dispersion to be less or equal ε .

Remark 2.5. In the strong asymptotic regime, the prefactor $\frac{1}{9}$ in Lemma 2.4 vanishes, i.e.

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) \gtrsim \frac{\log(n)}{n} \quad \text{for } n \to \infty.$$

This result can be deduced by considering the volume V_n of the largest empty box of the shape $B = (a, b) \times [0, 1]^{d-1}$, with 0 < a < b < 1. This quantity V_n is equivalent to the distribution of the maximal distance between adjacent points when distributing n + 1 random points on a circle with perimeter 1. We can use an asymptotic result by Schlemm [14, Cor. 1] on the largest gap of random points on a circle, namely that $(n + 1)V_n - \log(n + 1)$ converges to a standard Gumbel distribution with its expectation being the Euler-Mascheroni constant $\gamma \approx 0.5772$. Hence,

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) \ge \mathbb{E}V_n \simeq \frac{\gamma + \log(n+1)}{n+1} \simeq \frac{\log(n)}{n} \quad \text{for } n \to \infty.$$

2.2.2 Dimension-dependent lower bound

The proof of the lower bound w.r.t. the dimension is divided into two steps. First, we deduce a lower bound for the expected dispersion of n points in terms of the expected dispersion of d points, see Lemma 2.6. Thus, we reduce the problem to finding a lower bound for the expected dispersion of d points, which then is the goal of the second step, see Lemma 2.7. In the following proof, for $B \in \mathcal{B}$ and x_1, \ldots, x_ℓ with $\ell \in \mathbb{N}$, we use the notation

$$\operatorname{disp}_{|B}(x_1,\ldots,x_\ell) := \sup_{\substack{R \in \mathcal{B} \cap B \\ R \cap \{x_1,\ldots,x_\ell\} = \emptyset}} \lambda_d(R)$$

for the dispersion restricted to B. The following reduction lemma is a probabilistic version of [1, Lemma 1].

Lemma 2.6. For any $n, \ell \in \mathbb{N}$ we have

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) \ge \frac{\ell+1}{n+\ell+1} \ \mathbb{E}(\operatorname{disp}(X_1,\ldots,X_\ell)).$$

Proof. We start with a purely combinatorial argument, a version of the pigeonhole principle. If we split $[0,1]^d$ into m boxes B_1, \ldots, B_m of equal volume, then there is some $j \in \{1, \ldots, m\}$ such that B_j contains no more than $\lfloor n/m \rfloor$ of the points X_1, \ldots, X_n . Choosing $m = \lceil \frac{n+1}{\ell+1} \rceil = \lfloor \frac{n}{\ell+1} \rfloor + 1$, we have $\lfloor \frac{n}{m} \rfloor \leq \lfloor \frac{n}{n+1}(\ell+1) \rfloor \leq \ell$. For $k \in \mathbb{N}$, let $n_k \in \mathbb{N}$ be the time when B_j is hit by the sequence $(X_i)_{i \in \mathbb{N}} \subset [0, 1]^d$ for the k-th time (which for $X_1, X_2, \ldots \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$ almost surely happens). With $n_\ell \geq n$, by the choice of B_j , we have

$$\operatorname{disp}(X_1,\ldots,X_n) \ge \operatorname{disp}_{|B_j}(\{X_1,\ldots,X_n\} \cap B_j) \ge \operatorname{disp}_{|B_j}(X_{n_1},\ldots,X_{n_\ell}).$$

Let T be an affine transformation that maps B_j onto $[0,1]^d$. Then

$$\operatorname{disp}_{B_i}(X_{n_1},\ldots,X_{n_\ell}) = \lambda_d(B_j) \cdot \operatorname{disp}(TX_{n_1},\ldots,TX_{n_\ell}).$$

Recall that $X_1, X_2, \ldots \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$, hence, the points $TX_{n_1}, \ldots, TX_{n_\ell}$ are independent and uniformly distributed in $[0, 1]^d$. Taking the expectation and using $\lambda_d(B_j) = \frac{1}{m} \geq \frac{1}{n/(\ell+1)+1} = \frac{\ell+1}{n+\ell+1}$, yields the statement.

Having the previous lemma at hand, it is sufficient to provide a constant lower bound for the expected dispersion of d points. In a slightly more general way, we obtain the following.

Lemma 2.7. For any $d, \ell \in \mathbb{N}$ and $X_1, \ldots, X_\ell \stackrel{\text{iid}}{\sim} \text{Unif}([0, 1]^d)$ we have

 $\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_\ell)) \ge e^{-\ell/d}.$

Proof. For all $i \in \{1, \ldots, \ell\}$, let X_i^* denote the largest coordinate of X_i , i.e.,

$$X_i^* := \max\{X_i^{(1)}, \dots, X_i^{(d)}\}.$$

We choose $j^*(i) \leq d$ such that $X_i^{(j^*(i))} = X_i^*$. Let us consider the box

$$B = \prod_{j=1}^{d} [0, a_j) \, ,$$

where

$$a_j := \min(\{1\} \cup \{X_i^* \mid i \le \ell \text{ with } j^*(i) = j\})$$

This box is empty, since for all $i \leq \ell$ we have $X_i^{(j^*(i))} \geq a_{j^*(i)}$, and hence $X_i \notin B$. An illustration for the case d = 2 is provided in Figure 1. On the other hand, the volume of B is given by

$$\lambda^d(B) = \prod_{j=1}^d a_j = \prod_{i \in I} X_i^* \,,$$

where I is a suitable subset of $\{1, \ldots, \ell\}$. This yields

$$\operatorname{disp}(X_1,\ldots,X_\ell) \ge \prod_{i\in I} X_i^* \ge \prod_{i=1}^\ell X_i^*.$$

The random numbers X_i^* are independent and beta distributed with parameters $\alpha = d$ and $\beta = 1$, in particular, $\mathbb{E}(X_i^*) = 1 - 1/(d+1)$. Hence,

$$\mathbb{E}(\operatorname{disp}(X_1, \dots, X_\ell)) \ge \prod_{i=1}^{\ell} \mathbb{E}(X_i^*) = \left(1 - \frac{1}{d+1}\right)^{\ell} = \left(\frac{1}{1 + \frac{1}{d}}\right)^{\ell}$$
$$\ge \left(\frac{1}{\exp(1/d)}\right)^{\ell} = e^{-\ell/d}.$$

The proof of the lower bound follows by setting $\ell = d$ and combining the results of the two lemmas. We readily get

$$\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n)) \ge \frac{d+1}{\operatorname{e}(n+d+1)} > \frac{d}{2\operatorname{e} n}$$

where the last inequality follows from n > d. For $\varepsilon \in (0, \frac{1}{2e})$, the respective inverse lower bound $N(\varepsilon, d) \geq \frac{d}{2e\varepsilon}$ is straightforward, where the restriction on ε implies $N(\varepsilon, d) > d$.

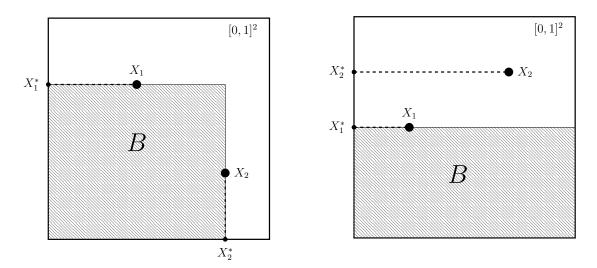


Figure 1: An illustration of the empty box construction from Lemma 2.7 for $d = \ell = 2$, featuring two different types of situations. In the left picture we have $B = [0, a_1) \times [0, a_2)$ with $X_1 = (0.4, 0.7)$, $j^*(1) = 2$, $a_1 = 0.7$ and $X_2 = (0.8, 0.3)$, $j^*(2) = 1$, $a_2 = 0.8$. In the right picture we have $B = [0, a_1) \times [0, a_2)$ with $X_1 = (0.25, 0.5)$, $j^*(1) = 2$, $a_1 = 0.5$ and $X_2 = (0.7, 0.75)$, $j^*(1) = 2$, $a_2 = 1$.

3 Notes and remarks

The dispersion of a point set, as defined in (1), has been introduced in [12], generalizing the work of [5]. The renewed interest in this quantity emerged from its appearance in the construction of algorithms for the approximation of rank-one tensors, see [2, 8, 11], where the dependence on the dimension is crucial. It is also related to the universal discretization problem, see [16], and the fixed volume discrepancy, see [17, 18].

The dispersion of a point set has also been studied on the torus instead of the unit cube, see for example [3, 19]. This setting can be described on the unit cube by choosing another set of test sets, namely

$$\widetilde{\mathcal{B}} := \left\{ \prod_{k=1}^{d} I_k(x,y) \mid x = (x^{(1)}, \dots, x^{(d)}), y = (y^{(1)}, \dots, y^{(d)}) \in [0,1]^d \cap \mathbb{Q}^d \right\},\$$

with

$$I_k(x,y) = \begin{cases} (x^{(k)}, y^{(k)}) & x^{(k)} < y^{(k)}, \\ [0,1] \setminus [y^{(k)}, x^{(k)}] & y^{(k)} \le x^{(k)}. \end{cases}$$

The set $\widetilde{\mathcal{B}}$ is called the test set of *periodic boxes*. Since the proof of the upper bound of the expected dispersion depends on \mathcal{B} only through the δ -cover, with the same arguments we can also derive an upper bound for the expected dispersion w.r.t. $\widetilde{\mathcal{B}}$ by using an appropriate *periodic* δ -cover. For $x_1, \ldots, x_n \in [0, 1]^d$ define

$$\widetilde{\operatorname{disp}}(x_1,\ldots,x_n) := \sup_{\substack{B \in \widetilde{\mathcal{B}} \\ B \cap \{x_1,\ldots,x_n\} = \emptyset}} \lambda_d(B).$$

With [13, Lemma 2], we obtain that there is a δ -cover $\widetilde{\Gamma}_{\delta}$ of $\widetilde{\mathcal{B}}$ with cardinality $|\widetilde{\Gamma}_{\delta}| \leq (4d\delta^{-1})^{2d}$, so that with $\delta = 2d/n$ we have

$$\mathbb{E}(\widetilde{\operatorname{disp}}(X_1,\ldots,X_n)) \leq \frac{5d}{n}\log(2n).$$

By the fact that $\mathcal{B} \subset \widetilde{\mathcal{B}}$ we obtain for any $x_1, \ldots, x_n \in [0, 1]^d$ that

$$\operatorname{disp}(x_1,\ldots,x_n) \leq \operatorname{disp}(x_1,\ldots,x_n),$$

hence, the lower bounds of Theorem 1.1 also carry over to $\mathbb{E}(\operatorname{disp}(X_1,\ldots,X_n))$. Here it is worth mentioning that the lower bound w.r.t. the dimension can also be deduced from [19, Theorem 1]. Thus, in this setting also a linear dimensiondependence is present in $\mathbb{E}(\widetilde{\operatorname{disp}}(X_1,\ldots,X_n))$. However, concerning the inverse of the expected dispersion in the periodic case, the precise growth w.r.t. the dimension remains open, we only know that it is between d and $d\log(d)$.

Acknowledgements

We thank Mario Ullrich and the referees for their valuable suggestions. Parts of the work have been done at the Dagstuhl seminar "Algorithms and Complexity for Continuous Problems" in August 2019, where we enjoyed the stimulating research environment. A. Hinrichs and D. Krieg are supported by the Austrian Science Fund (FWF) Project F5513-N26, which is a part of the Special Research Program "Quasi-Monte Carlo Methods: Theory and Applications". Daniel Rudolf gratefully acknowledges support of the Felix-Bernstein-Institute for Mathematical Statistics in the Biosciences (Volkswagen Foundation) and the Campus laboratory AIMS.

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