

COUNTABLE TENSOR PRODUCTS OF HERMITE SPACES AND SPACES OF GAUSSIAN KERNELS

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ABSTRACT. In recent years finite tensor products of reproducing kernel Hilbert spaces (RKHSs) of Gaussian kernels on the one hand and of Hermite spaces on the other hand have been considered in tractability analysis of multivariate problems. In the present paper we study countably infinite tensor products for both types of spaces. We show that the incomplete tensor product in the sense of von Neumann may be identified with an RKHS whose domain is a proper subset of the sequence space $\mathbb{R}^{\mathbb{N}}$. Moreover, we show that each tensor product of spaces of Gaussian kernels having square-summable shape parameters is isometrically isomorphic to a tensor product of Hermite spaces; the corresponding isomorphism is given explicitly, respects point evaluations, and is also an L^2 -isometry. This result directly transfers to the case of finite tensor products. Furthermore, we provide regularity results for Hermite spaces of functions of a single variable.

1. INTRODUCTION

This paper is motivated by the study of integration and L^2 -approximation in a tensor product setting for problems with a large or infinite number d of variables. Specifically, we consider Hilbert spaces $H(M)$ with reproducing kernels

$$M := \bigotimes_j m_j$$

of tensor product form. Here we have $j \in \{1, \dots, d\}$ if $d \in \mathbb{N}$ and $j \in \mathbb{N}$ if $d = \infty$, and we assume that either all of the kernels

$$m_j: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$$

are Hermite or all are Gaussian kernels. The same terminology, Hermite kernel or Gaussian kernel, will be used for the corresponding tensor product kernels M , and the reproducing kernel Hilbert spaces (RKHSs) with Hermite kernels are known as Hermite spaces, see Irrgeher and Leobacher (2015). Analogously, and despite a different use in stochastic analysis, the RKHSs of Gaussian kernels will be called Gaussian spaces throughout this paper.

For both types of kernels, Hermite and Gaussian, the functions in $H(M)$ are defined on the space \mathbb{R}^d if $d \in \mathbb{N}$ or on a subset of the sequence space $\mathbb{R}^{\mathbb{N}}$ if $d = \infty$. Moreover, the measure μ that defines the integral or the L^2 -norm is the corresponding product of the standard normal distribution μ_0 .

We are primarily interested in the case $d = \infty$. This is the technically more demanding case from which corresponding results for the case $d \in \mathbb{N}$ can be deduced easily. In the present paper we provide an analysis of the function spaces, while

the complexity of integration and L^2 -approximation will be studied in a follow-up paper.

Our approach is as follows. At first, we study the spaces $H(m_j)$ of functions of a single variable, and then we turn to the countable tensor product of these spaces, which allows to carry over the results for $d = 1$ to the tensor product Hilbert space in a canonical way. Finally, we identify the tensor product Hilbert space with an RKHS $H(M)$ for the tensor product kernel

$$M: \mathfrak{X} \times \mathfrak{X} \rightarrow \mathbb{R}$$

on a suitably chosen domain

$$\mathfrak{X} \subseteq \mathbb{R}^{\mathbb{N}}.$$

Let us refer to prior work on tractability of integration and L^2 -approximation in the case $d \in \mathbb{N}$, i.e., for finite tensor products of either Hermite or Gaussian kernels. Both types of kernels have been studied separately, so far. For Hermite spaces the first contribution is due to Irrgeher and Leobacher (2015), followed by Irrgeher et al. (2015), Irrgeher et al. (2016a), and Irrgeher et al. (2016b) with a focus on spaces of analytic functions, and by Dick et al. (2018) with a focus on spaces of finite smoothness. For Gaussian spaces the first contribution is due to Fasshauer et al. (2012), followed by Kuo et al. (2017), Sloan and Woźniakowski (2018), and Karvonen et al. (2021). To the best of our knowledge, the case $d = \infty$ has not been studied yet neither for Hermite nor for Gaussian kernels.

We show, in particular, that Hermite spaces and Gaussian spaces with $d = \infty$ are closely related in the following sense: Every Gaussian space with square-summable shape parameters is isometrically isomorphic to a Hermite space with an explicitly given isometry that respects point evaluations and is also an $L^2(\mu)$ -isometry. We add that the same holds true for every $d \in \mathbb{N}$. This result may explain an observation due to Karvonen et al. (2021, p. 2) concerning the analysis of integration and approximation problems, namely that techniques similar to those used in the Gaussian case are used in the Hermite case. A similar observation was made earlier by Kuo et al. (2017, p. 830), where the authors write that Gauss-Hermite quadratures used in Irrgeher et al. (2015) for integrands stemming from Hermite spaces work well also on Gaussian spaces, although the latter ones do not contain any non-trivial polynomial.

In the following we present the construction and a sketch of results for Hermite spaces. To define the Hermite kernels $m_j := k_j$ we employ the $L^2(\mu_0)$ -normalized Hermite polynomials h_ν of degree $\nu \in \mathbb{N}_0$, see Section 3.1 for the definition and basic properties. For any choice of Fourier weights $\alpha_{\nu,j} > 0$ with

$$\inf_{\nu \in \mathbb{N}_0} \alpha_{\nu,j} > 0$$

and

$$\sum_{\nu \in \mathbb{N}_0} \alpha_{\nu,j}^{-1} \cdot |h_\nu(x)|^2 < \infty$$

for all $x \in \mathbb{R}$ the corresponding Hermite kernel is given by

$$k_j(x, y) := \sum_{\nu \in \mathbb{N}_0} \alpha_{\nu,j}^{-1} \cdot h_\nu(x) \cdot h_\nu(y), \quad x, y \in \mathbb{R}.$$

A general set (H1)–(H3) of summability and monotonicity assumptions for the Fourier weights is presented in Section 3.2. In this introduction we focus on two

particular kinds of Fourier weights, namely the case of polynomial growth, where

$$\alpha_{\nu,j} := (\nu + 1)^{r_j}, \quad \nu, j \in \mathbb{N},$$

with $r_j > 0$, and the case of (sub-)exponential growth, where

$$\alpha_{\nu,j} := 2^{r_j \cdot \nu^{b_j}}, \quad \nu, j \in \mathbb{N},$$

with $r_j, b_j > 0$. The latter kind of Fourier weights with $b_j = 1$ for all $j \in \mathbb{N}$ is the proper choice when we study the relation between Hermite and Gaussian spaces, see Section 5.

Let us consider Hermite spaces of functions of a single variable. For any fixed $j \in \mathbb{N}$ the regularity of the functions from $H(k_j)$ is determined by the asymptotic behavior of $\alpha_{\nu,j}$ as $\nu \rightarrow \infty$, see Irrgeher and Leobacher (2015), Irrgeher et al. (2015), and Dick et al. (2018) for specific results. For Fourier weights with a polynomial growth the low regularity limit is $r_j = 1/2$, where we loose convergence of k_j . See Example 3.5 and cf. Dick et al. (2018, Sec. 2), where convergence is established for $r_j > 5/6$. For Fourier weights with a (sub-)exponential growth the elements of $H(k_j)$ belong to the Gevrey class of index $\max(1/(2b_j), 1)$; in particular, they are real analytic functions if $b_j \geq 1/2$. See Lemma 3.7 and cf. Irrgeher and Leobacher (2015, Prop. 3.7) and Irrgeher et al. (2015, Prop. 3), where analyticity is established for $b_j \geq 1$, as well as Irrgeher et al. (2016b, Rem. 1), where for $b_j \in]0, 1[$ the inclusion of $H(k_j)$ in the Gevrey class of index $1/b_j$ is claimed.

Furthermore, Hermite spaces with Fourier weights of polynomial growth with r_j varying in $]1/2, \infty[$ and of exponential growths with $b_j = 1$ and with r_j varying in $]0, \infty[$ form interpolation scales with equality of the norms due to quadratic interpolation and the original norm on the RKHS. See Remark 2.3 and Examples 3.5 and 3.6.

Next we present the main results for Hermite spaces of functions of infinitely many variables, where we require

$$r_j > 1/2 \quad \text{and} \quad \sum_{j \in \mathbb{N}} 2^{-r_j} < \infty$$

in the case of polynomial growth and

$$\sum_{j \in \mathbb{N}} 2^{-r_j} < \infty \quad \text{and} \quad \inf_{j \in \mathbb{N}} b_j > 0$$

in the case of (sub-)exponential growth. In both cases we additionally require

$$\sum_{j \in \mathbb{N}} |\alpha_{0,j} - 1| < \infty.$$

See Section 3.3. We show that the maximal domain \mathfrak{X} for the tensor product kernel $K := \bigotimes_{j \in \mathbb{N}} k_j$ is related to an intersection of weighted ℓ^2 -spaces and satisfies

$$\ell^\infty \subsetneq \mathfrak{X} \subsetneq \mathbb{R}^{\mathbb{N}}$$

as well as

$$\mu(\mathfrak{X}) = 1.$$

See Proposition 3.10 and Lemma 3.17. We conclude, in particular, that the Hermite spaces $H(K)$ of functions of infinitely many variables are never spaces of functions on the domain $\mathbb{R}^{\mathbb{N}}$.

In the case of exponential growth with

$$\inf_{j \in \mathbb{N}} b_j \geq 1$$

the maximal domain is explicitly given as

$$\mathfrak{X} = \left\{ \mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \sum_{j \in \mathbb{N}} 2^{-r_j} \cdot x_j^2 < \infty \right\}.$$

See Proposition 3.18 and cf. Proposition 3.19, which deals with the case of polynomial and sub-exponential growth.

Moreover, we show that the incomplete tensor product $H(\alpha_0^{-1/2})$, see von Neumann (1939) and Definition A.2, of the Hermite spaces $H(k_j)$ based on the constant functions $\alpha_{0,j}^{-1/2}$ as unit vectors may be identified with the Hermite space $H(K)$ on the maximal domain \mathfrak{X} in a canonical way. See Lemma 3.8 and Theorem A.6. We add that the tensor product norm on $H(K)$ respects the infinite-dimensional ANOVA decomposition of a function, i.e., different ANOVA components are pairwise orthogonal in $H(K)$. See Remark 3.14.

Next we come to Gaussian spaces. The Gaussian kernel $m_j := \ell_j$ with shape parameter $\sigma_j > 0$ is given by

$$\ell_j(x, y) := \exp(-\sigma_j^2 \cdot (x - y)^2), \quad x, y \in \mathbb{R}.$$

Gaussian spaces $H(\ell_j)$ and finite tensor products thereof are thoroughly analyzed in Steinwart et al. (2006), see also Steinwart and Christmann (2008, Sec. 4.4). In the following we sketch our main results for Gaussian spaces in the case $d = \infty$ with shape parameters $\sigma_j > 0$ satisfying

$$\sum_{j \in \mathbb{N}} \sigma_j^2 < \infty.$$

We show that the incomplete tensor product $G^{(\nu)}$ of the Gaussian spaces $H(\ell_j)$ based on the unit vectors $v_j(x) := \exp(-\sigma_j^2 \cdot x^2)$ may be identified with a Gaussian space $H(L)$ in a canonical way. The corresponding domain for the tensor product kernel $L := \bigotimes_{j \in \mathbb{N}} \ell_j$ is given by

$$\mathfrak{X} = \left\{ \mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \sum_{j \in \mathbb{N}} \sigma_j^2 \cdot x_j^2 < \infty \right\},$$

which is a proper subset of $\mathbb{R}^{\mathbb{N}}$, despite the convergence of $\prod_{j=1}^{\infty} \ell_j(x_j, y_j)$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$. See Lemma 4.2, Remark 4.6, and Theorem A.6. We add that $\mu(\mathfrak{X}) = 1$.

The identification of the Gaussian spaces $H(L)$ and the Hermite spaces $H(K)$ with incomplete tensor products allows to establish the following result via tensorization. For any Gaussian space $H(L)$ we consider the Hermite space $H(K)$ with Fourier weights

$$\alpha_{\nu,j} := \frac{1}{(1 - \beta_j) \cdot \beta_j^{\nu}}, \quad \nu \in \mathbb{N}_0, j \in \mathbb{N},$$

where

$$\beta_j := 1 - \frac{2}{1 + c_j^2} \quad \text{with} \quad c_j := (1 + 8\sigma_j^2)^{1/4}.$$

Moreover, we define

$$Qf(\mathbf{x}) := \prod_{j=1}^{\infty} c_j \cdot \exp\left(-\sum_{j \in \mathbb{N}} \frac{c_j^2 - 1}{4} x_j^2\right) \cdot f(c_1 x_1, c_2 x_2, \dots)$$

for $f \in L^2(\mu)$ and $\mathbf{x} \in \mathfrak{X}$. Then Q is an isometric isomorphism on $L^2(\mu)$ and $Q|_{H(K)}$ is an isometric isomorphism between $H(K)$ and $H(L)$. See Theorem 5.8. Another feature of Q that is highly relevant in the context of integration and approximation problems is the obvious fact that for every $\mathbf{x} \in \mathfrak{X}$ a single function value of $f \in H(K)$ suffices to compute $Qf(\mathbf{x})$ and vice versa.

This paper is organized as follows: Since all spaces of functions we consider are subspaces of (suitable) spaces of square-integrable functions, we discuss in Section 2 under which conditions subspaces of L^2 -spaces are reproducing kernel Hilbert spaces and what their kernels and their norms look like. Proposition 2.1 provides sufficient conditions and facilitates our later analysis of function spaces considerably. In Section 3 we define the Hermite spaces we want to study and present their essential properties. There and in the following two sections we first consider spaces of univariate functions and then turn to spaces of functions depending on infinitely many variables. The important example cases, where the Fourier weights exhibit polynomial or (sub-)exponential growth, are studied in Section 3.3. In Section 4 we present the essential well-known facts on Gaussian spaces of univariate functions, see Section 4.1, and then analyze Gaussian spaces of functions of infinitely many variables, see Section 4.2. Relations between Hermite and Gaussian spaces are established in Section 5. In the appendix we recall the construction and key properties of complete and incomplete tensor products of Hilbert spaces and discuss the two important cases of incomplete tensor products of reproducing kernel Hilbert spaces, see Section A.5, and of L^2 -spaces, see Section A.6.

Let us close the introduction with some remarks concerning notation, terminology, and conventions. We consider the Borel σ -algebra on \mathbb{R} and, in particular, the standard normal distribution μ_0 on this space. Moreover, we consider the corresponding product σ -algebra on $\mathbb{R}^{\mathbb{N}}$ and, in particular, the countable product μ of μ_0 on this space. For $E \neq \emptyset$ and $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ a function $K: E \times E \rightarrow \mathbb{K}$ is called a *reproducing kernel* if K is symmetric in the case $\mathbb{K} = \mathbb{R}$ and Hermitian in the case $\mathbb{K} = \mathbb{C}$ and positive definite in the following sense: For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in E$, $a_1, \dots, a_n \in \mathbb{K}$ we have $\sum_{i,j=1}^n \bar{a}_i a_j \cdot K(x_i, x_j) \geq 0$.

For any sequence $(a_j)_{j \geq J_0}$ in \mathbb{R} such that $(\prod_{j=J_0}^J a_j)_{J \geq J_0}$ is convergent we put

$$(1.1) \quad \prod_{j=J_0}^{\infty} a_j := \lim_{J \rightarrow \infty} \prod_{j=J_0}^J a_j.$$

The collection of arbitrary finite partial products of $(a_j)_{j \geq J_0}$ forms a net; the corresponding stronger notion of convergence is discussed in Section A.1.

We use the following notation for subspaces of $\mathbb{R}^{\mathbb{N}}$. The space of all sequences $\mathbf{x} := (x_j)_{j \in \mathbb{N}}$ such that $\{j \in \mathbb{N} : x_j \neq 0\}$ is finite is denoted by c_{00} , and ℓ^∞ denotes the space of all bounded sequences. For any sequence $\boldsymbol{\omega} := (\omega_j)_{j \in \mathbb{N}}$ of positive real numbers and $1 \leq p < \infty$ we use $\ell^p(\boldsymbol{\omega})$ to denote the space of all sequences \mathbf{x} such that

$$\sum_{j \in \mathbb{N}} \omega_j \cdot |x_j|^p < \infty.$$

For $n \in \mathbb{N}$ we use \mathbf{N}_n to denote the set of all sequences $(\nu_j)_{j \in \mathbb{N}}$ in \mathbb{N}_0 such that $\nu_j = 0$ for every $j > n$. Moreover, we put $\mathbf{N} := \bigcup_{n \in \mathbb{N}} \mathbf{N}_n$, i.e., $\mathbf{N} = c_{00} \cap \mathbb{N}_0^{\mathbb{N}}$.

We use U to denote the set of all finite subsets of \mathbb{N} .

2. L^2 -SUBSPACES AS REPRODUCING KERNEL HILBERT SPACES

Consider a space $L^2(\rho) := L^2(E, \rho)$ over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with respect to a measure ρ on a σ -algebra on a set E . Let N denote a countable set, and let $(\mathbf{e}_\nu)_{\nu \in N}$ denote a family of square-integrable functions on E such that the corresponding equivalence classes e_ν form an orthonormal basis of $L^2(\rho)$. Moreover, let $(\alpha_\nu)_{\nu \in N}$ denote a family of positive real numbers, which will be called *Fourier weights* in view of the following result that is essentially due to Gnewuch et al. (2019, Sec. 2.1).

Proposition 2.1. *Assume that*

$$(2.1) \quad \inf_{\nu \in N} \alpha_\nu > 0$$

and

$$(2.2) \quad \sum_{\nu \in N} \alpha_\nu^{-1} \cdot |\mathbf{e}_\nu(x)|^2 < \infty$$

for every $x \in E$. If $(c_\nu)_{\nu \in N} \in \mathbb{K}^N$ satisfies

$$\sum_{\nu \in N} \alpha_\nu \cdot |c_\nu|^2 < \infty,$$

then $\sum_{\nu \in N} c_\nu \cdot e_\nu$ converges in $L^2(\rho)$ and $\sum_{\nu \in N} c_\nu \cdot \mathbf{e}_\nu(x)$ is absolutely convergent for every $x \in E$. Moreover,

$$K(x, y) := \sum_{\nu \in N} \alpha_\nu^{-1} \cdot \mathbf{e}_\nu(x) \cdot \overline{\mathbf{e}_\nu(y)}, \quad x, y \in E,$$

defines a reproducing kernel on $E \times E$. The corresponding Hilbert space is given by

$$(2.3) \quad H(K) = \left\{ \sum_{\nu \in N} c_\nu \cdot \mathbf{e}_\nu : (c_\nu)_{\nu \in N} \in \mathbb{K}^N \text{ with } \sum_{\nu \in N} \alpha_\nu \cdot |c_\nu|^2 < \infty \right\}$$

and

$$\langle \mathfrak{f}, \mathfrak{g} \rangle_{H(K)} = \sum_{\nu \in N} \alpha_\nu \cdot \int_E \mathfrak{f} \cdot \overline{\mathbf{e}_\nu} d\rho \cdot \int_E \mathbf{e}_\nu \cdot \overline{\mathfrak{g}} d\rho, \quad \mathfrak{f}, \mathfrak{g} \in H(K).$$

Proof. The convergence of $\sum_{\nu \in N} c_\nu \cdot e_\nu$ and $\sum_{\nu \in N} c_\nu \cdot \mathbf{e}_\nu(y)$, as claimed, is easily verified.

The remaining part of the proof is based on the following facts. The linear subspace

$$(2.4) \quad \mathcal{H} := \left\{ f \in L^2(\rho) : \sum_{\nu \in N} \alpha_\nu \cdot |\langle f, e_\nu \rangle_{L^2(\rho)}|^2 < \infty \right\}$$

of $L^2(\rho)$, equipped with the scalar product

$$(2.5) \quad \langle f, g \rangle_{\mathcal{H}} := \sum_{\nu \in N} \alpha_\nu \cdot \langle f, e_\nu \rangle_{L^2(\rho)} \cdot \langle e_\nu, g \rangle_{L^2(\rho)}, \quad f, g \in \mathcal{H},$$

is a Hilbert space with a continuous embedding of norm $\sup_{\nu \in N} \alpha_\nu^{-1/2}$ into $L^2(\rho)$. The elements $h_\nu := \alpha_\nu^{-1/2} e_\nu$ with $\nu \in N$ form an orthonormal basis of \mathcal{H} .

We employ the first part of the proof to define a linear mapping $\Phi: \mathcal{H} \rightarrow \mathbb{K}^E$ by

$$\Phi f(x) := \sum_{\nu \in N} \langle f, e_\nu \rangle_{L^2(\rho)} \cdot \mathbf{e}_\nu(x), \quad x \in E,$$

i.e., Φf is the pointwise limit of the L^2 -Fourier partial sums of $f \in \mathcal{H}$. Obviously, $\Phi h_\nu = \alpha_\nu^{-1/2} \mathbf{e}_\nu$. The mapping Φ is injective, see Gnewuch et al. (2019, Rem. 2.6), so that $\Phi(\mathcal{H})$, equipped with the scalar product

$$\langle \Phi f, \Phi g \rangle = \langle f, g \rangle_{\mathcal{H}}, \quad f, g \in \mathcal{H},$$

is a Hilbert space of \mathbb{K} -valued functions on E . It follows that $(\Phi(\mathcal{H}), \langle \cdot, \cdot \rangle)$ is a reproducing kernel Hilbert space, see Gnewuch et al. (2019, Lem. 2.2), and its reproducing kernel is K , see Gnewuch et al. (2019, Lem. 2.1). Finally, observe that

$$\Phi(\mathcal{H}) = \left\{ \sum_{\nu \in N} c_\nu \cdot \mathbf{e}_\nu : (c_\nu)_{\nu \in N} \in \mathbb{K}^N \text{ with } \sum_{\nu \in N} \alpha_\nu \cdot |c_\nu|^2 < \infty \right\}. \quad \square$$

Subsequently, we do no longer distinguish between square-integrable functions on E and elements of $L^2(\rho)$; in particular, we identify \mathbf{e}_ν and e_ν .

Remark 2.2. Assume that (2.1) and (2.2) for every $x \in E$ are satisfied. According to Proposition 2.1, $H(K) \subseteq L^2(\rho)$ with a continuous embedding, which is compact if and only if for every $r > 0$ the set $\{\nu \in N : \alpha_\nu \leq r\}$ is finite. The proof of the proposition reveals that \mathcal{H} is a feature space and $\varphi: E \rightarrow \mathcal{H}$, given by $\varphi(x) := \sum_{\nu \in N} \alpha_\nu^{-1} \cdot e_\nu(x) \cdot e_\nu$, is a feature map of K . See, e.g., Steinwart and Christmann (2008, Sec. 4.1) for these concepts.

Additionally, assume that

$$e_{\nu^*} = 1$$

for some $\nu^* \in N$. Then

$$K^* := K - \alpha_{\nu^*}^{-1}$$

is a reproducing kernel, too. Furthermore,

$$\int_E f d\rho = 0$$

for every $f \in H(K^*)$, since $\int_E e_\nu d\rho = \langle e_\nu, e_{\nu^*} \rangle_{L^2(\rho)} = 0$ for $\nu \neq \nu^*$. In particular, this yields

$$H(1) \cap H(K^*) = \{0\},$$

and $\int_E f d\rho$ is the orthogonal projection of $f \in H(K)$ onto $H(1)$ in $H(K)$ as well as in $L^2(\rho)$.

Remark 2.3. We discuss interpolation of Hilbert spaces in the present framework. To this end we consider two families $(\alpha_{\nu,0})_{\nu \in N}$ and $(\alpha_{\nu,1})_{\nu \in N}$ of Fourier weights such that $\inf_{\nu \in N} \alpha_{\nu,0} > 0$ and $\{\nu \in N : \alpha_{\nu,1}/\alpha_{\nu,0} \leq r\}$ is finite for every $r > 0$. Clearly $\inf_{\nu \in N} \alpha_{\nu,1} > 0$, too. Analogously to (2.4) and (2.5) we obtain linear subspaces \mathcal{H}_0 and \mathcal{H}_1 of $L^2(\rho)$, based on $(\alpha_{\nu,0})_{\nu \in N}$ and $(\alpha_{\nu,1})_{\nu \in N}$, respectively. It follows that \mathcal{H}_1 is a dense subspace of \mathcal{H}_0 with a compact embedding.

For quadratic interpolation of the spaces \mathcal{H}_0 and \mathcal{H}_1 by means of the K -method and the J -method we obtain

$$K_{\theta,2}(\mathcal{H}_0, \mathcal{H}_1) = J_{\theta,2}(\mathcal{H}_0, \mathcal{H}_1) = \mathcal{H}$$

for every $0 < \theta < 1$, where \mathcal{H} is the linear subspace of $L^2(\rho)$ based on the Fourier weights

$$\alpha_\nu := \alpha_{\nu,1}^\theta \cdot \alpha_{\nu,0}^{1-\theta}, \quad \nu \in \mathbb{N}.$$

Furthermore, we have equality of norms in these spaces. See Chandler-Wilde et al. (2015, Thm. 3.4). Assume, additionally, that (2.2) is satisfied for $\alpha_\nu := \alpha_{\nu,0}$. Then the same conclusions hold true for the associated reproducing kernel Hilbert spaces.

3. HERMITE SPACES

3.1. Functions of a Single Variable. Here we consider the space $L^2(\mu_0) := L^2(\mathbb{R}, \mu_0)$ for the standard normal distribution μ_0 over $\mathbb{K} = \mathbb{R}$.

The normalized Hermite polynomials h_ν of degree $\nu \in \mathbb{N}_0$ form an orthonormal basis of $L^2(\mu_0)$ and can be obtained by orthonormalizing the monomials. In particular, we have $h_0(x) = 1$ and $h_1(x) = x$ for all $x \in \mathbb{R}$. An explicit representation for any $\nu \in \mathbb{N}_0$ is

$$(3.1) \quad h_\nu(x) = \sqrt{\nu!} \sum_{k=0}^{\lfloor \nu/2 \rfloor} (-1)^k \frac{x^{\nu-2k}}{2^k k! (\nu-2k)!}, \quad x \in \mathbb{R},$$

see Szegő (1975, Eqn. (5.5.4)). For basic properties of these functions we refer to, e.g., Bogachev (1998, Sec. 1.3). In the next lemma we collect some asymptotic properties of the Hermite polynomials.

Lemma 3.1. *For every $x \in \mathbb{R}$ we have*

$$(3.2) \quad \sup_{\nu \in \mathbb{N}_0} (\nu^{1/4} \cdot |h_\nu(x)|) < \infty$$

as well as Cramér's inequality

$$(3.3) \quad \sup_{\nu \in \mathbb{N}_0} |h_\nu(x)| \leq \exp(x^2/4).$$

For $x = 0$ we have

$$(3.4) \quad \inf_{\nu \in \mathbb{N}} (\nu^{1/4} \cdot |h_{2\nu}(x)|) > 0.$$

Proof. For every $x \in \mathbb{R}$ we have

$$(3.5) \quad \sup_{\nu \in \mathbb{N}_0} \frac{\Gamma(\nu/2 + 1)}{\Gamma(\nu + 1)} \cdot 2^{\nu/2} \cdot \sqrt{\nu!} \cdot |h_\nu(x)| < \infty,$$

see, e.g., Szegő (1975, Eqn. (8.22.8)). Moreover, Stirling's approximation yields the strong asymptotics

$$(3.6) \quad \frac{\Gamma(\nu/2 + 1)}{\Gamma(\nu + 1)} \cdot 2^{\nu/2} \cdot \sqrt{\nu!} \approx (\pi/2)^{1/4} \cdot \nu^{1/4}$$

as $\nu \rightarrow \infty$. Combine (3.5) and (3.6) to obtain (3.2). For Cramér's inequality (3.3) we refer to, e.g., Indritz (1961). For $x = 0$ and $\nu \in \mathbb{N}_0$ we have

$$(3.7) \quad |h_{2\nu}(x)| = \sqrt{(2\nu)!}/(\nu! \cdot 2^\nu),$$

see, e.g., Szegő (1975, Eqn. (5.5.5)). Furthermore, (3.6) implies

$$(3.8) \quad \frac{\nu! \cdot 2^\nu}{\sqrt{(2\nu)!}} \approx \pi^{1/4} \cdot \nu^{1/4}$$

as $\nu \rightarrow \infty$. Combine (3.7) and (3.8) to obtain (3.4). \square

We will also employ the following estimate (with a suboptimal constant).

Lemma 3.2. *For every $\nu \in \mathbb{N}_0$ and every $x \in \mathbb{R}$ we have*

$$|h_\nu(x)| \leq 2^\nu \max(1, |x|^\nu).$$

Proof. Due to (3.1) we have

$$|h_\nu(x)| \leq \sum_{k=0}^{\lfloor \nu/2 \rfloor} \frac{\sqrt{\nu!}}{2^k k! (\nu - 2k)!} \cdot \max(1, |x|^\nu).$$

Furthermore, the elementary estimate $(2k)! \leq 2^{2k}(k!)^2$ implies $\frac{\sqrt{\nu!}}{2^k k!} \leq \frac{\nu!}{(2k)!}$ for $k \leq \nu/2$, and therefore

$$\sum_{k=0}^{\lfloor \nu/2 \rfloor} \frac{\sqrt{\nu!}}{2^k k! (\nu - 2k)!} \leq \sum_{k=0}^{\lfloor \nu/2 \rfloor} \frac{\nu!}{(2k)! (\nu - 2k)!} \leq \sum_{k=0}^{\nu} \frac{\nu!}{k! (\nu - k)!} = 2^\nu. \quad \square$$

With $e_\nu = h_\nu$ and Fourier weights $\alpha_\nu \in]0, \infty[$ for $\nu \in \mathbb{N}_0$, the summability property (2.2) reads as

$$(3.9) \quad \sum_{\nu \in \mathbb{N}_0} \alpha_\nu^{-1} \cdot |h_\nu(x)|^2 < \infty.$$

It is closely related to the condition

$$(3.10) \quad \sum_{\nu \in \mathbb{N}} \alpha_\nu^{-1} \cdot \nu^{-1/2} < \infty.$$

Lemma 3.3. *If (3.10) is satisfied, then (3.9) holds for every $x \in \mathbb{R}$.*

Proof. The statement follows directly from (3.2). \square

Due to (3.4) the reverse implication holds true under mild additional assumptions. For instance, if (3.9) holds for $x = 0$ and $\alpha_\nu \leq \alpha_{\nu+1}$ holds for all ν sufficiently large, then (3.10) is satisfied.

Definition 3.4. Assume that $\inf_{\nu \in \mathbb{N}_0} \alpha_\nu > 0$ and that (3.9) is satisfied for every $x \in \mathbb{R}$. The reproducing kernel

$$(3.11) \quad k(x, y) := \sum_{\nu \in \mathbb{N}_0} \alpha_\nu^{-1} \cdot h_\nu(x) \cdot h_\nu(y), \quad x, y \in \mathbb{R},$$

is called a *Hermite kernel* and the Hilbert space $H(k)$ is called a *Hermite space* of functions of a single real variable.

Observe that Proposition 2.1 provides a characterization of the Hermite space $H(k)$ as a linear subspace of $L^2(\mu_0)$. For definition and terminology we refer to Irgeher and Leobacher (2015, Sec. 3.1) and Dick et al. (2018, Sec. 2), where $\sum_{\nu \in \mathbb{N}} \alpha_\nu^{-1} < \infty$ or $\sum_{\nu \in \mathbb{N}} \alpha_\nu \cdot \nu^{-1/6} < \infty$, respectively, is assumed, cf. (3.10) and Lemma 3.3.

Example 3.5. Let $r > 0$. If $\alpha_\nu := (\nu + 1)^r$ for $\nu \in \mathbb{N}_0$, then (3.10) is satisfied if and only if $r > 1/2$. Moreover, the spaces $H(k)$ with $r > 1/2$ form an interpolation scale, see Remark 2.3. In the particular case $r \in \mathbb{N}$ the space $H(k)$ is, up to equivalence of norms, the Sobolev space of all continuous functions in $L^2(\mu_0)$ with weak derivatives up to order r belonging to $L^2(\mu_0)$, see, e.g., Bogachev (1998, Prop. 1.5.4) and Dick et al. (2018, Sec. 2). Unfortunately, it seems to be unknown

if there is a corresponding natural interpolation scale of weighted Sobolev spaces of fractional order. The known interpolation results for weighted Sobolev spaces of functions on \mathbb{R} require weights that decrease slower at infinity than the Gaussian weight function, see Triebel (1978, Sec. 3.3.1). Hence an intrinsic characterization of the functions in $H(k)$ for non-integer r via fractional smoothness remains open.

Necessary as well as sufficient conditions, involving classical differentiability, for $f \in H(k)$ to hold true are presented in Irrgeher and Leobacher (2015, Sec. 3.1.1) in the case $r > 1$.

Example 3.6. Let $r, b > 0$. If $\alpha_0 := 1$ and $\alpha_\nu := 2^{r \cdot \nu^b}$ for $\nu \geq 1$, then (3.10) is satisfied in any case, and $H(k)$ consists of C^∞ -functions. Moreover, in the case $b = 1$ the spaces $H(k)$ with $r > 0$ form an interpolation scale, see Remark 2.3. If $b \geq 1$ then the elements of $H(k)$ are real analytic functions, see Irrgeher and Leobacher (2015, Prop. 3.7) for the case $b = 1$, which trivially extends to any $b \geq 1$. It is also observed in Irrgeher et al. (2016b, Rem. 1) without proof that, for $0 < b < 1$, the elements of $H(k)$ belong to the Gevrey class $G^\sigma(\mathbb{R})$ of index $\sigma := 1/b$. By definition, $G^\sigma(\mathbb{R})$ with index $\sigma \geq 1$ consists of all infinitely differentiable functions f on \mathbb{R} such that for any $R > 0$ there exists $c > 0$ with

$$|f^{(\ell)}(x)| \leq (c^{\ell+1} \ell!)^\sigma$$

for $\ell \in \mathbb{N}_0$ and $x \in [-R, R]$. The Gevrey class $G^1(\mathbb{R})$ coincides with the class of real analytic functions on \mathbb{R} , while $G^\sigma(\mathbb{R})$ contains compactly supported functions $f \neq 0$ for $\sigma > 1$, see Rodino (1993). The following lemma improves upon the known regularity results for functions in $H(k)$.

Lemma 3.7. *Let k be the Hermite kernel specified in Example 3.6 and let*

$$\sigma := \max(1, 1/2b).$$

Then

$$H(k) \subseteq G^\sigma(\mathbb{R}).$$

In particular, $H(k)$ contains only real analytic functions if $b \geq 1/2$.

Proof. The first part of the proof follows the argument of Irrgeher and Leobacher (2015, Prop. 3.7) and is based on the formula

$$h_\nu^{(\ell)} = \sqrt{\frac{\nu!}{(\nu - \ell)!}} h_{\nu - \ell}$$

for $\ell \leq \nu$ that allows it to directly relate the coefficients in the orthogonal expansion of $f^{(\ell)}$ with respect to the Hermite polynomials with the coefficients in the expansion of $f \in H(k)$. Then Cramér's inequality (3.3) together with the Cauchy-Schwarz inequality lead to the pointwise estimate

$$|f^{(\ell)}(x)| \leq \exp(x^2/4) \cdot \|f\|_{H(k)} \cdot \left(\sum_{\nu=\ell}^{\infty} \frac{\nu!}{(\nu - \ell)!} \alpha_\nu^{-1} \right)^{1/2}.$$

It follows that f belongs to the Gevrey class $G^\sigma(\mathbb{R})$ with index $\sigma \geq 1$ if there exists $c > 0$ such that

$$(3.12) \quad \sum_{\nu=\ell}^{\infty} \frac{\nu!}{(\nu - \ell)!} \alpha_\nu^{-1} \leq (c^{\ell+1} \ell!)^{2\sigma}$$

for $\ell \in \mathbb{N}_0$. We now use

$$\alpha_\nu^{-1} = 2^{-r \cdot \nu^b} \leq 2^r \cdot 2^{-r \cdot t^b}$$

for $t \in [\nu, \nu + 1]$ to estimate

$$\sum_{\nu=\ell}^{\infty} \frac{\nu!}{(\nu-\ell)!} \alpha_\nu^{-1} \leq \sum_{\nu=\ell}^{\infty} \nu^\ell \alpha_\nu^{-1} \leq 2^r \int_\ell^\infty t^\ell 2^{-r \cdot t^b} dt \leq c^{\ell+1} \int_0^\infty u^{(\ell+1)/b-1} e^{-u} du$$

with $c > 0$ depending only on r and b . The last integral is just $\Gamma\left(\frac{\ell+1}{b}\right)$. Now using Stirling's formula both as upper bound for $\Gamma\left(\frac{\ell+1}{b}\right)$ and as lower bound for $\ell!$ shows that there exists $c > 0$ depending only on r and b such that

$$\sum_{\nu=\ell}^{\infty} \frac{\nu!}{(\nu-\ell)!} \alpha_\nu^{-1} \leq (c^{\ell+1} \ell!)^{1/b}.$$

Now comparing with (3.12) proves the lemma. \square

3.2. Functions of Infinitely Many Variables. Our construction of Hermite spaces of functions of infinitely many real variables is based on Fourier weights $\alpha_{\nu,j} \in]0, \infty[$ for $\nu \in \mathbb{N}_0$ and $j \in \mathbb{N}$ with the following properties:

- (H1) $\sum_{\nu \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot \nu^{-1/2} < \infty$ for every $j \in \mathbb{N}$ and $\alpha_{\nu,j} \leq \alpha_{\nu+1,j}$ for all $\nu, j \in \mathbb{N}$,
- (H2) $\sum_{j \in \mathbb{N}} |\alpha_{0,j} - 1| < \infty$,
- (H3) there exists an integer $j_0 \in \mathbb{N}$ such that $\sum_{\nu \in \mathbb{N}, j \geq j_0} \alpha_{\nu,j}^{-1} < \infty$.

If (H1) is satisfied then we may consider the Hermite kernels

$$k_j(x, y) := \sum_{\nu \in \mathbb{N}_0} \alpha_{\nu,j}^{-1} \cdot h_\nu(x) \cdot h_\nu(y), \quad x, y \in \mathbb{R},$$

as well as the Hermite spaces $H(k_j)$ as linear subspaces of $L^2(\mu_0)$, see Proposition 2.1 and Lemma 3.3. Observe that

$$k_j^* := k_j - \alpha_{0,j}^{-1}$$

is a reproducing kernel, too, which follows from $h_0 = 1$. Cf. Remark 2.2. Property (H2) provides a certain amount of flexibility for the choice of the Fourier weights $\alpha_{0,j}$; values different from 1 naturally arise in Section 5.

In each of the spaces $H(k_j)$ we choose the constant function $\alpha_{0,j}^{-1/2}$ as unit vector and we study the corresponding incomplete tensor product

$$(3.13) \quad H(\alpha_0^{-1/2}) := \bigotimes_{j \in \mathbb{N}} (H(k_j))^{(\alpha_{0,j}^{-1/2})},$$

see Definition A.2.

We will employ Theorem A.6 to identify the space $H(\alpha_0^{-1/2})$ with a reproducing kernel Hilbert space of functions on a suitable subset \mathfrak{X} of $\mathbb{R}^{\mathbb{N}}$. For any non-empty domain $\mathfrak{X} \subseteq \mathbb{R}^{\mathbb{N}}$ that ensures the convergence of the partial products $\prod_{j=1}^J k_j(x_j, y_j)$ as $J \rightarrow \infty$ we consider the tensor product kernel K given by

$$(3.14) \quad K(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^{\infty} k_j(x_j, y_j), \quad \mathbf{x}, \mathbf{y} \in \mathfrak{X}.$$

For a finite subset \mathbf{u} of \mathbb{N} , we put

$$c_{\mathbf{u}} := \prod_{j \in \mathbf{u}} \alpha_{0,j} \cdot \prod_{j=1}^{\infty} \alpha_{0,j}^{-1},$$

as well as

$$k_u^*(\mathbf{x}, \mathbf{y}) := \prod_{j \in u} k_j^*(x_j, y_j), \quad \mathbf{x}, \mathbf{y} \in \mathfrak{X}.$$

Lemma 3.8. *Assume that (H1) and (H2) are satisfied. Then the maximal domain for K is given by*

$$(3.15) \quad \mathfrak{X} := \left\{ \mathbf{x} \in \mathbb{R}^{\mathbb{N}} : \sum_{j \in \mathbb{N}} k_j^*(x_j, x_j) < \infty \right\}.$$

Moreover,

$$(3.16) \quad K(\mathbf{x}, \mathbf{y}) = \sum_{u \in \mathcal{U}} c_u \cdot k_u^*(\mathbf{x}, \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathfrak{X},$$

with absolute convergence.

Proof. At first, we consider the particular case that $\alpha_{0,j} = 1$ for every $j \in \mathbb{N}$. For $J \in \mathbb{N}$ let \mathcal{U}_J denote the power set of $\{1, \dots, J\}$. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ we have

$$\prod_{j=1}^J k_j(x_j, y_j) = \prod_{j=1}^J (1 + k_j^*(x_j, y_j)).$$

Since

$$\ln \left(\prod_{j=1}^J (1 + k_j^*(x_j, y_j)) \right) = \sum_{j=1}^J \ln(1 + k_j^*(x_j, y_j))$$

and $t/2 \leq \ln(1+t) \leq t$ for all $0 \leq t \leq 1$, we conclude that $\prod_{j=1}^J k_j(x_j, y_j)$ converges as $J \rightarrow \infty$ if and only if $\mathbf{x} \in \mathfrak{X}$. Let $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$. Using

$$(3.17) \quad |k_j^*(x_j, y_j)| \leq \sqrt{k_j^*(x_j, x_j)} \cdot \sqrt{k_j^*(y_j, y_j)}$$

and the Cauchy Schwarz inequality for sums, we obtain

$$\begin{aligned} \sum_{u \in \mathcal{U}_J} \prod_{j \in u} |k_j^*(x_j, y_j)| &\leq \left(\sum_{u \in \mathcal{U}_J} \prod_{j \in u} k_j^*(x_j, x_j) \right)^{1/2} \cdot \left(\sum_{u \in \mathcal{U}_J} \prod_{j \in u} k_j^*(y_j, y_j) \right)^{1/2} \\ &= \left(\prod_{j=1}^J (1 + k_j^*(x_j, x_j)) \right)^{1/2} \cdot \left(\prod_{j=1}^J (1 + k_j^*(y_j, y_j)) \right)^{1/2}. \end{aligned}$$

Consequently,

$$\prod_{j=1}^J (1 + k_j^*(x_j, y_j)) = \sum_{u \in \mathcal{U}_J} \prod_{j \in u} k_j^*(x_j, y_j)$$

is absolutely convergent as $J \rightarrow \infty$. We conclude that \mathfrak{X} is the maximal domain for K , and (3.16) is satisfied with absolute convergence.

The general case is easily reduced to the particular case, since $\prod_{j=1}^J \alpha_{0,j}$ converges and $\lim_{j \rightarrow \infty} \alpha_{0,j} = 1$ due to (H2) and since

$$(3.18) \quad k_j = \alpha_{0,j}^{-1} \cdot (1 + \alpha_{0,j} \cdot k_j^*). \quad \square$$

Observe that \mathfrak{X} does not depend on the Fourier weights $\alpha_{0,j}$.

Definition 3.9. Assume that (H1) and (H2) are satisfied. The reproducing kernel K defined by (3.14) on the maximal domain \mathfrak{X} given by (3.15) is called a *Hermite kernel* and the Hilbert space $H(K)$ is called a *Hermite space* of functions of infinitely many real variables.

Let $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$. It follows from (3.17) that $\sum_{j \in \mathbb{N}} |k_j^*(x_j, y_j)| < \infty$. If (H1) and (H2) are satisfied, then the product in (3.14) is convergent in the sense of the definition in Section A.1, and $K(\mathbf{x}, \mathbf{y}) = 0$ is equivalent to the existence of $j \in \mathbb{N}$ with $k_j(x_j, y_j) = 0$.

The Hermite space $H(K)$ is, in the sense of Theorem A.6, the incomplete tensor product $H(\boldsymbol{\alpha}_0^{-1/2})$ of the Hermite spaces $H(k_j)$ based on the constant functions $\alpha_{0,j}^{-1/2}$ for $j \in \mathbb{N}$ as unit vectors as in (3.13).

We now analyze the maximal domain \mathfrak{X} in more detail. Put $\boldsymbol{\alpha}_\nu^{-1} := (\alpha_{\nu,j}^{-1})_{j \in \mathbb{N}}$ for $\nu \in \mathbb{N}$, and observe that (H3) implies $\sum_{j \in \mathbb{N}} \alpha_{\nu,j}^{-1} < \infty$. The space $\ell^{2\nu}(\boldsymbol{\alpha}_\nu^{-1})$ consists of all sequences $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\sum_{j \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot x_j^{2\nu} < \infty.$$

In the sequel, we consider the countable product μ of the standard normal distribution μ_0 on $\mathbb{R}^{\mathbb{N}}$.

Proposition 3.10. *Assume that (H1) is satisfied. Then we have*

$$\mathfrak{X} \subseteq \bigcap_{\nu \in \mathbb{N}} \ell^{2\nu}(\boldsymbol{\alpha}_\nu^{-1}) \subsetneq \mathbb{R}^{\mathbb{N}}.$$

Assume that (H1) and (H3) are satisfied. Then we have

$$\mu(\mathfrak{X}) = 1$$

and

$$\ell^\infty \subsetneq \mathfrak{X}.$$

Proof. Put $\ell^{2\nu} := \ell^{2\nu}(\boldsymbol{\alpha}_\nu^{-1})$, and observe that already $\ell^2 \subsetneq \mathbb{R}^{\mathbb{N}}$. We prove the remaining part of the first statement by induction.

Since $h_1(x_j) = x_j$ and $\alpha_{1,j}^{-1} \cdot |h_1(x_j)|^2 \leq k_j^*(x_j, x_j)$, we obtain $\mathfrak{X} \subseteq \ell^2$. Let $\mathbf{x} \in \mathfrak{X}$, let $\nu > 1$, and assume that $\mathbf{x} \in \ell^{2\kappa}$ holds for all $1 \leq \kappa < \nu$. Since h_ν is either an even or an odd polynomial, h_ν^2 is an even polynomial of degree 2ν and can thus be written as

$$h_\nu^2(x) = \sum_{\kappa=0}^{\nu} \beta_\kappa \cdot x^{2\kappa}$$

with suitable $\beta_\kappa \in \mathbb{R}$, where $\beta_\nu \neq 0$. Consequently,

$$x^{2\nu} = 1/\beta_\nu \cdot \left(|h_\nu(x)|^2 - \sum_{\kappa=0}^{\nu-1} \beta_\kappa \cdot x^{2\kappa} \right).$$

Due to the second condition in (H1) and our induction hypothesis we have for all $1 \leq \kappa < \nu$

$$\sum_{j \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot x_j^{2\kappa} \leq \sum_{j \in \mathbb{N}} \alpha_{\kappa,j}^{-1} \cdot x_j^{2\kappa} < \infty.$$

Furthermore,

$$\sum_{j \in \mathbb{N}} \alpha_{\nu, j}^{-1} \cdot |h_{\nu}(x_j)|^2 \leq \sum_{j \in \mathbb{N}} k_j^*(x_j, x_j) < \infty.$$

Altogether this yields

$$\sum_{j \in \mathbb{N}} \alpha_{\nu, j}^{-1} \cdot x_j^{2\nu} < \infty,$$

implying that $\mathbf{x} \in \ell^{2\nu}$, which proves the first statement.

Since

$$\int_{\mathbb{R}^{\mathbb{N}}} k_j^*(x_j, x_j) d\mu(\mathbf{x}) = \sum_{\nu \in \mathbb{N}} \alpha_{\nu, j}^{-1},$$

we get

$$\int_{\mathbb{R}^{\mathbb{N}}} \sum_{j \geq j_0} k_j^*(x_j, x_j) d\mu(\mathbf{x}) < \infty$$

from (H3) using the monotone convergence theorem. Hence $\sum_{j \in \mathbb{N}} k_j^*(x_j, x_j) < \infty$ holds μ -a.e., i.e., $\mu(\mathfrak{X}) = 1$. Combine Cramér's inequality, see Lemma 3.1, and (H3) to conclude that $\ell^{\infty} \subseteq \mathfrak{X}$. Finally, $\ell^{\infty} \neq \mathfrak{X}$, since $\mu(\ell^{\infty}) = 0$. \square

Recall that \mathbf{N} denotes the set of all sequences $\boldsymbol{\nu} := (\nu_j)_{j \in \mathbb{N}}$ in \mathbb{N}_0 such that $\{j \in \mathbb{N} : \nu_j \neq 0\}$ is finite. For $\boldsymbol{\nu} \in \mathbf{N}$ we set

$$h_{\boldsymbol{\nu}}(\mathbf{x}) := \prod_{j=1}^{\infty} h_{\nu_j}(x_j)$$

for $\mathbf{x} \in \mathfrak{X}$, which is well-defined since $h_0 = 1$. Moreover, we set

$$\alpha_{\boldsymbol{\nu}} := \prod_{j=1}^{\infty} \alpha_{\nu_j, j},$$

which is well-defined if (H2) is assumed.

In addition to the original Fourier weights we also consider

$$\alpha'_{\nu, j} := \begin{cases} 1, & \text{if } \nu = 0, \\ \alpha_{\nu, j}, & \text{otherwise,} \end{cases}$$

and we use $\alpha'_{\boldsymbol{\nu}}$, K' , and k'_j to denote the corresponding products and the corresponding reproducing kernels, respectively. The counterpart to (H2) is trivially satisfied for the new Fourier weights, and the counterparts to (H1) and (H3) follow from the respective properties of the original Fourier weights.

Lemma 3.11. *Assume that (H2) is satisfied. Then we have*

$$c_{\min} := \inf_{\boldsymbol{\nu} \in \mathbf{N}} \frac{\alpha_{\boldsymbol{\nu}}}{\alpha'_{\boldsymbol{\nu}}} = \prod_{j=1}^{\infty} \min(\alpha_{0, j}, 1) > 0$$

and

$$c_{\max} := \sup_{\boldsymbol{\nu} \in \mathbf{N}} \frac{\alpha_{\boldsymbol{\nu}}}{\alpha'_{\boldsymbol{\nu}}} = \prod_{j=1}^{\infty} \max(\alpha_{0, j}, 1) < \infty.$$

Proof. Use (H2) to conclude that

$$\inf_{\nu \in \mathbf{N}} \frac{\alpha_\nu}{\alpha'_\nu} = \lim_{J \rightarrow \infty} \left(\prod_{j=1}^J \min(\alpha_{0,j}, 1) \cdot \prod_{j=J+1}^{\infty} \alpha_{0,j} \right) = \prod_{j=1}^{\infty} \min(\alpha_{0,j}, 1) > 0.$$

The second statement is verified analogously. \square

Remark A.4 and Theorem A.7 show that the functions h_ν with $\nu \in \mathbf{N}$ form an orthonormal basis of $L^2(\mu) := L^2(\mathbb{R}^{\mathbf{N}}, \mu)$, which may obviously be identified with $L^2(\mathfrak{X}, \mu)$ if $\mu(\mathfrak{X}) = 1$. Together with the following lemma, Proposition 2.1 provides a characterization of the Hilbert space $H(K)$ as a linear subspace of $L^2(\mu)$.

Lemma 3.12. *Assume that (H1)–(H3) are satisfied. Then we have*

$$\inf_{\nu \in \mathbf{N}} \alpha_\nu > 0$$

and

$$(3.19) \quad \sum_{\nu \in \mathbf{N}} \alpha_\nu^{-1} \cdot |h_\nu(\mathbf{x})|^2 < \infty$$

for $\mathbf{x} \in \mathfrak{X}$ as well as

$$(3.20) \quad K(\mathbf{x}, \mathbf{y}) = \sum_{\nu \in \mathbf{N}} \alpha_\nu^{-1} \cdot h_\nu(\mathbf{x}) \cdot h_\nu(\mathbf{y})$$

for $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$.

Proof. Obviously, (H3) implies $\sum_{j \geq j_0} \alpha_{1,j}^{-1} < \infty$. We use the second condition in (H1) to conclude that there exists an integer $j_1 \geq j_0$ with

$$\inf_{j \geq j_1} \inf_{\nu \in \mathbf{N}} \alpha_{\nu,j} = \inf_{j \geq j_1} \alpha_{1,j} \geq 1.$$

Moreover,

$$c := \inf_{j=1, \dots, j_1-1} \inf_{\nu \in \mathbb{N}_0} \alpha_{\nu,j} > 0,$$

which follows from (H1). Together with (H2), cf. Lemma 3.11, this implies

$$\inf_{\nu \in \mathbf{N}} \alpha_\nu \geq c^{j_1-1} \cdot \prod_{j=j_1}^{\infty} \min(\alpha_{0,j}, 1) > 0.$$

Consider the particular case that $\alpha_{0,j} = 1$ for every $j \in \mathbb{N}$. Let $\mathbf{x} \in \mathbb{R}^{\mathbf{N}}$. According to Gnewuch et al. (2019, Lem. B.1), applied with $\beta_{\nu,j} := \alpha_{\nu,j}^{-1} \cdot |h_\nu(x_j)|^2$, (3.19) holds if and only if

$$\sum_{\nu, j \in \mathbb{N}} \alpha_{\nu,j}^{-1} \cdot |h_\nu(x_j)|^2 < \infty,$$

and the latter is equivalent to $\mathbf{x} \in \mathfrak{X}$. Let $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$. Applying Gnewuch et al. (2019, Lem. B.1) again, this time with $\beta_{\nu,j} := \alpha_{\nu,j}^{-1} \cdot h_\nu(x_j) \cdot h_\nu(y_j)$, we obtain (3.20).

The general case is easily reduced to the particular case by means of Lemma 3.11 and (3.18). \square

Lemma 3.13. *Assume that (H1) is satisfied. Then we have $H(k_j) = H(k'_j)$ for $j \in \mathbb{N}$ as vector spaces, and the corresponding identity maps have norms*

$$\|H(k_j) \hookrightarrow H(k'_j)\| = \min(\alpha_{0,j}, 1)^{-1/2} \quad \text{and} \quad \|H(k'_j) \hookrightarrow H(k_j)\| = \max(\alpha_{0,j}, 1)^{1/2}.$$

Assume that (H1)–(H3) are satisfied. Then we have $H(K) = H(K')$ as vector spaces, and the corresponding identity maps have norms

$$\|H(K) \hookrightarrow H(K')\| = c_{\min}^{-1/2} \quad \text{and} \quad \|H(K') \hookrightarrow H(K)\| = c_{\max}^{1/2}.$$

Proof. Let (H1) hold, and let $j \in \mathbb{N}$. Proposition 2.1 shows that $H(k_j) = H(k'_j)$ and confirms the claimed values of the norms of the embeddings.

Let now (H1)–(H3) be satisfied. We combine Proposition 2.1, Lemma 3.11, and Lemma 3.12 to obtain $H(K') = H(K)$ as well as the claimed values of the norms of the embeddings. \square

Remark 3.14. Assume that (H1)–(H3) are satisfied. For each $j \in \mathbb{N}$ we have

$$\int_{\mathbb{R}} k_j^*(x, y) d\mu_0(y) = 0$$

for all $x \in \mathbb{R}$, cf. Remark 2.2. This property induces a function space decomposition of the tensor product space $H(K)$ of ANOVA-type, cf. Baldeaux and Gnewuch (2014, Rem. 2.12). Let us make this more precise: Due to (3.16) we may write $H(K)$ as orthogonal sum

$$H(K) = \bigoplus_{\mathbf{u} \in \mathbf{U}} H(c_{\mathbf{u}} \cdot k_{\mathbf{u}}^*).$$

Note that due to $c_{\mathbf{u}} > 0$ we have $H(c_{\mathbf{u}} \cdot k_{\mathbf{u}}^*) = H(k_{\mathbf{u}}^*)$ as vector spaces for all $\mathbf{u} \in \mathbf{U}$. Hence each function $f \in H(K)$ can be uniquely represented as

$$f = \sum_{\mathbf{u} \in \mathbf{U}} f_{\mathbf{u}},$$

where $f_{\mathbf{u}} \in H(k_{\mathbf{u}}^*)$ for each $\mathbf{u} \in \mathbf{U}$, and its norm is given by

$$\|f\|_{H(K)}^2 = \sum_{\mathbf{u} \in \mathbf{U}} \|f_{\mathbf{u}}\|_{H(c_{\mathbf{u}} \cdot k_{\mathbf{u}}^*)}^2 = \sum_{\mathbf{u} \in \mathbf{U}} c_{\mathbf{u}}^{-1} \cdot \|f_{\mathbf{u}}\|_{H(k_{\mathbf{u}}^*)}^2.$$

This function decomposition is the infinite-dimensional *ANOVA-decomposition* of f . More precisely, we obtain

$$f_{\emptyset} = \int_{\mathfrak{X}} f d\mu$$

and, recursively, for arbitrary $\emptyset \neq \mathbf{u} \in \mathbf{U}$

$$f_{\mathbf{u}}(\mathbf{x}) = \int_{\mathbb{R}^{\mathbb{N} \setminus \mathbf{u}}} f(\mathbf{x}_{\mathbf{u}}, \mathbf{y}_{\mathbb{N} \setminus \mathbf{u}}) d\mu_0^{\mathbb{N} \setminus \mathbf{u}}(\mathbf{y}_{\mathbb{N} \setminus \mathbf{u}}) - \sum_{\mathbf{v} \subsetneq \mathbf{u}} f_{\mathbf{v}}(\mathbf{x}), \quad \mathbf{x} \in \mathfrak{X},$$

and

$$\|f\|_{L^2(\mu)}^2 = \sum_{\mathbf{u} \in \mathbf{U}} \|f_{\mathbf{u}}\|_{L^2(\mu)}^2$$

as well as

$$\sigma^2(f) = \sum_{\emptyset \neq \mathbf{u} \in \mathbf{U}} \sigma^2(f_{\mathbf{u}}),$$

where $\sigma^2(\cdot)$ denotes the variance. In particular, for $f \in H(K)$ the ANOVA-components $f_{\mathbf{u}}$ with $\mathbf{u} \in \mathbf{U}$ are orthogonal in $H(K)$ as well as in $L^2(\mu)$, and each function $f_{\mathbf{u}}$ depends only on the variables x_j with $j \in \mathbf{u}$.

Remark 3.15. Infinite tensor products K of reproducing kernels $k_j: D \times D \rightarrow \mathbb{R}$ provide a convenient setting for the study of computational problems for functions of infinitely many variables, and most often the whole Cartesian product $D^{\mathbb{N}}$ is the natural domain for the kernel K . The complexity of problems of this kind has first been analyzed in Hickernell and Wang (2001), Hickernell et al. (2010), Kuo et al. (2010), and we refer to, e.g., Gnewuch et al. (2019) for recent results and references. So far two different types of spaces of functions of infinitely many variables have been studied: First, *weighted spaces*, where the weights model the importance of different groups of variables. This kind of weights was first introduced in Sloan and Woźniakowski (1998). Second, *spaces of increasing smoothness* based on Fourier weights, which is the approach we follow in this paper. The effect of increasing smoothness on the tractability of multivariate problems was first studied in Papageorgiou and Woźniakowski (2010).

Assume that $\alpha_{0,j} = 1$ for every $j \in \mathbb{N}$. Due to the representation (3.16) the kernel K is the superposition of finite tensor products of reproducing kernels k_j^* . In such a setting the *multivariate decomposition method* has been established as a powerful generic algorithm for integration and L^2 -approximation, if additionally all of the k_j^* are weighted anchored kernels, which leads to a so-called anchored function space decomposition. By definition, k_j^* is *anchored* at $x \in D$ if

$$k_j^*(x, x) = 0.$$

We refer to the pioneering papers Kuo et al. (2010) and Plaskota and Wasilkowski (2011), and to Gilbert et al. (2018) for a recent contribution.

Observe, however, that in the present setting of Hermite spaces none of the kernels k_j^* is anchored at any point $x \in \mathbb{R}$, since $h_1(x) = x$ for every $x \in \mathbb{R}$ and $h_2(0) \neq 0$. As explained in Remark 3.14, instead of having an anchored function space decomposition, $H(K)$ has an ANOVA-decomposition. This prohibits the application of the standard error analysis of the multivariate decomposition method developed in Plaskota and Wasilkowski (2011) for integration and in Wasilkowski (2012) for L^2 -approximation and of standard arguments to derive lower bounds for deterministic algorithms on $H(K)$.

In the ANOVA setting the multivariate decomposition method has been analyzed directly in Dick and Gnewuch (2014) for integration in the randomized setting and indirectly via suitable embedding theorems in Gnewuch et al. (2017, 2019) for integration and L^2 -approximation in the randomized and the deterministic setting.

While Dick and Gnewuch (2014) and Gnewuch et al. (2017) treat weighted spaces $H(K)$ and allow for domains \mathfrak{X} of K that are proper subsets of $D^{\mathbb{N}}$, the paper Gnewuch et al. (2019) studies spaces of increasing smoothness whose kernels are defined on the whole Cartesian product $D^{\mathbb{N}}$. It is important to note that Hermite spaces $H(K)$ are never spaces of functions on the domain $\mathbb{R}^{\mathbb{N}}$, see Proposition 3.10. In a forthcoming paper we will develop the approach from Gnewuch et al. (2019) further and apply it, in particular, to Hermite spaces $H(K)$.

3.3. Two Examples. In the sequel we consider two particular kinds of Fourier weights $\alpha_{\nu,j}$ and the corresponding Hermite spaces $H(k_j)$ and $H(K)$, cf. Examples 3.5 and 3.6.

(PG) Let $r_j > 1/2$ for $j \in \mathbb{N}$ such that

$$(3.21) \quad \sum_{j \in \mathbb{N}} 2^{-r_j} < \infty.$$

The Fourier weights with a *polynomial growth* are given by

$$\alpha_{\nu,j} := (\nu + 1)^{r_j}$$

for $\nu, j \in \mathbb{N}$ and by any choice of $\alpha_{0,j}$ satisfying (H2).

(EG) Let $r_j > 0$ for $j \in \mathbb{N}$ such that (3.21) is satisfied. Moreover, let $b_j > 0$ such that

$$\inf_{j \in \mathbb{N}} b_j > 0.$$

The Fourier weights with a *(sub-)exponential growth* are given by

$$\alpha_{\nu,j} := 2^{r_j \cdot \nu^{b_j}}$$

for $\nu, j \in \mathbb{N}$ and by any choice of $\alpha_{0,j}$ satisfying (H2).

Finite tensor products of Hermite spaces with different kinds of Fourier weights have been studied, e.g., in the following papers. Fourier weights with a polynomial growth are considered in Irrgeher and Leobacher (2015) for $r_j > 1$ and in Dick et al. (2018) for $r_j \in \mathbb{N}$. For Fourier weights with a (sub-)exponential growth we refer to Irrgeher and Leobacher (2015), Irrgeher et al. (2015) as well as Irrgeher et al. (2016b), and Irrgeher et al. (2016a) for the cases $b_j = 1$, $b_j \geq 1$, and $\inf_{j \in \mathbb{N}} b_j > 0$, respectively.

Remark 3.16. Let

$$\hat{r} := \liminf_{j \rightarrow \infty} \frac{r_j}{\ln(j)}.$$

Then $\hat{r} > 1/\ln(2)$ is a sufficient, and $\hat{r} \geq 1/\ln(2)$ is a necessary condition for (3.21) to hold. See, e.g., Gnewuch et al. (2019, Lem. B.3).

Lemma 3.17. *In both cases, (PG) and (EG), we have (H1)–(H3).*

Proof. Due to the considerations in Examples 3.5 and 3.6, respectively, (H1) is satisfied in both cases, and (H2) is satisfied by assumption.

For proving (H3) we use

$$\sum_{k \geq k_0} k^{-\tau} \leq k_0^{-\tau} + \int_{k_0}^{\infty} k^{-\tau} d\tau = k_0^{-\tau} \cdot \left(1 + \frac{k_0}{\tau - 1}\right)$$

for $\tau > 1$ and $k_0 \in \mathbb{N}$.

Consider the case (PG). Here we choose $j_0 \in \mathbb{N}$ such that $r_j \geq 2$ for every $j \geq j_0$. It follows that

$$\sum_{\nu \in \mathbb{N}} \alpha_{\nu,j}^{-1} = \sum_{\nu \in \mathbb{N}} (\nu + 1)^{-r_j} \leq 2^{-r_j} \cdot 3$$

for $j \geq j_0$, which yields (H3), as claimed.

Consider the case (EG). Here we put $b := \inf_{j \in \mathbb{N}} b_j > 0$ and choose $j_0 \geq 2$ and $c > 0$ such that $2^{r_j} \geq j^c$ for every $j \geq j_0$, see Remark 3.16. It follows that

$$\alpha_{\nu,j} = 2^{r_j \cdot \nu^{b_j}} \geq 2^{r_j \cdot \nu^b} \geq j^{c \cdot \nu^b}$$

for $j \geq j_0$ and $\nu \in \mathbb{N}$. We choose $\nu_0 \in \mathbb{N}$ such that $c \cdot \nu_0^b > 1$. For $\nu \geq \nu_0$ this yields

$$\sum_{j \geq j_0} \alpha_{\nu,j}^{-1} \leq j_0^{-c \cdot \nu^b} \cdot \left(1 + \frac{j_0}{c \cdot \nu_0^b - 1}\right),$$

while

$$\sum_{j \geq j_0} \alpha_{\nu,j}^{-1} \leq \sum_{j \geq j_0} 2^{-r_j} < \infty$$

for $1 \leq \nu < \nu_0$. Since

$$\sum_{\nu \in \mathbb{N}} j_0^{-c \cdot \nu^b} < \infty,$$

we obtain (H3), as claimed. \square

According to Proposition 3.10 and Lemma 3.17, the maximal domain \mathfrak{X} satisfies

$$\mathfrak{X} \subseteq \bigcap_{\nu \in \mathbb{N}} \ell^{2\nu}(\alpha_\nu^{-1}) \subseteq \ell^2(\alpha_1^{-1})$$

in both cases, (PG) and (EG).

Proposition 3.18. *In the case (EG) with exponential growth, i.e., if $b_j \geq 1$ for every $j \in \mathbb{N}$, we have*

$$\mathfrak{X} = \bigcap_{\nu \in \mathbb{N}} \ell^{2\nu}(\alpha_\nu^{-1}) = \ell^2(\alpha_1^{-1}).$$

Proof. It suffices to show that

$$\ell^2(\alpha_1^{-1}) \subseteq \mathfrak{X}.$$

Since \mathfrak{X} becomes larger if we increase the b_j , $j \in \mathbb{N}$, and $\ell^2(\alpha_1^{-1})$ is independent of the choice of the b_j , $j \in \mathbb{N}$, we actually only need to consider the case where $b_j = 1$ for every $j \in \mathbb{N}$. Let $\mathbf{x} \in \ell^2(\alpha_1^{-1})$. To prove $\mathbf{x} \in \mathfrak{X}$ it suffices to prove

$$(3.22) \quad \sum_{j \geq j_1} k_j^*(x_j, x_j) < \infty$$

for some $j_1 \in \mathbb{N}$. For all $j \in \mathbb{N}$ we put $z_j := \max(1, |x_j|)$. Due to Lemma 3.2 there exists a $C > 0$ such that $|h_\nu(x_j)|^2 \leq C^\nu |z_j|^{2\nu}$ for all $\nu \in \mathbb{N}$. Therefore we obtain, for all $j_1 \in \mathbb{N}$,

$$\begin{aligned} \sum_{j \geq j_1} k_j^*(x_j, x_j) &= \sum_{j \geq j_1} \sum_{\nu \in \mathbb{N}} \alpha_{\nu, j}^{-1} \cdot |h_\nu(x_j)|^2 \\ &\leq \sum_{j \geq j_1} \sum_{\nu \in \mathbb{N}} 2^{-r_j \nu} C^\nu \cdot |z_j|^{2\nu} \\ &\leq \sum_{\nu \in \mathbb{N}} \left(C \sum_{j \geq j_1} 2^{-r_j} \cdot |z_j|^2 \right)^\nu. \end{aligned}$$

Since $\mathbf{x} \in \ell^2(\alpha_1^{-1})$ and (3.21) is satisfied, we have $\sum_{j \in \mathbb{N}} 2^{-r_j} |z_j|^2 < \infty$. Hence we may choose j_1 to be large enough to yield

$$\sum_{j \geq j_1} 2^{-r_j} |z_j|^2 < C^{-1}.$$

This choice obviously ensures (3.22). \square

Proposition 3.19. *Consider the case (PG) or the case (EG) with sub-exponential growth, i.e., $\limsup_{j \rightarrow \infty} b_j < 1$. If $\hat{r} > 1/\ln(2)$ then we have*

$$\bigcap_{\nu \in \mathbb{N}} \ell^{2\nu}(\alpha_\nu^{-1}) \subsetneq \bigcap_{\nu=1}^{\nu_0} \ell^{2\nu}(\alpha_\nu^{-1})$$

for every $\nu_0 \in \mathbb{N}$.

Proof. Since $\hat{r} > 1/\ln(2)$, we may choose $0 < \varepsilon < 1$ and $j_0 \in \mathbb{N}$ such that

$$2^{r_j} \geq j^{1+\varepsilon}$$

for $j \geq j_0$. In the case (EG) with sub-exponential growth we may furthermore assume

$$b_j \leq 1 - \varepsilon$$

for $j \geq j_0$. Moreover, we choose $1 < a < 1 + \varepsilon$, and we put $\delta := 1 - a/(1 + \varepsilon) > 0$. Let $\nu_0 \in \mathbb{N}$. For $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ with

$$x_j^{2\nu_0} = 2^{r_j} \cdot j^{-a}$$

we show that

$$(3.23) \quad \mathbf{x} \in \bigcap_{\nu=1}^{\nu_0} \ell^{2\nu}(\boldsymbol{\alpha}_\nu^{-1}),$$

but

$$(3.24) \quad \mathbf{x} \notin \ell^{2k\nu_0}(\boldsymbol{\alpha}_{k\nu_0}^{-1})$$

for $k \in \mathbb{N}$ sufficiently large.

Let $j \geq j_0$ and $1 \leq \nu \leq \nu_0$. Since $|x_j| \geq 1$ and $\alpha_{\nu,j} \geq 2^{r_j}$, we obtain

$$\sum_{j \geq j_0} \alpha_{\nu,j}^{-1} \cdot x_j^{2\nu} \leq \sum_{j \geq j_0} 2^{-r_j} \cdot x_j^{2\nu_0} < \infty.$$

Consequently, we have (3.23).

Let $j \geq j_0$ and $k \in \mathbb{N}$. Note that

$$x_j^{2k\nu_0} = 2^{r_j k} \cdot j^{-ak} \geq 2^{r_j k \delta}.$$

Consider the case (EG) with sub-exponential growth. Here we have

$$\alpha_{k\nu_0,j}^{-1} \geq 2^{-r_j \cdot (k\nu_0)^{1-\varepsilon}}.$$

For every $k \in \mathbb{N}$ with

$$k\delta \geq (k\nu_0)^{1-\varepsilon}$$

this implies

$$\alpha_{k\nu_0,j}^{-1} \cdot x_j^{2k\nu_0} \geq 2^{r_j \cdot (k\delta - (k\nu_0)^{1-\varepsilon})} \geq 1,$$

which completes the proof of (3.24) in this case.

Consider the case (PG). Here we have

$$\alpha_{k\nu_0,j}^{-1} = (1 + k\nu_0)^{-r_j}$$

by definition. For every $k \in \mathbb{N}$ with

$$2^{k\delta} \geq 1 + k\nu_0$$

this implies

$$\alpha_{k\nu_0,j}^{-1} \cdot x_j^{2k\nu_0} \geq \left(2^{k\delta} \cdot (1 + k\nu_0)^{-1}\right)^{r_j} \geq 1,$$

which completes the proof of (3.24) in the case (PG). \square

In the cases (PG) and (EG) with sub-exponential growth we do not know if $\mathfrak{X} = \bigcap_{\nu \in \mathbb{N}} \ell^{2\nu}(\boldsymbol{\alpha}_\nu^{-1})$ or if there is another simple characterization of this domain.

4. GAUSSIAN SPACES

Now we study Hilbert spaces of Gaussian kernels, which we shortly address as Gaussian spaces. As before, we take $\mathbb{K} = \mathbb{R}$.

4.1. Functions of a Single Variable. At first we consider spaces of univariate functions.

Definition 4.1. Let $\sigma > 0$. The reproducing kernel ℓ_σ given by

$$\ell_\sigma(x, y) := \exp(-\sigma^2 \cdot (x - y)^2), \quad x, y \in \mathbb{R},$$

is called a *Gaussian kernel* and the Hilbert space $H(\ell_\sigma)$ is called a *Gaussian space* of functions of a single real variable with *shape parameter* σ .

We collect some facts about the spaces $H(\ell_\sigma)$ from Steinwart et al. (2006), see also Steinwart and Christmann (2008, Sec. 4.4). Each function $f \in H(\ell_\sigma)$ is the real part of an entire function g restricted to the real line, where g belongs to the complex reproducing kernel Hilbert space with kernel ℓ_σ extended to \mathbb{C} in the obvious way. In particular, if f is constant on any open non-empty interval, then f is the zero function. Moreover, the functions

$$e_{\nu, \sigma}(x) := \frac{1}{\sqrt{\nu!}} (\sqrt{2}\sigma)^\nu x^\nu \cdot \ell_\sigma(x, 0), \quad x \in \mathbb{R},$$

with $\nu \in \mathbb{N}_0$ form an orthonormal basis of $H(\ell_\sigma)$. It is easily verified that

$$(4.1) \quad \ell_\sigma(x, y) = \sum_{\nu \in \mathbb{N}_0} e_{\nu, \sigma}(x) \cdot e_{\nu, \sigma}(y)$$

with absolute convergence for all $x, y \in \mathbb{R}$. If $0 < \sigma < \sigma'$, then $H(\ell_\sigma) \subsetneq H(\ell_{\sigma'})$ with a non-compact continuous embedding of norm $\sqrt{\sigma'/\sigma}$. Observe that the identity (4.1) for the Gaussian kernel is reminiscent of definition (3.11) of the Hermite kernel. Based on Mehler's formula instead of (4.1), a characterization of $H(\ell_\sigma)$ as an L^2 -subspace in the sense of Proposition 2.1 will be given in Section 5.1.

4.2. Functions of Infinitely Many Variables. Gaussian spaces of functions of infinitely many real variables are based on a sequence $\boldsymbol{\sigma} := (\sigma_j)_{j \in \mathbb{N}}$ of shape parameters $\sigma_j > 0$; for the corresponding Gaussian kernels and basis functions in the univariate case we use the short hands $\ell_j := \ell_{\sigma_j}$ and $e_{\nu, j} := e_{\nu, \sigma_j}$.

We proceed similar to Section 3.2, but we will encounter some important differences along the way. First of all, $1 \notin H(\ell_j)$. In each of the spaces $H(\ell_j)$ we therefore choose the unit vector

$$v_j := e_{0, j},$$

i.e., $v_j(x) = \exp(-\sigma_j^2 \cdot x^2)$, and we study the corresponding incomplete tensor product

$$G^{(\boldsymbol{v})} := \bigotimes_{j \in \mathbb{N}} (H(\ell_j))^{(v_j)}.$$

We will employ Theorem A.6 again to identify the space $G^{(\boldsymbol{v})}$ with a reproducing kernel Hilbert space of functions on a suitable subset of $\mathbb{R}^{\mathbb{N}}$. In the present case we even have the convergence of the partial products $\prod_{j=1}^J \ell_j(x_j, y_j)$ as $J \rightarrow \infty$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{\mathbb{N}}$. Even more, the products $\prod_{j \in \mathbb{N}} \ell_j(x_j, y_j)$ converge in the sense of

Section A.1, since $\ell_j(x_j, y_j) \in]0, 1]$. Hence we may study the tensor product kernel L given by

$$(4.2) \quad L(\mathbf{x}, \mathbf{y}) := \prod_{j=1}^{\infty} \ell_j(x_j, y_j), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}.$$

One might expect that $G^{(\nu)}$ may be identified with $H(L)$ in a canonical way, analogously to the result for Hermite spaces.

For any reproducing kernel M on $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and any non-empty set $\mathfrak{Y} \subseteq \mathbb{R}^{\mathbb{N}}$ we use $M_{\mathfrak{Y}}$ to denote the restriction $M|_{\mathfrak{Y} \times \mathfrak{Y}}$.

The elementary tensors

$$e_{\nu} := \bigotimes_{j \in \mathbb{N}} e_{\nu_j, j}$$

with $\nu \in \mathcal{N}$ form an orthonormal basis of $G^{(\nu)}$, see Remark A.4. We will identify the tensor product space $G^{(\nu)}$ with a reproducing kernel Hilbert space $H(L_{\mathfrak{Y}})$ for a suitable non-empty domain $\mathfrak{Y} \subseteq \mathbb{R}^{\mathbb{N}}$ via the linear mapping

$$\Phi: G^{(\nu)} \rightarrow \mathbb{R}^{\mathfrak{Y}},$$

given by

$$(4.3) \quad \Phi f(\mathbf{x}) = \sum_{\nu \in \mathcal{N}} \langle f, e_{\nu} \rangle_{G^{(\nu)}} \cdot \prod_{j=1}^{\infty} e_{\nu_j, j}(x_j)$$

for $\mathbf{x} \in \mathfrak{Y}$, cf. Theorem A.6. More precisely, Φ should, of course, be well-defined and should induce an isometric isomorphism between $G^{(\nu)}$ and $H(L_{\mathfrak{Y}})$. It turns out, in particular, that we cannot achieve the latter goal with $\mathfrak{Y} = \mathbb{R}^{\mathbb{N}}$; instead, as shown below, $\mathfrak{Y} = \mathfrak{X}$ with

$$(4.4) \quad \mathfrak{X} := \ell^2(\boldsymbol{\sigma}^2)$$

is the proper choice. By definition, \mathfrak{X} consists of all sequences $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ such that

$$\sum_{j \in \mathbb{N}} \sigma_j^2 \cdot x_j^2 < \infty.$$

Obviously,

$$L(\mathbf{x}, \mathbf{y}) = \begin{cases} \exp(-\sum_{j \in \mathbb{N}} \sigma_j^2 \cdot (x_j - y_j)^2), & \text{if } \mathbf{x} - \mathbf{y} \in \mathfrak{X}, \\ 0, & \text{otherwise,} \end{cases}$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$.

Lemma 4.2. *For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$ we have*

$$\sum_{j \in \mathbb{N}} |\ell_j(x_j, y_j) - 1| < \infty \quad \Leftrightarrow \quad \mathbf{x} - \mathbf{y} \in \mathfrak{X}.$$

In particular,

$$\sum_{j \in \mathbb{N}} |v_j(x_j) - 1| < \infty \quad \Leftrightarrow \quad \mathbf{x} \in \mathfrak{X}.$$

Proof. Since $\prod_{j \in \mathbb{N}} \ell_j(x_j, y_j)$ converges and $\ell_j(x_j, y_j) \neq 0$ for every $j \in \mathbb{N}$, the convergence of $\sum_{j \in \mathbb{N}} |\ell_j(x_j, y_j) - 1|$ is equivalent to $L(\mathbf{x}, \mathbf{y}) \neq 0$, i.e., to $\mathbf{x} - \mathbf{y} \in \mathfrak{X}$. The second statement is just a special case of the first statement, namely with $\mathbf{y} := 0$. \square

Lemma 4.2 and Theorem A.6 immediately imply that Φ induces an isometric isomorphism between the incomplete tensor product $G^{(\nu)}$ and the reproducing kernel Hilbert space $H(L_{\mathfrak{X}})$. In this sense $H(L_{\mathfrak{X}})$ is the incomplete tensor product of the Gaussian spaces $H(\ell_j)$ based on the unit vectors $v_j = e_{0,j}$ for $j \in \mathbb{N}$.

Definition 4.3. Let $\sigma := (\sigma_j)_{j \in \mathbb{N}}$ be a sequence of shape parameters $\sigma_j > 0$. The reproducing kernel $L_{\mathfrak{X}}$ defined by (4.2) on the domain \mathfrak{X} given by (4.4) is called a *Gaussian kernel* and the Hilbert space $H(L_{\mathfrak{X}})$ is called a *Gaussian space* of functions of infinitely many real variables.

Larger domains $\mathfrak{Y} \supseteq \mathfrak{X}$ in the definition of Φ in (4.3) are discussed in the sequel. The definition of \tilde{L} in the following lemma is the counterpart to the representation (4.1) of ℓ_j in the univariate case. Moreover, the lemma ensures that Φ is well-defined for any choice of \mathfrak{Y} .

Lemma 4.4. Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^{\mathbb{N}}$. The series

$$\tilde{L}(\mathbf{x}, \mathbf{y}) := \sum_{\nu \in \mathbf{N}} \prod_{j=1}^{\infty} e_{\nu_j, j}(x_j) \cdot e_{\nu_j, j}(y_j)$$

is absolutely convergent, and

$$\tilde{L}(\mathbf{x}, \mathbf{y}) = \begin{cases} L(\mathbf{x}, \mathbf{y}), & \text{if } \mathbf{x}, \mathbf{y} \in \mathfrak{X}, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Let $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and $n \in \mathbb{N}$, and let \mathbf{N}_n denote the set of all sequences $\nu \in \mathbf{N}$ with $\nu_j = 0$ for every $j > n$. Since (4.1) yields $\sum_{\nu \in \mathbf{N}_0} |e_{\nu_j, j}(x)|^2 = 1$ for $j \in \mathbb{N}$ and $x \in \mathbb{R}$, we obtain

$$\sum_{\nu \in \mathbf{N}_n} \prod_{j=1}^n |e_{\nu_j, j}(x_j)|^2 = 1.$$

It follows that

$$\sum_{\nu \in \mathbf{N}_n} \prod_{j=1}^{\infty} |e_{\nu_j, j}(x_j)|^2 = \prod_{j=n+1}^{\infty} |e_{0, j}(x_j)|^2 = \exp\left(-2 \sum_{j=n+1}^{\infty} \sigma_j^2 x_j^2\right),$$

and hereby

$$\sum_{\nu \in \mathbf{N}} \prod_{j=1}^{\infty} |e_{\nu_j, j}(x_j)|^2 = \lim_{n \rightarrow \infty} \sum_{\nu \in \mathbf{N}_n} \prod_{j=1}^{\infty} |e_{\nu_j, j}(x_j)|^2 = \begin{cases} 1, & \text{if } \mathbf{x} \in \mathfrak{X}, \\ 0, & \text{otherwise.} \end{cases}$$

This yields the absolute convergence, as claimed, and $\tilde{L}(\mathbf{x}, \mathbf{y}) = 0$ if $\mathbf{x} \notin \mathfrak{X}$ or $\mathbf{y} \notin \mathfrak{X}$.

Next we verify $\tilde{L}(\mathbf{x}, \mathbf{y}) = L(\mathbf{x}, \mathbf{y})$ for $\mathbf{x}, \mathbf{y} \in \mathfrak{X}$. In the latter case

$$\sum_{\nu \in \mathbf{N}_n} \prod_{j=1}^{\infty} e_{\nu_j, j}(x_j) \cdot e_{\nu_j, j}(y_j) = \prod_{j=1}^n \ell_j(x_j, y_j) \cdot \exp\left(-\sum_{j=n+1}^{\infty} \sigma_j^2 \cdot (x_j^2 + y_j^2)\right).$$

In the limit $n \rightarrow \infty$ we obtain the claim. \square

Because of its series representation, the function \tilde{L} is a reproducing kernel on $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$. Since $L - \tilde{L} = L \cdot \mathbf{1}_{\mathfrak{Z} \times \mathfrak{Z}}$ with $\mathfrak{Z} := \mathbb{R}^{\mathbb{N}} \setminus \mathfrak{X}$, the function $L - \tilde{L}$ is a reproducing kernel on $\mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, too.

Proposition 4.5. *Assume that*

$$\mathfrak{X} \subsetneq \mathfrak{Y} \subseteq \mathbb{R}^N.$$

Then

- (i) $H(L_{\mathfrak{Y}})$ is the orthogonal sum of the closed, proper subspaces $H(\tilde{L}_{\mathfrak{Y}})$ and $H(L_{\mathfrak{Y}} - \tilde{L}_{\mathfrak{Y}})$,
- (ii) $H(\tilde{L}_{\mathfrak{Y}}) = \{f \in H(L_{\mathfrak{Y}}) : f_{\mathfrak{Y} \setminus \mathfrak{X}} = 0\}$,
- (iii) Φ is an isometric isomorphism between $G^{(\nu)}$ and $H(\tilde{L}_{\mathfrak{Y}})$.

Proof. For $\mathbf{x} \in \mathfrak{Y} \setminus \mathfrak{X}$ we have $\tilde{L}(\mathbf{x}, \mathbf{x}) = 0$ and $(L - \tilde{L})(\mathbf{x}, \mathbf{x}) = 1$, while $\tilde{L}(\mathbf{x}, \mathbf{x}) = 1$ and $(L - \tilde{L})(\mathbf{x}, \mathbf{x}) = 0$ for $\mathbf{x} \in \mathfrak{X}$, see Lemma 4.4. It follows that $H(\tilde{L}_{\mathfrak{Y}})$ and $H(L_{\mathfrak{Y}} - \tilde{L}_{\mathfrak{Y}})$ are non-trivial, but have a trivial intersection. This yields (i) and (ii).

Let $\mathbf{x} \in \mathfrak{Y} \setminus \mathfrak{X}$. Note that $\tilde{L}(\mathbf{x}, \mathbf{x}) = 0$ is equivalent to

$$\prod_{j=1}^{\infty} e_{\nu_j, j}(x_j) = 0$$

for every $\nu \in \mathbf{N}$. It follows that $\Phi f(\mathbf{x}) = 0$ for every $f \in G^{(\nu)}$. We conclude that $G^{(\nu)}$ is identified with $H(\tilde{L}_{\mathfrak{Y}})$ via Φ , as claimed in (iii). \square

Remark 4.6. Lemma 4.4 and Proposition 4.5 show that the proper choice of domain of the reproducing kernel L defined in (4.2) is $\mathfrak{X} := \ell^2(\boldsymbol{\sigma}^2)$, since this is the maximal domain such that the canonical mapping $\Phi: G^{(\nu)} \rightarrow \mathbb{R}^{\mathfrak{X}}$ defined in (4.3) defines an isometric isomorphism between $G^{(\nu)}$ and $\Phi(G^{(\nu)}) = H(L_{\mathfrak{X}}) = H(\tilde{L}_{\mathfrak{X}})$. If we consider strictly larger domains \mathfrak{Y} , then we still have $\Phi(G^{(\nu)}) = H(\tilde{L}_{\mathfrak{Y}})$, but $H(\tilde{L}_{\mathfrak{Y}}) \neq H(L_{\mathfrak{Y}})$.

5. ISOMORPHISMS BETWEEN GAUSSIAN SPACES AND HERMITE SPACES

5.1. Functions of a Single Variable. It is known that the Gaussian kernel ℓ_{σ} with shape parameter $\sigma > 0$ can be represented in terms of the Hermite polynomials h_{ν} , see, e.g., Rasmussen and Williams (2006, Sec. 4.3.1) and Fasshauer et al. (2012, Sec. 3). A starting point to derive such a representation is Mehler's formula

$$\exp\left(\frac{1}{1 - \beta^2} \cdot (\beta \cdot xy - \frac{1}{2}\beta^2 \cdot (x^2 + y^2))\right) = m(x, y)$$

with

$$m(x, y) := (1 - \beta^2)^{1/2} \cdot \sum_{\nu \in \mathbb{N}_0} \beta^{\nu} \cdot h_{\nu}(x) \cdot h_{\nu}(y),$$

which holds for all $x, y \in \mathbb{R}$ and $\beta \in]0, 1[$, see Erdélyi et al. (1953, Eqn. (22) on p. 194). Note that

$$(5.1) \quad \sum_{\nu \in \mathbb{N}_0} \beta^{\nu} \cdot |h_{\nu}(x)|^2 < \infty$$

for every $x \in \mathbb{R}$ already follows from Cramér's inequality, see Lemma 3.1. We add that Mehler's formula also yields a closed form representation of Hermite kernels in the case (EG) with $b = 1$, see Irrgeher and Leobacher (2015, p. 186).

Using

$$\beta \cdot xy - \frac{1}{2}\beta^2 \cdot (x^2 + y^2) = -\frac{1}{2}\beta \cdot (x - y)^2 + \frac{1}{2}\beta(1 - \beta) \cdot (x^2 + y^2)$$

and introducing a scaling parameter $c > 0$, Mehler's formula may be rewritten as

$$\exp\left(-\frac{c^2\beta}{2(1-\beta^2)} \cdot (x-y)^2\right) = \exp(-\tau \cdot (x^2 + y^2)) \cdot m(cx, cy)$$

with

$$\tau := \frac{c^2\beta}{2(1+\beta)}.$$

For $f \in L^2(\mu_0)$ and $x \in \mathbb{R}$ we define

$$qf(x) := c^{1/2} \cdot \exp(-\tau \cdot x^2) \cdot f(cx).$$

If

$$(5.2) \quad \frac{c^2\beta}{2(1-\beta^2)} = \sigma^2$$

then

$$\ell_\sigma(x, y) = \frac{(1-\beta^2)^{1/2}}{c} \cdot \sum_{\nu \in \mathbb{N}_0} \beta^\nu \cdot qh_\nu(x) \cdot qh_\nu(y)$$

for all $x, y \in \mathbb{R}$.

Lemma 5.1. *The mapping q is an isometric isomorphism on $L^2(\mu_0)$ if and only if*

$$(5.3) \quad \tau = \frac{c^2 - 1}{4}.$$

Proof. Let $t: \mathbb{R} \rightarrow \mathbb{R}$ be given by $t(x) := cx$. The image measure $t\mu_0$ of μ_0 with respect to t is the normal distribution with zero mean and variance c^2 . The density of $t\mu_0$ with respect to μ_0 is therefore given by the density ratio

$$\varphi(x) := c^{-1} \cdot \exp\left(-\frac{x^2}{2c^2}\right) \cdot \exp\left(\frac{x^2}{2}\right) = c^{-1} \cdot \exp\left(\frac{c^2 - 1}{2c^2} \cdot x^2\right).$$

For $f \in L^2(\mu_0)$ this yields

$$\begin{aligned} \int_{\mathbb{R}} |qf(x)|^2 d\mu_0(x) &= c \cdot \int_{\mathbb{R}} \exp(-2\tau/c^2 \cdot |t(x)|^2) \cdot |f(t(x))|^2 d\mu_0(x) \\ &= c \cdot \int_{\mathbb{R}} \varphi(x) \cdot \exp(-2\tau/c^2 \cdot |x|^2) \cdot |f(x)|^2 d\mu_0(x). \end{aligned}$$

We conclude that q is an isometry on $L^2(\mu_0)$ if and only if

$$\frac{c^2 - 1}{2c^2} = \frac{2\tau}{c^2},$$

which is equivalent to (5.3). Assume that (5.3) is satisfied. It follows that

$$qf(x) = |\varphi(t(x))|^{-1/2} \cdot f(t(x)).$$

For $g \in L^2(\mu_0)$ and $f(x) := |\varphi(x)|^{1/2} \cdot g(t^{-1}(x))$ we obtain $f \in L^2(\mu_0)$ as well as $qf = g$. \square

Lemma 5.2. *For every $\sigma > 0$ the unique solution $(\beta, c) \in]0, 1[\times]0, \infty[$ of (5.2) and (5.3) is given by*

$$(5.4) \quad c = (1 + 8\sigma^2)^{1/4} \quad \text{and} \quad \beta = 1 - \frac{2}{1 + c^2}.$$

Proof. Let $(\beta, c) \in]0, 1[\times]0, \infty[$ satisfy (5.2) and (5.3). From (5.3) we obtain

$$c^2 = \frac{1 + \beta}{1 - \beta},$$

and together with (5.2) this yields

$$\frac{\beta}{2(1 - \beta)^2} = \sigma^2.$$

Since $\beta \mapsto \beta/(1 - \beta)^2$ is strictly increasing on $]0, 1[$, we conclude that the first component β of the solution of (5.2) and (5.3) is uniquely determined. Due to (5.2) the same holds true for the second component c .

Conversely, it is straightforward to verify that c and β according to (5.4) belong to $]0, \infty[$ and $]0, 1[$, respectively, and satisfy (5.2) and (5.3). \square

In the sequel we assume that c and β are chosen according to (5.4), and we introduce the Fourier weights

$$\alpha_\nu := \frac{1}{(1 - \beta) \cdot \beta^\nu}, \quad \nu \in \mathbb{N}_0.$$

Since

$$\frac{(1 - \beta^2)^{1/2}}{c} = 1 - \beta,$$

we obtain the representation

$$(5.5) \quad \ell_\sigma(x, y) = \sum_{\nu \in \mathbb{N}_0} \alpha_\nu^{-1} \cdot qh_\nu(x) \cdot qh_\nu(y), \quad x, y \in \mathbb{R},$$

of the Gaussian kernel ℓ_σ from Mehler's formula and Lemma 5.2. Cf. Fasshauer et al. (2012, p. 257), who consider the normal distribution with zero mean and variance $1/2$, instead of the standard normal distribution μ_0 , together with unnormalized Hermite polynomials.

Due to Lemma 5.1 and Lemma 5.2 the functions qh_ν with $\nu \in \mathbb{N}_0$ form an orthonormal basis of $L^2(\mu_0)$, and (5.1) corresponds to (2.2). We conclude that Proposition 2.1, with the orthonormal basis $(qh_\nu)_{\nu \in \mathbb{N}_0}$, provides a characterization of $H(\ell_\sigma)$ as a linear subspace of $L^2(\mu_0)$.

We relate the Gaussian space $H(\ell_\sigma)$ to the Hermite space $H(k_\sigma)$ with

$$(5.6) \quad k_\sigma(x, y) := \sum_{\nu \in \mathbb{N}_0} \alpha_\nu^{-1} \cdot h_\nu(x) \cdot h_\nu(y), \quad x, y \in \mathbb{R}.$$

Proposition 5.3. *The mapping $q|_{H(k_\sigma)}$ is an isometric isomorphism between $H(k_\sigma)$ and $H(\ell_\sigma)$.*

Proof. We use Lemma 5.1 and Lemma 5.2 as well as (5.5) and Proposition 2.1. Let $c_\nu \in \mathbb{R}$, $\nu \in \mathbb{N}_0$, satisfy $\sum_{\nu \in \mathbb{N}_0} \alpha_\nu \cdot |c_\nu|^2 < \infty$. Then $f := \sum_{\nu \in \mathbb{N}_0} c_\nu \cdot h_\nu$ is in $H(k_\sigma)$, and the sum converges in $L^2(\mu_0)$ and absolutely at every point $x \in \mathbb{R}$. Observe that the pointwise convergence is preserved by q , and therefore $qf(x) = \sum_{\nu \in \mathbb{N}_0} c_\nu \cdot qh_\nu(x)$. Moreover, $qf = \sum_{\nu \in \mathbb{N}_0} c_\nu \cdot qh_\nu$ with convergence in $L^2(\mu_0)$. It follows that $qf \in H(\ell_\sigma)$ and $\|f\|_{H(k_\sigma)} = \|qf\|_{H(\ell_\sigma)}$. Identity (2.3), applied to $H(k_\sigma)$ as well as to $H(\ell_\sigma)$, yields that $q(H(k_\sigma)) = H(\ell_\sigma)$. \square

Remark 5.4. We combine Proposition 2.1, Lemma 5.1, Lemma 5.2, and Proposition 5.3 to conclude that

$$\begin{array}{ccc}
H(k_\sigma) & \hookrightarrow & L^2(\mu_0) \\
\downarrow q|_{H(k_\sigma)} & & \downarrow q \\
H(\ell_\sigma) & \hookrightarrow & L^2(\mu_0)
\end{array}$$

is a commutative diagram of bounded linear operators with isometric isomorphisms in the vertical direction.

We close this subsection by commenting on inclusion relations between Gaussian spaces and Hermite spaces.

Remark 5.5. Consider an arbitrary Hermite kernel k and an arbitrary Gaussian kernel ℓ_σ . Since

$$|f(x) - f(y)|^2 \leq 2 \|f\|_{H(\ell_\sigma)}^2 \cdot (1 - \ell_\sigma(x, y))$$

for all $f \in H(\ell_\sigma)$ and $x, y \in \mathbb{R}$, the functions from $H(\ell_\sigma)$ are bounded. Therefore the space $H(\ell_\sigma)$ does not contain any non-trivial polynomial, while $H(k)$ obviously contains the whole space Π of polynomials. In particular,

$$H(k) \not\subseteq H(\ell_\sigma).$$

Observe that ℓ_σ is translation-invariant. General conditions for translation-invariant kernels ℓ that guarantee $H(\ell) \cap \Pi = \{0\}$ have recently been established in Dette and Zhigljavsky (2021) and Karvonen (2021).

Remark 5.6. We briefly discuss inclusions of Gaussian spaces in Hermite spaces. More precisely, we consider Hermite spaces $H(k)$ with Fourier weights $\alpha_\nu := \beta^{-\nu}$ for any $\beta \in]0, 1[$. The following holds true for every $\sigma > 0$. There exist $0 < \beta^{(1)} \leq \beta^{(2)} < 1$ such that

$$H(\ell_\sigma) \not\subseteq H(k) \quad \text{if } 0 < \beta < \beta^{(1)}$$

and

$$H(\ell_\sigma) \subsetneq H(k) \quad \text{if } \beta^{(2)} < \beta < 1.$$

In particular, for β according to (5.4), where $H(k) = H(k_\sigma)$ as vector spaces and $q|_{H(k_\sigma)}$ is an isometric isomorphism between $H(k_\sigma)$ and $H(\ell_\sigma)$, we have $\beta < \beta^{(1)}$, and therefore

$$H(\ell_\sigma) \not\subseteq H(k_\sigma).$$

So far, our analysis only yields values $\beta^{(1)} < \beta^{(2)}$; therefore we skip the proof.

5.2. Functions of Infinitely Many Variables. As in Section 4.2, we consider again a sequence σ of shape parameters $\sigma_j > 0$ and the corresponding sequence of Gaussian kernels ℓ_j . We study Gaussian spaces and Hermite spaces of functions of infinitely many variables with Fourier weights given by

$$\alpha_{\nu, j} := \frac{1}{(1 - \beta_j) \cdot \beta_j^\nu}, \quad \nu \in \mathbb{N}_0, \quad j \in \mathbb{N},$$

where

$$c_j := (1 + 8\sigma_j^2)^{1/4} \quad \text{and} \quad \beta_j := 1 - \frac{2}{1 + c_j^2},$$

cf. Lemma 5.2, under the assumption

$$(5.7) \quad \sum_{j \in \mathbb{N}} \sigma_j^2 < \infty.$$

Note that (5.7) is equivalent to

$$(5.8) \quad \sum_{j \in \mathbb{N}} |c_j - 1| < \infty.$$

As in Section 4.2, we consider the domain $\mathfrak{X} := \ell^2(\boldsymbol{\sigma}^2)$.

We will establish the counterpart to Remark 5.4 via tensorization and identification of the corresponding incomplete tensor product spaces with the reproducing kernel Hilbert spaces $H(K)$ and $H(L_{\mathfrak{X}})$ or the space $L^2(\mu)$, respectively.

Lemma 5.7. *Assume that (5.7) is satisfied. Then we have (H1)–(H3), and \mathfrak{X} is the maximal domain of the kernel K and satisfies $\mu(\mathfrak{X}) = 1$.*

Proof. From (5.7) we obtain the weak asymptotics

$$\alpha_{1,j}^{-1} = (1 - \beta_j) \cdot \beta_j \asymp \beta_j \asymp \sigma_j^2,$$

the convergence $\sum_{j \in \mathbb{N}} \beta_j < \infty$, and $\ell^2(\boldsymbol{\alpha}_1^{-1}) = \ell^2(\boldsymbol{\sigma}^2)$ for $\boldsymbol{\alpha}_1^{-1} := (\alpha_{1,j}^{-1})_{j \in \mathbb{N}}$.

Let $r_j > 0$ be defined by $\beta_j = 2^{-r_j}$. Note that the Fourier weights

$$\beta_j^{-\nu} = 2^{r_j \cdot \nu}, \quad \nu \in \mathbb{N}_0, \quad j \in \mathbb{N},$$

are of the form (EG) with $b_j := 1$. Due to Lemma 3.17 we have (H1)–(H3) in this case, and $\ell(\boldsymbol{\sigma}^2)$ is the maximal domain of the corresponding Hermite kernel, see Proposition 3.18.

It remains to consider $\alpha_{\nu,j} = (1 - \beta_j)^{-1} \cdot \beta_j^{-\nu}$ instead of $\beta_j^{-\nu}$. Since

$$0 < \inf_{j \in \mathbb{N}} (1 - \beta_j) \leq \sup_{j \in \mathbb{N}} (1 - \beta_j) = 1,$$

(H1), (H3), and the maximal domain are not affected by this change. Since

$$\alpha_{0,j} - 1 = \frac{\beta_j}{1 - \beta_j},$$

we also have (H2). Due to Proposition 3.10 we have $\mu(\mathfrak{X}) = 1$. \square

Put

$$c_* := \prod_{j=1}^{\infty} c_j > 1,$$

cf. (5.8). Let $t: \mathbb{R}^{\mathbb{N}} \rightarrow \mathbb{R}^{\mathbb{N}}$ and $\varphi: \mathfrak{X} \rightarrow [0, \infty[$ be given by

$$t(\mathbf{x}) := (c_1 x_1, c_2 x_2, \dots)$$

for $\mathbf{x} \in \mathbb{R}^{\mathbb{N}}$ and by

$$\varphi(\mathbf{x}) := c_*^{-1} \cdot \exp\left(\sum_{j \in \mathbb{N}} \frac{c_j^2 - 1}{2c_j^2} x_j^2\right)$$

for $\mathbf{x} \in \mathfrak{X}$. For $f \in L^2(\mu)$ and $\mathbf{x} \in \mathfrak{X}$ we define

$$Qf(\mathbf{x}) := \left(\varphi(t(\mathbf{x}))\right)^{-1/2} \cdot f(t(\mathbf{x})) = c_*^{1/2} \cdot \exp\left(-\sum_{j \in \mathbb{N}} \frac{c_j^2 - 1}{4} x_j^2\right) \cdot f(t(\mathbf{x})).$$

Theorem 5.8. *Assume that (5.7) is satisfied. Then the mapping Q is an isometric isomorphism on $L^2(\mu)$, and its restriction $Q|_{H(K)}$ is an isometric isomorphism between $H(K)$ and $H(L_{\mathfrak{X}})$.*

Proof. At first we aim at the space $L^2(\mu)$ and the mapping Q . Let

$$L^{(\mathbf{1})} := \bigotimes_{j \in \mathbb{N}} (L^2(\mu_0))^{(1)} \quad \text{and} \quad L^{(\mathbf{w})} := \bigotimes_{j \in \mathbb{N}} (L^2(\mu_0))^{(w_j)}$$

denote the incomplete tensor products of the spaces $L^2(\mu_0)$ based on the constant function 1 as the unit vector for every $j \in \mathbb{N}$ and based on the unit vectors

$$w_j := q_j 1$$

for $j \in \mathbb{N}$, respectively. We use Lemma 5.1 and Lemma 5.2 to conclude that the tensor product

$$q := \bigotimes_{j \in \mathbb{N}} q_j : L^{(\mathbf{1})} \rightarrow L^{(\mathbf{w})}$$

of the operators $q_j : L^2(\mu_0) \rightarrow L^2(\mu_0)$ is well-defined and an isometric isomorphism.

By definition $w_j(x) = c_j^{1/2} \cdot \exp(-(c_j^2 - 1)/4 \cdot x^2)$. We use $0 < w_j \leq c_j^{1/2}$ and $\|w_j\|_{L^2(\mu_0)} = 1$ to obtain $\langle w_j, 1 \rangle_{L^2(\mu_0)} = \|w_j\|_{L^1(\mu_0)}$ and

$$1 \geq \|w_j\|_{L^1(\mu_0)} \geq c_j^{-1/2} \cdot \int_{\mathbb{R}} w_j^2 d\mu_0 = c_j^{-1/2}.$$

Therefore

$$(5.9) \quad \sum_{j \in \mathbb{N}} |\langle w_j, 1 \rangle_{L^2(\mu_0)} - 1| \leq \sum_{j \in \mathbb{N}} (1 - c_j^{-1/2}) < \infty,$$

cf. (5.8). The estimate (5.9) together with (A.2) implies

$$L^{(\mathbf{w})} = L^{(\mathbf{1})}.$$

Let $\Psi : L^2(\mu) \rightarrow L^{(\mathbf{1})}$ denote the isometric isomorphism according to Theorem A.7 and

$$\tilde{Q} := \Psi^{-1} \circ q \circ \Psi : L^2(\mu) \rightarrow L^2(\mu).$$

We claim that $\tilde{Q} = Q$. Recall that the functions h_{ν} with $\nu \in \mathbf{N}$ form an orthonormal basis of $L^2(\mu)$. By definition of q and h_{ν} ,

$$(q \circ \Psi)h_{\nu} = q \bigotimes_{j \in \mathbb{N}} h_{\nu_j} = \bigotimes_{j \in \mathbb{N}} q_j h_{\nu_j}.$$

We combine (5.9) and Theorem A.7 to obtain

$$\tilde{Q}h_{\nu}(\mathbf{x}) = \prod_{j=1}^{\infty} q_j h_{\nu_j}(x_j) = Qh_{\nu}(\mathbf{x})$$

for ρ -a.e. $\mathbf{x} \in \mathfrak{X}$. For $f \in L^2(\mu)$ we have

$$\tilde{Q}f = \sum_{\nu \in \mathbf{N}} \langle f, h_{\nu} \rangle_{L^2(\mu)} \cdot \tilde{Q}h_{\nu} = \sum_{\nu \in \mathbf{N}} \langle f, h_{\nu} \rangle_{L^2(\mu)} \cdot Qh_{\nu}$$

with convergence in $L^2(\mu)$. Observe that Q preserves pointwise convergence. Redefining the functions h_{ν} on sets of measure zero we also have

$$Qf = \sum_{\nu \in \mathbf{N}} \langle f, h_{\nu} \rangle_{L^2(\mu)} \cdot Qh_{\nu}$$

with pointwise convergence along a subsequence. It follows that

$$Qf = \tilde{Q}f \in L^2(\mu),$$

as claimed, and therefore Q is an isometric isomorphism on $L^2(\mu)$.

Proceeding analogously to the first part of the proof we now aim at the spaces $H(K)$ and $H(L_{\mathfrak{X}})$ and the mapping $Q|_{H(K)}$. Let

$$H(\alpha_0^{-1/2}) := \bigotimes_{j \in \mathbb{N}} (H(k_j))^{\alpha_{0,j}^{-1/2}}$$

denote the incomplete tensor products of the spaces $H(k_j)$ based on the constant functions $\alpha_{0,j}^{-1/2}$ for $j \in \mathbb{N}$ as unit vectors. Furthermore, let

$$G^{(\mathbf{v})} := \bigotimes_{j \in \mathbb{N}} (H(\ell_j))^{(v_j)} \quad \text{and} \quad G^{(\widehat{\mathbf{w}})} := \bigotimes_{j \in \mathbb{N}} (H(\ell_j))^{(\widehat{w}_j)}$$

denote the incomplete tensor products of the spaces $H(\ell_j)$ based on the vectors $v_j := \ell_j(\cdot, 0)$ and

$$\widehat{w}_j := q_j \alpha_{0,j}^{-1/2} = \alpha_{0,j}^{-1/2} w_j$$

for $j \in \mathbb{N}$, respectively. We use Lemma 5.3 to conclude that the tensor product

$$\widehat{q} := \bigotimes_{j \in \mathbb{N}} q_j|_{H(k_j)} : H(\alpha_0^{-1/2}) \rightarrow G^{(\widehat{\mathbf{w}})}$$

of the operators $q_j|_{H(k_j)} : H(k_j) \rightarrow H(\ell_j)$ is well-defined and an isometric isomorphism.

Since

$$\langle \widehat{w}_j, v_j \rangle_{H(\ell_j)} = \widehat{w}_j(0) = \alpha_{0,j}^{-1/2} \cdot (q_j 1)(0) = (c_j / \alpha_{0,j})^{1/2},$$

we obtain

$$(5.10) \quad \sum_{j \in \mathbb{N}} |\langle \widehat{w}_j, v_j \rangle_{H(\ell_j)} - 1| \leq \sum_{j \in \mathbb{N}} |(c_j^{1/2} - 1) / \alpha_{0,j}^{1/2}| + \sum_{j \in \mathbb{N}} |\alpha_{0,j}^{-1/2} - 1| < \infty,$$

cf. (5.8) and (H2). The estimate (5.10) together with (A.2) implies

$$G^{(\widehat{\mathbf{w}})} = G^{(\mathbf{v})}.$$

Let $\Phi_1 : H(\alpha_0^{-1/2}) \rightarrow H(K)$ and $\Phi_2 : G^{(\mathbf{v})} \rightarrow H(L_{\mathfrak{X}})$ denote the isometric isomorphisms according to Theorem A.6 and

$$\widehat{Q} := \Phi_2 \circ \widehat{q} \circ \Phi_1^{-1} : H(K) \rightarrow H(L_{\mathfrak{X}}).$$

We claim that $\widehat{Q} = Q|_{H(K)}$. Recall that the functions $\alpha_{\nu}^{-1/2} h_{\nu}$ with $\nu \in \mathbf{N}$ form an orthonormal basis of $H(K)$. By definition of \widehat{q} and h_{ν} ,

$$(\widehat{q} \circ \Phi_1^{-1}) h_{\nu} = \widehat{q} \bigotimes_{j \in \mathbb{N}} h_{\nu_j} = \bigotimes_{j \in \mathbb{N}} q_j h_{\nu_j}.$$

We combine (5.10) and Theorem A.6 to obtain

$$\widehat{Q} h_{\nu}(\mathbf{x}) = \prod_{j=1}^{\infty} q_j h_{\nu_j}(x_j) = Q h_{\nu}(\mathbf{x})$$

for every $\mathbf{x} \in \mathfrak{X}$. For $f \in H(K)$ and $\mathbf{x} \in \mathfrak{X}$ this implies

$$\begin{aligned} \widehat{Q} f(\mathbf{x}) &= \sum_{\nu \in \mathbf{N}} \alpha_{\nu}^{-1} \cdot \langle f, h_{\nu} \rangle_{H(K)} \cdot \widehat{Q} h_{\nu}(\mathbf{x}) = \sum_{\nu \in \mathbf{N}} \alpha_{\nu}^{-1} \cdot \langle f, h_{\nu} \rangle_{H(K)} \cdot Q h_{\nu}(\mathbf{x}) \\ &= Q f(\mathbf{x}), \end{aligned}$$

which completes the proof. \square

Remark 5.9. According to Proposition 2.1 and Theorem 5.8,

$$\begin{array}{ccc}
H(K) & \hookrightarrow & L^2(\mu) \\
\downarrow Q|_{H(K)} & & \downarrow Q \\
H(L_{\mathfrak{X}}) & \hookrightarrow & L^2(\mu)
\end{array}$$

is a commutative diagram of bounded linear operators with isometric isomorphisms in the vertical direction.

Another important feature of Q is the obvious fact that for every $\mathbf{x} \in \mathfrak{X}$ a single function value of $f \in H(K)$ suffices to compute $Qf(\mathbf{x})$ and vice versa.

These results allow to transfer computational problems from the space $H(L_{\mathfrak{X}})$ onto the space $H(K)$, as it will be shown in a forthcoming paper, cf. Remark 3.15. In this approach it is preferable to replace the Fourier weight $\alpha_{0,j}$ by one in the definition of each of the kernels k_j ; the corresponding tensor product kernel K' has already been considered in Section 3.2. We obtain $H(K) = H(K')$ as vector spaces and, as a minor change, that $Q|_{H(K')}$ is an isomorphism between $H(K')$ and $H(L_{\mathfrak{X}})$ with norm $\prod_{j=1}^{\infty} (1 - \beta_j)^{-1/2}$; its inverse has norm one. See Lemma 3.13.

Remark 5.10. There is a one-to-one correspondence between the scale of Hermite spaces $H(K)$ appearing in Theorem 5.8 and the scale of Hermite spaces with Fourier weights of exponential growth (EG) with $b_j = 1$ for all $j \in \mathbb{N}$. Indeed, the change of weights $(1 - \beta_j)^{-1} \beta_j^{-\nu} \mapsto \beta_j^{-\nu}$ for $\nu \in \mathbb{N}_0$ and $j \in \mathbb{N}$ leaves the vector spaces $H(K)$ invariant and leads to different, but equivalent norms, cf. the proof of Lemma 5.7. It is easy to see that the mapping between the scales of Hermite spaces induced by this change of weights is well-defined and bijective.

APPENDIX A. COUNTABLE TENSOR PRODUCTS OF HILBERT SPACES

Tensor products of arbitrary families of Hilbert spaces have been introduced (actually named complete direct products) and thoroughly studied in von Neumann (1939). In the present paper we are interested in countably infinite tensor products, and thus we consider a sequence $(H_j)_{j \in \mathbb{N}}$ of Hilbert spaces $H_j \neq \{0\}$ over the same scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

As shown in Sections A.2 and A.3, the main results from von Neumann (1939) on complete and incomplete tensor products of Hilbert spaces can be represented within the framework of reproducing kernel Hilbert spaces in a concise way. Tensor products of bounded linear operators are discussed in Section A.4.

In Sections A.5 and A.6 we consider particular cases, where the results are due to Rüdßmann (2020) and Guichardet (1969), respectively. In Sections A.5 we study tensor products of reproducing kernel Hilbert spaces and their relation to Hilbert spaces with reproducing kernels of tensor product form. In Sections A.6 we consider tensor products of L^2 -spaces and their relation to L^2 -spaces with respect to the corresponding product measures.

A.1. Convergence of Products. We discuss the convergence of infinite products in \mathbb{K} , see von Neumann (1939, Chap. 2) and cf. (1.1). Let $(z_j)_{j \in \mathbb{N}}$ denote a sequence in \mathbb{K} . We say that the product $\prod_{j \in \mathbb{N}} z_j$ is *convergent* with value $z \in \mathbb{K}$, if the following holds for every $\varepsilon > 0$. There exists a finite set $I_0 \subseteq \mathbb{N}$ such that $|\prod_{j \in I} z_j - z| \leq \varepsilon$ holds for every finite set $I_0 \subseteq I \subseteq \mathbb{N}$. The value of a convergent product is uniquely determined; hence we put $\prod_{j \in \mathbb{N}} z_j := z$, given convergence with value z .

In the case of convergence we obviously also have convergence of $(\prod_{j=1}^J z_j)_{J \in \mathbb{N}}$, as required in (1.1), and $\prod_{j \in \mathbb{N}} z_j = \prod_{j=1}^{\infty} z_j$.

Moreover, we say that $\prod_{j \in \mathbb{N}} z_j$ is *quasi-convergent*, if $\prod_{j \in \mathbb{N}} |z_j|$ converges. The latter is necessary, but not sufficient for convergence of $\prod_{j \in \mathbb{N}} z_j$, and for non-convergent, but quasi-convergent products we put $\prod_{j \in \mathbb{N}} z_j := 0$. The following properties are equivalent:

- (i) $\prod_{j \in \mathbb{N}} z_j$ is convergent and $\prod_{j \in \mathbb{N}} z_j \neq 0$,
- (ii) $\prod_{j \in \mathbb{N}} z_j$ is quasi-convergent and $\prod_{j \in \mathbb{N}} z_j \neq 0$,
- (iii) $\sum_{j \in \mathbb{N}} |z_j - 1| < \infty$ and $z_j \neq 0$ for every $j \in \mathbb{N}$.

Furthermore, $\sum_{j \in \mathbb{N}} |z_j - 1| < \infty$ implies $\lim_{J \rightarrow \infty} \prod_{j=J}^{\infty} z_j = 1$. If we have $z_j \in \mathbb{R}$ with $z_j \geq 1$ for every $j \in \mathbb{N}$, then the convergence of $\prod_{j \in \mathbb{N}} z_j$ is obviously equivalent to the convergence of $(\prod_{j=1}^J z_j)_{J \in \mathbb{N}}$.

A.2. The Complete Tensor Product. In the sequel we use the notation $\mathbf{f} = (f_j)_{j \in \mathbb{N}}$ with $f_j \in H_j$ for elements $\mathbf{f} \in \times_{j \in \mathbb{N}} H_j$. For the construction of the complete tensor product of the spaces H_j we define

$$C := \left\{ \mathbf{f} \in \times_{j \in \mathbb{N}} H_j : \prod_{j \in \mathbb{N}} \|f_j\|_{H_j} \text{ converges} \right\}.$$

For $\mathbf{f}, \mathbf{g} \in C$ the product $\prod_{j \in \mathbb{N}} \langle f_j, g_j \rangle_{H_j}$ is quasi-convergent, see von Neumann (1939, Lem. 2.5.2). Hence we may define a mapping $\mathcal{K}: C \times C \rightarrow \mathbb{K}$ by

$$\mathcal{K}(\mathbf{g}, \mathbf{f}) := \prod_{j \in \mathbb{N}} \langle f_j, g_j \rangle_{H_j}, \quad \mathbf{f}, \mathbf{g} \in C,$$

and von Neumann (1939, Lem. 3.4.1) implies that \mathcal{K} is a reproducing kernel.

Cf. von Neumann (1939, Def. 3.5.1) for the following definition.

Definition A.1. The *complete tensor product* of the spaces H_j is the Hilbert space

$$H := \bigotimes_{j \in \mathbb{N}} H_j := H(\mathcal{K})$$

with reproducing kernel \mathcal{K} .

For any $\mathbf{f} \in C$ the function

$$\bigotimes_{j \in \mathbb{N}} f_j := \mathcal{K}(\cdot, \mathbf{f}) \in H$$

is called an *elementary tensor*. Its norm is $\|\bigotimes_{j \in \mathbb{N}} f_j\|_H = \prod_{j \in \mathbb{N}} \|f_j\|_{H_j}$, and the span of elementary tensors is dense in H .

It is desirable to work on smaller domains $\tilde{C} \subsetneq C$ consisting of sequences that are easier to handle. If the restriction mapping $f \mapsto f|_{\tilde{C}}$ defined on $H(\mathcal{K})$ is injective, it is actually already an isometric isomorphism onto $H(\mathcal{K}|_{\tilde{C} \times \tilde{C}})$. Hence we may identify in a natural way the reproducing kernel Hilbert spaces $H = H(\mathcal{K})$ and $H(\mathcal{K}|_{\tilde{C} \times \tilde{C}})$.

In this sense, it suffices to consider the elements of H on the domain

$$C_0 := \left\{ \mathbf{f} \in \times_{j \in \mathbb{N}} H_j : \sum_{j \in \mathbb{N}} \left| \|f_j\|_{H_j} - 1 \right| < \infty \right\} \subsetneq C.$$

Indeed, for $\mathbf{f} \in C$ we have $\mathcal{K}(\mathbf{f}, \mathbf{f}) \neq 0$ if and only if $\mathbf{f} \in C_0$ and $f_j \neq 0$ for every $j \in \mathbb{N}$. In particular, $\mathcal{K}(\cdot, \mathbf{f}) = 0$ for $\mathbf{f} \notin C_0$, so that all elements $g \in H$ vanish on $C \setminus C_0$.

Consider $\mathbf{f} \in C$ and a sequence $(z_j)_{j \in \mathbb{N}}$ in \mathbb{K} with a convergent product. For $z := \prod_{j \in \mathbb{N}} z_j$ and \mathbf{f}' given by $f'_j := z_j \cdot f_j$ we have $\mathbf{f}' \in C$ and

$$z \cdot \mathcal{K}(\cdot, \mathbf{f}) = \mathcal{K}(\cdot, \mathbf{f}'),$$

see von Neumann (1939, Lem. 3.3.6). This property allows to further reduce the domain, namely to consider the elements of H only on

$$V := \{\mathbf{f} \in \times_{j \in \mathbb{N}} H_j : \|f_j\|_{H_j} = 1 \text{ for every } j \in \mathbb{N}\} \subsetneq C_0.$$

Indeed, for $\mathbf{f} \in C$ with $\mathcal{K}(\cdot, \mathbf{f}) \neq 0$ and $\mathbf{f}' \in V$ given by

$$(A.1) \quad f'_j := \|f_j\|_{H_j}^{-1} \cdot f_j$$

we obtain

$$\mathcal{K}(\cdot, \mathbf{f}) = \prod_{j \in \mathbb{N}} \|f_j\|_{H_j} \cdot \mathcal{K}(\cdot, \mathbf{f}').$$

A.3. The Incomplete Tensor Product. Incomplete tensor products are subspaces of H and constructed in the following way. The property

$$\sum_{j \in \mathbb{N}} |\langle f_j, g_j \rangle_{H_j} - 1| < \infty$$

for $\mathbf{f}, \mathbf{g} \in C_0$ defines an equivalence relation on C_0 , see von Neumann (1939, Lem. 3.3.3). Observe that $\mathbf{g} \in C_0$ has at most a finite number of components $g_j = 0$. If we replace all of them by non-zero elements of the corresponding Hilbert spaces H_j , we obtain $\mathbf{f} \in C_0$, equivalent to \mathbf{g} , with $\mathcal{K}(\cdot, \mathbf{f}) \neq 0$. For $\mathbf{f} \in C_0$ with $\mathcal{K}(\cdot, \mathbf{f}) \neq 0$ and $\mathbf{f}' \in V$ given by (A.1) we have the equivalence of \mathbf{f} and \mathbf{f}' . For a sequence $\mathbf{v} \in V$ of unit vectors $v_j \in H_j$ we consider its equivalence class

$$C^{(\mathbf{v})} := \left\{ \mathbf{f} \in C_0 : \sum_{j \in \mathbb{N}} |\langle v_j, f_j \rangle_{H_j} - 1| < \infty \right\}.$$

Observe that $\{C^{(\mathbf{v})} : \mathbf{v} \in V\}$ is a partition of the set C_0 . Cf. von Neumann (1939, Def. 4.1.1) for the following definition.

Definition A.2. The *incomplete tensor product* of the spaces H_j based on $\mathbf{v} \in V$ is the closed linear subspace

$$H^{(\mathbf{v})} := \bigotimes_{j \in \mathbb{N}} H_j^{(v_j)} := \overline{\text{span}}\{\mathcal{K}(\cdot, \mathbf{f}) : \mathbf{f} \in C^{(\mathbf{v})}\}$$

of H , equipped with the induced norm.

It follows that for every $\mathbf{w} \in V$

$$(A.2) \quad \mathbf{w} \in C^{(\mathbf{v})} \quad \Rightarrow \quad H^{(\mathbf{v})} = H^{(\mathbf{w})}$$

and

$$\mathbf{w} \notin C^{(\mathbf{v})} \quad \Rightarrow \quad H^{(\mathbf{v})} \perp H^{(\mathbf{w})}.$$

The latter orthogonality property follows from $\mathcal{K}(\mathbf{g}, \mathbf{f}) = 0$ for all non-equivalent $\mathbf{f}, \mathbf{g} \in C_0$. Consequently, it suffices to consider the elements of $H^{(\mathbf{v})}$ on the domain $C^{(\mathbf{v})}$, since every $g \in H^{(\mathbf{v})}$ vanishes on $C \setminus C^{(\mathbf{v})}$.

The observations above yield the following representation of H as an orthogonal sum of Hilbert spaces: If $\mathcal{R} \subseteq V$ is a system of representers of the equivalence classes $C^{(v)}$, $v \in V$, then

$$H = \bigoplus_{v \in \mathcal{R}} H^{(v)}.$$

The following approximation is employed in the proof of von Neumann (1939, Lem. 4.1.2). Because of its particular relevance for this paper we present the result together with a proof.

Lemma A.3. *Let $\mathbf{f} \in C^{(v)}$. For $\mathbf{f}_J \in C^{(v)}$ given by*

$$f_{J,j} := \begin{cases} f_j, & \text{if } j < J, \\ v_j, & \text{otherwise,} \end{cases}$$

we have

$$\lim_{J \rightarrow \infty} \|\mathcal{K}(\cdot, \mathbf{f}) - \mathcal{K}(\cdot, \mathbf{f}_J)\|_H = 0.$$

Proof. Let $\mathbf{f} \in C^{(v)}$. Assume at first that $\mathbf{f} \in V$. Then we obtain

$$1/2 \cdot \|\mathcal{K}(\cdot, \mathbf{f}) - \mathcal{K}(\cdot, \mathbf{f}_J)\|_H^2 = 1 - \Re \prod_{j \geq J} \langle v_j, f_j \rangle_{H_j}.$$

This yields the convergence as claimed.

Assume now that $\mathcal{K}(\cdot, \mathbf{f}) \neq 0$. Then we put $z_j := \|f_j\|_{H_j}$. For $\mathbf{f}' \in V$ according to (A.1) we already know that $\mathcal{K}(\cdot, \mathbf{f}'_J)$ converges to $\mathcal{K}(\cdot, \mathbf{f}')$ in H . Furthermore,

$$\prod_{j \in \mathbb{N}} z_j \cdot \|\mathcal{K}(\cdot, \mathbf{f}') - \mathcal{K}(\cdot, \mathbf{f}'_J)\|_H = \left\| \mathcal{K}(\cdot, \mathbf{f}) - \prod_{j \geq J} z_j \cdot \mathcal{K}(\cdot, \mathbf{f}_J) \right\|_H$$

and

$$\left\| \mathcal{K}(\cdot, \mathbf{f}_J) - \prod_{j \geq J} z_j \cdot \mathcal{K}(\cdot, \mathbf{f}_J) \right\|_H^2 = \left| 1 - \prod_{j=J}^{\infty} z_j \right|^2 \cdot \prod_{j=1}^{J-1} z_j^2.$$

This yields the convergence as claimed, since $\sum_{j \in \mathbb{N}} |z_j - 1| < \infty$ also in this case.

It remains to observe that $\mathcal{K}(\cdot, \mathbf{f}_J) = 0$ for J sufficiently large, if $\mathcal{K}(\cdot, \mathbf{f}) = 0$. \square

Subsequently, we discuss further properties of incomplete tensor products, which may be derived with the help of Lemma A.3 in a straightforward way.

First of all,

$$H^{(v)} = \overline{\text{span}}\{\mathcal{K}(\cdot, \mathbf{f}) : \mathbf{f} \in C_{\star}^{(v)}\},$$

where

$$C_{\star}^{(v)} := \{\mathbf{f} \in C_0 : \{j \in \mathbb{N} : f_j \neq v_j\} \text{ is finite}\} \subsetneq C^{(v)}.$$

Consequently, it suffices to consider the elements of $H^{(v)}$ on the domain $C_{\star}^{(v)}$, since

$$(A.3) \quad g(\mathbf{f}) = \lim_{J \rightarrow \infty} g(\mathbf{f}_J)$$

for every $g \in H^{(v)}$ and every $\mathbf{f} \in C^{(v)}$.

As a closed linear subspace of a reproducing kernel Hilbert space, $H^{(v)}$ is a reproducing kernel Hilbert space, too. Moreover, the reproducing kernel $\mathcal{K}^{(v)} : C \times C \rightarrow \mathbb{K}$ of $H^{(v)}$ satisfies

$$\mathcal{K}^{(v)}(\mathbf{g}, \mathbf{f}) = \begin{cases} \mathcal{K}(\mathbf{g}, \mathbf{f}), & \text{if } \mathbf{f}, \mathbf{g} \in C^{(v)}, \\ 0, & \text{otherwise.} \end{cases}$$

Observe that $\mathcal{K}(\mathbf{g}, \mathbf{f}) = \prod_{j \in \mathbb{N}} \langle f_j, g_j \rangle_{H_j}$ is convergent, and not just quasi-convergent, for $\mathbf{f}, \mathbf{g} \in C^{(\mathbf{v})}$.

Instead of $\mathcal{K}^{(\mathbf{v})}$ and $H^{(\mathbf{v})}$ one may actually consider the kernel

$$\mathcal{K}_\star^{(\mathbf{v})} := \mathcal{K}|_{C_\star^{(\mathbf{v})} \times C_\star^{(\mathbf{v})}}$$

and the space $H(\mathcal{K}_\star^{(\mathbf{v})})$, since the restriction $g \mapsto g|_{C_\star^{(\mathbf{v})}}$ defines an isometric isomorphism between $H^{(\mathbf{v})}$ and $H(\mathcal{K}_\star^{(\mathbf{v})})$. The corresponding inverse extension map is essentially given by (A.3).

Remark A.4. Consider any choice of orthonormal bases $(e_{\nu,j})_{\nu \in N_j}$ in each of the spaces H_j , and assume that $0 \in N_j$ for notational convenience. Let \mathcal{N} denote the set of all sequences $\boldsymbol{\nu} := (\nu_j)_{j \in \mathbb{N}}$ with $\nu_j \in N_j$ for every $j \in \mathbb{N}$ and with $\{j \in \mathbb{N} : \nu_j \neq 0\}$ being finite. If

$$e_{0,j} = v_j$$

for every $j \in \mathbb{N}$, then the elementary tensors

$$(A.4) \quad e_\boldsymbol{\nu} := \bigotimes_{j \in \mathbb{N}} e_{\nu_j, j}$$

with $\boldsymbol{\nu} \in \mathcal{N}$ form an orthonormal basis of $H^{(\mathbf{v})}$.

A.4. Tensor Products of Operators. We now consider two incomplete tensor products

$$H^{(\mathbf{v})} := \bigotimes_{j \in \mathbb{N}} H_j^{(v_j)} \quad \text{and} \quad G^{(\mathbf{w})} := \bigotimes_{j \in \mathbb{N}} G_j^{(w_j)}$$

together with a sequence of bounded linear operators $T_j: H_j \rightarrow G_j$. We assume that $T_j v_j = w_j$ for every $j \in \mathbb{N}$ and that $\prod_{j \in \mathbb{N}} \|T_j\|$ converges. Observe that $T_j v_j = w_j$ implies $\|T_j\| \geq 1$. Letting

$$T \bigotimes_{j \in \mathbb{N}} f_j = \bigotimes_{j \in \mathbb{N}} T_j f_j$$

for elementary tensors $\bigotimes_{j \in \mathbb{N}} f_j$ with $\mathbf{f} \in C_\star^{(\mathbf{v})}$ and extending this linearly defines a linear operator from the span $H_0^{(\mathbf{v})}$ of these elementary tensors to $G^{(\mathbf{w})}$. This follows similarly as the corresponding statement for finite algebraic tensor products, see, e.g., Hackbusch (2012, Sec. 4.3.6). It is readily checked that this operator is bounded, in fact it has norm

$$\|T\| = \prod_{j \in \mathbb{N}} \|T_j\|.$$

Again, this follows similarly as for finite Hilbert space tensor products, see, e.g., Hackbusch (2012, Prop. 4.127). Since $H_0^{(\mathbf{v})}$ is dense in $H^{(\mathbf{v})}$, T uniquely extends to a linear operator $T: H^{(\mathbf{v})} \rightarrow G^{(\mathbf{w})}$ with the same norm. This operator is called the *tensor product* of the operators T_j and denoted by $\bigotimes_{j \in \mathbb{N}} T_j$. If, additionally, each operator T_j is an isomorphism and $\prod_{j \in \mathbb{N}} \|T_j^{-1}\|$ converges as well, then $T: H^{(\mathbf{v})} \rightarrow G^{(\mathbf{w})}$ is an isomorphism with inverse $T^{-1} = \bigotimes_{j \in \mathbb{N}} T_j^{-1}$.

A.5. Incomplete Tensor Products of Reproducing Kernel Hilbert Spaces.

Here we consider the particular case

$$H_j := H(k_j), \quad j \in \mathbb{N},$$

with reproducing kernels $k_j: D_j \times D_j \rightarrow \mathbb{K}$. For every $j \in \mathbb{N}$ we assume that there exists a point $x \in D_j$ with $k_j(\cdot, x) \neq 0$, which is equivalent to $H_j \neq \{0\}$. We put $D := \times_{j \in \mathbb{N}} D_j$.

To $\mathbf{x} \in D$ we associate

$$\tau(\mathbf{x}) := (k_j(\cdot, x_j))_{j \in \mathbb{N}} \in \times_{j \in \mathbb{N}} H_j,$$

and we put

$$\mathfrak{X}_0 := \left\{ \mathbf{x} \in D : \sum_{j \in \mathbb{N}} |k_j(x_j, x_j) - 1| < \infty \right\}$$

as well as

$$\mathfrak{X}^{(v)} := \left\{ \mathbf{x} \in \mathfrak{X}_0 : \sum_{j \in \mathbb{N}} |v_j(x_j) - 1| < \infty \right\}$$

for $v \in V$. Let $\mathbf{x} \in D$. Observe that $\mathbf{x} \in \mathfrak{X}_0$ if and only if $\tau(\mathbf{x}) \in C_0$. Moreover, $\mathbf{x} \in \mathfrak{X}^{(v)}$ if and only if $\tau(\mathbf{x}) \in C^{(v)}$.

The following result is due to Rübmann (2020, Lem. 4.9).

Lemma A.5. *If $\mathfrak{X}^{(v)} \neq \emptyset$, then*

$$H^{(v)} = \overline{\text{span}}\{\mathcal{K}(\cdot, \tau(\mathbf{x})) : \mathbf{x} \in \mathfrak{X}^{(v)}\}.$$

Proof. Put $\mathcal{H} := \text{span}\{\mathcal{K}(\cdot, \tau(\mathbf{x})) : \mathbf{x} \in \mathfrak{X}^{(v)}\}$. It suffices to prove the following result. For every $\mathbf{f} \in C_\star^{(v)} \cap v$ there exists a sequence $(f_J)_{J \in \mathbb{N}}$ in \mathcal{H} such that

$$\lim_{J \rightarrow \infty} \|\mathcal{K}(\cdot, \mathbf{f}) - f_J\|_H = 0.$$

For a fixed choice of $\mathbf{x} \in \mathfrak{X}^{(v)}$ with $k_j(\cdot, x_j) \neq 0$ for every $j \in \mathbb{N}$ we define $\mathbf{w} \in C^{(v)} \cap v$ by

$$w_j := \|k_j(\cdot, x_j)\|_{H_j}^{-1} \cdot k_j(\cdot, x_j).$$

Since $\mathbf{w} \in C^{(w)}$, we get $C^{(v)} = C^{(w)}$. Furthermore, we define $\mathbf{f}_J \in C_\star^{(w)} \cap v$ for $J \in \mathbb{N}$ by

$$f_{J,j} := \begin{cases} f_j, & \text{if } j \leq J, \\ w_j, & \text{otherwise.} \end{cases}$$

Lemma A.3 yields

$$\lim_{J \rightarrow \infty} \|\mathcal{K}(\cdot, \mathbf{f}) - \mathcal{K}(\cdot, \mathbf{f}_J)\|_H = 0.$$

Fix $J \in \mathbb{N}$ and consider the norm given by $\|(h_1, \dots, h_J)\| := \max_{j=1, \dots, J} \|h_j\|_{H_j}$ on $H_1 \times \dots \times H_J$. By $\zeta_J(h_1, \dots, h_J) = \mathcal{K}(\cdot, \mathbf{g})$ with

$$g_j := \begin{cases} h_j, & \text{if } j \leq J, \\ w_j, & \text{otherwise,} \end{cases}$$

we obtain a continuous mapping ζ_J from $H_1 \times \dots \times H_J$ into H . For every $j \leq J$ we choose a sequence $(h_{j,n})_{n \in \mathbb{N}}$ in $\text{span}\{k_j(\cdot, x) : x \in D_j\}$ with $\lim_{n \rightarrow \infty} \|f_j - h_{j,n}\|_{H_j} = 0$. It follows that

$$\lim_{n \rightarrow \infty} \|\zeta_J(h_{1,n}, \dots, h_{J,n}) - \mathcal{K}(\cdot, \mathbf{f}_J)\|_H = 0.$$

Finally, $\zeta_J(h_{1,n}, \dots, h_{J,n}) \in \mathcal{H}$, which completes the proof. \square

Assume that $\mathfrak{X}_0 \neq \emptyset$. In this case

$$K(\mathbf{x}, \mathbf{y}) := \mathcal{K}(\tau(\mathbf{x}), \tau(\mathbf{y})) = \prod_{j \in \mathbb{N}} k_j(x_j, y_j), \quad \mathbf{x}, \mathbf{y} \in \mathfrak{X}_0,$$

yields a reproducing kernel $K: \mathfrak{X}_0 \times \mathfrak{X}_0 \rightarrow \mathbb{K}$ of tensor product form. The complete tensor product H of the reproducing kernel Hilbert spaces $H(k_j)$ is a feature space and $\mathbf{x} \mapsto \mathcal{K}(\cdot, \tau(\mathbf{x}))$ with $\mathbf{x} \in \mathfrak{X}_0$ is a feature map of K . Consequently,

$$H(K) = \{g \circ \tau|_{\mathfrak{X}_0} : g \in H\}$$

and

$$\|f\|_{H(K)} = \min\{\|g\|_H : g \circ \tau|_{\mathfrak{X}_0} = f\}.$$

Assume that $\mathfrak{X}^{(v)} \neq \emptyset$. Then analogous results hold true for the tensor product kernel

$$K^{(v)} := K|_{\mathfrak{X}^{(v)} \times \mathfrak{X}^{(v)}}$$

and the incomplete tensor product $H^{(v)}$ of the reproducing kernel Hilbert spaces $H(k_j)$.

In the sense of the following result, which is due to Rüßmann (2020, Thm. 4.10), $H^{(v)}$ is the reproducing kernel Hilbert space with the tensor product kernel $K^{(v)}$.

Theorem A.6. *Let $v \in V$. If $\mathfrak{X}^{(v)} \neq \emptyset$, then*

$$\Phi: H^{(v)} \rightarrow \mathbb{K}^{\mathfrak{X}^{(v)}},$$

given by

$$\Phi g(\mathbf{x}) := g(\tau(\mathbf{x})), \quad \mathbf{x} \in \mathfrak{X}^{(v)},$$

induces an isometric isomorphism between $H^{(v)}$ and $H(K^{(v)})$. In particular, for $\mathbf{f} \in C^{(v)}$ and $\mathbf{x} \in \mathfrak{X}^{(v)}$ the product $\prod_{j \in \mathbb{N}} f_j(x_j)$ converges and

$$\left(\Phi \bigotimes_{j \in \mathbb{N}} f_j \right) (\mathbf{x}) = \prod_{j \in \mathbb{N}} f_j(x_j).$$

Proof. Due to Lemma A.5 the mapping Φ is injective, and thus it induces an isometric isomorphism as claimed. Furthermore, for $\mathbf{f} \in C^{(v)}$ and $\mathbf{x} \in \mathfrak{X}^{(v)}$,

$$\bigotimes_{j \in \mathbb{N}} f_j(\tau(\mathbf{x})) = \langle \mathcal{K}(\cdot, \mathbf{f}), \mathcal{K}(\cdot, \tau(\mathbf{x})) \rangle_H = \prod_{j \in \mathbb{N}} f_j(x_j)$$

with convergence as claimed, since $\tau(\mathbf{x}) \in C^{(v)}$. \square

A.6. Incomplete Tensor Products of L^2 -Spaces. Here we consider the particular case

$$H_j := L^2(\rho_j) := L^2(D_j, \rho_j), \quad j \in \mathbb{N},$$

for a sequence $(\rho_j)_{j \in \mathbb{N}}$ of probability measures on σ -algebras on sets D_j . Let $\rho := \times_{j \in \mathbb{N}} \rho_j$ denote the product measure on the product σ -algebra on $D := \times_{j \in \mathbb{N}} D_j$. Recall that we identify square-integrable functions and the elements of the corresponding L^2 -space. We now show that the incomplete tensor product $H^{(1)}$ based on the constant function 1 as the unit vector for every $j \in \mathbb{N}$ is, in a canonical way, isometrically isomorphic to $L^2(\rho) := L^2(D, \rho)$.

We introduce the following notation. For $\mathbf{f} \in C^{(1)}$ and $J \in \mathbb{N}$ we consider the function $\mathbf{x} \mapsto \prod_{j=1}^J f_j(x_j)$, which belongs to $L^2(\rho)$. In the case of convergence in

$L^2(\rho)$ as $J \rightarrow \infty$, we use $\prod_{j=1}^{\infty} f_j$ to denote the limit; to indicate this convergence we say that $\prod_{j=1}^{\infty} f_j$ exists in $L^2(\rho)$.

See Guichardet (1969, Corollary 6) for the following result.

Theorem A.7. *For every $\mathbf{f} \in C^{(1)}$ we have existence of $\prod_{j=1}^{\infty} f_j$ in $L^2(\rho)$. Moreover, the mapping $\Psi: L^2(\rho) \rightarrow \mathbb{K}^C$ given by*

$$\Psi\gamma(\mathbf{g}) := \int_D \gamma \cdot \prod_{j=1}^{\infty} \overline{g_j} d\rho, \quad \mathbf{g} \in C^{(u)},$$

and $\Psi\gamma(\mathbf{g}) := 0$ for $\mathbf{g} \in C \setminus C^{(1)}$ defines an isometric isomorphism between $L^2(\rho)$ and $H^{(1)}$. In particular,

$$(A.5) \quad \Psi^{-1} \bigotimes_{j \in \mathbb{N}} f_j = \prod_{j=1}^{\infty} f_j$$

for $\mathbf{f} \in C^{(1)}$.

Proof. The existence of $\prod_{j=1}^{\infty} f_j$ in $L^2(\rho)$ trivially holds true for $\mathbf{f} \in C_{\star}^{(1)}$. Let $\psi: C_{\star}^{(1)} \rightarrow L^2(\rho)$ be given by $\psi\mathbf{f} := \prod_{j=1}^{\infty} f_j$. Since $\langle \psi\mathbf{f}, \psi\mathbf{g} \rangle_{L^2(\rho)} = \mathcal{K}(\mathbf{g}, \mathbf{f})$ for $\mathbf{f}, \mathbf{g} \in C_{\star}^{(1)}$, the mapping ψ is a feature map and $L^2(\rho)$ is a feature space for the reproducing kernel $\mathcal{K}_{\star}^{(1)}$. The space $H(\mathcal{K}_{\star}^{(1)})$ may therefore be described via the linear mapping $\Psi_1: L^2(\rho) \rightarrow \mathbb{K}^{C_{\star}^{(1)}}$, given by

$$\Psi_1\gamma(\mathbf{g}) := \langle \gamma, \psi\mathbf{g} \rangle_{L^2(\rho)} = \int_D \gamma \cdot \prod_{j=1}^{\infty} \overline{g_j} d\rho, \quad \mathbf{g} \in C_{\star}^{(1)}.$$

Since $\text{span } \psi(C_{\star}^{(1)})$ is a dense subset of $L^2(\rho)$, the mapping Ψ_1 is injective. We conclude that Ψ_1 induces an isometric isomorphism between $L^2(\rho)$ and $H(\mathcal{K}_{\star}^{(1)})$, which will also be denoted by Ψ_1 .

Recall that $h \mapsto h|_{C_{\star}^{(1)}}$ defines an isometric isomorphism between $H^{(1)}$ and $H(\mathcal{K}_{\star}^{(1)})$. Its inverse $\Psi_2: H(\mathcal{K}_{\star}^{(1)}) \rightarrow H^{(1)}$ is given by (A.3), i.e., by the extension

$$\Psi_2 h(\mathbf{g}) := \begin{cases} \lim_{J \rightarrow \infty} h(\mathbf{g}_J), & \text{if } \mathbf{g} \in C^{(1)}, \\ 0, & \text{if } \mathbf{g} \in C \setminus C^{(1)}. \end{cases}$$

Altogether, we obtain the isometric isomorphism $\Psi := \Psi_2 \circ \Psi_1: L^2(\rho) \rightarrow H^{(1)}$.

Let $\gamma \in L^2(\rho)$. By definition of Ψ and \mathbf{g}_J ,

$$\Psi\gamma(\mathbf{g}) = \begin{cases} \lim_{J \rightarrow \infty} \int_D \gamma \cdot \prod_{j=1}^J \overline{g_{j,j}} d\rho, & \text{if } \mathbf{g} \in C^{(1)}, \\ 0, & \text{if } \mathbf{g} \in C \setminus C^{(1)}. \end{cases}$$

In the particular case $\gamma := \prod_{j=1}^{\infty} f_j$ with $\mathbf{f} \in C_{\star}^{(1)}$ we get

$$\Psi\gamma(\mathbf{g}) = \lim_{J \rightarrow \infty} \int_D \prod_{j=1}^J f_j(x_j) \cdot \overline{g_j}(x_j) d\rho(\mathbf{x}) = \mathcal{K}(\mathbf{g}, \mathbf{f}) = \bigotimes_{j \in \mathbb{N}} f_j(\mathbf{g})$$

for $\mathbf{g} \in C^{(1)}$, i.e., (A.5) is satisfied for $\mathbf{f} \in C_*^{(1)}$. In the general case $\mathbf{f} \in C^{(1)}$ this implies

$$\Psi^{-1} \bigotimes_{j \in \mathbb{N}} f_{J,j}(\mathbf{x}) = \prod_{j=1}^{\infty} f_{J,j}(\mathbf{x}) = \prod_{j=1}^J f_j(x_j)$$

for ρ -a.e. $\mathbf{x} \in D$. Moreover, $\bigotimes_{j \in \mathbb{N}} f_{J,j}$ converges to $\bigotimes_{j \in \mathbb{N}} f_j$ in $H^{(1)}$ as $J \rightarrow \infty$, see Lemma A.3. It follows that

$$\lim_{J \rightarrow \infty} \left\| \prod_{j=1}^{\infty} f_{J,j} - \Psi^{-1} \bigotimes_{j \in \mathbb{N}} f_j \right\|_{L^2(\rho)} = 0,$$

i.e., $\prod_{j=1}^{\infty} f_j$ exists in $L^2(\rho)$ and (A.5) is satisfied for $\mathbf{f} \in C^{(1)}$. Consequently, with $\mathbf{g} \in C^{(1)}$ in place of \mathbf{f} ,

$$\Psi\gamma(\mathbf{g}) = \lim_{J \rightarrow \infty} \int_D \gamma \cdot \prod_{j=1}^{\infty} \overline{g_{J,j}} d\rho = \int_D \gamma \cdot \prod_{j=1}^{\infty} \overline{g_j} d\rho$$

for $\gamma \in L^2(\rho)$. □

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REFERENCES

- J. Baldeaux and M. Gnewuch. Optimal randomized multilevel algorithms for infinite-dimensional integration on function spaces with ANOVA-type decomposition. *SIAM J. Numer. Anal.*, 52:1128–1155, 2014.
- V. I. Bogachev. *Gaussian Measures*, volume 62 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1998.
- S. N. Chandler-Wilde, D. P. Hewett, and A. Moiola. Interpolation of Hilbert and Sobolev spaces: quantitative estimates and counterexamples. *Mathematika*, 61:414–443, 2015.
- H. Dette and A. Zhigljavsky. Reproducing kernel Hilbert spaces, polynomials, and the classical moment problem. *SIAM/ASA J. Uncertain. Quantif.*, 9:1589–1614, 2021.
- J. Dick and M. Gnewuch. Optimal randomized changing dimension algorithms for infinite-dimensional integration on function spaces with ANOVA-type decomposition. *J. Approx. Theory*, 184:111–145, 2014.
- J. Dick, C. Irrgeher, G. Leobacher, and F. Pillichshammer. On the optimal order of integration in Hermite spaces with finite smoothness. *SIAM J. Numer. Anal.*, 56:684–707, 2018.
- A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi. *Higher transcendental functions. Vol. II*. McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953. Based, in part, on notes left by Harry Bateman.
- G. Fasshauer, F. J. Hickernell, and H. Woźniakowski. On dimension-independent rates of convergence for function approximation with Gaussian kernels. *SIAM J. Numer. Anal.*, 50:247–271, 2012.

- A. D. Gilbert, F. Y. Kuo, D. Nuyens, and G. W. Wasilkowski. Efficient implementations of the multivariate decomposition method for approximating infinite-variate integrals. *SIAM J. Scient. Comput.*, 40:A3240–A3266, 2018.
- M. Gnewuch, M. Hefter, A. Hinrichs, and K. Ritter. Embeddings of weighted Hilbert spaces and applications to multivariate and infinite-dimensional integration. *Journal of Approximation Theory*, 222:8–39, 2017.
- M. Gnewuch, M. Hefter, A. Hinrichs, K. Ritter, and G. W. Wasilkowski. Embeddings for infinite-dimensional integration and L_2 -approximation with increasing smoothness. *J. Complexity*, 54:101406, 2019.
- A. Guichardet. *Tensor Products of C^* -Algebras, Part II. Infinite Tensor Products*, volume 13 of *Lecture Notes Series*. Aarhus Universitet, 1969.
- W. Hackbusch. *Tensor spaces and numerical tensor calculus*, volume 42 of *Springer Series in Computational Mathematics*. Springer, Heidelberg, 2012.
- F. J. Hickernell and X. Wang. The error bounds and tractability of quasi-Monte Carlo algorithms in infinite dimensions. *Math. Comp.*, 71:1614–1661, 2001.
- F. J. Hickernell, T. Müller-Gronbach, B. Niu, and K. Ritter. Multi-level Monte Carlo algorithms for infinite-dimensional integration on $\mathbb{R}^{\mathbb{N}}$. *J. Complexity*, 26(3):229–254, 2010.
- J. Indritz. An inequality for Hermite polynomials. *Proc. Amer. Math. Soc.*, 12:981–983, 1961.
- C. Irrgeher and G. Leobacher. High-dimensional integration on the \mathbb{R}^d , weighted Hermite spaces, and orthogonal transforms. *J. Complexity*, 31:174–205, 2015.
- C. Irrgeher, P. Kritzer, G. Leobacher, and F. Pillichshammer. Integration in Hermite spaces of analytic functions. *J. Complexity*, 31:380–404, 2015.
- C. Irrgeher, P. Kritzer, F. Pillichshammer, and H. Woźniakowski. Tractability of multivariate approximation defined over Hilbert spaces with exponential weights. *J. Approx. Theory*, 207:301–338, 2016a.
- C. Irrgeher, P. Kritzer, F. Pillichshammer, and H. Woźniakowski. Approximation in Hermite spaces of smooth functions. *J. Approx. Theory*, 207:98–126, 2016b.
- T. Karvonen. On non-inclusion of certain functions in reproducing kernel Hilbert spaces. arXiv:2102.10628, 2021.
- T. Karvonen, C. J. Oates, and M. Girolami. Integration in reproducing kernel Hilbert spaces of Gaussian kernels. arXiv:2004.12654v2, 2021.
- F. Y. Kuo, I. H. Sloan, G. W. Wasilkowski, and H. Woźniakowski. Liberating the dimension. *J. Complexity*, 26:422–454, 2010.
- F. Y. Kuo, I. H. Sloan, and H. Woźniakowski. Multivariate integration for analytic functions with Gaussian kernels. *Math. Comp.*, 86:829–853, 2017.
- A. Papageorgiou and H. Woźniakowski. Tractability through increasing smoothness. *J. Complexity*, 26(5):409–421, 2010.
- L. Plaskota and G. W. Wasilkowski. Tractability of infinite-dimensional integration in the worst case and randomized settings. *J. Complexity*, 27:505–518, 2011.
- C. E. Rasmussen and C. K. I. Williams. *Gaussian Processes for Machine Learning*. MIT Press, Cambridge, 2006.
- L. Rodino. *Linear partial differential operators in Gevrey spaces*. World Scientific Publishing Co., Inc., River Edge, NJ, 1993.
- R. Rößmann. Tensor products of Hilbert spaces. Master’s thesis, Department of Mathematics, TU Kaiserslautern, 2020.
- I. H. Sloan and H. Woźniakowski. When are quasi-Monte Carlo algorithms efficient for high dimensional integrals? *J. Complexity*, 14:1–33, 1998.
- I. H. Sloan and H. Woźniakowski. Multivariate approximation for analytic functions with Gaussian kernels. *J. Complexity*, 45:1–21, 2018.
- I. Steinwart and A. Christmann. *Support vector machines*. Information Science and Statistics. Springer-Verlag, New York, 2008.

- I. Steinwart, D. Hush, and C. Scovel. An explicit description of the reproducing kernel Hilbert space of Gaussian RBF kernels. *IEEE Trans. Inform. Theory*, 52:4635–4663, 2006.
- G. Szegő. *Orthogonal Polynomials*. American Mathematical Society, Providence, R.I., fourth edition, 1975. American Mathematical Society, Colloquium Publications, Vol. XXIII.
- H. Triebel. *Interpolation theory, function spaces, differential operators*, volume 18 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam-New York, 1978.
- J. von Neumann. On infinite direct products. *Compositio Math.*, 6:1–77, 1939.
- G. W. Wasilkowski. Liberating the dimension for L_2 -approximation. *J. Complexity*, 28(3):304–319, 2012.

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