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# **RBF-FD** formulas and convergence properties

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### Abstract

The local RBF is becoming increasingly popular as an alternative to the global version that su ers from ill-conditioning. In this paper, we study analytically the convergence behavior of the local RBF method as a function of the number of nodes employed in the scheme, the nodal distance, and the shape parameter. We derive exact formulas for the rst and second derivatives in one dimension, and for the Laplacian in two dimensions. Using these formulas we compute Taylor expansions for the error. From this analysis, we nd that there is an optimal value of the shape parameter for which the error is minimum. This optimal parameter is independent of the nodal distance. Our theoretical results are corroborated by numerical experiments.

Key words: Radial basis functions; mesh-free

# 1 Introduction

Radial Basis Functions (RBF) originate as a very e cient technique for interpolation of multidimensional scattered data (see [8] and references therein). Later, it became popular as a truly *mesh-free* method for the solution of partial di erential equations (PDEs) on irregular domains. This application of RBFs was rst proposed by Edward Kansa [14,15] and it is based on enforcing collocation of the PDE in a set of scattered nodes, to compute a global solution in the space spanned by a set of identical RBFs translated to a set

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of RBF centers. The main advantages of the method are ease of programming and potential spectral accuracy, but its main drawback is ill-conditioning of the resulting linear system. To overcome this drawback a *local* version of the method was later proposed by several authors [3,24,26] simultaneously. The idea of the *local RBF* method, is to sacri ce the spectral accuracy inherent to the *global* method, in order to have a sparse better-conditioned linear system capable of solving large multidimensional PDEs. Another advantage of the local version of the method is its suitability for problems with discontinuous boundary conditions [1,5].

The local RBF method can also be considered as a generalization of the classical nite di erence (FD) method to scattered node layouts. In classical nite di erences, derivatives of a function u at a given point are approximated as linear combinations of the values of u at some surrounding nodes. In 1-D, for example, the kth-derivative at node  $x_i$  is approximated by

$$\frac{d^k u}{dx^k} \sum_{x=x_i}^N w_{ji}^{(k)} u(x_i) \qquad j = 1 \qquad N$$

where  $x_i$  is a set of surrounding nodes which usually are equispaced. The unknown weights  $w_{ji}^{(k)}$  are usually computed using polynomial interpolation [9]. These 1-D formulas can be combined to create FD formulas for partial derivatives in two or more dimensions, provided that the nodes in the stencil are located on some kind of structured grid, which severely limits the geometric exibility of the method. In the case of RBF nite di erence formulas (*RBF-FD*) this restriction is eliminated since the weights are obtained by RBF interpolation on the set of surrounding nodes.

Once the weights for the derivatives appearing in the PDE have been determined for each scattered node, the di erential operator is enforced at each of those nodes. This procedure leads to a sparse, linear system of equations whose solution yields the approximate values of u at the nodes. This local RBF method has been successfully applied to solve a variety of problems [1,4,5,18,22,24,25].

However, papers addressing the convergence properties of the method are more scarce. It is well known that the local method lacks the spectral accuracy of the global RBF method, but the exact dependence of the error with average distance between nodes h, shape parameter c, and number of supporting nodes N, is not known. We mention, though, that Ding et. al. [6] carried out numerical experiments using Poisson's equation on an equispaced grid to experimentally determine these dependencies. They found an error estimate  $O((h \ c)^n)$ in which n is a constant dependent on the number of nodes N used in the formulas (n19 for 6N9, n36 for 9 < N27, n49 for 27 < N34).

Fornberg and coworkers [7,10] analyzed the behavior of RBF interpolants in the limit of increasingly at radial functions (c ). They found that in the 1-D case, with very simple requirements on the basis functions, the interpolants converge to the Lagrange interpolating polynomial and, therefore, in this limit *RBF-FD* di erentiation is equivalent to the standard nite di erence method. Wright and Fornberg [27] used Hermite RBF interpolation method to derive new nite di erence formulas (*RBF-HFD*) which also include a linear combination of derivatives at some surrounding nodes. They used cardinal RBF interpolants to derive *RBF-FD* and *HFD* formulas in some simple cases and studied their behavior in the limit of at basis functions. They also analyzed numerically the dependence of the error on the shape parameter by using them to solve some simple elliptic PDE problems.

In this work we address the convergence properties of RBF-FD formulas on equispaced grids and analyze the dependence of the error with nodal distance h, shape parameter c, and number of supporting nodes N. The main result of our study is to analytically show the existence of an optimal value of the shape parameter that minimizes the truncation error. The optimal value is independent of the nodal distance and only depends on the value of the function and its derivatives.

The paper is organized as follows. In Section 2 we describe the RBF-FD formulas and how to determine the unknown weighting coe cients. In Section 3 we use Taylor series expansion in the limit c h to derive closed form expressions of the weighting coe cients for rst and second order derivatives. A series expansion in powers of h leads to closed form expressions for the error as a function of h and c. In Section 4 we derive the corresponding expressions for the error of RBF-FD formulas to approximate the Laplacian. The results of Sections 3 and 4 are used in Section 5 to derive the optimal value of the shape parameter. Section 6 extends these results to the case of non-equispaced nodes. Finally, we summarize the main results of this work in Section 7.

# 2 RBF-FD formulation

In this section we describe how the RBF-FD formulas are derived and how the weights can be exactly computed. Consider a stencil consisting of N scattered nodes  $\mathbf{x}_1$   $\mathbf{x}_N$ , and a di erential operator  $\mathcal{L}$ . For a given node, say  $\mathbf{x}_1$ , the objective is to approximate  $\mathcal{L}u(\mathbf{x}_1)$  as a linear combination of the values of u at the N scattered nodes, so that

$$\mathcal{L}u(\mathbf{x}_1) \qquad \stackrel{N}{\underset{i=1}{\overset{i}{\longrightarrow}}} u(\mathbf{x}_i) \tag{1}$$

To determine the weighting coe cients  $_i$ , a set of base functions  $_i(\mathbf{x})$ , i = 1 N are required. In that base,

$$\mathcal{L}_{j}(\mathbf{x}_{1}) = \bigvee_{i=1}^{N} {}_{i}{}_{j}(\mathbf{x}_{i}) \qquad j = 1 \ 2 \qquad N$$

$$(2)$$

This is a system of N linear equations on N unknowns whose solution yields the unknown weighting coe cients  $_i$ . In the following we will use multiquadrics as RBFs,

$$_i(\mathbf{x}) = \overline{c^2 + \mathbf{x} + \mathbf{x}_i}$$

where c is the shape parameter. As c increases the multiquadrics becomes increasingly at and this has an important e ect in the accuracy of the approximation. The general behavior is such that the larger the shape parameter c, the smaller the approximation error. However, the multiquadric RBF approximation su ers from a trade-o principle [21], i.e. increasing the shape parameter to improve the accuracy results in a more ill-conditioned matrix and, therefore, to a signi cant increase of rounding errors.

## **3** One-dimensional RBF-FD Formulas

In this section we show how to derive the exact RBF-FD formulas for rst and second derivatives. We compute the limit of these formulas for c h, and perform a Taylor expansion of the error in powers of h.

## 3.1 First Derivative

Consider an RBF-FD approximation to the  $\$  rst derivative using N = 3 equispaced nodes. In this case,

$$u(x_1) = {}_1 u(x_1 \quad h) + {}_2 u(x_1) + {}_3 u(x_1 + h)$$
(3)

Substituting function u by multiquadrics radial basis functions centered at  $x_1 - h$ ,  $x_1$ , and  $x_1 + h$ , results in the following linear system of equations,

$$\frac{h}{\overline{h^2 + c^2}} = c_{-1} + \overline{h^2 + c^2}_{-2} + \overline{4h^2 + c^2}_{-3}$$

$$0 = \overline{h^2 + c^2}_{-1} + c_{-2} + \overline{h^2 + c^2}_{-3}$$
(4)

node	N = 3	N = 4	N = 5	N = 6
$ \begin{array}{c cc} x_1 & 2h \\ x_1 & h \end{array} $	$\frac{1}{1}$ 1 + $\frac{h^2}{1}$	$\frac{1}{1}$ 1 + $\frac{5h^2}{2}$	$\begin{array}{c c} \frac{1}{12h} & 1 + \frac{8h^2}{c^2} \\ \frac{2}{c^2} & 1 + \frac{2h^2}{c^2} \end{array}$	$\begin{array}{c c} \frac{1}{20h} & 1 + \frac{1294h^2}{77c^2} \\ \frac{1}{1} & 1 + \frac{887h^2}{77c^2} \end{array}$
$x_1$	$2h$ $2c^2$ 0	$\begin{array}{ccc} 3h & 2c^2 \\ \frac{1}{2h} & 1 & \frac{3h^2}{c^2} \end{array}$	$3h$ $c^2$ $0$	$ \begin{array}{cccccccccccccccccccccccccccccccccccc$
$x_1 + h$	$\frac{1}{2h}$ 1 + $\frac{h^2}{2c^2}$	$\frac{1}{h}  1  \frac{h^2}{2c^2}$	$\frac{2}{3h}$ 1 + $\frac{2h^2}{c^2}$	$\frac{1}{h}$ 1 $\frac{73 h^2}{154 c^2}$
$x_1 + 2h$		$\frac{1}{6h}  1 + \frac{h^2}{c^2}$	$\frac{1}{12h}$ 1 + $\frac{8h^2}{c^2}$	$\begin{bmatrix} \frac{1}{4h} & 1 + \frac{334h^2}{77c^2} \\ 1 & 2223h^2 \end{bmatrix}$
$x_1 + 3h$ Table 1				$\frac{1}{30h}$ 1 + $\frac{2223h}{154c^2}$

RBF-FD coe cients for rst derivative in the limit c = h.

$$\frac{h}{\overline{h^2 + c^2}} = \overline{4h^2 + c^2}_1 + \overline{h^2 + c^2}_2 + c_3$$

whose solution is,

$$a_{1} = a_{3} = \frac{1}{4h} \frac{1 + 1 + \frac{4h^{2}}{c^{2}}}{1 + \frac{h^{2}}{c^{2}}} \qquad a_{2} = 0$$

In the limit when c = h

$$_{1} = _{3} = \frac{1}{2h} + \frac{h^{2}}{2c^{2}} = 0$$
 (5)

which coincides with the standard central di erence approximation to the rst derivative with a correction term of order  $h^2 c^2$  (see Table 1).

Including additional nodes simply leads to larger linear systems to determine the coe cients of the RBF-FD formulas for the rst derivatives. Using a symbolic language (such as Mathematica or Maple) it is possible to derive the exact formulas for the coe cients with up to at least six equispaced nodes (the formula for six nodes computed with Mathematica is 45 pages long). More useful is to compute the Taylor series expression when c = h. These results are shown in Table 1 for terms up to  $c^{-2}$ .

It is interesting to compute the errors resulting from these formulas. For instance, in the case of N = 3, introducing the values of the coe cients given by (5) into (3), and expanding  $u(x_1 + h)$  and  $u(x_1 - h)$  results in,

$$\hat{u}'(x_1) = \frac{1}{2h} \left( 1 + \frac{h^2}{2c^2} \right) \left[ u(x_1 + h) - u(x_1 - h) \right] =$$

$$= \frac{1}{2h} \left( 1 + \frac{h^2}{2c^2} \right) \left[ 2h u'(x_1) + \frac{h^3}{3} u'''(x_1) + \dots \right] \Rightarrow$$

$$\Rightarrow \quad \epsilon_3(x_1) \equiv \hat{u}'(x_1) - u'(x_1) \approx \frac{h^2}{6} u'''(x_1) + \frac{h^2}{2c^2} u'(x_1) \tag{6}$$

Thus, it is second order in h, like the standard central difference formula, and second order in  $(h/c)^2$ .



Fig. 1. Error in approximation of first derivative with RBF-FD formula for three equispaced nodes.  $u(x) = \exp(-x^2)$ , x = 1. Left: c/h dependence for h = 0.01. Right: h dependence for c = 10. Dashed line equation (6).

We can check these results numerically by computing the error in approximating the first derivative of  $u = \exp(-x^2)$  at x = 1. Figure 1 shows the error as a function of c/h (left) and as a function of h (right). Both figures show that the numerical results (in solid lines) closely agree with equation (6) (in dot-dashed lines) until a critical value of c/h is reached ( $c/h \approx 5000$ ) when the linear system (4) becomes ill conditioned and rounding errors deteriorate the accuracy of the solution. For small values of c/h, the contribution of the second term in (6),  $\frac{h^2}{2c^2}u'(x_1)$ , is dominant and the error shown in the left side of Figure 1 decreases as  $(1/c)^2$ . For large values of c/h, the contribution of the first term,  $\frac{h^2}{6}u'''(x_1)$ , is dominant and the error approaches a constant value. In the case of the error dependence with h, shown in the right side of the figure, the first term is dominant throughout. Notice that the extra parameter c makes it possible to minimize the approximation error given by equation (6). In fact, for this simple case the error is zero for  $c^2 = -3 u'(x_1) / u'''(x_1)$ . If  $u'(x_1)$  and  $u'''(x_1)$  have opposite signs then  $c^2$  is positive and there is a real positive value of c for which the error is zero (see Section 5).

We can repeat the same procedure for the case N = 4. After some algebra

$$\epsilon_4(x_1) \approx -\frac{h^3}{12} u^{(IV)}(x_1) - \frac{h^3}{c^2} u''(x_1) + \frac{3h^3}{4c^4} u(x_1), \qquad (7)$$

where  $\epsilon_4(x_1) = \hat{u}'(x_1) - u'(x_1)$  for N = 4. Notice that in this case, the leading contribution of the terms of order  $O(h/c)^2$  in the expansion of the coefficients (see Table 1) cancels out and it is necessary to include terms of order  $O(h/c)^4$  in those expansions.



Fig. 2. Error in approximation of first derivative with RBF-FD formula with four equispaced nodes.  $u(x) = \exp(-x^2)$ , x = 1. Left: c/h dependence for h = 0.01. Right: h dependence for c = 10. Dashed line equation (7).

Figure 2 shows the error in the approximation of the first derivative of  $u = \exp(-x^2)$  at x = 1 using the RBF-FD formula for four equispaced nodes and compares it to the error given by equation (7). As was the case with three nodes, there is a critical value of the shape parameter c above which the system becomes ill conditioned leading to high errors. Notice also that for small values of c/h the contribution of the second term,  $\frac{3h^3}{4c^4}$ , is dominant

and the error shown in the left side of Figure 2 decreases as  $(1 \ c)^4$ . For large values of  $c \ h$  the contribution of the rst term,  $\frac{h^3}{12} u^{(IV)}(x_1)$ , is dominant and the error approaches a constant value. There is an intermediate region around  $c \ h = 10^2$ , where the three terms are comparable.

The corresponding results for N = 5 and N = 6 are

$${}_{5}(x_{1}) \qquad \frac{h^{4}}{30} u^{(V)}(x_{1}) \qquad \frac{4 h^{4}}{3 c^{2}} u^{-}(x_{1}) \qquad \frac{5 h^{4}}{2 c^{4}} u^{-}(x_{1})$$
(8)

and

$${}_{6}(x_{1}) \qquad \frac{h^{5}}{60} u^{(VI)}(x_{1}) + \frac{37 h^{5}}{28 c^{2}} u^{(IV)}(x_{1}) + \frac{255 h^{5}}{28 c^{4}} u^{(x_{1})} \frac{165 h^{5}}{28 c^{6}} u(x_{1})$$

$$(9)$$

Thus, the errors of the RBF-FD formulas for N nodes can be written as

$$_{N}(x_{1}) = h^{N-1} \prod_{m=0}^{(N+k-1)-2} \frac{A_{m}}{c^{2m}} u^{(N-2m)}(x_{1})$$
 (10)

where  $A_m$  are constants which depend on N, and k = 0 if N odd and k = 1 if N even. There are additional terms not included in this formula which are  $O(h^{N+1-k})$ . Thus, for the smaller values of c h the last term in the above expression is dominant and the error behaves as

$${}_{N}(x_{1}) = \begin{array}{c} O(h \ c)^{N-1} u (x_{1}) & \text{if } N \text{ odd} \\ O(h^{N-1} \ c^{N}) u(x_{1}) & \text{if } N \text{ even} \end{array}$$
(11)

For large values of c h the rst term in the above expression is dominant and the error approaches a value independent of c. This value coincides with the corresponding standard nite di erence error. For intermediate values of c hsome of the other terms might become dominant, depending on the particular function u and the value of h used (see for instance Figure 3).

## 3.2 Second Derivative

Analogously, we derive the RBF-FD approximation to the second derivative using three equispaced nodes. In this case

node	N = 3	N = 4	N = 5	N = 6
$x_1  2h$			$\frac{1}{12h^2}$ 1 + $\frac{74h^2}{7c^2}$	$\frac{1}{12h^2}$ 1 + $\frac{5906h^2}{385c^2}$
$x_1$ h	$\frac{1}{h^2}  1 + \frac{h^2}{c^2}$	$\frac{1}{h^2}$ 1 + $\frac{13 h^2}{6 c^2}$	$\frac{4}{3 h^2}  1 + \frac{37 h^2}{14 c^2}$	$\frac{4}{3 h^2}  1 + \frac{1273 h^2}{308 c^2}$
$x_1$	$\frac{2}{h^2}  1 + \frac{h^2}{c^2}$	$\frac{2}{h^2}$ 1 + $\frac{11h^2}{4c^2}$	$\frac{5}{2h^2}$ 1 + $\frac{74h^2}{35c^2}$	$\frac{5}{2h^2}  1 + \frac{1426h^2}{385c^2}$
$x_1 + h$	$\frac{1}{h^2}  1 + \frac{h^2}{c^2}$	$\frac{1}{h^2}$ 1 + $\frac{9 h^2}{2 c^2}$	$\frac{4}{3h^2}  1 + \frac{37h^2}{14c^2}$	$\frac{4}{3h^2}  1 + \frac{433h^2}{77c^2}$
$x_1 + 2h$		$\frac{7}{6c^2}$ 1 $\frac{13h^2}{7c^2}$	$\frac{1}{12h^2}  1 + \frac{74h^2}{7c^2}$	$\frac{1}{12h^2}  1 + \frac{2650h^2}{77c^2}$
$x_1 + 3h$				$\frac{153}{385 c^2}$

Table 2

RBF-FD coe cients for second derivative in the limit ch.

$$\frac{d^2 u}{dx^2}(x_1) = {}_1 u(x_1 \quad h) + {}_2 u(x_1) + {}_3 u(x_1 + h)$$
(12)

The RBF-FD formula can be obtained by substituting function u by multiquadrics radial basis functions centered at  $x_1 = h$ ,  $x_1$ ,  $x_1 + h$ . Solving the resulting linear system leads to

$${}_{1} = {}_{3} = \frac{2 + \frac{h^{2}}{c^{2}} + 2}{4 h^{2} + 2 \frac{1 + 4 \frac{h^{2}}{c^{2}}}{1 + 4 \frac{h^{2}}{c^{2}}} + 5 \frac{h^{2}}{c^{2}} + 2 \frac{h^{4}}{c^{4}}}{4 h^{2} + 1 + \frac{h^{2}}{c^{2}}}$$
$${}_{2} = \frac{2 + \frac{h^{2}}{c^{2}} + 2}{1 + 4 \frac{h^{2}}{c^{2}}} + 3 \frac{h^{2}}{c^{2}}}{2 h^{2} + 1 + \frac{h^{2}}{c^{2}}}$$

In the limit when ch,

$$_{1} = _{3} = \frac{1}{h^{2}} \quad 1 + \frac{h^{2}}{c^{2}} \qquad _{2} = \frac{2}{h^{2}} \quad 1 + \frac{h^{2}}{c^{2}}$$
(13)

which again coincides with the standard central di erence approximation to the second derivative with a correction term of order  $h^2 c^2$ . Table 2 shows the corresponding results for other values of N.

Introducing the values of the coe cients given by (13) into (12), and expanding  $u(x_1 + h)$  and  $u(x_1 - h)$  we discrete corresponding error

$$\hat{\epsilon}_3(x_1) \approx \frac{h^2}{12} u^{(IV)}(x_1) + \frac{h^2}{c^2} u''(x_1) - \frac{3h^2}{4c^4} u(x_1), \qquad (14)$$

where  $\hat{\epsilon}_3(x_1) = \hat{u}''(x_1) - u''(x_1)$  for N = 3. The same error dependence applies for N = 4 ( $\hat{\epsilon}_4(x_1) \approx \hat{\epsilon}_3(x_1)$ ). For N = 5

$$\hat{\epsilon}_{5}(x_{1}) \approx -\frac{h^{4}}{90} u^{(VI)}(x_{1}) - \frac{37 h^{4}}{42 c^{2}} u^{(IV)}(x_{1}) - \frac{85 h^{4}}{14 c^{4}} u''(x_{1}) + \frac{55 h^{4}}{14 c^{6}} u(x_{1}), \qquad (15)$$

and the same dependence is obtained for N = 6 ( $\hat{\epsilon}_6(x_1) \approx \hat{\epsilon}_5(x_1)$ ).



Fig. 3. Error in approximation of second derivative with RBF-FD formula with five equispaced nodes. Left: c/h dependence for h = 0.04. Right: h dependence for c = 0.1. Dot-dashed line equation (15). Dashed lines: each of the terms in equation (15).

In general, the error associated to the RBF-FD formulas using N nodes can be written as

$$\epsilon_N(x_1) \approx h^{N+k-2} \sum_{m=0}^{(N+k)/2} \frac{A_m}{c^{2m}} u^{(N+k-2m)}(x_1)$$
 (16)

where  $A_m$  are constants which depend on N, and k = 0 if N even and k = 1 if N odd. There are additional terms not included in this formula which are  $O(h^{N+2k-1})$ .

As an example, Figure 3 shows the error in the approximation of the second derivative of  $u = \exp(x^2)$  at x = 1 using the RBF-FD formula for ve equispaced nodes and compares it to the error given by equation (15). Again, the error predicted by the equation (in dot-dashed lines) closely agrees with the actual numerical error (in solid lines) until a critical value of the shape parameter is reached above which the system becomes ill-conditioned. Also shown in the left side of the gure (thin dashed lines), are the contributions of each one of the four terms appearing in equation (15) to the total error  $_5(x_1)$ . For the smaller values of c h the contribution of the last term is dominant and, therefore, the error decreases as  $(1 \ c)^6$ . For larger values of c h the contribution of the rst term is dominant and, therefore,  $_5(x_1)$  approaches a constant (this is not observed in the numerical results because those large values of c h lie in the ill-conditioned region). For intermediate values of c h, there is a region where the second term is dominant and the error decreases as  $(1 \ c)^2$ .

## 4 Two-dimensional RBF-FD Formulas

In this section we use the same procedure of the previous Section to derive RBF-FD formulas for the Laplacian. We compute the limit of these formulas for c h, and perform a Taylor expansion of the error in powers of h.

## 4.1 Laplacian

To compute the errors for the RBF-FD formulas of the Laplacian we can proceed as in the previous section by computing the exact values of the coe cients with a symbolic program (Mathematica) and using these values to perform a Taylor series expansion for the corresponding errors. We take the nodes from a regular, equispaced grid, following the same order convention used in reference [27] which is shown in Figure 4.

However, this procedure is only possible for a small number of nodes for which Mathematica is able to calculate the solution. For instance, in the case of the RBF-FD formula for ve nodes, the coe cients in the limit c h are

$$_{0} = \frac{4}{h^{2}} = \frac{10}{3c^{2}}$$
  $_{i} = \frac{1}{h^{2}} + \frac{5}{6c^{2}}$   $_{i} = 1$  4

and the error of the approximation is given by



Fig. 4. Order of nodes in equispaced stencil.

$$5 \qquad \frac{h^2}{12} \quad u^{(4\ 0)}(\mathbf{x}_1) + u^{(0\ 4)}(\mathbf{x}_1) + \frac{5\ h^2}{6\ c^2} \quad u^{(2\ 0)}(\mathbf{x}_1) + u^{(0\ 2)}(x_1) \\ \qquad \frac{7\ h^2}{6\ c^4}\ u(\mathbf{x}_1) \quad (17)$$

where  $u^{(m\,n)}$  denotes the partial derivative of function u with respect to x, m times and respect to y, n times. For six nodes there is an additional coe cient whose value is  $_{5} = (16 \ 3) h^{2} c^{4}$ . This results in a small change of the error, so that

6 5 
$$\frac{16 h^4}{3 c^4} u^{(1 1)}(\mathbf{x}_1)$$

Analogously, for N = 7 nodes,

$${}_{0} = \frac{4}{h^{2}} = \frac{6}{c^{2}} = {}_{1} = \frac{1}{h^{2}} + \frac{5}{6c^{2}} = {}_{2} = \frac{1}{h^{2}} + \frac{7}{2c^{2}} = {}_{3} = {}_{4} = \frac{1}{h^{2}} + \frac{13}{6c^{2}} = {}_{5} = {}_{6} = -\frac{4}{3c^{2}}$$

and the error is

7 5 
$$\frac{4h^3}{3c^2}u^{(1\,2)}(\mathbf{x}_1) = \frac{4h^3}{3c^4}u^{(1\,0)}(\mathbf{x}_1) = \frac{2h^4}{3c^2}u^{(2\,2)}(\mathbf{x}_1)$$

For N > 7 the computational requirements to obtain closed form solutions for the coe cients and for the error using Mathematica are too high. However, it is possible to derive numerically the dependence of the error with h, c and with the partial derivatives of the function by choosing appropriately the function to approximate. For instance, to determine the coe cient of  $u^{(1\ 2)}$  in the Laplacian with N nodes, one can use the corresponding RBF-FD formula to compute numerically the Laplacian of  $u(\mathbf{x}) = x y^2$  at  $\mathbf{x}_1 = (0\ 0)$  for di erent values of h and c. Fitting the results to a power dependence with h and c determines the exact form of the coe cient of  $u^{(1\ 2)}$ . In this way, we derive formulas for the error for any number of nodes N.

The more interesting results are those for N = 9 and N = 13 since then the symmetries with respect to x and y increase the accuracy of the approximation. For these particular values,

$${}_{9} \qquad \frac{1}{12} \quad u^{(4\ 0)}(\mathbf{x}_{1}) + u^{(0\ 4)}(\mathbf{x}_{1}) \qquad \frac{1}{5} \quad u^{(2\ 2)}(\mathbf{x}_{1}) \quad h^{2} + + 0 \quad 47 \quad u^{(2\ 0)}(\mathbf{x}_{1}) + u^{(0\ 2)}(x_{1}) \quad \frac{h^{2}}{c^{2}} \qquad \frac{2\ h^{2}}{3\ c^{4}} \quad u(\mathbf{x}_{1})$$
(18)

and

$$\begin{array}{rcl} {}_{13} & \frac{1}{90} & u^{(6\ 0)}(\mathbf{x}_1) + u^{(0\ 6)}(\mathbf{x}_1) & h^4 \\ & 0\ 93\ u^{(4\ 0)}(\mathbf{x}_1) & 0\ 5\ u^{(2\ 2)}(x_1) + 0\ 93\ u^{(0\ 4)}(\mathbf{x}_1) & \frac{h^4}{c^2} \\ & 4\ 4\ u^{(2\ 0)}(\mathbf{x}_1) + u^{(0\ 2)}(\mathbf{x}_1) & \frac{h^4}{c^4} + 5\ 2\ u(\mathbf{x}_1)\frac{h^4}{c^6} \end{array}$$
(19)

The error dependence with h and c of the RBF-FD approximation of the Laplacian with 5-8 nodes is identical to leading order. The nine nodes formula also has the same dependence  $O(h^2)$  although the coe cients are di erent. For thirteen nodes the error dependence is  $O(h^4)$  (see Section 4.2). For h = c the general behavior of the error dependence of the N nodes RBF-FD formula for the Laplacian is

$${}_{N}(\mathbf{x}_{1}) \qquad h^{p} \sum_{\substack{m=0 \ r=0}}^{p\ 2+1\ p\ 2+1\ m} \frac{A_{m\ r}}{c^{2\ m}} u^{(p+2\ 2\ (m+r)\ 2r)}(\mathbf{x}_{1})$$
(20)

where  $A_{m\,r}$  are constants which depend on N, and p is the smallest even number that satis es

$$(p \quad 1)^2 + 4 \quad N \quad (p+1)^2 + 3$$



Fig. 5. Relative error in the approximation of the Laplacian with RBF-FD formula as a function of the number of nodes N.  $\mathbf{x}_1 = (0.1, 0.2), c = 0.2$ . Left; h = 0.025. Right; h = 0.01.

4.2 Numerical experiments

In this Section we carry out numerical experiments to compute the error of the RBF-FD formulas for the Laplacian, and use them to check the analytical results derived in the previous section. As a first experiment we use the same functions analyzed in reference [6]. Figure 5 shows the dependence of the relative error at the point  $\mathbf{x}_1 = (0.1, 0.2)$  with the number of nodes for the case c = 0.2 and  $h_1 = 0.025$  (left side) and  $h_2 = 0.01$  (right side) ( $h_1$  and c are the parameters chosen in Figure 5 of reference [6]). The functions and the corresponding symbols used are

$$\begin{split} \Box &\to u_1 = \frac{3}{4} \exp\left(-\frac{(9\,x-2)^2 + (9\,y-2)^2}{4}\right) + \frac{3}{4} \exp\left(-\frac{(9\,x+1)^2}{49} - \frac{(9\,y+1)}{10}\right) \\ &\quad \frac{1}{2} \exp\left(-\frac{(9\,x-7)^2 + (9\,y-3)^2}{4}\right) - \frac{2}{10} \exp\left(-(9\,x-4)^2 - (9\,x-7)^2\right), \\ \triangle &\to u_2 = \left(1 - \frac{x}{2}\right)^6 \left(1 - \frac{y}{2}\right)^6 + 1000 \left(1 - x\right)^3 x^3 \left(1 - y\right)^3 y^3 + y^6 \left(1 - \frac{x}{2}\right)^6 + x^6 \left(1 - \frac{y}{2}\right)^6, \\ \nabla &\to u_3 = \sin\left(\pi x\right) \sin\left(\pi y\right), \\ \diamond &\to u_4 = x^2 + y^2. \end{split}$$

For clarity of the figure we do not include the results of the analytical expressions of the error, but it should be remarked that they closely agree with the numerical results. Notice the existence of plateaus where the errors are approximately constant separated by transition regions where the errors decrease rapidly. This is the same behavior shown in Figure 5 of reference [6], although both results are not identical. In fact, Figure 5 shows the relative error in approximating the Laplacian with RBF-FD formulas at a specific location  $\mathbf{x}_1$ , while Figure 5 of reference [6] shows the infinity norm of the relative error in the solution of Poisson equation with the local RBF method.



Fig. 6. Relative error in the approximation of the Laplacian with RBF-FD formula as a function of h.  $\mathbf{x}_1 = (0.1, 0.2), c = 0.2$ . N = 5 - 9 dot-dashed, N = 10 - 12 solid, N = 13 - 25 dotted, N = 26 - 28 dashed, N = 29 - 33 dotted.

The behavior observed in Figure 5 can be better understood by considering the error dependence with h shown in Figure 6. This figure is similar to Figure 7 of reference [6], and shows very similar behavior. Notice that if  $h \ll c$ , the error is  $O(h^2)$  for N = 5-12,  $O(h^4)$  for N = 13-28,  $O(h^6)$  for N = 29-33. This is the expected error dependence according to equation (20). If h = O(c) terms of higher order in h which are neglected in equation (20) become important and introduce a correction in the results. This is the reason why plateaus in Figure 5 are much more constant for  $h = h_2$  than for  $h = h_1$ . Notice that for  $(p + 1)^2 + 1 \leq N \leq (p + 1)^2 + 3$  the correction is not negligible and the error formula (20) is not valid. This is due to the fact that the layout symmetry is lost along the x and y axes.

Finally, to analyze the dependence of the error with shape parameter c, we consider the function

$$u(\mathbf{x}) = \exp\left[-\left(x - \frac{1}{4}\right)^2 - \left(y - \frac{1}{2}\right)^2\right] \cos(2\pi y) \sin(\pi x), \quad (21)$$

which was used by Wright and Fornberg [27] in their analysis of the solution of elliptic PDEs with RBF-FD and RBF-HFD formulas.



Fig. 7. Error in approximation of Laplacian at  $\mathbf{x}_1 = (0, 0)$  with N = 5 RBF-FD formula as a function of the inverse of the shape parameter 1/c. From top to bottom, h = 0.2, h = 0.1, h = 0.05, h = 0.02, h = 0.01, h = 0.005. •; numerical results. •; equation(17).

Figure 7 shows the error as a function of c for different values of h. It is equivalent to Figure 2 of reference [27] and shows a very similar behavior. As before, it should be remarked that both results should not be identical since Figure 7 shows the error in approximating the Laplacian with RBF-FD formulas at a specific location  $\mathbf{x}_1$ , while Figure 2 of reference [27] shows the infinity norm of the error in the solution of Poisson equation with the local RBF method. Notice that there is a value of the shape parameter for which the error is minimum. This value is approximately constant except for large values of h. Notice also that for c large and h small the resulting linear system becomes ill-conditioned and rounding error deteriorates the accuracy of the solution.

## 5 Optimal Shape Parameter

Several observations regarding the dependence of the error of the RBF-FD formulas with respect to shape parameter c are readily apparent from Figures 1 to 3 and 7:

• The error decreases with increasing c as some power which depends on the value of c/h.

For large values of c, the conventional nite di erence formulas are recovered as it was shown in [7,10], and the error approaches a constant value which is the error of conventional nite di erences.

There is a range of values of c for which the error of the RBF-FD formulas is smaller than the error of conventional nite di erences.

There is an *optimal* value of the shape parameter for which the error is minimum.

Notice also that the optimal c is either a value for which  $d_N dc$  is zero (Figure 2) or a value at which  $_N = 0$  (Figures 1, 3 and 7).

Since we have derived closed form expressions for the error of RBF-FD formulas, it is possible to compute in each case the optimal shape parameter cprovided that the value of the function and its derivatives are known. Equations (10), (16) and (20) have the general form

$$_{N}(x_{1}) \qquad h^{p} \prod_{m=0}^{M} \frac{a_{m}(x_{1})}{c^{2m}}$$
 (22)

where  $a_m$  are constants which depend on the derivatives and values of the particular function at  $x_1$ . Denoting z = 1  $c^2$ , the optimal shape parameter is obtained from the positive real roots of the polynomials

$$a_1 + 2a_2z + + Ma_Mz^{M-1} = 0 (23)$$

which implies  $d_N dc = 0$ , or

$$a_0 + a_1 z + a_2 z^2 + \dots + a_M z^M = 0 (24)$$

which implies N = 0. Solution of these two polynomials results in 2M = 1 roots for z = 1  $c^2$ . It is important to remark that the optimal shape parameter c only depends on the value of the function and its derivatives at the node. Therefore, to rst order, it is independent of the mesh size h. For larger values of h there is a correction term of order O(h).

For instance, let us consider the RBF-FD approximation of the second derivative of  $u = \exp(x^2)$  at x = 1 with ve equispaced nodes, which is shown in Figure 3. The coe cients of the polynomials are given by (15), so that

$$a_{0} = \frac{1}{90} u^{(VI)}(x_{1}) \qquad a_{1} = \frac{37}{42} u^{(IV)}(x_{1})$$
$$a_{2} = \frac{85}{14} u(x_{1}) \qquad a_{3} = \frac{55}{14} u(x_{1})$$

In this case, the two roots of the rst polynomial (23) are complex (1 03 0 658 *i*), and the three roots of the second polynomial (24) are two complex (1 482 1 383 *i*) and one real, z = 0 1266. Thus, the optimal shape parameter is c = 1 0 1266 = 2 81, which is shown as a vertical dash-dotted line in Figure 3. The optimal shape parameters for the rst derivative using three and four nodes can be analogously computed (c = 1 2247 and c = 0 8666, respectively) and are also shown with vertical lines in Figures 1 and 2.

In the case of the N = 5 8 Laplacian RBF-FD formula,

$$a_{0} = \frac{1}{12} \quad u^{(4\ 0)}(\mathbf{x}_{1}) + u^{(0\ 4)}(\mathbf{x}_{1}) \qquad a_{1} = \frac{5}{6} \quad u^{(2\ 0)}(\mathbf{x}_{1}) + u^{(0\ 2)}(x_{1})$$
$$a_{2} = -\frac{7}{6}u(\mathbf{x}_{1})$$

The solution of equations (23) and (24) in terms of the derivatives are,

$$(c)^2 = \frac{14\,u(\mathbf{x}_1)}{5\,d_2} \tag{25}$$

$$(c)^{2} = \frac{14 u(\mathbf{x}_{1})}{5 d_{2}} \frac{14 u(\mathbf{x}_{1})}{25 d_{2}^{2} + 14 d_{4} u(\mathbf{x}_{1})}$$
(26)

where,

$$d_2 = u^{(2\ 0)}(\mathbf{x}_1) + u^{(0\ 2)}(x_1) \qquad d_4 = u^{(4\ 0)}(\mathbf{x}_1) + u^{(0\ 4)}(\mathbf{x}_1)$$

Thus, for the ve nodes RBF-FD approximation to the Laplacian of function (21) at  $\mathbf{x}_1 = (0 \ 0)$ ,

$$a_0 = \frac{1}{12} (71\ 8014) = 5\ 9835$$
  $a_1 = \frac{5}{6} 2\ 2984 = 1\ 9153$   $a_2 = 0$ 

Solution of equation (24) gives  $z = a_0 a_1 = 3$  1239, and the optimal shape parameter is, therefore 1  $c = \overline{31239} = 17675$  which is shown in Figure 7.

As a last example we consider the function

$$u(\mathbf{x}) = \frac{25}{25 + (x \quad 0 \ 2)^2 + 2 \ y^2}$$

which is the solution of the problem described in Section 5.2 of reference [27].



Fig. 8. Error in approximation of Laplacian at  $\mathbf{x}_1 = (0, 0)$  with N = 9 RBF-FD formula as a function of the inverse of the shape parameter 1/c. h = 0.1. •; numerical results.  $\circ$ ; equation(18).

Figure 8 shows the error in approximation of the Laplacian with the nine nodes RBF-FD formula as a function of the inverse of the shape parameter. This dependence is very similar to that observed in Figure 4 of reference [27]. Also shown is the optimal value of the shape parameter which, using equation (18), results in  $1/c^* = \sqrt{0.0685} = 0.2617$ .

The problem of how to select appropriate values for the shape parameter has been of primary concern both from the theoretical and from the applications point of view. For the global RBF method [14,15], it has been often assumed that the value of the shape parameter c should vary linearly with node spacing h. For instance, for interpolation problems, Hardy [12] suggests the use of c = 0.815 d, where d is the average distance to the nearest neighborst nei bor (d = h for equispaced nodes). Franke [11] on the other hand recommends  $c = 1.25 D/\sqrt{N}$ , where D is the diameter of the smallest circle containing all data points ( $c = 1.25\sqrt{2}h$  for equispaced nodes). Other authors proposed techniques to select good values of the shape parameter [2,17,20]. With regards to the solution of PDEs, the work of Huang et.al. [13] using arbitrary precision computations, is of particular relevance. From their numerical results they derive a formula for the error dependence on shape parameter cand nodal spacing h. From this formula they obtain the optimal value of the shape parameter that minimizes the error;  $c = -\log \lambda/(3 a h)$ , where a and  $\lambda$ are constants that depend of the problem.

However, our results show that, at least for the local RBF method, the value of c is independent of h. Nevertheless, it should be pointed out that, in practical applications, the node density is often increased (h decreased) in regions where the solution varies rapidly. In these boundary layer type regions, the solution varies in small characteristic lengths  $(l \ L)$ . Thus,  $d_2 = O(L \ l)^2$ ,  $d_4 = O(L \ l)^4$  and, therefore, from (25)-(26) the optimal shape parameter is  $c = O(l \ L)$ . In those regions, therefore, the shape parameter should be taken small not because h is small, but because the solution varies rapidly.

# 6 Unstructured nodes

In this Section we extend our results to the relevant case of unstructured nodes. For instance, in the case of three non-equispaced nodes  $[x_1 \quad h \quad x_1 \quad x_1 + \quad h]$ , the coe cients of the RBF-FD formula for the rst derivative in the limit  $c \quad h$  are

$${}_{1} = \frac{1}{h(1+)} {}^{1} + \frac{1}{2} \frac{h^{2}}{c^{2}} {}_{2} = \frac{1}{h} {}^{1} + \frac{1}{2} \frac{h^{2}}{c^{2}} {}_{3} = \frac{1}{h(1+)} {}^{1} + \frac{1}{2} \frac{h^{2}}{c^{2}}$$
(27)

which coincides with the standard 3-node nite di erence approximation to the rst derivative with a correction term of order  $h^2 c^2$ . Also notice that for = 1 we recover the results of equation (5). The corresponding error of approximation is

$${}_{3}(x_{1}) \qquad \frac{h^{2}}{6} u (x_{1}) + \frac{h^{2}}{2 c^{2}} u (x_{1}) + (1) \frac{h^{3}}{24} u^{(IV)}(x_{1}) + (1) \frac{h^{3}}{8 c^{4}} u(x_{1})$$
(28)

which coincides with equation (6) for = 1. Similarly, for N = 4 nodes  $[x_1 \quad h \quad x_1 \quad x_1 + \ _1h \quad x_1 + (\ _1 + \ _2)h]$  the error is

$${}_{4}(x_{1}) = {}_{1}\left( {}_{1}+{}_{2}\right) \frac{h^{3}}{24} u^{(IV)}(x_{1}) = {}_{1}\left( {}_{1}+{}_{2}\right) \frac{h^{3}}{2c^{2}} u(x_{1}) + \\ + {}_{3}{}_{1}\left( {}_{1}+{}_{2}\right) \frac{h^{3}}{8c^{4}} u(x_{1})$$
(29)

which coincides with equation (7) for  $_1 = 1, _2 = 1$ .



Fig. 9. Error in approximation of first derivative with RBF-FD formula for three non-equispaced nodes as a function of c/h.  $u(x) = \exp(-x^2)$ , x = 1, h = 0.01.

Figure 9 shows the error as a function of c/h in the approximation of the first derivative with RBF-FD formula corresponding to three non-equispaced nodes. The results correspond to the numerical solution. The analytical results corresponding to equation (28) are not shown for clarity of the figure but they coincide with the numerical results. It can be observed that for large values of c/h the error of standard finite difference formulas is recovered. For smaller values of c/h the error decreases as  $(h/c)^2$ . Notice also that the optimal value of the shape parameter is independent of  $\lambda$ . This is to be expected since all the terms of order  $h^2$  in equation (28) contain the factor  $\lambda$ , and therefore this factor disappears when equating the error to zero. Similarly, the optimal value of the shape parameter in the case of four non-equispaced nodes is also independent of  $\lambda$  since all the terms of order  $h^3$  in equation (29) contain the factor  $\lambda_1$  ( $\lambda_1 + \lambda_2$ ).

In the case of the second derivative, the coefficients of the RBF-FD formula in the limit  $c \gg h$  using three non-equispaced nodes  $[x_1 - h, x_1, x_1 + \lambda h]$  are

$$\alpha_{1} = \frac{2}{h^{2} (1+\lambda)} \left( 1 - \frac{\lambda (\lambda^{2} + \lambda)}{2} \frac{h^{2}}{c^{2}} \right),$$

$$\alpha_{2} = -\frac{2}{h^{2} \lambda} \left( 1 - \frac{\lambda^{2} - 4\lambda + 1}{2\lambda} \frac{h^{2}}{c^{2}} \right),$$

$$\alpha_{3} = \frac{2}{h^{2} \lambda (1+\lambda)} \left( 1 + \frac{3\lambda - 1}{2} \frac{h^{2}}{c^{2}} \right).$$
(30)

The corresponding formula for the approximation error is

$$\epsilon_{3}(x_{1}) \approx \frac{\lambda - 1}{3} h \, u'''(x_{1}) + (\lambda - 1) \, \frac{h}{c^{2}} \, u'(x_{1}) + \\ + \left[\lambda \, (\lambda - 1) + 1\right] \, \frac{h^{2}}{12} \, u^{(IV)}(x_{1}) + \lambda \frac{h^{2}}{c^{2}} \, u''(x_{1}) + \left[\lambda \, (\lambda - 5) + 1\right] \, \frac{h^{2}}{4 \, c^{4}} \, u(x_{1}) \,. \tag{31}$$

This formula coincide with equation (14) when  $\lambda = 1$ . Notice that if  $\lambda \neq 1$  the dependence of the error with h is only first order. Notice also that, to first order, the optimal value of the shape parameter,  $c^*$ , is independent of  $\lambda$  since all the terms of order h in equation (31) contain the same factor  $(\lambda - 1)$ .

Similar formulas can be derived for approximating the first and second derivatives with more nodes. For instance, the 4-node RBF-FD approximation to the second derivative using nodes  $[x_1 - h, x_1, x_1 + \lambda_1 h, x_1 + (\lambda_1 + \lambda_2)h]$  is

$$\epsilon_4(x_1) \approx \left[\lambda_2 - \lambda_1 \left(\lambda_1 + \lambda_2 - 2\right)\right] \left[\frac{h^2}{12} u^{(IV)}(x_1) + \frac{h^2}{c^2} u''(x_1) - \frac{3h^2}{4c^4} u(x_1)\right].$$
(32)

As in previous cases the value of  $c^*$  is independent of the location of the nodes and of the local distance h. In 1D this result is general.



Fig. 10. Optimal value of the shape parameter in the approximation of the Laplacian of function (21) at  $\mathbf{x}_1 = (0, 0)$  with N = 5 non-equispaced RBF-FD formula.

However, in 2D the value of  $c^*$  depends on the location of the nodes in the stencil but not on the nodal distance h. Consider for instance the equispaced 5-node stencil approximation of the Laplacian, in which we move the location of one node. Thus, the coordinates of the five nodes are  $[(x_1, y_1), (x_1, y_1 +$ 

h)  $(x_1 + h \ y_1)$   $(x_1 \ y_1 \ h)$   $(x_1 \ h \ y_1)]$ . Figure 10 shows the value of c as a function of corresponding to the Laplacian of function (21). For = 1, the value of c for equispaced nodes is recovered ( $c = 1 \ 1 \ 7675 = 0 \ 5658$ ). For other values of the value of c varies continuously. Notice that when  $c \ h$ , the value of c is independent of h. For c = O(h) there are corrections of higher order that come into play. In the case of fully arbitrary nodes the analysis is more complex but can be carried out in the same manner described in Section 4.

To understand the relationship between standard – nite di–erences and RBF-FD formulas, consider the function value at a node  $\mathbf{x}_i$  expressed by a Taylor expansion

$$u(\mathbf{x}_i) = u(\mathbf{x}_1) + u(\mathbf{x}_1) \quad \mathbf{x}_i + \frac{1}{2} \quad {}^{2}u(\mathbf{x}_1) : (\mathbf{x}_i \quad \mathbf{x}_i^T) + e_i$$

where  $\mathbf{x}_i = \mathbf{x}_i$   $\mathbf{x}_1$ , and  $e_i$  is the error in the expansion. Here, we have denoted the matrix scalar product by : . A linear combination with coe cients i i=1 N equals

$$\overset{N}{\underset{i=1}{\overset{i}{=}}} u(\mathbf{x}_{i}) = u(\mathbf{x}_{1}) \overset{N}{\underset{i=1}{\overset{i}{=}}} + u(\mathbf{x}_{1}) \overset{N}{\underset{i=2}{\overset{i}{=}}} + \frac{1}{2} \overset{2}{\overset{2}{\underbrace{u(\mathbf{x}_{1}):}}} \overset{N}{\underset{i=2}{\overset{i}{=}}} (\mathbf{x}_{i} \mathbf{x}_{i}^{T}) + \overset{N}{\underset{i=2}{\overset{i}{=}}} i e_{i}$$

This FD formula approximates the Laplacian to rst order exactly (i.e.  $\sum_{i=1}^{N} u(\mathbf{x}_i) = u(\mathbf{x}_1)$ ) for constant, linear and quadratic functions, provided that the coe cients satisfy the following conditions:

$$\sum_{i=1}^{N} \sum_{i=2}^{N} \sum_{i=2}^{N} \sum_{i=2}^{N} \left( \mathbf{x}_{i} \quad \mathbf{x}_{i}^{T} \right)_{i} = 2I$$

$$(33)$$

These are a total of 6 conditions which have to be satis ed for the approximation to be *consistent* [23]. In matrix form, with  $\mathbf{x}_i = (x_i \ y_i)$ , we can write (33) as

1	1	1	1	1		0
0	$x_2$	$x_3$	$x_N$	2		0
0	$y_2$	$y_3$	$y_N$	3	_	0
0	$x_2y_2$	$x_3y_3$	$x_N y_N$	4	_	0
0	$x_{2}^{2}$	$x_{3}^{2}$	$x_N^2$			2
0	$y_2^2$	$y_{3}^{2}$	$y_N^2$	N		2

Thus, if six nodes are used in the stencil and the matrix has full rank, there is a unique set of coe cients  $_i$  that satisfy the constraints (33). If N < 6 there is no solution and if N > 6 there are in nitely many solutions. In this case, a unique set of coe cients can be derived, for instance, by the generalized nite di erence method (GFDM) [19] or by moving least squares methods [16]. If Taylor series is carried out until next order and the FD formula is required to be exact also for cubic functions, then four additional constraints have to be satis ed (corresponding to the coe cients of  $u^{(3 \ 0)} u^{(0 \ 3)} u^{(2 \ 1)} u^{(1 \ 2)}$ ). Thus, a unique solution will exist for N = 10. In general, if the system is full rank a unique solution of order p exists for N = (p+2)(p+3) - 2 (so called triangle numbers).

With RBF-FD this limitation does not exist. In fact, adding a new node to an existing stencil also adds the corresponding RBF to the basis of the functional space. Therefore the matrix associated to system (2) is always square and, provided it is of full rank, it has a unique solution. For values of N for which the standard nite di erence formulation has a unique solution, the coe cients of RBF-FD in the limit care identical to the coe cients of standard nite di erences. Thus, the order of RBF-FD formulas coincide with the order of the corresponding nite di erence formulas (order 1 for 6 N9, order 2 for 1014, order 3 for 15 NN20).

Consider, for instance, the case of 6 nodes. Following the same procedure described in Section 4, the error of approximation in the limit c h can be expressed as

$${}_{6} {}_{6} {}_{h} {}_{A_{0\,0}} u^{(3\,0)}(\mathbf{x}_{1}) + A_{0\,1} u^{(2\,1)}(\mathbf{x}_{1}) + A_{0\,2} u^{(1\,2)}(\mathbf{x}_{1}) + + A_{0\,3} u^{(0\,3)}(\mathbf{x}_{1}) + \frac{h}{c^{2}} {}_{A_{1\,0}} u^{(1\,0)}(\mathbf{x}_{1}) + A_{1\,1} u^{(0\,1)}(x_{1})$$
(34)

where the coe cients  $A_{ij}$  are constants which can be computed for a given node distribution. Notice that the error is of order h, like with standard nite di erences. The coe cients  $A_{0i}$  satisfy the compatibility constraints (33) and therefore coincide with the coe cients of the standard 6-node FD formula.



Fig. 11. Approximation error for Laplacian of function (21) at  $\mathbf{x}_1 = (0, 0)$  with N = 6 non-equispaced RBF-FD formula.  $\mathbf{x} [(0,0), (-1.17h, 0.72h), (-0.82h, -1.21h), (0.4h, -0.5h), (1.16h, 0.28h), (0, 1.19h)]$ . Left: dependence with c/h (h = 0.01). Right: dependence with h (c = 2). Solid line; numerical results. Dot-dashed line: equation (34). Dashed line: finite differences. Dotted line: optimal value  $c^*$ .

Figure 11 compares the analytical approximation of the error given by equation (34) (dot-dashed line) with the actual numerical error (solid line) for a specific node distribution shown in the right side of the figure. Similarly to what was observed for the case of equispaced nodes (Figure 7) there is an optimal value of the shape parameter ( $c^* = 0.1754$ ) for which the error becomes zero. This value is shown by a dotted line in the left side of the figure. To the left of that minimum the error decreases as  $c^{-2}$  and to the right of that minimum it approaches the error corresponding to standard finite differences. As before, the value of  $c^*$  is simply obtained by equating to zero equation (34).

The right side of Figure 11 compares the dependence of the error with h given by equation (34) to the numerically computed dependence. To obtain the numerical dependence with h we use the same distribution shown in the inset of the figure but vary its scale with h. Both results are in good agreement until the onset of ill-conditioning and show that the error reduction is O(h).

Similar results can be obtained for any value of N. In fact, it is possible to derive a general formula for the error in approximating the Laplacian with N non-equispaced nodes in the limit  $c \gg h$ . This formula is the analogous of equation (20) for unstructured grids;

where k = 0 if p even and k = 1 if p odd, and

$$\frac{(p+2)(p+3)}{2} \qquad N < \frac{(p+3)(p+4)}{2}$$

Equation (35) has the same form than equation (22) and, therefore, the procedure described in Section 5 for computing the optimal value of the shape parameter is also applicable to the case of non-equispaced nodes.

## 7 Conclusions

We have derived series solutions in powers of the shape parameter c, and nodal distance h, for the error in approximating di erential operators with RBF-FD formulas at a certain location  $\mathbf{x}_1$ . The main conclusions of our work are the following:

RBF-FD formulas approach conventional nite di erence formulas in the limit of in nitely at basis functions  $(c \ h)$ .

For each formula, there is a range of values of the shape parameter for which RBF-FD formulas are signi cantly more accurate than the corresponding conventional nite di erence formulas.

In the case of equispaced nodes, Ding *et.al.* [6] concluded that the error dependence with c and h of the local multiquadric-based di erential quadrature (LMQDQ) method for the Laplacian is  $_{N} = O(h c)^{n}$ , with n = 1.9 for 6 = N = 9, n = 3.6 for 9 < N = 27, n = 4.9 for 27 < N = 34. However, we nd that  $_{N} = O(h^{p} c^{q})$ , where p is only a function of N, and q is a function of N, h, and the value of the function and its derivatives at  $\mathbf{x}_{1}$ .

For equispaced nodes; p = 2 for 5 N 12, p = 4 for 13 N 28, and p = 6 for 29 N 52, ....

For non-equispaced nodes; p = 1 for 6 N 9, p = 2 for 10 N 14, p = 3 for 15 N 20, ....

There are specic values of N for which the error is signicantly smaller than the error for N = 1. These values should be used in practical applications. For equispaced nodes;  $N = (p - 1)^2 + 4$ , where the order p is any even number. For non-equispaced nodes; N = (p + 2) (p + 3) 2 where the order p is any integer.

For each RBF-FD formula there is an optimal value of the shape parameter, c for which the error is minimum. This value is independent of h and only depends on the value of the function and its derivatives at  $\mathbf{x}_1$ .

It should be pointed out, that in order to use the optimal value of the shape parameter at each location, it is necessary to know the value of the function and its derivatives which, in practical cases, it is not known a priori. However, in the solution of linear elliptic problems with the RBF-FD method, one could rst compute an approximate solution using a constant value of the shape parameter, and then use this approximate solution to compute the optimal value of the shape parameter at each node. With these values a new more accurate solution can be computed applying again the RBF-FD method. In non-linear problems, where some type of iterative procedure is needed, the updating of the shape parameter at each location can be e ciently incorporated into the iterative algorithm.

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