# Capacitance matrix technique for avoiding spurious eigenmodes in the solution of hydrodynamic stability problems by Chebyshev collocation method 

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#### Abstract

We present a simple technique for avoiding physically spurious eigenmodes that often occur in the solution of hydrodynamic stability problems by the Chebyshev collocation method. The method is demonstrated on the solution of the OrrSommerfeld equation for plane Poiseuille flow. Following the standard approach, the original fourth order differential equation is factorised into two second-order equations using a vorticity-type auxiliary variable with unknown boundary values which are then eliminated by a capacitance matrix approach. However the elimination is constrained by the conservation of the structure of matrix eigenvalue problem, it can be done in two basically different ways. A straightforward application of the method results in a couple of physically spurious eigenvalues which are either huge or close to zero depending on the way the vorticity boundary conditions are eliminated. The zero eigenvalues can be shifted to any prescribed value and thus removed by a slight modification of the second approach.


Keywords: spurious eigenvalue; Chebyshev collocation method; hydrodynamic stability

## 1. Introduction

Spectral methods are known to achieve exponential convergence rate [3], which makes them particularly useful for solving numerically demanding differential eigenvalue problems which arise in hydrodynamic stability analysis [12]. Unfortunately, besides providing accurate and efficient solutions for a certain number of leading eigenvalues, spectral methods often produce physically spurious unstable modes, which cannot be removed by increasing the numerical resolution [8]. For detailed discussion of these modes we refer to Boyd [2]. Such physically spurious eigenvalues can appear in all types of spectral methods including Galerkin [16], tau [4] and collocation approximations [3], unless some kind of ad hoc approach is applied to avoid them. In the Galerkin method, spurious eigenvalues can be removed by using the basis functions also as the test functions instead of separate Chebyshev polynomials [17]. A number of approaches avoiding spurious eigenvalues have also been found for the tau method [7, 11, 10]. The same can be achieved also for the collocation (or pseudospectral) method by using two distinct interpolating polynomials [9]. Following the approach of McFadden et al. 11] for the tau method, Huang and Sloan [9]
use a Lagrange interpolating polynomial for second-order terms which is by two orders lower than the Hermite interpolant used for other terms. The choice of the latter polynomial depends on the particular combination of the boundary conditions for the problem to be solved [15, p. 493].

The objective of this paper is to present a simple method avoiding spurious eigenmodes in the Chebyshev collocations method which uses only the Lagrange interpolating polynomial applicable to general boundary conditions. Our approach is based on the capacitance matrix technique which is used to eliminate fictitious boundary conditions for a vorticity-type auxiliary variable. The elimination can be performed in two basically different ways which respectively produce a pair of infinite and zero spurious eigenvalues. The latter can be shifted to any prescribed value by a simple modification of the second approach. The main advantage of our method is not only its simplicity but also applicability to more general problems with complicated boundary conditions.

The paper is organised as follows. In the next section we introduce the OrrSommerfeld problem for plane Poiseuille flow, which is a standard test case for this type of method. Section 3 presents the basics of the Chebyshev collocation method that we use. The elimination of the vorticity boundary conditions, which constitutes the basis of our method, is performed in Sec. 4 Section 5 contains numerical results for the Orr-Sommerfeld problem of plane Poiseuille flow. The paper is concluded by a summary of results in Sec. 6

## 2. Hydrodynamic stability problem

The method will be developed by considering the standard hydrodynamic stability problem of plane Poiseuille flow of an incompressible liquid with density $\rho$ and kinematic viscosity $\nu$ driven by a constant pressure gradient $\nabla p_{0}=-\boldsymbol{e}_{x} P_{0}$ in the gap between two parallel walls located $z= \pm h$ in the Cartesian system of coordinates with the $x$ and $z$ axes directed streamwise and transverse to the walls, respectively. The velocity distribution $\boldsymbol{v}(\boldsymbol{r}, t)$ is governed by the NavierStokes equation

$$
\begin{equation*}
\partial_{t} \boldsymbol{v}+(\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{v}=-\rho^{-1} \boldsymbol{\nabla} p+\nu \boldsymbol{\nabla}^{2} \boldsymbol{v} \tag{1}
\end{equation*}
$$

and subject to the incompressiblity constraint $\boldsymbol{\nabla} \cdot \boldsymbol{v}=0$. Subsequently, all variables are non-dimensionalised by using $h$ and $h^{2} / \nu$ as the length and time scales, respectively. Note that instead of the commonly used maximum flow velocity, we employ the viscous diffusion speed $\nu / h$ as the characteristic velocity. This non-standard choice will allow us to test our numerical method against the analytical eigenvalue solution for a quiescent liquid.

The problem above admits a rectilinear base flow $\boldsymbol{v}_{0}(z)=\operatorname{Re} \bar{u}(z) \boldsymbol{e}_{x}$, where $\bar{u}(z)=1-z^{2}$ is the parabolic velocity profile and $R e=U_{0} h / \nu$ is the Reynolds number defined in terms of the maximum flow velocity $U_{0}=2 P_{0} h^{2} / \rho \nu$. Stability of this base flow is analysed with respect to small-amplitude perturbations $\boldsymbol{v}_{1}(\boldsymbol{r}, t)$ by searching the velocity as $\boldsymbol{v}=\boldsymbol{v}_{0}+\boldsymbol{v}_{1}$. Since the base flow is invariant in both $t$ and $\boldsymbol{x}=(x, y)$, perturbation can be sought as a Fourier mode

$$
\begin{equation*}
\boldsymbol{v}_{1}(\boldsymbol{r}, t)=\hat{\boldsymbol{v}}(z) \mathrm{e}^{\lambda t+\mathrm{i} \boldsymbol{k} \cdot \boldsymbol{x}}+\text { c.c. } \tag{2}
\end{equation*}
$$

defined by a complex amplitude distribution $\hat{\boldsymbol{v}}(z)$, temporal growth rate $\lambda$ and the wave vector $\boldsymbol{k}=(\alpha, \beta)$. The incompressiblity constraint, which takes the
form $\boldsymbol{D} \cdot \hat{\boldsymbol{v}}=0$, where $\boldsymbol{D} \equiv \boldsymbol{e}_{z} \frac{d}{d z}+\mathrm{i} \boldsymbol{k}$ is a spectral counterpart of the nabla operator, is satisfied by expressing the component of the velocity perturbation in the direction of the wave vector as $\hat{u}_{\|}=\boldsymbol{e}_{\|} \cdot \hat{\boldsymbol{v}}=\mathrm{i} k^{-1} \hat{w}^{\prime}$, where $\boldsymbol{e}_{\|}=\boldsymbol{k} / k$ and $k=|\boldsymbol{k}|$. Taking the curl of the linearised counterpart of Eq. (11) to eliminate the pressure gradient and then projecting it onto $\boldsymbol{e}_{z} \times \boldsymbol{e}_{\| 1}$, after some transformations we obtain the Orr-Sommerfeld equation

$$
\begin{equation*}
\lambda \boldsymbol{D}^{2} \hat{w}=\boldsymbol{D}^{4} \hat{w}+\mathrm{i} \alpha \operatorname{Re}\left(\bar{u}^{\prime \prime}-\bar{u} \boldsymbol{D}^{2}\right) \hat{w} \tag{3}
\end{equation*}
$$

which is written in a non-standard form corresponding to our choice of the characteristic velocity. Note that the Reynolds number appears in this equation as a factor at the convective term rather than its reciprocal at the viscous term as in the standard form. As a result, the growth rate $\lambda$ differs by a factor $R e$ from its standard definition. The same difference, in principle, applies also to the velocity perturbation amplitude which, however, is not important as long as only the linear stability is concerned. In this form, Eq. (3) admits a regular analytical solution at $R e=0$, which is used as a benchmark for the numerical solution in Sec. 5

The no-slip and impermeability boundary conditions require

$$
\begin{equation*}
\hat{w}=\hat{w}^{\prime}=0 \quad \text { at } \quad z= \pm 1 \tag{4}
\end{equation*}
$$

Because three control parameters $R e$ and $(\alpha, \beta)$ appear in Eq. (3) as only two combinations $\alpha R e$ and $\alpha^{2}+\beta^{2}$, solutions for oblique modes with $\beta \neq 0$ are equivalent to the transverse ones with $\beta=0$ and a larger $\alpha$ and, thus, a smaller $R e$ which keep both parameter combinations constant [6]. Therefore, it is sufficient to consider only the transverse perturbations $(k=\alpha)$.

The first step in avoiding spurious eigenvalues in the discretizied version of Eq. (3) to be derived in the following section is to represent Eq. (3) as a system of two second-order equations [8]

$$
\begin{align*}
\lambda \hat{\zeta} & =D^{2} \hat{\zeta}+\mathrm{i} \alpha \operatorname{Re}\left(\bar{u}^{\prime \prime} \hat{w}-\bar{u} \hat{\zeta}\right)  \tag{5}\\
\hat{\zeta} & =D^{2} \hat{w} \tag{6}
\end{align*}
$$

where $\hat{\zeta}$ is a vorticity-type auxiliary variable which has no explicit boundary conditions.

## 3. Chebyshev collocation method

The problem is solved numerically using a collocation method with $N+1$ Chebyshev-Gauss-Lobatto nodes

$$
\begin{equation*}
z_{i}=\cos (i \pi / N), \quad i=0, \cdots, N \tag{7}
\end{equation*}
$$

at which the discretizied solution $(\hat{w}, \hat{\zeta})\left(z_{i}\right)=\left(w_{i}, \zeta_{i}\right)=(\mathbf{w}, \boldsymbol{\zeta})$ and its derivatives are sought. The latter are expressed in terms of the former by using the so-called differentiation matrices, which for the first and second derivatives are denoted by $D_{i, j}^{(1)}$ and $D_{i, j}^{(2)}$ with explicit expressions given in the Appendix. Requiring Eqs. (56) to be satisfied at the internal collocation points $0<i<N$
and the boundary conditions (4) at the boundary points $i=0, N$, the following system of $2 N$ algebraic equations is obtained for the same number of unknowns

$$
\begin{align*}
\lambda \boldsymbol{\zeta}_{0} & =\mathbf{A} \boldsymbol{\zeta}_{0}+\mathbf{B} \zeta_{1}+\mathbf{g}_{0}  \tag{8}\\
\boldsymbol{\zeta}_{0} & =\mathbf{A} \mathbf{w}_{0}  \tag{9}\\
\mathbf{0}_{1} & =\mathbf{C w}_{0} \tag{10}
\end{align*}
$$

where $\mathbf{0}$ is the zero matrix and the subscripts 0 and 1 denote the parts of the solution at the inner and boundary collocation points, respectively; $\mathbf{w}_{1}=\mathbf{0}_{1}$ due to the first boundary condition (4) and

$$
\begin{equation*}
g_{i}=\mathrm{i} \alpha \operatorname{Re}\left(\bar{u}_{i}^{\prime \prime} w_{i}-\bar{u}_{i} \zeta_{i}\right) \tag{11}
\end{equation*}
$$

The matrices

$$
\begin{array}{ll}
A_{i, j}=\left(\boldsymbol{D}^{2}\right)_{i, j}, & 0<(i, j)<N \\
B_{i, j}=\left(\boldsymbol{D}^{2}\right)_{i, j}, & 0<i<N, j=0, N \tag{13}
\end{array}
$$

represent the parts of the collocation approximation of the operator

$$
\begin{equation*}
\left(\boldsymbol{D}^{2}\right)_{i, j}=D_{i, j}^{(2)}-\alpha^{2} I_{i, j} \tag{14}
\end{equation*}
$$

using the inner and boundary points, respectively; $I_{i, j}$ is the unity matrix. Equation (10) is a discretizied version of the second boundary condition (4) imposed on $\hat{w}^{\prime}$ which is defined by the matrix

$$
\begin{equation*}
C_{i j}=D_{i, j}^{(1)}, \quad i=0, N ; 0<j<N \tag{15}
\end{equation*}
$$

Our goal is to reduce Eqs. (8) to the standard matrix eigenvalue problem for $\hat{w}_{0}$. First, $\boldsymbol{\zeta}_{0}$ is eliminated from Eq. (8) by using Eq. (9), which results in

$$
\begin{equation*}
\lambda \mathbf{A} \mathbf{w}_{0}=\mathbf{A}^{2} \mathbf{w}_{0}+\mathbf{B} \zeta_{1}+\mathbf{g}_{0} \tag{16}
\end{equation*}
$$

Next, we can use Eq. (10) to eliminate $\boldsymbol{\zeta}_{1}$ from the equation above. This, as shown in the next section, can be done in two basically different ways.

## 4. Elimination of the vorticity boundary values

In order to eliminate $\boldsymbol{\zeta}_{1}$ from Eq. (16) using Eq. (10) we employ a modified capacitance (or influence) matrix method. For the basics of this method, see [13, p. 178] and references therein. Modifications to the method are due to the structure of the matrix eigenvalue problem which needs to be conserved in the elimination process. The general capacitance matrix approach suggests to express $\mathbf{w}_{0}$ from Eq. (16) and then to substitute it into Eq. (10), which then would result in a system of linear equations for $\zeta_{1}$. However, as noted above, the elimination procedure must be linear in $\lambda$ for the eigenvalue problem structure to be conserved. It means that $\mathbf{w}_{0}$ can be expressed either from the right or left hand side of Eq. (16) but not from the combination of both sides as in the standard capacitance matrix approach for the time stepping schemes.

Our first approach is to express $\mathbf{w}_{0}$ from the r.h.s. of Eq. (16) by inverting $\mathbf{A}^{2}$ and then substituting it into the boundary condition (10), which results in

$$
\begin{equation*}
\mathbf{C w}_{0}=\mathbf{C A}^{-2}\left(\lambda \mathbf{A} \mathbf{w}_{0}-\mathbf{B} \boldsymbol{\zeta}_{1}-\mathbf{g}_{0}\right)=\mathbf{0}_{1} \tag{17}
\end{equation*}
$$

Next, solving the equation above for

$$
\begin{equation*}
\boldsymbol{\zeta}_{1}=\left(\mathbf{C A}^{-2} \mathbf{B}\right)^{-1} \mathbf{C A}^{-2}\left(\lambda \mathbf{A} \mathbf{w}_{0}-\mathbf{g}_{0}\right) \tag{18}
\end{equation*}
$$

and substituting it into Eq. (16), we obtain

$$
\begin{equation*}
\lambda \mathbf{E A} \mathbf{w}_{0}=\left(\mathbf{A}^{2}+\mathbf{E G}\right) \mathbf{w}_{0} \tag{19}
\end{equation*}
$$

where $\mathbf{G w}_{0}=\mathbf{g}_{0}$ and

$$
\begin{equation*}
\mathbf{E}=\mathbf{I}-\mathbf{B}\left(\mathbf{C A}^{-2} \mathbf{B}\right)^{-1} \mathbf{C A}^{-2} \tag{20}
\end{equation*}
$$

It is important to notice that $\mathbf{E B}=0 \mathbf{B}$, which means that $\mathbf{E}$ is singular. Namely, it has a zero eigenvalue of multiplicity two corresponding to two eigenvectors represented by the columns of B. Representing Eq. (19) as

$$
\begin{equation*}
\left(\mathbf{A}^{2}+\mathbf{E G}\right)^{-1} \mathbf{E} \mathbf{A} \mathbf{w}_{0}=\lambda^{-1} \mathbf{w}_{0}, \tag{21}
\end{equation*}
$$

which is a standard eigenvalule problem for $\lambda^{-1}$, it is obvious that zero eigenvalues of $\mathbf{E}$ result in two zero eigenvalues $\lambda^{-1}$, which in turn correspond to infinite eigenvalues $\lambda$ of the original Eq. (19). A way to avoid these spurious eigenvalues is described below.

Alternative approach to eliminate $\boldsymbol{\zeta}_{1}$ is to express $\lambda \mathbf{w}_{0}$ from the l.h.s. of Eq. (16) by inverting $\mathbf{A}$ and then substituting it into the boundary condition (10), which results in

$$
\begin{equation*}
\lambda \mathbf{C} \mathbf{w}_{0}=\mathbf{C A}^{-1}\left(\mathbf{A}^{2} \mathbf{w}_{0}+\mathbf{B} \zeta_{1}+\mathbf{g}_{0}\right)=\mathbf{0}_{1} . \tag{22}
\end{equation*}
$$

This equation can be solved for $\zeta_{1}$ similarly to Eq. (17) as

$$
\begin{equation*}
\boldsymbol{\zeta}_{1}=\left(\mathbf{C A}^{-1} \mathbf{B}\right)^{-1} \mathbf{C A}^{-1}\left(\mathbf{A}^{2}+\mathbf{G}\right) \mathbf{w}_{0} \tag{23}
\end{equation*}
$$

which substituted in Eq. (16) leads to

$$
\begin{equation*}
\lambda \mathbf{A} \mathbf{w}_{0}=\mathbf{F}\left(\mathbf{A}^{2}+\mathbf{G}\right) \mathbf{w}_{0} \tag{24}
\end{equation*}
$$

where the transformation matrix

$$
\begin{equation*}
\mathbf{F}=\mathbf{I}-\mathbf{B}\left(\mathbf{C A}^{-1} \mathbf{B}\right)^{-1} \mathbf{C A}^{-1} \tag{25}
\end{equation*}
$$

is singular with two zero eigenvalues because it satisfies $\mathbf{F B}=0 \mathbf{B}$ similarly to the $\mathbf{E}$ considered above. In contrast to the previous eigenvalue problem defined by Eq. (19), now the singular transformation matrix appears on the r.h.s. of Eq. (24) and thus it produces two zero rather than infinite eigenvalues $\lambda$.

It is important to notice that zero eigenvalues represent an alternative solution to Eq. (22), which can be satisfied not only by the boundary condition (10) but also by $\lambda=0$. Consequently, these spurious eigenvalues can be shifted from zero to any value $\lambda_{0}$ by subtracting $\lambda_{0} \mathbf{C w}_{0}$ from both sides of Eq. (22), which obviously does not affect the true eigenmodes satisfying Eq. (10). As a result we obtain

$$
\begin{equation*}
\boldsymbol{\zeta}_{1}=\left(\mathbf{C A}^{-1} \mathbf{B}\right)^{-1} \mathbf{C A}^{-1}\left(\mathbf{A}\left(\mathbf{A}-\lambda_{0} \mathbf{I}\right)+\mathbf{G}\right) \mathbf{w}_{0} \tag{26}
\end{equation*}
$$

which substituted in Eq. (16) leads to the following standard eigenvalue problem

$$
\begin{equation*}
\lambda \mathbf{w}_{0}=\left(\mathbf{A}^{-1} \mathbf{F}\left(\mathbf{A}\left(\mathbf{A}-\lambda_{0} \mathbf{I}\right)+\mathbf{G}\right)+\lambda_{0} \mathbf{I}\right) \mathbf{w}_{0} \tag{27}
\end{equation*}
$$

The complex matrix eigenvalue problems above are solved using the LAPACK's ZGEEV routine [1].

| $N=8$ | $N=16$ | $N=32$ | Exact |
| :---: | :---: | :---: | :---: |
| $5.4285 \times 10^{16}$ | $1.2597 \times 10^{17}$ | $3.5670 \times 10^{16}$ | - |
| $3.1699 \times 10^{16}$ | $-1.1600 \times 10^{17}$ | $-6.0842 \times 10^{18}$ | - |
| -9.3120595 | -9.3137399 | -9.3137399 | -9.3137399 |
| -20.709030 | -20.570571 | -20.570571 | -20.570571 |
| -39.297828 | -38.947806 | -38.947789 | -38.947789 |
| -66.057825 | -60.054233 | -60.055435 | -60.055435 |
| -73.670710 | -88.285123 | -88.299997 | -88.299997 |
|  | -119.43366 | -119.27480 | -119.27480 |
|  | -157.89593 | -157.38866 | -157.38866 |
|  | -199.64318 | -198.23234 | -198.23234 |
|  | -226.99053 | -246.21576 | -246.21576 |
|  | -384.38914 | -296.92876 | -296.92874 |
|  | -409.06660 | -354.78191 | -354.78176 |
|  | -961.90740 | -415.36266 | -415.36420 |
|  | -961.99676 | -483.07721 | -483.08684 |
|  |  | -553.58796 | -553.53879 |
|  |  | -711.16464 | -711.45255 |
|  |  | $\vdots$ | $\vdots$ |

Table 1: The eigenvalues found numerically by solving Eq. (21) (method I) with various number of collocation points $N$ for $\alpha=1$ and $R e=0$. The exact eigenvalues values are the roots of the characteristic equation resulting from analytical solution of Eq. (3) for $R e=0$.

## 5. Numerical results

In order to validate the approach developed above we start with $R e=0$ for which Eq. (3) can easily be solved analytically leading to the characteristic equation

$$
\begin{equation*}
\frac{\tanh (k)}{\tan \left(\sqrt{k^{2}-\lambda}\right)}= \pm\left(\frac{k}{\sqrt{k^{2}-\lambda}}\right)^{ \pm 1} \tag{28}
\end{equation*}
$$

which defines two branches of eigenvalues $\lambda$ for the even and odd modes corresponding to the plus and minus signs in the above expression. The eigenvalues resulting from Eq. (21), which represents our first approach, are shown in Table 1 for various numbers of collocation points along with the exact solution defined by Eq. (28). As seen, this approach indeed produces a couple of huge spurious eigenvalues, which are due to the singularity of the transformation matrix $\mathbf{E}$ (20) pointed out above. At the same time, the numerical solution accurately reproduces the leading eigenvalues of the exact solution. The accuracy, however, decreases down the spectrum so that only a half of the exact eigenvalues are reproduced by the numerical solution. The other half are numerically spurious eigenvalues which are due to the discretization of the problem [2].

Our second approach defined by Eq. (24) produces exactly the same eigenvalues as the first one for the given $N$ except for the two spurious eigenvalues which are now machine-size zeros rather than infinities. Using the modification of the second approach defined by Eq. (27), these zero eigenvalues can be shifted to any prescribed value $\lambda_{0}$ without affecting other eigenvalues. Further we use $\lambda_{0}=4(N / 4)^{4}$ which shifts the two physically spurious eigenvalues to the region of numerically spurious eigenvalues located in the lower part of spectrum. The


Figure 1: Relative variation of leading eigenvalues with the number of collocation points $N$ for $\alpha=1, R e=0$ and $R e=10^{4}$.

| $N$ | $c$ |
| :---: | :---: |
| 16 | $(0.23272286,0.00922887)$ |
| 20 | $(0.23814366,0.00566303)$ |
| 24 | $(0.23842504,0.00282919)$ |
| 28 | $(0.23735182,0.00357013)$ |
| 32 | $(0.23747200,0.00372519)$ |
| 36 | $(0.23752527,0.00375400)$ |
| 40 | $(0.23752494,0.00373917)$ |
| 44 | $(0.23752716,0.00374012)$ |
| 48 | $(0.23752633,0.00373961)$ |
| 52 | $(0.23752653,0.00373969)$ |
| 56 | $(0.23752648,0.00373967)$ |
| 60 | $(0.23752649,0.00373967)$ |
| 64 | $(0.23752649,0.00373967)$ |

Table 2: Phase velocity $c=-\mathrm{i} \lambda /(\operatorname{Re} k)$ of the most unstable mode depending on the number of collocation points $N$ for $\alpha=1$ and $R e=10^{4}$.
variation of the five leading eigenvalues with the number of collocation points $N$ plotted in Fig. 1 shows an exponential convergence rate characteristic for the spectral numerical methods [3].

Next, we consider the solution of Eq. (27) for $R e=10^{4}$ and $\alpha=1$, which is a standard test case for the linear stability analysis of plane Poiseuille flow. The leading eigenvalue for this case is shown in table 2 in terms of the commonly used phase velocity $c=-\mathrm{i} \lambda / \operatorname{Re} k$. For $N \gtrsim 60$ the solution is seen to converge to the reference value obtained in [12] using a tau method with $M \gtrsim 30$ even Chebyshev polynomials. As seen in Fig. 11 however the convergence rate for $R e=10^{4}$ is somewhat slower than for $R e=0$, it is still exponential with the final accuracy comparable to the previous case.

The number of collocation points can be reduced by a half by considering even and odd modes separately as done in [12]. In our case, this would require substitution of differentiation matrices (29)30) for general functions with their half-size counterparts for even and odd functions, which, however, lies outside the scope of this paper.

## 6. Summary and conclusions

We have developed a simple technique for avoiding physically spurious eigenmodes in the solution of hydrodynamic stability problems by the Chebyshev collocation method, which was demonstrated on the Orr-Sommerfeld equation for plane Poiseuille flow. The method is based on the factorisation of the original fourth order differential equation into two second-order equations using a vorticity-type auxiliary variable which has no explicit boundary conditions. The main element of the method is the elimination of the vorticity boundary values by using a capacitance matrix approach to obtain a standard matrix eigenvalue problem. Although the elimination is constrained by of the structure of eigenvalue problem, it can be still done in two basically different ways. Both approaches result in couple of physically spurious eigenvalues, which are either huge or close to zero depending on the way the vorticity boundary values are eliminated. We showed that these spurious eigenvalues are due to the double singularity of the transformation matrices which eliminate the vorticity boundary conditions by multiplying either the stiffness or mass matrices of the original generalised eigenvalue problem. By a slight modification of the second approach, the zero eigenvalues can be shifted to any prescribed value and thus moved to the region of numerically spurious eigenvalues at the end of spectrum.

The main advantage of our method is not only its simplicity but also its applicability to more general stability problems with complex boundary conditions involving several variables. An example of such a problem is that of 3D linear stability of MHD duct flow using a non-standard vector stream function and vorticity formulation, which results in the coupling of the stream function components through the boundary conditions 14]. In this case neither Galerkin nor collocation method with the ad hoc approach of Huang and Sloan [9] is applicable because no simple basis functions satisfying the boundary conditions can be constructed.

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## Appendix

The differentiation matrices for the first and second derivatives at Chebyshev-Gauss-Lobatto nodes (7) are defined as follows [13, pp. 393-394]:

$$
D_{i, j}^{(1)}= \begin{cases}\frac{2 N^{2}+1}{6} & j=0  \tag{29}\\ \frac{c_{j}}{c_{i}} \frac{(-1)^{i+j}}{\left.c_{j}-z_{i}\right)} & i \neq j \\ -\frac{z_{j}}{2\left(1-z_{j}^{2}\right)} & 0<i=j<N \\ -\frac{2 N^{2}+1}{6} & j=N\end{cases}
$$

and

$$
D_{i, j}^{(2)}=\left(D_{i, j}^{(1)}\right)^{2}= \begin{cases}\frac{(-1)^{i+j}}{c_{j}} \frac{z_{i}^{2}+z_{i} z_{j}-2}{\left(1-z_{j}^{2}\right)\left(z_{i}-z_{j}\right)^{2}} & 0<i \neq j<N  \tag{30}\\ -\frac{\left(N^{2}-1\right)\left(1-z_{i}^{2}\right)+3}{3\left(1-z_{j}^{2}\right)^{2}} & 0<i=j<N \\ \frac{2}{3} \frac{(-1)^{j}}{c_{j}} \frac{\left(2 N^{2}+1\right)\left(1-z_{j}\right)-6}{\left(1-z_{j}\right)^{2}} & j \neq i=0 \\ \frac{2}{3} \frac{(-1)^{j+N}}{c_{j}} \frac{\left(2 N^{2}+1\right)\left(1+z_{j}\right)-6}{\left(1+z_{j}\right)^{2}} & j \neq i=N \\ \frac{N^{4}-1}{15} & i=j=0, N,\end{cases}
$$

where $c_{i}=1$ for $0<i<N$ and $c_{i}=2$ for $i=0, N$.

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